# RANDOMIZED DOUGLAS-RACHFORD METHODS FOR LINEAR SYSTEMS: IMPROVED ACCURACY AND EFFICIENCY 

DEREN HAN, YANSHENG SU, AND JIAXIN XIE


#### Abstract

The Douglas-Rachford (DR) method is a widely used method for finding a point in the intersection of two closed convex sets (feasibility problem). However, the method converges weakly and the associated rate of convergence is hard to analyze in general. In addition, the direct extension of the DR method for solving more-than-twosets feasibility problems, called the $r$-sets-DR method, is not necessarily convergent. To improve the efficiency of the optimization algorithms, the introduction of randomization and the momentum technique has attracted increasing attention. In this paper, we propose the randomized $r$-sets- DR ( RrDR ) method for solving the feasibility problem derived from linear systems, showing the benefit of the randomization as it brings linear convergence in expectation to the otherwise divergent $r$-sets-DR method. Furthermore, the convergence rate does not depend on the dimension of the coefficient matrix. We also study $\operatorname{RrDR}$ with heavy ball momentum and establish its accelerated rate. Numerical experiments are provided to confirm our results and demonstrate the notable improvements in accuracy and efficiency of the DR method, brought by the randomization and the momentum technique.


## 1. Introduction

1.1. Problem setup. Consider the large-scale system of linear equations

$$
\begin{equation*}
A x=b, \tag{1}
\end{equation*}
$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m}$. The problem of solving the linear system (1) arises in various fields of science and engineering, such as optimal control [60], machine learning [19], signal processing [15], and partial differential equations [59]. Throughout this paper, we assume that this linear system is consistent and refer to a certain solution of (1) as $x^{*}$.

Let $a_{1}^{\top}, a_{2}^{\top}, \ldots, a_{m}^{\top}$ denote the rows of $A$ and let $b=\left(b_{1}, \ldots, b_{m}\right)^{\top}$. Then (1) can be rewritten as the following feasibility problem

Find $x^{*} \in C=\bigcap_{i=1}^{m} C_{i}, \quad$ where $C_{i}:=\left\{x:\left\langle a_{i}, x\right\rangle=b_{i}\right\}$.

[^0]A classic approach to solving (2) is the projection method. However, it can be costly to compute the projection onto the intersection $C$. So a more practical strategy is to successively project the current iterate onto a single feasible set $C_{i}$ at each iteration, where $C_{i}$ is chosen in a certain manner. There are numerous such methods, for instance, the Dykstra's method [25], the von Neumann method [5], and the Douglas-Rachford (DR) method [22]. In this paper, we focus on the DR method.
1.2. Douglas-Rachford method. The Douglas-Rachford method [47] is a notable splitting approach for finding zeros of the sum of maximal monotone operators. It has already been applied to various optimization problems whose objective function is a sum of proper closed convex functions; see [2,26,46,47]. Since feasibility problems are special cases where the operators are normal cones, it implies that the DR method can be applied to them. For any closed set $C_{i}$, let $P_{C_{i}}$ denote the orthogonal projection operator onto $C_{i}$ and $R_{C_{i}}:=2 P_{C_{i}}-I$ denote the reflection operator over $C_{i}$, where $I$ denotes the identity operator. Specifically, when $C_{i}=\left\{x:\left\langle a_{i}, x\right\rangle=b_{i}\right\}$ is a hyperplane, for any $x \in \mathbb{R}^{n}$ we have

$$
P_{C_{i}}(x)=x-\frac{\left\langle a_{i}, x\right\rangle-b_{i}}{\left\|a_{i}\right\|_{2}^{2}} a_{i}, \quad \text { and } \quad R_{C_{i}}(x)=x-2 \frac{\left\langle a_{i}, x\right\rangle-b_{i}}{\left\|a_{i}\right\|_{2}^{2}} a_{i} .
$$

The canonical DR method is restricted to dealing with only two sets, namely, finding a feasible point in the intersection of $C_{1}$ and $C_{2}$. Starting from a proper $x^{0}$, it iterates with the format

$$
x^{k+1}=\frac{1}{2}\left(I+R_{C_{2}} R_{C_{1}}\right)\left(x^{k}\right),
$$

which is illustrated in Figure 1. For this reason, the DR scheme is also known as reflect-reflect-average [11] or averaged alternating reflections [6] in the literature.


Figure 1. One step of the Douglas-Rachford method: Reflect-reflectaverage; $x^{k+1}=\frac{1}{2}\left(x^{k}+z^{k}\right)=\frac{1}{2}\left(I+R_{C_{2}} R_{C_{1}}\right)\left(x^{k}\right)$.

However, when it comes to the $m$-sets $(m>2)$ convex feasibility problem, the direct extension of the canonical Douglas-Rachford method $x^{k+1}=\frac{1}{2}\left(I+R_{C_{m}} \cdots R_{C_{2}} R_{C_{1}}\right)\left(x^{k}\right)$ may fail even in simple instances. See an example from Artacho, Borwein, and Tam [2] in Figure 2. To overcome this problem, Borwein and Tam [10] devised the cyclic DR method by using only two sets at a time

$$
\begin{equation*}
x^{k+1}=\frac{1}{2}\left(I+R_{C_{t_{k+1}}} R_{C_{t_{k}}}\right)\left(x^{k}\right), \tag{3}
\end{equation*}
$$

where $t_{k}=(k \bmod m)+1$. This pattern can be extended by employing more sets at each time, and such ideas are detailed as the cyclic r-sets-DR method $[1,17]$. While the conditions for weak convergence of the canonical DR method and the cyclic $r$-sets-DR method have been established, it remains difficult to obtain effective theoretical estimates of their convergence rate. Due to the complexity of the required computations, existing estimates are not comparable to those of other state-of-the-art iterative methods [4, 6, 8 , $14,29,48]$. Given this situation, we intend to introduce modern optimization techniques to improve the properties of the DR method.


Figure 2. Failure of the 3 -sets-DR iteration: The iteration $x^{k}:=\left(\frac{1}{2} I+\right.$ $\left.\frac{1}{2} R_{C_{3}} R_{C_{2}} R_{C_{1}}\right)^{k}\left(x^{0}\right)$ may cycle.
1.3. Our contribution. In this paper, we investigate the DR method with randomization for solving feasibility problems derived from linear systems. The main contributions of this work are as follows.

1. We introduce the randomization technique to the $r$-sets-DR (RrDR) method for solving the feasibility problem (2) and demonstrate that this approach is effective in simplifying the analysis of the $r$-sets-DR method and endows the otherwise divergent $r$-sets-DR method with a linear convergence. Specifically, we prove that the expected
norm of the error $\mathbb{E}\left[\left\|x^{k}-x^{*}\right\|_{2}^{2}\right]$ of $\operatorname{RrDR}$ converges linearly, with the convergence rate depending only on the singular values of $A$ and the relaxation parameter of the algorithm, but not on the size of $A$.
2. We then focus on a variant of the $\operatorname{RrDR}$ method with momentum ( mRrDR ), which is inspired by the success of Polyak's heavy ball momentum method [50, 53, 61]. Although the expected norm of the error $\mathbb{E}\left[\left\|x^{k}-x^{*}\right\|_{2}^{2}\right]$ of mRrDR also converges linearly, we find that its convergence rate is weaker than that of the $\operatorname{RrDR}$ method. Therefore, we also consider the norm of the expected error $\left\|\mathbb{E}\left[x^{k}-x^{*}\right]\right\|_{2}^{2}$, in terms of which mRrDR shows superiority over RrDR. To the best of our knowledge, this is the first study to investigate the momentum variants of the $r$-sets-DR method.
3. Furthermore, we also compare our DR-originated method with other state-of-theart iterative methods for linear systems. Moveover, we even show the superiority of the mRrDR over the built-in function of Matlab pinv and lsqminnorm when the number of the rows of $A$ is sufficiently larger than the number of columns.
1.4. Notations. We use $\mathbb{Z}_{+}$to denote the set of positive integers. For any random variables $\xi$ and $\zeta$, we use $\mathbb{E}[\xi]$ and $\mathbb{E}\left[\xi \mid \zeta=\zeta_{0}\right]$ to denote the expectation of $\xi$ and the conditional expectation of $\xi$ given $\zeta=\zeta_{0}$. For vector $x \in \mathbb{R}^{n}$, we use $x_{i}, x^{\top}$, and $\|x\|_{2}$ to denote the $i$-th entry, the transpose, and the Euclidean norm of $x$, respectively. For matrix $A \in \mathbb{R}^{m \times n}$, we use $a_{i}, A_{j}, A^{\top}, A^{\dagger},\|A\|_{2},\|A\|_{F}, \operatorname{Row}(A)$, and $\operatorname{Range}(A)$ to denote the $i$-th row, $j$-th column, the transpose, the Moore-Penrose pseudoinverse, the spectual norm, the Frobenius norm, the row space, and the range space of $A$, repectively. We use $A=U \Sigma V^{\top}$ to denote the singular value decomposition (SVD) of $A$, where $U \in \mathbb{R}^{m \times m}, \Sigma \in \mathbb{R}^{m \times n}$, and $V \in \mathbb{R}^{n \times n}$. The nonzero singular values of $A$ are $\sigma_{1}(A) \geq \sigma_{2}(A) \geq \ldots \geq \sigma_{t}(A):=\sigma_{\min }(A)>0$, where $t$ is the rank of $A$ and $\sigma_{\min }(A)$ is the smallest nonzero singular value of $A$.

Throughout this paper, we use $x^{*}$ to denote a certain solution of the linear system (1), and for any $x^{0} \in \mathbb{R}^{n}$, we set $x_{0}^{*}:=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}$ and $x_{L N}^{*}:=A^{\dagger} b$. We mention that $x_{0}^{*}$ is the orthogonal projection of $x^{0}$ onto the set $\left\{x \in \mathbb{R}^{n} \mid A x=b\right\}$, and $x_{L N}^{*}$ is the least-norm solution of the linear system.
1.5. Organization. The remainder of the paper is organized as follows. In Section 2 we will give a review of the related work. Section 3 and Section 4 describe randomized $r$-sets-Douglas-Rachford method and its momentum variant, respectively. Section 5 reports the
mentioned numerical experiments and Section 6 concludes the paper. Proofs of all main results are provided in the appendix.

## 2. Related work

One of the most widely used projection solvers for (1) is the Kaczmarz method [42], which can be recognized as a special kind of the Dykstra's method. Starting from $x^{0} \in \mathbb{R}^{n}$, the canonical Kaczmarz method constructs $x^{k+1}$ by

$$
x^{k+1}=P_{C_{i_{k}}}\left(x^{k}\right)=x^{k}-\frac{\left\langle a_{i_{k}}, x^{k}\right\rangle-b_{i_{k}}}{\left\|a_{i_{k}}\right\|_{2}^{2}} a_{i_{k}},
$$

where $i_{k}$ is cyclically selected from $\{1, \cdots, m\}$. The sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ converges to $x_{0}^{*}$ but the convergence rate is hard to obtain. In the seminal paper [70], Strohmer and Vershynin first investigated the randomized Kaczmarz (RK) method. Specifically, they proved that if $i_{k}$ is selected with probability $\operatorname{Pr}\left(i_{k}=i\right)=\frac{\left\|a_{i}\right\|_{2}^{2}}{\|A\|_{F}^{2}}$, the method converges linearly

$$
\mathbb{E}\left[\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}\right] \leq\left(1-\frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{k}\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2}
$$

Subsequently, there has been a significant amount of work on the development of Kaczmarztype methods, with references available in [3, 23, 24, $33,34,39,41,45,62,67,69,74]$. Among these methods, we pay extra attention to the reflection Kaczmarz method studied by Steinerberger in [68], which constructs $x^{k+1}$ via

$$
x^{k+1}=R_{C_{i_{k}}}\left(x^{k}\right)=x^{k}-2 \frac{\left\langle a_{i_{k}}, x^{k}\right\rangle-b_{i_{k}}}{\left\|a_{i_{k}}\right\|_{2}^{2}} a_{i_{k}} .
$$

Although the generated sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ does not converge to $x^{*}$ as their distance $\| x^{k}$ $x^{*}\left\|_{2} \equiv\right\| x^{0}-x^{*} \|_{2}$ remains constant, Steinerberger [68] proved the sublinear convergence of the average $\frac{1}{k} \sum_{i=1}^{k} x^{i}$. Moreover, the author noted that with a particular restart strategy, the algorithm can reach the same complexity as the RK method. In fact, our work is inspired by such reflection approaches [66,68], while instead of using the orthogonal projections or reflections only, we study the DR method which incorporates the reflection-average approach.

Recently, Hu and Cai [40] studied the canonical randomized Douglas-Rachford (RDR) method in a simple case where $r=2$. They proved that $\mathbb{E}\left[x^{k}\right] \rightarrow x^{*}$ as $k \rightarrow \infty$, however, without convergence rates analysis. We also note that such convergence does not lead to
$\mathbb{E}\left[\left\|x^{k}-x^{*}\right\|_{2}^{2}\right] \rightarrow 0$. As an exemplification [72], consider $x^{k}=x^{*}+r^{k}$, where $r^{k}$ are drawn i.i.d. from $N(0, I / \sqrt{n})$. Such a sequence satisfies $\mathbb{E}\left[x^{k}\right]=x^{*}$, while $\mathbb{E}\left[\left\|x^{k}-x^{*}\right\|_{2}^{2}\right] \equiv 1$.

In recent years, the momentum acceleration technique has been recognized as an effective approach to improving the performance of optimization methods. For instance, the stochastic gradient descent (SGD) method [64] can be incorporated with the heavy ball momentum [61], deriving the well-known stochastic heavy ball momentum (SHBM) method [30,65] with enhanced performance for solving large-scale optimization problems. The RK method, as a special case of SGD (see Section 3), was studied in the momentum framework [50,53]. Specifically, Loizou and Richtárik [50] investigated the SHBM method for solving stochastic problems that are reformulated from consistent linear systems, and established the global, non-asymptotic linear convergence rates of the proposed methods. In [34], Gower and Richtárik developed the sketch-and-project method, a versatile randomized iterative method which includes the RK algorithm as a special case, for solving consistent linear systems. The momentum variants of the sketch-and-project method has been investigated in [51,63]. Enlightened by the success of the heavy ball momentum technique in these stochastic methods, we intend to employ such momentum acceleration in the randomized DR method.

We note that another popular momentum acceleration is the Nesterov's momentum [57,58], leading to the famous accelerated gradient descent (AGD) method [9]. Recently, variants of Nesterov's momentum has also been introduced for the acceleration of stochastic optimization algorithms [44]. In [49], Liu and Wright applied the acceleration scheme of Nesterov to the RK method. It has also been applied to the sampling Kaczmarz Motzkin (SKM) algorithm for linear feasibility problems [54, 55].

## 3. Randomized Douglas-Rachford method

In this section, we propose our randomized Douglas-Rachford method for solving linear systems and analyze its convergence property. Actually, for the canonical DR algorithm, there are a lot of modifications and relaxations in the literature. An important and useful one is the approach studied by Eckstein and Bertsekas [26], known as the generalized Douglas-Rachford (GDR) method

$$
x^{k+1}=\left((1-\alpha) I+\alpha R_{C_{2}} R_{C_{1}}\right)\left(x^{k}\right), \alpha \in(0,1) .
$$

It reduces to the DR method when $\alpha=\frac{1}{2}$. The significance of introducing the relaxation parameter $\alpha$ is that, in general, the performance of the optimization algorithms can be practically accelerated under overrelaxation conditions ( $\alpha>\frac{1}{2}$ ) [56, 70].

Based on the variants of the DR method for $m$-sets condition discussed in Section 1.2, we consider a more universal setting of the DR method, called the extrapolated $r$-sets-DR method. At the $k$-th iteration, the algorithm chooses $r$ sets $C_{j_{k_{1}}}, \ldots, C_{j_{k_{r}}}$ and updates $x^{k}$ via

$$
x^{k+1}=\left((1-\alpha) I+\alpha R_{C_{j_{k_{r}}}} R_{C_{j_{k_{r-1}}}} \ldots R_{C_{j_{k_{1}}}}\right)\left(x^{k}\right),
$$

where $\alpha \in(0,1)$ is the extrapolation or relaxation parameter. The criterion of the selection of the sets $C_{j_{k_{1}}}, \cdots, C_{j_{k_{r}}}$ is where the randomization is adopted. We prove that if the set $C_{i}$ is selected with probability proportional to $\left\|a_{i}\right\|_{2}^{2}$, the method converges linearly in expectation.

A description of the iterative procedure is presented in Algorithm 1. To demonstrate the algorithm in a simple and straightforward form, the description is in terms of $a_{1}, \ldots, a_{m}$ and $b=\left(b_{1}, \ldots, b_{m}\right)^{\top}$ rather than the operators $R_{C_{i}}, i=1, \ldots, m$. It can be seen that not only the canonical $\operatorname{RDR}$ method is included in this framework with $r=2$ and $\alpha=\frac{1}{2}$, but also the RK method can be recovered with $r=1$ and $\alpha=\frac{1}{2}$.

Additionally, we assume that $\operatorname{rank}(A) \geq 2$. Suppose that $\operatorname{rank}(A)=1$ and $r$ is an even number, now the extrapolated $r$-sets-DR method just reflects the iteration back and forth through the same hyperplane. So the iteration sequence of the method satisfies $x^{0}=x^{1}=$ $\ldots=x^{k}$ and the method fails. In fact, $\operatorname{rank}(A)=1$ is a trivial case where one can get the solution by only one step of projection.

Finally, we give some comments on the connection between Algorithm 1 and the SGD method [38,52,64]. The RK method can be viewed as SGD applied to the following leastsquares problem

$$
\begin{equation*}
\min _{x \in \mathbb{R}^{n}} f(x):=\frac{1}{2 m}\|A x-b\|_{2}^{2}=\frac{1}{m} \sum_{i=1}^{m} f_{i}(x) \tag{4}
\end{equation*}
$$

where $f_{i}(x)=\frac{1}{2}\left(\left\langle a_{i}, x\right\rangle-b_{i}\right)^{2}$. For convenience, we assume that $A \in \mathbb{R}^{m \times n}$ is normalized such that $\left\|a_{i}\right\|_{2}^{2}=1$. The SGD method solves (4) using unbiased estimates for the gradient of the objective function. Particularly, one can employ $\nabla f_{i}(x)$ since $\mathbb{E}\left[\nabla f_{i}(x)\right]=\nabla f(x)$.

```
Algorithm 1 Randomized \(r\)-sets-Douglas-Rachford (RrDR) method
Input: \(A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m}, r \in \mathbb{Z}_{+}, k=0\), extrapolation/relaxation parameter \(\alpha \in(0,1)\)
    and an initial \(x^{0} \in \mathbb{R}^{n}\).
        1: Set \(z_{0}^{k}:=x^{k}\).
        2: for \(\ell=1, \ldots, r\) do
    3: Select \(j_{k_{\ell}} \in\{1, \ldots, m\}\) with probability \(\operatorname{Pr}\left(\right.\) row \(\left.=j_{k_{\ell}}\right)=\frac{\left\|a_{j_{k_{k}}}\right\|_{2}^{2}}{\|A\|_{F}^{2}}\).
    4: Compute
        \(z_{\ell}^{k}:=z_{\ell-1}^{k}-2 \frac{\left\langle a_{j_{k_{\ell}}}, z_{\ell-1}^{k}\right\rangle-b_{j_{k_{\ell}}}}{\left\|a_{j_{k_{\ell}}}\right\|_{2}^{2}} a_{j_{k_{\ell}}}\).
    5: end for
    6: Update
        \(x^{k+1}:=(1-\alpha) x^{k}+\alpha z_{r}^{k}\).
    7: If the stopping rule is satisfied, stop and go to output. Otherwise, set \(k=k+1\) and
        return to Step 1.
Output: The approximate solution \(x^{k}\).
```

At the $k$-th iteration, SGD draws $\nabla f_{i_{k}}(x)$ and updates $x^{k}$ via

$$
\begin{equation*}
x^{k+1}=x^{k}-\lambda_{k} \nabla f_{i_{k}}\left(x^{k}\right), \tag{5}
\end{equation*}
$$

where $\lambda_{k}$ is an appropriately chosen stepsize. Since here $\nabla f_{i_{k}}\left(x^{k}\right)=\left(\left\langle a_{i_{k}}, x^{k}\right\rangle-b_{i_{k}}\right) a_{i_{k}}$, one can recover the RK method by applying SGD to the problem (4). Based on this fact, we reconsider Algorithm 1 with $\alpha=\frac{1}{2}$ and $r=2$, where we have $z_{1}^{k}=x^{k}-2 \nabla f_{j_{k_{1}}}\left(x^{k}\right)$, $z_{2}^{k}=z_{1}^{k}-2 \nabla f_{j_{k_{2}}}\left(z_{1}^{k}\right)$, and hence

$$
x^{k+1}=x^{k}-\nabla f_{j_{k_{1}}}\left(x^{k}\right)-\nabla f_{j_{k_{2}}}\left(x^{k}-2 \nabla f_{j_{k_{1}}}\left(x^{k}\right)\right) .
$$

In fact, the last term $\nabla f_{j_{k_{2}}}\left(x^{k}-2 \nabla f_{j_{k_{1}}}\left(x^{k}\right)\right)$ is known as the stochastic extragradient (SEG) [32]. Therefore, we state that Algorithm 1 can be regarded as a combination of SGD and SEG.
3.1. Convergence of iterates. In this subsection, we analyze the convergence properties of Algorithm 1. The first result demonstrates that the expected norm of the error $\mathbb{E}\left[\| x^{k}-\right.$ $\left.x_{0}^{*} \|_{2}^{2}\right]$ converges linearly.

Theorem 3.1. Suppose that the linear system $A x=b$ is consistent, $\alpha \in(0,1)$, and $x^{0} \in \mathbb{R}^{n}$ is an arbitrary initial vector. Let $x_{0}^{*}=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}$. Then the iteration sequence
$\left\{x^{k}\right\}_{k=0}^{\infty}$ generated by Algorithm 1 satisfies

$$
\mathbb{E}\left[\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}\right] \leq\left(\alpha^{2}+(1-\alpha)^{2}+2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)^{k}\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2}
$$

Remark 3.2. If the initial vector $x^{0} \in \operatorname{Row}(A)$, then we have $x_{0}^{*}=A^{\dagger} b=x_{L N}^{*}$. This implies that the iteration sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ generated by Algorithm 1 now converges to the least-norm solution $x_{L N}^{*}$. Additionally, we note that the upper bound in Theorem 3.1 is tight. Please refer to Remark 7.4 for more details.

Remark 3.3. We note that the almost sure convergence of the iterates of the stochastic algorithms has been extensively studied [13, 20, 65]. By utilizing Lemma 2.1 in [13] and Theorem 3.1, it can be easily deduced that the iteration sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ generated by Algorithm 1 converges almost surely to $x_{0}^{*}$.

Remark 3.4. Let $\widetilde{\alpha}:=2 \alpha \in(0,2)$. If $r=1$, then Algorithm 1 becomes $x^{k+1}=x^{k}-$ $\widetilde{\alpha} \frac{\left\langle a_{i_{k}}, x^{k}\right\rangle-b_{i_{k}}}{\left\|a_{i_{k}}\right\|_{2}^{2}} a_{i_{k}}$ and Theorem 3.1 leads to

$$
\mathbb{E}\left[\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}\right] \leq\left(1-\frac{\widetilde{\alpha}(2-\widetilde{\alpha}) \sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{k}\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2}
$$

which is exactly the conclusion obtained in [16, Theorem 1] for the RK method with relaxation.

We now make a comparison between our method and the RK method in terms of convergence rate. Since the computational cost of Algorithm 1 at each step is about $r$-times as expensive as that of the RK method, the comparison can be expressed as

$$
\begin{equation*}
\left(1-\frac{4 \alpha(1-\alpha) \sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r} \leq \alpha^{2}+(1-\alpha)^{2}+2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r} \tag{6}
\end{equation*}
$$

Indeed, let $p=1-\frac{4 \alpha(1-\alpha) \sigma_{\text {min }}^{2}(A)}{\|A\|_{F}^{2}}$ and $q=1-2 \frac{\sigma_{\text {min }}^{2}(A)}{\|A\|_{F}^{2}}$, then for any fixed $\alpha \in(0,1)$, one can verify that

$$
g(r):=\frac{\alpha^{2}+(1-\alpha)^{2}}{p^{r}}+2 \alpha(1-\alpha)\left(\frac{q}{p}\right)^{r}
$$

is monotonically increasing, i.e. $g(r) \geq g(1)=1$ so that (6) holds. This implies that the RK method is theoretically better than Algorithm 1. Nevertheless, numerical experiments demonstrate that, with an appropriate $r>1$, Algorithm 1 is more efficient and requires fewer row-actions than the RK method (see Section 5.2.2).

Next, let us consider the convergence of the norm of the expected error $\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2}$.

Theorem 3.5. Suppose that the linear system $A x=b$ is consistent, $x^{0} \in \mathbb{R}^{n}$ is an arbitrary initial vector, and the relaxation parameter satisfies

$$
0<\alpha<\min \left\{1, \frac{1}{1-\left(1-2 \sigma_{\max }^{2}(A) /\|A\|_{F}^{2}\right)^{r}}\right\}
$$

Let $x_{0}^{*}=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}$. Then the iteration sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ generated by Algorithm 1 satisfies

$$
\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2} \leq\left((1-\alpha)+\alpha\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)^{2 k}\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2} .
$$

Since $(1-\alpha)+\alpha\left(1-2 \sigma_{\text {min }}^{2}(A) /\|A\|_{F}^{2}\right)^{r} \in(0,1)$, we know that $\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2} \rightarrow 0$ as $k \rightarrow \infty$. One may be confused by the similarity between the quantity $\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2}$ in Theorem 3.5 and $\mathbb{E}\left[\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}\right]$ in Theorem 3.1. Actually, the convergence of the former is much weaker than that of the later. Supposing $\mathbb{E}[x]$ is bounded for all $x \in \mathbb{R}^{n}$, by definition, we have

$$
\mathbb{E}\left[\left\|x-x_{0}^{*}\right\|_{2}^{2}\right]=\left\|\mathbb{E}\left[x-x_{0}^{*}\right]\right\|_{2}^{2}+\mathbb{E}\left[\|x-\mathbb{E}[x]\|_{2}^{2}\right]
$$

which implies that the convergence of $\mathbb{E}\left[\left\|x-x_{0}^{*}\right\|_{2}^{2}\right]$ leads to that of $\left\|\mathbb{E}\left[x-x_{0}^{*}\right]\right\|_{2}^{2}$, but not vice versa. The reason why it is also considered is that we aim to systematically compare $\operatorname{RrDR}$ method with its momentum variant on both quantities in Section 4.
3.2. Convergence direction. Inspired by the recent work of Steinerberger [67], in this subsection we consider the convergence direction of Algorithm 1. The following result shows different convergence rates along different singular vectors of $A$.

Theorem 3.6. Suppose that $x^{*}$ is a solution of the linear system $A x=b$ and $\alpha \in(0,1)$. For any $v_{i}$ being the (right) singular vector of $A$ associated to the singular value $\sigma_{i}(A)$, the iteration sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ generated by Algorithm 1 satisfies

$$
\mathbb{E}\left[\left\langle x^{k}-x^{*}, v_{i}\right\rangle\right]=\left((1-\alpha)+\alpha\left(1-\frac{2 \sigma_{i}^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)^{k}\left\langle x^{0}-x^{*}, v_{i}\right\rangle
$$

Remark 3.7. Unlike the cases in Theorem 3.1 and 3.5, $x^{*}$ in Theorem 3.6 is an arbitrary solution of the linear system. Certainly, one can take $x_{0}^{*}$. We also note that such convergence is weaker than that of $\mathbb{E}\left[\left\|x^{k}-x^{*}\right\|_{2}^{2}\right]$. Consider the example given in Section 1.2 where $x^{k}=x^{*}+r^{k}$ with $r^{k}$ being drawn i.i.d. from $N(0, I / \sqrt{n})$. Such a sequence satisfies $\mathbb{E}\left[\left\langle x^{k}-x^{*}, v_{i}\right\rangle\right]=0$, while $\mathbb{E}\left[\left\|x^{k}-x^{*}\right\|_{2}^{2}\right] \equiv 1$.

Theorem 3.6 exhibits that the RrDR method converges exponentially along different singular directions of $A$ at different rates depending on the singular values. It accounts for the typical semiconvergence phenomenon. That is, the residual $\left\|A x^{k}-b\right\|_{2}^{2}$ decays faster at the beginning, but then gradually stagnates. Recently, the semiconvergence phenomenon has been exploited by Wu and Xiang [73] for the randomized row iterative method [34]. They generalized the study in [41] and split the total error into the low- and high-frequency solution spaces. In the literature, several acceleration techniques have been proposed to avoid semiconvergence phenomenon, for instance, the weighted version [69] and momentum acceleration technique [50]. In this paper, we will introduce the momentum acceleration technique to the RrDR method.

## 4. Momentum acceleration

In this section, we provide the momentum induced $\operatorname{RrDR}(m R r D R)$ method for solving feasibility problem derived from linear systems. First, we give a short description of the heavy ball momentum method. Consider the unconstrained minimization problem $\min _{x \in \mathbb{R}^{n}} f(x)$, where $f$ is a differentiable convex function. To solve the problem, the gradient descent method with momentum (HBM) of Polyak [61] takes the form

$$
\begin{equation*}
x^{k+1}=x^{k}-\alpha \nabla f\left(x^{k}\right)+\beta\left(x^{k}-x^{k-1}\right), \tag{7}
\end{equation*}
$$

where $\alpha>0$ is the stepsize, $\beta$ is the momentum parameter, and $\nabla f\left(x^{k}\right)$ denotes the gradient of $f$ at $x^{k}$. When $\beta=0$, the method reduces to the gradient descent method. If the full gradient in (7) is replaced by the unbiased estimate of the gradient, then it becomes the stochastic HBM (SHBM) method. In [31], the authors showed that the deterministic HBM method converges globally and sublinearly for smooth and convex functions. For the SHBM, one may refer to $[30,65]$ for more discussions.

Inspired by the success of the SHBM method, in this section we incorporate the HBM into our RrDR method, obtaining the mRrDR method described in Algorithm 2. To the best of our knowledge, this is the first time that momentum variants of the $r$-sets-DR method are investigated. In the rest of this section, we will study the convergence properties of the proposed mRrDR method.
4.1. Convergence of iterates. We have the following convergence result for Algorithm 2.

Algorithm 2 Randomized $r$-sets-Douglas-Rachford with momentum (mRrDR)
Input: $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m}, r \in \mathbb{Z}_{+}, k=1$, extrapolation/relaxation parameter $\alpha$, the heavy ball momentum parameter $\beta$, and initial vectors $x^{1}, x^{0} \in \mathbb{R}^{n}$.

1: Set $z_{0}^{k}:=x^{k}$.
2: for $\ell=1, \ldots, r$ do
3: $\quad$ Select $j_{k_{\ell}} \in\{1, \ldots, m\}$ with probability $\operatorname{Pr}\left(\right.$ row $\left.=j_{k_{\ell}}\right)=\frac{\left\|a_{j_{k_{k}}}\right\|_{2}^{2}}{\|A\|_{F}^{2}}$.
4: Compute

$$
z_{\ell}^{k}:=z_{\ell-1}^{k}-2 \frac{\left\langle a_{j_{k_{\ell}}}, z_{\ell-1}^{k}\right\rangle-b_{j_{k_{\ell}}}}{\left\|a_{j_{k_{\ell}}}\right\|_{2}^{2}} a_{j_{k_{\ell}}} .
$$

5: end for
6: Update

$$
x^{k+1}:=(1-\alpha) x^{k}+\alpha z_{r}^{k}+\beta\left(x^{k}-x^{k-1}\right) .
$$

7: If the stopping rule is satisfied, stop and go to output. Otherwise, set $k=k+1$ and return to Step 1.
Output: The approximate solution $x^{k}$.

Theorem 4.1. Suppose that the linear system $A x=b$ is consistent, $x^{1}=x^{0} \in \mathbb{R}^{n}$ are arbitrary initial vectors, and $x_{0}^{*}=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}$. Assume $0<\alpha<1, \beta \geq 0$ and that

$$
\gamma_{1}:=\alpha^{2}+(1-\alpha)^{2}+(2 \alpha(1-\alpha)+3 \alpha \beta)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}+2 \beta^{2}+3(1-\alpha) \beta
$$

and

$$
\gamma_{2}:=2 \beta^{2}+(1-\alpha) \beta+\alpha \beta\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}
$$

satisfy $\gamma_{1}+\gamma_{2}<1$. Then the iteration sequence $\left\{x^{k}\right\}_{k=0}^{\infty}$ generated by Algorithm 2 satisfies

$$
\mathbb{E}\left[\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2}\right] \leq q^{k}(1+\tau)\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2}, \forall k \geq 0,
$$

where $q=\frac{\gamma_{1}+\sqrt{\gamma_{1}^{2}+4 \gamma_{2}}}{2}$ and $\tau=q-\gamma_{1} \geq 0$. Moreover, $\gamma_{1}+\gamma_{2} \leq q<1$.

We here provide one approach to choose the parameters $\alpha$ and $\beta$. Specifically, letting

$$
\tau_{1}:=4(1-\alpha)+4 \alpha\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r} \text { and } \tau_{2}:=2 \alpha(1-\alpha)\left(1-\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)
$$

if the parameters $\alpha$ and $\beta$ are chosen such that

$$
0<\alpha<1 \text { and } 0 \leq \beta<\frac{1}{8}\left(\sqrt{\tau_{1}^{2}+16 \tau_{2}}-\tau_{1}\right)
$$

then it can be verified that $\gamma_{1}+\gamma_{2}<1$.
Next, we compare the convergence rates obtained in Theorems 3.1 and 4.1. From the definition of $\gamma_{1}$ and $\gamma_{2}$, we know that the convergence rate $q(\beta)$ in Theorem 4.1 can be
viewed as a function of $\beta$. Since $\beta \geq 0$, we have $\tau_{2} \geq 0$. As $\gamma_{1}+\gamma_{2}<1$, it implies that $\gamma_{1} \gamma_{2}+\gamma_{2}^{2}=\gamma_{2}\left(\gamma_{1}+\gamma_{2}\right) \leq \gamma_{2}$. Therefore, $\gamma_{1}^{2}+4 \gamma_{2} \geq\left(\gamma_{1}+2 \gamma_{2}\right)^{2}$ and hence,

$$
\begin{aligned}
q(\beta) & \geq \gamma_{1}+\gamma_{2}=4 \beta^{2}+\tau_{1} \beta-\tau_{2}+1 \geq 1-\tau_{2} \\
& =q(0)=\alpha^{2}+(1-\alpha)^{2}+2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r} .
\end{aligned}
$$

Clearly, the lower bound on $q$ is an increasing function of $\beta$, which implies that for any $\beta$ the rate is always inferior to that of Algorithm 1 in Theorem 3.1.
4.2. Accelerated linear rate of expected iterates. To theoretically exhibit the enhancement of the heavy ball momentum, we now show that with a proper selection of the relaxation parameter $\alpha$ and momentum parameter $\beta$, Algorithm 2 enjoys an accelerated linear convergence rate in terms of $\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2}$.

Theorem 4.2. Suppose that the linear system $A x=b$ is consistent, $x^{1}=x^{0} \in \mathbb{R}^{n}$ are arbitrary initial vectors, and $x_{0}^{*}=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}$. Let $\left\{x^{k}\right\}_{k=0}^{\infty}$ be the iteration sequence in Algorithm 2. Assume that the relaxation parameter

$$
0<\alpha<\min \left\{1, \frac{1}{1-\left(1-2 \sigma_{\max }^{2}(A) /\|A\|_{F}^{2}\right)^{r}}\right\}
$$

and the momentum parameter

$$
\left(1-\sqrt{\alpha\left(1-\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)}\right)^{2}<\beta<1 .
$$

Then there exists a constant $c>0$ such that

$$
\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2} \leq \beta^{k} c .
$$

Remark 4.3. Note that the convergence factor in Theorem 4.2 is equal to the value of the momentum parameter $\beta$. Theorem 3.5 shows that Algorithm 1 (without momentum) converges with iteration complexity

$$
O\left(\log \left(\varepsilon^{-1}\right)\left(\alpha\left(1-\left(1-2 \sigma_{\min }^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)\right)^{-1}\right)
$$

In contrast, based on Theorem 4.2 we have, for $\beta=\left(1-\sqrt{0.99 \alpha\left(1-\left(1-2 \sigma_{\min }^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)}\right)^{2}$, the iteration complexity of Algorithm 2 is

$$
O\left(\log \left(\varepsilon^{-1}\right) \sqrt{\left(0.99 \alpha\left(1-\left(1-2 \sigma_{\min }^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)\right)^{-1}}\right),
$$

which is a quadratic improvement on the above conclusion.

## 5. NuMERICAL EXPERIMENTS

In this section, we study the computational behavior of the two proposed algorithms, $\operatorname{RrDR}$ and mRrDR. In particular, we focus mainly on the evaluation of the performance of $m R r D R$. We compare mRrDR with some of the state-of-the-art methods, namely, RK [70], RGS [35, 45], and RP-ADMM [71]. Moreover, we also compare the implementation of $m R r D R$ with the built-in function pinv and lsqminnorm in Matlab.

All the methods are implemented in Matlab R2019b for Windows 10 on a desktop PC with the $\operatorname{Intel}(\mathrm{R})$ Core(TM) i7-10710U CPU @ 1.10 GHz and 16 GB memory.
5.1. Numerical setup. We mainly use the following three types of data for our test.

Synthetic data. For synthetic data, given different values of $\|A\|_{F}^{2} / \sigma_{\text {min }}^{2}(A)$, we generate a group of matrices $A$. We then generate the exact solution $x^{*}$ by $x^{*}=\left(A^{\top} w\right) /\left\|A^{\top} w\right\|_{2}$ with a random $w \in \mathbb{R}^{m}$ to ensure that it lies in the $\operatorname{Row}(A)$ and $b=A x^{*}$ to ensure the consistency of the system. The synthetic data are designed to investigate the influence of the rate coefficient on the convergence process.

Real-world data. The real-world data are available via the SuiteSparse Matrix Collection [43] and LIBSVM [18]. In our experiments, we only use the matrices $A$ of the datasets. If $m<n$, then we use $A^{\top}$ as the coefficient matrix. Similarly, to ensure the consistency of the linear system, we first generate the solution by $x^{*}=\left(A^{\top} w\right) /\left\|A^{\top} w\right\|_{2}$ and then set $b=A x^{*}$.

Average consensus. Suppose $G=(V, E)$ is an undirected connected network with the vertex set $V=\left\{v_{1}, v_{2}, \cdots, v_{n}\right\}$ and the edge set $E(|E|=m)$. In the average consensus (AC) problem, each vertex $v_{i} \in V$ owns a private value $c_{i} \in \mathbb{R}$, and the goal of the problem is to compute the average of the private values of each vertex of the network, $\bar{c}:=\frac{1}{n} \sum_{i} c_{i}$, where only communication between neighbours is allowed. The problem is fundamental in distributed computing and multiagent systems $[12,51]$, and has many applications such as PageRank, coordination of autonomous agents, and rumor spreading in social networks. Recently, under an appropriate setting, the famous randomized pairwise gossip algorithm [12] for solving the AC problem has been proved to be equivalent to the RK method. One may refer to [51] for more details.

For the synthetic data and real data, the initial vector is chosen as $x^{0}=0$ (or $x^{1}=x^{0}=0$ for mRrDR ). For the AC problem, the initial vector is chosen as $x^{0}=c$ (or $x^{1}=x^{0}=c$ for mRrDR ). The computations are terminated once the relative solution error (RSE), defined as $\mathrm{RSE}=\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2} /\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2}$, is less than a specific error tolerance or the number of iteration exceeds a certain limit. All the results below are averaged over 10 trials.
5.2. Optimal selection of the parameters. As observed in the convergence theorems presented in sections 3 and 4 , the parameters of $\operatorname{RrDR}$ and mRrDR have an influence on the convergence rate. In this subsection, we aim to find relatively favorable choices via numerical tests. All computations are terminated once RSE $<10^{-12}$.
5.2.1. Choice of $\alpha$ and $\beta$. In this subsection, we demonstrate the computational behavior of mRrDR with respect to different parameters $\alpha$ and $\beta$. (Note that for $\beta=0$ it is equivalent to RrDR .) We measure the performance of the method with respect to the number of iterations. From Figures 3, 4, 5, and 6, it is obvious that the introduction of momentum term leads to an improvement in the performance of $\operatorname{RrDR}$. More specifically, from the figures we observe the following.

1. The momentum technique can improve the convergence behavior of the method. For any fixed value of $\alpha$, with appropriate momentum parameters $0<\beta \leq 0.9$, mRrDR typically converges faster than its non-momentum variant RrDR.
2. For the $\operatorname{RrDR}$ method $(\beta=0)$, a larger value of $\alpha$ can help achieve a faster convergence. This is consistent with the observation in the literature that the overrelaxation parameter is more advisable for better performance [56, 70].
3. For different values of $\|A\|_{F}^{2} / \sigma_{\min }^{2}(A)$ (well- or ill-conditioned linear systems), and different choices of $r,(\alpha, \beta)=(0.5,0.4)$ is typically a good option for a sufficient fast convergence of mRrDR .
5.2.2. Choice of $r$. In this subsection, we investigate the computational performance of mRrDR with respect to the parameter $r$. We plot the performance of the method in terms of the number of row actions. Figures 7 and 8 illustrate our experimental results. Note again that $\beta=0$ represents the $\operatorname{RrDR}$ method and $r=1$ represents the RK method.


Figure 3. Performance of mRrDR with different parameters $\alpha$ and $\beta$ for consistent linear systems with Gaussian matrix $A$, where $r=2$. The title of each plot indicates the dimensions of the matrix $A$ and the value of $\|A\|_{F}^{2} / \sigma_{\text {min }}^{2}(A)$.


Figure 4. Performance of mRrDR with different parameters $\alpha$ and $\beta$ for consistent linear systems with Gaussian matrix $A$, where $r=3$. The title of each plot indicates the dimensions of the matrix $A$ and the value of $\|A\|_{F}^{2} / \sigma_{\text {min }}^{2}(A)$.


Figure 5. Performance of mRrDR with different parameters $\alpha$ and $\beta$ on real data from SuiteSparse Matrix Collection [43]. Franz1: $(m, n)=(2240,768)$, WorldCites: $(m, n)=(315,100)$, nemsafm: $(m, n)=(2348,334)$, where $r=3$.


Figure 6. Performance of $m R r D R$ with different parameters $\alpha$ and $\beta$ on the AC problem, where $r=2$.

It is can be seen that a larger $r$ may lead to slower convergence and $r \in[1,10]$ is a good option. Besides, with an appropriate $r$, the mRrDR method may converge faster than the mRK method $(r=1)$. For example, the first row in Figure 7 shows that mRrDR with $r=4$ performs better than the mRK method.
5.3. Comparison to the cyclic DR method. In this subsection, we compare the mRrDR method to the cyclic DR method (3) to demonstrate the effectiveness of randomization. During the test, mRrDR is implemented with $r=2$ and $(\alpha, \beta)=(0.5,0)$ or $(\alpha, \beta)=$ $(0.5,0.4)$. For the cyclic DR method (3), we set $r=2$ and $\alpha=0.5$. From Figure 9 , we can observe the significant improvement in efficiency that randomization and momentum bring.
5.4. Comparison to the other methods. We now compare mRrDR to other related methods for solving linear systems, including RK [70], randomized Gauss-Seidel (RGS)


Figure 7. Performance of mRrDR with different values of $r . r=1$ refers to the mRK method. The parameters $\alpha=0.5, \beta=0$. The title of each plot indicates the test data. The number of row actions is employed to illustrate the evolutions for the different settings of mRrDR .


Figure 8. Performance of mRrDR with different values of $r . r=1$ refers to the mRK method. The parameters $\alpha=0.5, \beta=0.4$. The title of each plot indicates the dimensions of the matrix $A$ and the value of $\|A\|_{F}^{2} / \sigma_{\min }^{2}(A)$. The number of row actions is employed to illustrate the evolutions for the different settings of mRrDR.


Figure 9. Comparison of $\operatorname{mRrDR}(r=2, \alpha=0.5)$ and the cyclic $\operatorname{DR}$ ( $\alpha=0.5$ ), where Franz1 from SuiteSparse Matrix Collection [43], aloi and heart-scale from LIBSVM [18]. We stop the algorithms if RSE $<10^{-12}$ or if the number of iteration exceeds a certain limit.
method [35, 45], and randomly permuted alternating direction method of multipliers (RPADMM) [71]. The RGS, also known as the randomized coordinate descent (RCD) method, updates with the following iterative strategy:

$$
x^{k+1}:=x^{k}-\frac{A_{j_{k}}^{\top}\left(A x^{k}-b\right)}{\left\|A_{j_{k}}\right\|_{2}^{2}} e_{j_{k}},
$$

where $j_{k} \in\{1,2, \ldots, n\}$ is selected with probability $\operatorname{Pr}\left(j_{k}=j\right)=\frac{\left\|A_{j}\right\|_{2}^{2}}{\|A\|_{F}^{2}}$, and $A_{j}, j=1, \ldots, n$ represent the columns of $A$ and $e_{j}$ is a column vector with the $j$-th entry being one and all other entries being zero.

The RP-ADMM tackles the following constrained problem

$$
\min _{x \in \mathbb{R}^{n}} 0 \text { subject to } A x=b
$$

whose augmented Lagrangian function is defined by

$$
\mathcal{L}(x ; \mu):=-\mu^{\top}(A x-b)+\frac{\rho}{2}\|A x-b\|_{2}^{2}
$$

where $\rho>0$ is a given penalty parameter and $\mu$ denotes the Lagrangian multiplier. Then the RP-ADMM method proceeds as follows: Given an approximation $\left(x^{k}, \mu^{k}\right)$, it picks a permutation $\sigma$ of $\{1, \ldots, n\}$ uniformly at random, and constructs $\left(x^{k+1}, \mu^{k+1}\right)$ from $\left(x^{k}, \mu^{k}\right)$ via

$$
\left(x^{k+1}\right)_{\sigma(i)}=\arg \min _{x_{\sigma(i)}} \mathcal{L}\left(\left(x^{k+1}\right)_{\sigma(1)}, \ldots,\left(x^{k+1}\right)_{\sigma(i-1)}, x_{\sigma(i)},\left(x^{k}\right)_{\sigma(i+1)}, \ldots,\left(x^{k}\right)_{\sigma(n)} ; \mu^{k}\right)
$$

and $\mu^{k+1}=\mu^{k}-\left(A x^{k+1}-b\right)$. We note that the alternating direction method of multipliers (ADMM) is equivalent to the DR method $[26,36]$ in the sense that the sequences generated by both algorithms coincide with a careful choice of starting point [7, Remark 3.14].

Since there are seldom differences in the numerical performance for $r \in[1,10]$, here we only show the cases where $r=2$. For the RP-ADMM method, during our test, we set $\rho=1$ and the initial vector $\mu^{0}=0$. Figures 10 and 11 summarize the results of the experiment. It can be seen that mRrDR is more efficient than the other considered methods.


Figure 10. Performance of RK, mRrDR, RGS, and RP-ADMM for synthetic data. We stop the algorithms if $\operatorname{RSE}<10^{-12}$ or if the number of iterations exceeds a certain limit.
5.5. Comparison to pinv and lisqminnorm. In this subsection, we compare the performance of mRrDR with Matlab functions pinv and lsqminnorm. To easily obtain the leastnorm solution, we first generate full column rank coefficient matrices as follows. For given


Figure 11. Performance of RK, mRrDR, RGS, and RP-ADMM for real world data and the AC problem, where crew1 and model1 from SuiteSparse Matrix Collection [43], ijcnn1 from LIBSVM [18]. We stop the algorithms if $\operatorname{RSE}<10^{-12}$ or if the number of iterations exceeds a certain limit.
$m \geq n$, and $\kappa>1$, we set $A=U D V^{\top}$, where $U \in \mathbb{R}^{m \times n}, D \in \mathbb{R}^{n \times n}$, and $V \in \mathbb{R}^{n \times n}$. Using Matlab notation, these matrices are generated by [U, ~] = qr ( $\operatorname{randn}(m, n), 0),[V, \sim]=$ $\operatorname{qr}(\operatorname{randn}(\mathrm{n}, \mathrm{n}), 0)$, and $\mathrm{D}=\operatorname{diag}(1+(\kappa-1) . * \operatorname{rand}(\mathrm{n}, 1))$. Note that the condition number of $A$ is now upper bounded by $\kappa$. Next, we generate the solution vector $x^{*}$ by setting $x^{*}=\operatorname{randn}(\mathrm{n}, 1)$, and then we calculate $b=A x^{*}$ to obtain the right-hand side vector of the linear system. It can be observed that $x^{*}$ is the desired unique solution of the constructed linear system.

Figure 12 illustrates our experimental results with fixed $n$. The mRrDR method is implemented with $r=1$ and $(\alpha, \beta)=(0.5,0.4)$, or $r=2$ and $(\alpha, \beta)=(0.5,0.4)$. We terminate the mRrDR method if the accuracy of its approximate solution is comparable to that of the approximate solution obtained using pinv and lsqminnorm. In Figure 12, we plot the computing time against the increasing number of rows. It can be observed that when the number of rows exceeds certain thresholds, mRrDR outperforms pinv and lsqminnorm. We can also find that the performance of the mRrDR method is more sensitive to the increase of the condition number $\kappa$, as the convergence bound implies.


Figure 12. Figures depict the CPU time (in seconds) vs increasing number of rows. The title of each plot indicates the values of $n$ and $\kappa$.

## 6. Concluding Remarks

In this work, we studied the $r$-sets-Douglas-Rachford method enriched with randomization and heavy ball momentum for solving linear systems. We proved global linear convergence rates of the method as well as an accelerated linear rate in terms of the norm of expected error. Our convergence analysis showed the effectiveness of randomization in simplifying the analysis of the DR method and making the divergent $r$-sets-DR method converge linearly. We corroborated our theoretical results with extensive experimental testing and confirmed the better performance of the mRrDR method.

There are still many possible future venues of research. A bunch of advanced schemes for the selection of sets to project have been investigated in the literature of the Kaczamarz method, such as the greedy selection rule [3], its weighted variant in [69] and the approach with sampling in [21]. These criteria are convenient to be adopted to the DR context for further improvement in efficiency. Moreover, the linear systems arising in practical problems are very likely to be inconsistent due to noise, which contradicts the basic assumption in this paper. The extended randomized Kaczmarz [23, 75] was proposed for such cases. It should also be a valuable topic to explore the extensions of DR methods for inconsistent linear systems.

## References

[1] Francisco J Aragón Artacho, Yair Censor, and Aviv Gibali. The cyclic Douglas-Rachford algorithm with $r$-sets-Douglas-Rachford operators. Optim. Methods Softw., 34(4):875-889, 2019.
[2] Francisco J Aragón Artacho, Jonathan M Borwein, and Matthew K Tam. Recent results on DouglasRachford methods for combinatorial optimization problems. J. Optim. Theory Appl., 163(1):1-30, 2014.
[3] Zhong-Zhi Bai and Wen-Ting Wu. On greedy randomized Kaczmarz method for solving large sparse linear systems. SIAM J. Sci. Comput., 40(1):A592-A606, 2018.
[4] Heinz H. Bauschke, J.Y. Bello Cruz, Tran T.A. Nghia, Hung M. Phan, and Xianfu Wang. The rate of linear convergence of the Douglas-Rachford algorithm for subspaces is the cosine of the Friedrichs angle. J. Approx. Theory, 185:63-79, 2014.
[5] Heinz H Bauschke and Jonathan M Borwein. On the convergence of von Neumann's alternating projection algorithm for two sets. Set-Valued Var. Anal., 1(2):185-212, 1993.
[6] Heinz H. Bauschke, Patrick L. Combettes, and D.Russell Luke. Finding best approximation pairs relative to two closed convex sets in Hilbert spaces. J. Approx. Theory, 127(2):178-192, 2004.
[7] Heinz H Bauschke and Valentin R Koch. Projection methods: Swiss army knives for solving feasibility and best approximation problems with halfspaces. Contemp. Math., 636:1-40, 2015.
[8] Heinz H Bauschke, Dominikus Noll, and Hung M Phan. Linear and strong convergence of algorithms involving averaged nonexpansive operators. J. Math. Anal. Appl., 421(1):1-20, 2015.
[9] Amir Beck and Marc Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems. SIAM J. Imaging Sci., 2(1):183-202, 2009.
[10] J. M. Borwein and M. K. Tam. A cyclic Douglas-Rachford iteration scheme. J. Optim. Theory Appl., 160(1):1-29, 2014.
[11] Jonathan M Borwein and Brailey Sims. The Douglas-Rachford algorithm in the absence of convexity. In Fixed-point algorithms for inverse problems in science and engineering, pages 93-109. Springer, 2011.
[12] Stephen Boyd, Arpita Ghosh, Balaji Prabhakar, and Devavrat Shah. Randomized gossip algorithms. IEEE Trans. Inform. Theory, 52(6):2508-2530, 2006.
[13] Luis Briceño-Arias, Julio Deride, and Cristian Vega. Random activations in primal-dual splittings for monotone inclusions with a priori information. J. Optim. Theory Appl., pages 1-26, 2022.
[14] Minh N. Bùi and Patrick L. Combettes. The Douglas-Rachford algorithm converges only weakly. SIAM J. Control Optim., 58(2):1118-1120, 2020.
[15] Charles Byrne. A unified treatment of some iterative algorithms in signal processing and image reconstruction. Inverse Probl., 20(1):103-120, 2003.
[16] Yong Cai, Yang Zhao, and Yuchao Tang. Exponential convergence of a randomized Kaczmarz algorithm with relaxation. In Proceedings of the 2011 2nd International Congress on Computer Applications and Computational Science, pages 467-473. Springer, 2012.
[17] Yair Censor and Rafiq Mansour. New Douglas-Rachford algorithmic structures and their convergence analyses. SIAM J. Optim., 26(1):474-487, 2016.
[18] Chih-Chung Chang and Chih-Jen Lin. Libsvm: a library for support vector machines. ACM transactions on intelligent systems and technology (TIST), 2(3):1-27, 2011.
[19] Kai-Wei Chang, Cho-Jui Hsieh, and Chih-Jen Lin. Coordinate descent method for large-scale 12-loss linear support vector machines. J. Mach. Learn. Res., 9(7):1369-1398, 2008.
[20] Patrick L Combettes and Jean-Christophe Pesquet. Stochastic quasi-Fejér block-coordinate fixed point iterations with random sweeping. SIAM J. Optim., 25(2):1221-1248, 2015.
[21] Jesus A De Loera, Jamie Haddock, and Deanna Needell. A sampling Kaczmarz-Motzkin algorithm for linear feasibility. SIAM J. Sci. Comput., 39(5):S66-S87, 2017.
[22] Jim Douglas and H. H. Rachford. On the numerical solution of heat conduction problems in two and three space variables. Trans. Amer. Math. Soc., 82(2):421-439, 1956.
[23] Kui Du, Wu-Tao Si, and Xiao-Hui Sun. Randomized extended average block Kaczmarz for solving least squares. SIAM J. Sci. Comput., 42(6):A3541-A3559, 2020.
[24] Yi-Shu Du, Ken Hayami, Ning Zheng, Keiichi Morikuni, and Jun-Feng Yin. Kaczmarz-type inneriteration preconditioned flexible GMRES methods for consistent linear systems. SIAM J. Sci. Comput., 43(5):S345-S366, 2021.
[25] Richard L. Dykstra. An algorithm for restricted least squares regression. J. Amer. Statist. Assoc., 78(384):837-842, 1983.
[26] Jonathan Eckstein and Dimitri P Bertsekas. On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators. Math. Program., 55(1):293-318, 1992.
[27] Saber Elaydi. An introduction to difference equations. Springer New York, NY, 1996.
[28] Jay P Fillmore and Morris L Marx. Linear recursive sequences. SIAM Rev., 10(3):342-353, 1968.
[29] A. Galántai. On the rate of convergence of the alternating projection method in finite dimensional spaces. J. Math. Anal. Appl., 310(1):30-44, 2005.
[30] Guillaume Garrigos and Robert M Gower. Handbook of convergence theorems for (stochastic) gradient methods. arXiv preprint arXiv:2301.11235, 2023.
[31] Euhanna Ghadimi, Hamid Reza Feyzmahdavian, and Mikael Johansson. Global convergence of the heavy-ball method for convex optimization. In 2015 European control conference (ECC), pages 310315. IEEE, 2015.
[32] Eduard Gorbunov, Hugo Berard, Gauthier Gidel, and Nicolas Loizou. Stochastic extragradient: General analysis and improved rates. In International Conference on Artificial Intelligence and Statistics, pages 7865-7901. PMLR, 2022.
[33] Robert M Gower, Denali Molitor, Jacob Moorman, and Deanna Needell. On adaptive sketch-and-project for solving linear systems. SIAM J. Matrix Anal. Appl., 42(2):954-989, 2021.
[34] Robert M. Gower and Peter Richtárik. Randomized iterative methods for linear systems. SIAM J. Matrix Anal. Appl., 36(4):1660-1690, 2015.
[35] Michael Griebel and Peter Oswald. Greedy and randomized versions of the multiplicative Schwarz method. Linear Algebra Appl., 437(7):1596-1610, 2012.
[36] DeRen Han. A survey on some recent developments of alternating direction method of multipliers. $J$. Oper. Res. Soc. China, 10(1):1-52, 2022.
[37] Deren Han and Jiaxin Xie. On pseudoinverse-free randomized methods for linear systems: Unified framework and acceleration. arXiv preprint arXiv:2208.05437, 2022.
[38] Moritz Hardt, Ben Recht, and Yoram Singer. Train faster, generalize better: Stability of stochastic gradient descent. In Proc. 33th Int. Conf. Machine Learning, pages 1225-1234. PMLR, 2016.
[39] Ahmed Hefny, Deanna Needell, and Aaditya Ramdas. Rows versus columns: Randomized Kaczmarz or Gauss-Seidel for ridge regression. SIAM J. Sci. Comput., 39(5):S528-S542, 2017.
[40] Leyu Hu and Xingju Cai. Convergence of a randomized Douglas-Rachford method for linear system. Numer. Algebra Control Optim., 10(4):463-474, 2020.
[41] Yuling Jiao, Bangti Jin, and Xiliang Lu. Preasymptotic convergence of randomized Kaczmarz method. Inverse Probl., 33(12):125012, 2017.
[42] S Karczmarz. Angenäherte auflösung von systemen linearer glei-chungen. Bull. Int. Acad. Pol. Sic. Let., Cl. Sci. Math. Nat., pages 355-357, 1937.
[43] Mohsen Kolodziej, Scott P fnd Aznaveh, Matthew Bullock, Jarrett David, Timothy A Davis, Matthew Henderson, Yifan Hu, and Read Sandstrom. The suitesparse matrix collection website interface. J. Open Source Softw., 4(35):1244, 2019.
[44] Guanghui Lan. First-order and stochastic optimization methods for machine learning. Springer, 2020.
[45] Dennis Leventhal and Adrian S Lewis. Randomized methods for linear constraints: convergence rates and conditioning. Math. Oper. Res., 35(3):641-654, 2010.
[46] Guoyin Li and Ting Kei Pong. Douglas-Rachford splitting for nonconvex optimization with application to nonconvex feasibility problems. Math. Program., 159(1):371-401, 2016.
[47] Scott B. Lindstrom and Brailey Sims. Survey: Sixty years of Douglas-Rachford. J. Aust. Math. Soc., 110(3):333-370, 2021.
[48] P. L. Lions and B. Mercier. Splitting algorithms for the sum of two nonlinear operators. SIAM J. Numer. Anal., 16(6):964-979, 1979.
[49] Ji Liu and Stephen Wright. An accelerated randomized Kaczmarz algorithm. Math. Comp., 85(297):153178, 2016.
[50] Nicolas Loizou and Peter Richtárik. Momentum and stochastic momentum for stochastic gradient, newton, proximal point and subspace descent methods. Comput. Optim. Appl., 77(3):653-710, 2020.
[51] Nicolas Loizou and Peter Richtárik. Revisiting randomized gossip algorithms: General framework, convergence rates and novel block and accelerated protocols. IEEE Trans. Inform. Theory, 67(12):83008324, 2021.
[52] Anna Ma and Deanna Needell. Stochastic gradient descent for linear systems with missing data. Numer. Math. Theory Methods Appl., 12(1):1-20, 2019.
[53] Md Sarowar Morshed, Sabbir Ahmad, et al. Stochastic steepest descent methods for linear systems: Greedy sampling \& momentum. arXiv preprint arXiv:2012.13087, 2020.
[54] Md Sarowar Morshed, Md Saiful Islam, et al. Accelerated sampling Kaczmarz-Motzkin algorithm for the linear feasibility problem. J. Global Optim., 77(2):361-382, 2020.
[55] Md Sarowar Morshed, Md Saiful Islam, and Md Noor-E-Alam. Sampling Kaczmarz-Motzkin method for linear feasibility problems: generalization and acceleration. Math. Program., pages 1-61, 2021.
[56] Ion Necoara. Faster randomized block Kaczmarz algorithms. SIAM J. Matrix Anal. Appl., 40(4):14251452, 2019.
[57] Yurii Nesterov. A method for solving the convex programming problem with convergence rate $\mathrm{O}\left(1 / k^{2}\right)$. In Dokl. akad. nauk Sssr, volume 269, pages 543-547, 1983.
[58] Yurii Nesterov. Introductory lectures on convex optimization: A basic course, volume 87. Springer Science \& Business Media, 2003.
[59] Maxim A Olshanskii and Eugene E Tyrtyshnikov. Iterative methods for linear systems: theory and applications. SIAM, 2014.
[60] Andrei Patrascu and Ion Necoara. Nonasymptotic convergence of stochastic proximal point methods for constrained convex optimization. J. Mach. Learn. Res., 18(1):7204-7245, 2017.
[61] Boris T Polyak. Some methods of speeding up the convergence of iteration methods. Comput. Math. Math. Phys., 4(5):1-17, 1964.
[62] Meisam Razaviyayn, Mingyi Hong, Navid Reyhanian, and Zhi-Quan Luo. A linearly convergent doubly stochastic Gauss-Seidel algorithm for solving linear equations and a certain class of over-parameterized optimization problems. Math. Program., 176(1):465-496, 2019.
[63] Peter Richtárik and Martin Takácv. Stochastic reformulations of linear systems: Algorithms and convergence theory. SIAM J. Matrix Anal. Appl., 41(2):487-524, 2020.
[64] Herbert Robbins and Sutton Monro. A stochastic approximation method. Ann. Math. Statistics, pages 400-407, 1951.
[65] Othmane Sebbouh, Robert M Gower, and Aaron Defazio. Almost sure convergence rates for stochastic gradient descent and stochastic heavy ball. In Conference on Learning Theory, pages 3935-3971. PMLR, 2021.
[66] Changpeng Shao. A deterministic Kaczmarz algorithm for solving linear systems. SIAM J. Matrix Anal. Appl., 44(1):212-239, 2023.
[67] Stefan Steinerberger. Randomized Kaczmarz converges along small singular vectors. SIAM J. Matrix Anal. Appl., 42(2):608-615, 2021.
[68] Stefan Steinerberger. Surrounding the solution of a linear system of equations from all sides. Quart. Appl. Math., 79:419-429, 2021.
[69] Stefan Steinerberger. A weighted randomized Kaczmarz method for solving linear systems. Math. Comp., 90:2815-2826, 2021.
[70] Thomas Strohmer and Roman Vershynin. A randomized Kaczmarz algorithm with exponential convergence. J. Fourier Anal. Appl., 15(2):262-278, 2009.
[71] Ruoyu Sun, Zhi-Quan Luo, and Yinyu Ye. On the efficiency of random permutation for ADMM and coordinate descent. Math. Oper. Res., 45(1):233-271, 2020.
[72] Stephen Wright and Ching-Pei Lee. Analyzing random permutations for cyclic coordinate descent. Math. Comp., 89(325):2217-2248, 2020.
[73] Nian-Ci Wu and Hua Xiang. Convergence analyses based on frequency decomposition for the randomized row iterative method. Inverse Probl., 37(10):105004, 2021.
[74] Zi-Yang Yuan, Lu Zhang, Hongxia Wang, and Hui Zhang. Adaptively sketched Bregman projection methods for linear systems. Inverse Probl., 38(6):065005, 2022.
[75] Anastasios Zouzias and Nikolaos M. Freris. Randomized extended Kaczmarz for solving least squares. SIAM J. Matrix Anal. Appl., 34(2):773-793, 2013.

## 7. Appendix. Proof of the main results

For any $i \in\{1, \ldots, m\}$, we set

$$
T_{C_{i}}:=I-2 \frac{a_{i} a_{i}^{\top}}{\left\|a_{i}\right\|_{2}^{2}}
$$

It is easy to verify that $T_{C_{i}} T_{C_{i}}=I$, and thus for any $y \in \mathbb{R}^{n},\left\|T_{C_{i}} y\right\|_{2}=\|y\|_{2}$.
7.1. Proof of Theorems 3.1 and 4.1. We first prove some crucial facts about the algorithms.

Lemma 7.1. Let $\left\{z_{r}^{k}\right\}_{k=0}^{\infty}$ be the iteration sequence generated by Algorithm 1 or Algorithm
2. Then

$$
\left\|z_{r}^{k}-x^{*}\right\|_{2}=\left\|x^{k}-x^{*}\right\|_{2}
$$

Proof. Note that

$$
\begin{align*}
z_{r}^{k}-x^{*} & =z_{r-1}^{k}-x^{*}-2 \frac{\left\langle a_{j_{k_{r}}}, z_{r-1}^{k}\right\rangle-b_{j_{k_{r}}}}{\left\|a_{j_{k_{r}}}\right\|_{2}^{2}} a_{j_{k_{r}}}=z_{r-1}^{k}-x^{*}-2 \frac{\left\langle a_{j_{k_{r}}}, z_{r-1}^{k}-x^{*}\right\rangle}{\left\|a_{j_{k_{r}}}\right\|_{2}^{2}} a_{j_{k_{r}}} \\
& =\left(I-2 \frac{a_{j_{k_{r}}} a_{j_{k_{r}}}^{\top}}{\left\|a_{j_{k_{r}}}\right\|_{2}^{2}}\right)\left(z_{r-1}^{k}-x^{*}\right)=T_{C_{j_{k_{r}}}}\left(z_{r-1}^{k}-x^{*}\right)  \tag{8}\\
& =T_{C_{j_{k_{r}}}} T_{C_{j_{k_{r-1}}}} \cdots T_{C_{j_{k_{1}}}}\left(z_{0}^{k}-x^{*}\right)=T_{C_{j_{k_{r}}}} T_{C_{j_{k_{r-1}}}} \cdots T_{C_{j_{k_{1}}}}\left(x^{k}-x^{*}\right) .
\end{align*}
$$

Note that for any $y \in \mathbb{R}^{n}$, it holds that $\|y\|_{2}=\left\|T_{C_{j_{k r}}} T_{C_{j_{k_{r-1}}}} \cdots T_{C_{j_{k_{1}}}} y\right\|_{2}$. Hence, we have

$$
\left\|z_{r}^{k}-x^{*}\right\|_{2}=\left\|T_{C_{j_{k r}}} T_{C_{j_{k_{r-1}}}} \cdots T_{C_{j_{k_{1}}}}\left(x^{k}-x^{*}\right)\right\|_{2}=\left\|x^{k}-x^{*}\right\|_{2}
$$

as desired.
Lemma 7.2. Let $\left\{x^{k}\right\}_{k=0}^{\infty}$ and $\left\{z_{r}^{k}\right\}_{k=0}^{\infty}$ be the sequences generated by Algorithm 1 or Algorithm 2. Then

$$
\left\|(1-\alpha) x^{k}+\alpha z_{r}^{k}-x^{*}\right\|_{2}^{2}=\left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle z_{r}^{k}-z^{*}, x^{k}-x^{*}\right\rangle .
$$

Proof. We have

$$
\begin{aligned}
\left\|(1-\alpha) x^{k}+\alpha z_{r}^{k}-x^{*}\right\|_{2}^{2} & =\left\|(1-\alpha)\left(x^{k}-x^{*}\right)+\alpha\left(z_{r}^{k}-x^{*}\right)\right\|_{2}^{2} \\
& =\left\|(1-\alpha)\left(x^{k}-x^{*}\right)\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle x^{k}-x^{*}, z_{r}^{k}-x^{*}\right\rangle+\left\|\alpha\left(z_{r}^{k}-x^{*}\right)\right\|_{2}^{2} \\
& =\left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle z_{r}^{k}-x^{*}, x^{k}-x^{*}\right\rangle,
\end{aligned}
$$

where the last equality follows from Lemma 7.1.

Lemma 7.3. Let $\left\{x^{k}\right\}_{k=0}^{\infty}$ be the sequences generated by Algorithm 1 or Algorithm $2\left(x^{1}=\right.$ $\left.x^{0}\right)$ and $x_{0}^{*}=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}$. Then $x^{k}-x_{0}^{*} \in \operatorname{Row}(A)$.

Proof. Since Algorithm 1 is a special case of Algorithm 2 with $\beta=0$, we only concentrate on the proof for Algorithm 2. We first prove that $x^{k} \in x^{0}+\operatorname{Row}(A)$ for any $k \geq 0$ by induction. Initially, $x^{1}=x^{0} \in x^{0}+\operatorname{Row}(A)$. If $x^{k} \in x^{0}+\operatorname{Row}(A)$ holds for all $k \leq t(t \geq 1)$, then

$$
z_{r}^{t}=x^{t}-2 \frac{\left\langle a_{j_{t_{1}}}, x^{t}-x^{*}\right\rangle}{\left\|a_{j_{1}}\right\|_{2}^{2}} a_{j_{t_{1}}}-\cdots-2 \frac{\left\langle a_{j_{t_{r}}}, z_{r-1}^{t}-x^{*}\right\rangle}{\left\|a_{j_{t_{r}}}\right\|_{2}^{2}} a_{j_{t_{r}}} \in x^{0}+\operatorname{Row}(A)
$$

and $x^{t}-x^{t-1} \in \operatorname{Row}(A)$. Therefore,

$$
\begin{aligned}
x^{t+1} & =(1-\alpha) x^{t}+\alpha z_{r}^{t}+\beta\left(x^{t}-x^{t-1}\right) \\
& \in(1-\alpha)\left(x^{0}+\operatorname{Row}(A)\right)+\alpha\left(x^{0}+\operatorname{Row}(A)\right)+\operatorname{Row}(A) \\
& =x^{0}+\operatorname{Row}(A) .
\end{aligned}
$$

Thus $x^{k} \in x^{0}+\operatorname{Row}(A)$ holds for all $k \geq 0$. Note that $x_{0}^{*}=A^{\dagger} b+\left(I-A^{\dagger} A\right) x^{0}=$ $A^{\dagger}\left(b-A x^{0}\right)+x^{0} \in x^{0}+\operatorname{Row}(A)$, we arrive at the conclusion $x^{k}-x_{0}^{*} \in \operatorname{Row}(A)$.

Now we are ready to prove the main results. In fact, Theorem 3.1 can be directly derived from Theorem 4.1 by letting $\beta=0$. We include an individual proof of Theorem 3.1 for readability, meanwhile showing the tightness of the convergence rate in a clear way.

Proof of Theorem 3.1. From Lemma 7.2 and taking the conditional expectation under the probability $\operatorname{Pr}\left(\right.$ row $\left.=j_{k_{\ell}}\right)=\frac{\left\|a_{j_{k}}\right\|_{2}^{2}}{\|A\|_{F}^{2}}$, we get

$$
\begin{align*}
& \underset{j_{k}, \ldots, j_{k_{1}}}{\mathbb{E}}\left[\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2} \mid x^{k}\right]=\underset{j_{k_{r}}, \ldots, j_{k_{1}}}{\mathbb{E}}\left[\left\|(1-\alpha) x^{k}+\alpha z_{r}^{k}-x_{0}^{*}\right\|_{2}^{2} \mid x^{k}\right]  \tag{9}\\
= & \left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha) \underset{j_{k_{r}}, \ldots, j_{k_{1}}}{\mathbb{E}}\left[\left\langle T_{C_{j_{k_{r}}}} T_{C_{j_{k_{r-1}}}} \ldots T_{C_{j_{k_{1}}}}\left(x^{k}-x_{0}^{*}\right), x^{k}-x_{0}^{*}\right\rangle\right] \\
= & \left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle\underset{j_{k_{r}}}{\mathbb{E}}\left[T_{C_{j_{k_{r}}}}\right] \ldots \underset{j_{k_{1}}}{\mathbb{E}}\left[T_{C_{j_{k_{1}}}}\right]\left(x^{k}-x_{0}^{*}\right), x^{k}-x_{0}^{*}\right\rangle \\
= & \left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x_{0}^{*}\right), x^{k}-x_{0}^{*}\right\rangle .
\end{align*}
$$

The first equality follows from Step 6 in Algorithm 1; the second equality follows from (8); the third equality follows from the linearity of the expectation and the independence of
$j_{k_{1}}, \ldots, j_{k_{r}}$; the last equality follows from the fact that for any $\ell \in\{1, \ldots, r\}$,

$$
\begin{equation*}
\underset{j_{k_{\ell}}}{\mathbb{E}}\left[T_{C_{j_{k_{\ell}}}}\right]=\sum_{i=1}^{m} \frac{\left\|a_{i}\right\|_{2}^{2}}{\|A\|_{F}^{2}}\left(I-2 \frac{a_{i} a_{i}^{\top}}{\left\|a_{i}\right\|_{2}^{2}}\right)=I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}} \tag{10}
\end{equation*}
$$

By Lemma 7.3, we know that $x^{k}-x_{0}^{*} \in \operatorname{Row}(A)=\operatorname{Range}\left(A^{\top}\right)$. Then we have

$$
\begin{equation*}
\left(x^{k}-x_{0}^{*}\right)^{\top}\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x_{0}^{*}\right) \leq\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2} \tag{11}
\end{equation*}
$$

which implies

$$
\underset{j_{k_{r}}, \ldots, j_{k_{1}}}{\mathbb{E}}\left[\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2} \mid x^{k}\right] \leq\left(\alpha^{2}+(1-\alpha)^{2}+2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}
$$

Taking the expectation over the entire history we have

$$
\mathbb{E}\left[\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2}\right] \leq\left(\alpha^{2}+(1-\alpha)^{2}+2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right) \mathbb{E}\left[\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}\right]
$$

By induction on the iteration index $k$, we can obtain the desired result.
Remark 7.4. If $\sigma_{1}(A)=\sigma_{\min }(A)$, that is, all nonzero singular values of $A$ are equal, then the inequality in (11) becomes equality. As a result, the upper bound in Theorem 3.1 is also equality, indicating that the upper bound in Theorem 3.1 is tight.

To prove Theorem 4.1, the following result is required.
Lemma 7.5 ([37], Lemma 8.1). Fix $F^{1}=F^{0} \geq 0$ and let $\left\{F^{k}\right\}_{k \geq 0}$ be a sequence of nonnegative real numbers satisfying the relation

$$
F^{k+1} \leq \gamma_{1} F^{k}+\gamma_{2} F^{k-1}, \quad \forall k \geq 1
$$

where $\gamma_{2} \geq 0, \gamma_{1}+\gamma_{2}<1$ and at least one of the coefficients $\gamma_{1}, \gamma_{2}$ is positive. Then the sequence satisfies the relation

$$
F^{k+1} \leq q^{k}(1+\tau) F^{0}, \quad \forall k \geq 0,
$$

where $q=\frac{\gamma_{1}+\sqrt{\gamma_{1}^{2}+4 \gamma_{2}}}{2}$ and $\tau=q-\gamma_{1} \geq 0$. Moreover, $q \geq \gamma_{1}+\gamma_{2}$, with equality if and only if $\gamma_{2}=0$.

Now, we are going to prove Theorem 4.1.

Proof of Theorem 4.1. First, we have

$$
\begin{align*}
& \left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2}=\left\|(1-\alpha) x^{k}+\alpha z_{r}^{k}+\beta\left(x^{k}-x^{k-1}\right)-x_{0}^{*}\right\|_{2}^{2}  \tag{12}\\
& =\underbrace{\left\|(1-\alpha) x^{k}+\alpha z_{r}^{k}-x_{0}^{*}\right\|_{2}^{2}}_{\text {(a) }}+\underbrace{2 \beta\left\langle(1-\alpha) x^{k}+\alpha z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle}_{\text {(b) }}+\underbrace{\beta^{2}\left\|x^{k}-x^{k-1}\right\|_{2}^{2}}_{\text {(c) }} .
\end{align*}
$$

We now analyze the three expressions (a), (b), (c) separately. From Lemma 7.2, we have

$$
\text { (a) }=\left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle z_{k}^{r}-x_{0}^{*}, x^{k}-x_{0}^{*}\right\rangle .
$$

We now bound the second expression. First, we have

$$
\text { (b) } \begin{aligned}
& =2(1-\alpha) \beta\left\langle x^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle+2 \alpha \beta\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle \\
& =2(1-\alpha) \beta\left\langle x^{k}-x_{0}^{*}, x^{k}-x_{0}^{*}\right\rangle+2(1-\alpha) \beta\left\langle x^{k}-x_{0}^{*}, x_{0}^{*}-x^{k-1}\right\rangle+2 \alpha \beta\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle \\
& =2(1-\alpha) \beta\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2(1-\alpha) \beta\left\langle x^{k}-x_{0}^{*}, x_{0}^{*}-x^{k-1}\right\rangle+2 \alpha \beta\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle .
\end{aligned}
$$

Noting that $2\left\langle x^{k}-x_{0}^{*}, x_{0}^{*}-x^{k-1}\right\rangle \leq\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2}$, which implies

$$
\text { (b) } \leq 3(1-\alpha) \beta\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+(1-\alpha) \beta\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha \beta\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle \text {. }
$$

The third expression can be bounded by

$$
\text { (c) } \leq 2 \beta^{2}\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \beta^{2}\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2} .
$$

By substituting all the bounds into (12), we obtain

$$
\begin{aligned}
\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2} \leq & \left(\alpha^{2}+(1-\alpha)^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha(1-\alpha)\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x_{0}^{*}\right\rangle \\
& +3(1-\alpha) \beta\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+(1-\alpha) \beta\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2}+2 \alpha \beta\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle \\
& +2 \beta^{2}\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+2 \beta^{2}\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2} \\
\leq & \left(\alpha^{2}+(1-\alpha)^{2}+3(1-\alpha) \beta+2 \beta^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+\left((1-\alpha) \beta+2 \beta^{2}\right)\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2} \\
& +2 \alpha(1-\alpha)\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x_{0}^{*}\right\rangle+2 \alpha \beta\left\langle z_{r}^{k}-x_{0}^{*}, x^{k}-x^{k-1}\right\rangle
\end{aligned}
$$

Now taking the conditional expectation under the probability $\operatorname{Pr}\left(\right.$ row $\left.=j_{k_{\ell}}\right)=\frac{\left\|a_{j_{k}}\right\|_{2}^{2}}{\|A\|_{F}^{2}}$, we get

$$
\begin{align*}
\mathbb{E}\left[\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2} \mid x^{k}\right] \leq & \left(\alpha^{2}+(1-\alpha)^{2}+3(1-\alpha) \beta+2 \beta^{2}\right)\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2} \\
& +\left((1-\alpha) \beta+2 \beta^{2}\right)\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2} \\
& +\underbrace{2 \alpha(1-\alpha)\left\langle\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x_{0}^{*}\right), x^{k}-x_{0}^{*}\right\rangle}_{\text {© }}  \tag{13}\\
& +\underbrace{2 \alpha \beta\left\langle\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x_{0}^{*}\right), x^{k}-x^{k-1}\right\rangle}_{\text {© }} .
\end{align*}
$$

Similar to the argument in (11), we know that

$$
\text { (d) } \leq 2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}
$$

For expression (e), we have

$$
\begin{aligned}
(\mathrm{e} & =2 \alpha \beta\left\langle\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x_{0}^{*}\right), x^{k}-x_{0}^{*}\right\rangle+2 \alpha \beta\left\langle\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x_{0}^{*}\right), x_{0}^{*}-x^{k-1}\right\rangle \\
& \leq 3 \alpha \beta\left\|\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r / 2}\left(x^{k}-x_{0}^{*}\right)\right\|_{2}^{2}+\alpha \beta\left\|\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r / 2}\left(x_{0}^{*}-x^{k-1}\right)\right\|_{2}^{2} \\
& \leq 3 \alpha \beta\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}+\alpha \beta\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2},
\end{aligned}
$$

where the first inequality follows from $2\langle v, u\rangle \leq\|v\|_{2}^{2}+\|u\|_{2}^{2}$ and the second inequality follows from the fact that $x^{k}-x_{0}^{*} \in \operatorname{Row}(A), x^{k-1}-x_{0}^{*} \in \operatorname{Row}(A)$, and (11). Hence

$$
\begin{aligned}
& \mathbb{E}\left[\left\|x^{k+1}-x_{0}^{*}\right\|_{2}^{2} \mid x^{k}\right] \leq \underbrace{\left(2 \beta^{2}+(1-\alpha) \beta+\alpha \beta\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}\right)}_{\gamma_{2}}\left\|x^{k-1}-x_{0}^{*}\right\|_{2}^{2} \\
& +\underbrace{\left(\alpha^{2}+(1-\alpha)^{2}+(2 \alpha(1-\alpha)+3 \alpha \beta)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}+2 \beta^{2}+3(1-\alpha) \beta\right)}_{\gamma_{1}}\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}
\end{aligned}
$$

By taking expectation over the entire history, and letting $F^{k}:=\mathbb{E}\left[\left\|x^{k}-x_{0}^{*}\right\|_{2}^{2}\right]$, we get the relation

$$
\begin{equation*}
F^{k+1} \leq \gamma_{1} F^{k}+\gamma_{2} F^{k-1} \tag{14}
\end{equation*}
$$

Noting that the requirements for the coefficients $\gamma_{1}$ and $\gamma_{2}$ in Lemma 7.5 are fulfilled, i.e. $\gamma_{2} \geq 0, \gamma_{1}+\gamma_{2}<1$ and at least one of the coefficients $\gamma_{1}, \gamma_{2}$ is positive. Indeed, since $\alpha \in(0,1)$ and $\beta \geq 0$, we know that $\gamma_{2} \geq 0$. If $\gamma_{2}=0$, then $\beta=0$ and now we have

$$
\gamma_{1}=\alpha^{2}+(1-\alpha)^{2}+2 \alpha(1-\alpha)\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r}>0
$$

which implies that at least one of the coefficients $\gamma_{1}, \gamma_{2}$ is positive. The condition $\gamma_{1}+\gamma_{2}<1$ holds by the assumption. Then apply Lemma 7.5 to (14), one can get the theorem.

Remark 7.6. In (13), the inequality $2\langle v, u\rangle \leq\|v\|_{2}^{2}+\|u\|_{2}^{2}$ was used to estimate the expression (e). For future work, one may improve this estimate by using the parameterized Young's inequality $2\langle v, u\rangle \leq \varepsilon\|v\|_{2}^{2}+\frac{1}{\varepsilon}\|u\|_{2}^{2}$ and optimizing $\varepsilon$ over $\varepsilon>0$.
7.2. Proof of Theorems 3.5, 4.2, and 3.6. The following lemmas are useful for our proof.

Lemma 7.7. Consider the singular value decomposition $A=U \Sigma V^{\top}$ and let $\left\{x^{k}\right\}_{k=0}^{\infty}$ be the sequences generated by Algorithm 2 or Algorithm $1(\beta=0)$. Then

$$
V^{\top} \mathbb{E}\left[x^{k+1}-x^{*}\right]=\left((1-\alpha+\beta) I+\alpha\left(I-\frac{2 \Sigma^{\top} \Sigma}{\|A\|_{F}^{2}}\right)^{r}\right) V^{\top} \mathbb{E}\left[x^{k}-x^{*}\right]-\beta V^{\top} \mathbb{E}\left[x^{k-1}-x^{*}\right] .
$$

Proof. First, we have

$$
\begin{aligned}
x^{k+1}-x^{*} & =(1-\alpha) x^{k}+\alpha z_{r}^{k}+\beta\left(x^{k}-x^{k-1}\right)-x^{*} \\
& =(1-\alpha)\left(x^{k}-x^{*}\right)+\alpha\left(z_{r}^{k}-x^{*}\right)+\beta\left(x^{k}-x^{k-1}\right) .
\end{aligned}
$$

Taking expectations, we have

$$
\begin{aligned}
\mathbb{E}\left[x^{k+1}-x^{*} \mid x^{k}\right] & =(1-\alpha)\left(x^{k}-x^{*}\right)+\alpha \mathbb{E}\left[z_{r}^{k}-x^{*} \mid x^{k}\right]+\beta\left(x^{k}-x^{k-1}\right) \\
& =(1-\alpha)\left(x^{k}-x^{*}\right)+\alpha\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\left(x^{k}-x^{*}\right)+\beta\left(x^{k}-x^{k-1}\right) \\
& =\left((1-\alpha+\beta) I+\alpha\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\right)\left(x^{k}-x^{*}\right)-\beta\left(x^{k-1}-x^{*}\right),
\end{aligned}
$$

where the second equality follows from (10). Taking the expectations again, we have

$$
\begin{equation*}
\mathbb{E}\left[x^{k+1}-x^{*}\right]=\left((1-\alpha+\beta) I+\alpha\left(I-2 \frac{A^{\top} A}{\|A\|_{F}^{2}}\right)^{r}\right) \mathbb{E}\left[x^{k}-x^{*}\right]-\beta \mathbb{E}\left[x^{k-1}-x^{*}\right] . \tag{15}
\end{equation*}
$$

Plugging $A^{\top} A=V \Sigma^{\top} \Sigma V^{\top}$ into (15), and multiplying both sides form the left by $V^{\top}$, we can get the lemma.

Lemma 7.8. Suppose the nonzero singular values of $A \in \mathbb{R}^{m \times n}$ are $\sigma_{1}(A) \geq \sigma_{2}(A) \geq \ldots \geq$ $\sigma_{t}(A)=\sigma_{\min }^{2}(A)$ and $r \in \mathbb{Z}_{+}$. Then for any $1 \leq i \leq t$,

$$
\begin{equation*}
\left(1-2 \frac{\sigma_{i}^{2}(A)}{\|A\|_{F}^{2}}\right)^{r} \leq\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{r} \tag{16}
\end{equation*}
$$

Proof. Since

$$
1-2 \frac{\sigma_{i}^{2}(A)}{\|A\|_{F}^{2}} \leq 1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}
$$

we know that (16) holds provided that $r$ is odd. Next, we consider the case where $r$ is even. Since
$\left(1-2 \frac{\sigma_{i}^{2}(A)}{\|A\|_{F}^{2}}\right)^{2}-\left(1-2 \frac{\sigma_{\min }^{2}(A)}{\|A\|_{F}^{2}}\right)^{2}=\frac{4\left(\sigma_{\min }^{2}(A)-\sigma_{i}^{2}(A)\right)\left(\|A\|_{F}^{2}-\sigma_{\min }^{2}(A)-\sigma_{i}^{2}(A)\right)}{\|A\|_{F}^{4}} \leq 0$, we know that $\left|1-2 \frac{\sigma_{i}^{2}(A)}{\mid A \|_{F}^{2}}\right| \leq\left|1-2 \frac{\sigma_{\text {min }}^{2}(A)}{\|A\|_{F}^{2}}\right|$. This implies that (16) holds for the case where $r$ is even.

Lemma 7.9 ( $[27,28])$. Consider the second degree linear homogeneous recurrence relation:

$$
r^{k+1}=\gamma_{1} r^{k}+\gamma_{2} r^{k-1}
$$

with initial conditions $r^{0}, r^{1} \in \mathbb{R}$. Assume that the constant coefficients $\gamma_{1}$ and $\gamma_{2}$ satisfy the inequality $\gamma_{1}^{2}+4 \gamma_{2}<0$ (the roots of the characteristic equation $t^{2}-\gamma_{1} t-\gamma_{2}=0$ are imaginary). Then there are complex constants $c_{0}$ and $c_{1}$ (depending on the initial conditions $r_{0}$ and $r_{1}$ ) such that:

$$
r^{k}=2 M^{k}\left(c_{0} \cos (\theta k)+c_{1} \sin (\theta k)\right)
$$

where $M=\left(\sqrt{\frac{\gamma_{1}^{2}}{4}+\frac{\left(-\gamma_{1}^{2}-4 \gamma_{2}\right)}{4}}\right)=\sqrt{-\gamma_{2}}$ and $\theta$ is such that $\gamma_{1}=2 M \cos (\theta)$ and $\sqrt{-\gamma_{1}^{2}-4 \gamma_{2}}=$ $2 M \sin (\theta)$.

We first prove Theorem 4.2.
Proof of Theorem 4.2. Set $s^{k}:=V^{\top} \mathbb{E}\left[x^{k}-x_{0}^{*}\right]$. Then from Lemma 7.7 we have

$$
s^{k+1}=\left((1-\alpha+\beta) I+\alpha\left(I-\frac{2 \Sigma^{\top} \Sigma}{\|A\|_{F}^{2}}\right)^{r}\right) s^{k}-\beta s^{k-1},
$$

which can be rewritten in a coordinate form as follows:

$$
\begin{equation*}
s_{i}^{k+1}=\left((1-\alpha+\beta)+\alpha\left(1-2 \sigma_{i}^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right) s_{i}^{k}-\beta s_{i}^{k-1}, \quad \forall i=1,2, \ldots, n, \tag{17}
\end{equation*}
$$

where $s_{i}^{k}$ indicates the $i$-th coordinate of $s^{k}$.
We now consider two cases: $\sigma_{i}(A)=0$ or $\sigma_{i}(A)>0$.

If $\sigma_{i}(A)=0$, then (17) takes the form:

$$
s_{i}^{k+1}=(1+\beta) s_{i}^{k}-\beta s_{i}^{k-1}
$$

Since $x^{1}-x_{0}^{*}=x^{0}-x_{0}^{*}=A^{\dagger}\left(A x^{0}-b\right)$, we have $s_{i}^{0}=s_{i}^{1}=v_{i}^{\top} A^{\dagger}\left(A x^{0}-b\right)=0$. So

$$
s_{i}^{k}=0 \text { for all } k \geq 0 .
$$

If $\sigma_{i}(A)>0$. We use Lemma 7.9 to establish the desired bound. By the selection constraints of $\alpha$, one can verify that $(1-\alpha)+\alpha\left(1-2 \sigma_{i}^{2}(A) /\|A\|_{F}^{2}\right)^{r} \geq 0$ and since $\beta \geq 0$, we have $(1-\alpha)+\beta+\alpha\left(1-2 \sigma_{i}^{2}(A) /\|A\|_{F}^{2}\right)^{r} \geq 0$ and hence

$$
\begin{aligned}
\gamma_{1}^{2}+4 \gamma_{2} & =\left(1-\alpha+\beta+\alpha\left(1-2 \sigma_{i}^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)^{2}-4 \beta \\
& \leq\left(1-\alpha+\beta+\alpha\left(1-2 \sigma_{\min }^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)^{2}-4 \beta \\
& <0,
\end{aligned}
$$

where the first inequality follows from Lemma 7.8 and the last inequality follows from the assumption that $\left(1-\sqrt{\alpha\left(1-\left(1-2 \sigma_{\min }^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)}\right)^{2}<\beta<1$. Using Lemma 7.9, the following bound can be deduced

$$
s_{i}^{k}=2\left(-\gamma_{2}\right)^{k / 2}\left(c_{0} \cos (\theta k)+c_{1} \sin (\theta k)\right) \leq 2 \beta^{k / 2} p_{i}
$$

where $p_{i}$ is a constant depending on the initial conditions (we can simply choose $p_{i}=$ $\left|c_{0}\right|+\left|c_{1}\right|$.) Now put the two cases together, for all $k \geq 0$ we have

$$
\begin{aligned}
\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2} & =\left\|V^{\top} \mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2}=\left\|s^{k}\right\|^{2}=\sum_{i=1}^{n}\left(s_{i}^{k}\right)^{2}=\sum_{i: \sigma_{i}(A)=0}\left(s_{i}^{k}\right)^{2}+\sum_{i: \sigma_{i}(A)>0}\left(s_{i}^{k}\right)^{2} \\
& =\sum_{i: \sigma_{i}(A)>0}\left(s_{i}^{k}\right)^{2} \leq \sum_{i: \sigma_{i}(A)>0} 4 \beta^{k} p_{i}^{2}=\beta^{k} c,
\end{aligned}
$$

where $c=4 \sum_{i: \lambda_{i}>0} p_{i}^{2}$.
Proof of Theorem 3.5. Theorem 3.5 can be directly derived from Lemma 7.7. Indeed, using similar arguments as that in the proof of Theorem 4.2, we can get

$$
\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2}=\sum_{i: \sigma_{i}(A)>0}\left(s_{i}^{k}\right)^{2},
$$

where $s_{i}^{k}=\left((1-\alpha)+\alpha\left(1-2 \sigma_{i}^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right) s_{i}^{k-1}, i=1, \ldots, n$. It follows from Lemma 7.8, we have

$$
\left\|\mathbb{E}\left[x^{k}-x_{0}^{*}\right]\right\|_{2}^{2} \leq\left(1-\alpha+\alpha\left(1-2 \sigma_{\min }^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right)^{2 k}\left\|x^{0}-x_{0}^{*}\right\|_{2}^{2} .
$$

This completes the proof of the theorem.

Proof of Theorem 3.6. By Lemma 7.7 and using similar arguments as that in the proof of Theorem 4.2 and (17), we know that

$$
s_{i}^{k+1}=\left((1-\alpha)+\alpha\left(1-2 \sigma_{i}^{2}(A) /\|A\|_{F}^{2}\right)^{r}\right) s_{i}^{k}, \quad \forall i=1,2, \ldots, n,
$$

where $s_{i}^{k+1}=\mathbb{E}\left[\left\langle x^{k+1}-x^{*}, v_{i}\right\rangle\right]$ and hence the theorem holds.

LMiB of the Ministry of Education, School of Mathematical Sciences, Beihang University, Beijing, 100191, China.

Email address: handr@buaa.edu.cn

School of Mathematical Sciences, Beihang University, Beijing, 100191, China.
Email address: suyansheng@buaa.edu.cn
LMIB of the Ministry of Education, School of Mathematical Sciences, Beihang University, Beijing, 100191, China.

Email address: xiejx@buaa.edu.cn


[^0]:    Key words: Douglas-Rachford; Randomization; Heavy ball momentum; Convergence rate; Linear systems; Kaczmarz method

    Mathematics subject classification (2020): 90C25, 65F10, 65F20, 68W20

