SYSTEM OF PHASE OSCILLATORS WITH DIAGONALIZABLE INTERACTION

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Abstract. We consider a system of N phase oscillators having randomly distributed natural frequencies and diagonalizable interactions among the oscillators. We show that in the limit of $N \to \infty$, all solutions of such a system are incoherent with probability one for any strength of coupling, which implies that there is no sharp transition from incoherence to coherence as the coupling strength is increased, in striking contrast to Kuramoto's (special) oscillator system.

Key words. Network of phase oscillators, Kuramoto model

AMS subject classifications. 34C15, 37N25, 37N20

1. Introduction. Synchronization of coupled oscillators is a ubiquitous phenomenon in natural and artificial systems. Examples include synchronization of pacemaker cells of the heart [11, 12], rhythmic activities in the brain [4, 13], synchronous flashing of fireflies [1, 2], arrays of lasers [8, 9], and superconducting Josephson junctions [18, 19]. Characterization of the phenomenon using mathematical models has been a topic of great interest for researchers in various scientific and engineering disciplines.

Wiener [16, 17], who recognized the ubiquity of synchronization phenomena in the real world, made a first attempt at characterization using the Fourier integrals. A more successful approach was taken by Winfree [20], who used a population of interacting limit-cycle oscillators to describe synchronization properties. He realized that if the interactions among the oscillators are weak and the oscillators are nearly identical, the separation of fast and slow timescales leads to a reduced model that can be expressed solely in terms of the phase of each oscillator. Kuramoto [10] put this idea on firmer foundation by employing a perturbation method to show that the reduced equation has a universal form. His analysis of this model in the case of mean-field coupling kicked off an avalanche of theoretical investigations of his model and its generalizations.

More generally and rigorously, if each oscillator has an exponentially stable limit-cycle and interactions among them are weak, the reduced phase equation can be shown (see Theorem 9.1 [6, p. 253]) to have the form

$$\dot{\theta} = \omega + \varepsilon f(\theta), \quad \theta \in \mathbb{T}^N, \tag{1.1}$$

where $\omega \in \mathbb{R}^N$ is the vector of natural frequencies of the oscillators that are coupled to one another through the interaction function $f : \mathbb{T}^N \to \mathbb{R}^N$, and $\varepsilon > 0$ represents the overall strength of the coupling. The universal form of the interaction function derived by Kuramoto [10] under additional assumption that the oscillators are almost identical, corresponds to the choice

$$f_i(\theta) = \sum_{j=1}^N h_{ij}(\theta_j - \theta_i), \quad i = 1, \dots, N, \tag{1.2}$$

where $f(\theta) = (f_1(\theta), \dots, f_N(\theta))^T$. The mean-field model he studied results when $h_{ij}(x) = \sin(x)/N$ for all i, j.

Let $\theta(t)$ be a solution of the equation (1.1). The oscillators i and j are said to be *locked* if $\lim_{t \to \infty} \theta_i(t)/\theta_j(t) = 1$. The solution is said to be *coherent* if all pairs of oscillators are locked. If none of the oscillator pairs are locked, the solution is *incoherent*. A solution that is neither coherent nor incoherent is called *partially coherent*. The main conclusion of Kuramoto's work [10] on his mean-field model is that in the limit of $N \to \infty$, there exists a critical coupling strength ε_c such that for $\varepsilon < \varepsilon_c$ the solution is incoherent, but for $\varepsilon > \varepsilon_c$ partially coherent solutions appear, for which the fraction of locked oscillator pairs is nonzero. Although his result was important since this behavior closely resembles the phase transition phenomena widely observed in statistical physics, his analysis is heuristic and makes assumptions about the symmetry of the distribution of natural frequencies, which might not be necessary for the results [14].

In this paper, we consider a class of *diagonalizable* interaction functions, in which separation of variables is possible after an appropriate coordinate transformation. This allows us to prove rigorously that for the system (1.1) with a generic diagonalizable interaction function, if the solution is partially coherent, then it is almost surely coherent. This, together with the fact that the probability of having a coherent solution goes to zero in the limit of $N \to \infty$, leads to our main conclusion. Namely, for any $\varepsilon > 0$, the solution is almost surely incoherent, in the limit of $N \to \infty$. Our result shows that a diagonalizable system of phase oscillators cannot exhibit sudden transition from incoherence to coherence, in sharp contrast to the mean-field model of Kuramoto. This implies that for the system (1.1) to exhibit a phase transition, the interaction function f that cannot be diagonalized.

There is an alternative rigorous approach to Kuramoto's mean-field model, in which the partial differential equation for the density of oscillators with certain frequency, which is obtained by taking the continuum limit $N \to \infty$, is studied to analyze the stability of the solutions. See [15] for an excellent review in this direction.

The approach taken here is similar to that in [5, p. 80]. However, some conclusions made there might be misleading or lack detailed analysis. This paper is intended to correct and clarify these points.

The rest of the paper is organized as following. In §2, we introduce an appropriate change of variables to separate a time-like variable from the rest of the system. In §3, we define diagonalizable interaction and show how complete separation of variables can be achieved. We also establish some properties of diagonalizable systems. Then, in §4, we introduce randomness of the natural frequencies of the oscillators and state our main results. Finally, we discuss some approximate behavior of the system for large N in §6, and §7 is reserved for concluding remarks.

2. Separation of the Time-like Variable. In this and the following sections, we consider the system (1.1) of N phase oscillators where the natural frequency vector ω and the coupling strength ε are fixed (nonrandom) constants. We will consider ω to be a random vector in §4 in order to make probabilistic statements about the system.

Let us suppose that the interaction function f satisfies two conditions
(C1) $\mathbf{1}^T f(\theta) = \mathbf{0}$ for all $\theta \in \mathbb{T}^N$ and
(C2) $f(v\mathbf{1} + \theta) = f(\theta)$ for all $\theta \in \mathbb{T}^N$,
where $\mathbf{1} = (1/\sqrt{N}, \dots, 1/\sqrt{N})^T$. The condition (C1) says that the interaction function is orthogonal to the vector $\mathbf{1}$. The second condition (C2) expresses the translation invariance of f along the direction of $\mathbf{1}$. If, for example, the interaction function has

the form (1.2), these conditions are satisfied if the functions h_{ij} are odd. In particular, the mean-field model of Kuramoto does satisfy these conditions.

Under conditions (C1) and (C2), the system (1.1) can be separated into two independent systems—one for the time-like variable and the other for the phase deviations.

Let W be an $N \times (N-1)$ matrix whose columns, denoted by W_j, $j = 1, \ldots, N-1$, form an orthonormal basis of the subspace $\mathbf{1}^\perp \equiv \{x \in \mathbb{R}^N : \mathbf{1}^T x = 0\}$. In other words, W is an $N \times (N-1)$ matrix that satisfies $\mathbf{1}^T W = \mathbf{0}$ and $W^T W = I_{N-1}$, where I_{N-1} is the $(N-1) \times (N-1)$ identity matrix. Then, the change of variable

$$\theta = v\mathbf{1} + Wu \tag{2.1}$$

takes the system (1.1) into two systems:

$$\dot{v} = \mathbf{1}^T \omega, \tag{2.2}$$
$$\dot{u} = W^T \omega + \varepsilon W^T f(Wu) \tag{2.3}$$

that can be solved separately.

Systems satisfying the conditions (C1) and (C2) arise in mathematical neuroscience [5, 6], in which θ often takes the form $\omega t + \phi$ in the limit $t \to \infty$, where ω is the vector of carrier frequencies and ϕ is the vector of phase deviations. The equation (2.3), in some sense, governs the behavior of the phase deviations.

The solution to the equation (2.2) is $v(t) = v(0) + (\mathbf{1}^T \omega)t$, and hence the variable v is time-like if $\mathbf{1}^T \omega \neq 0$, or equivalently, if the average natural frequency $\sum_i \omega_i / N$ is nonzero. Thus, the behavior of the solution of (1.1) is essentially determined by (2.3).

Recall that the solution is called coherent if $\lim_{t \to \infty} \theta_i(t)/\theta_j(t) = 1$. This can be rephrased in terms of the vector $\mu \equiv \lim_{t \to \infty} u(t)/t$ of output frequencies of the u-equation (2.3), if it exists.

LEMMA 2.1. *Let $u(t)$ be a solution of (2.3) and suppose $\mu \equiv \lim_{t \to \infty} u(t)/t$ exists. Then, the solution of (1.1) is coherent if and only if $\mu = 0$.*

Proof. Let $\Omega = \lim_{t \to \infty} \theta(t)/t = (\mathbf{1}^T \omega)\mathbf{1} + W\mu$. Then, $\mu = 0$ implies that $\Omega = (\mathbf{1}^T \omega)\mathbf{1}$, which in turn implies that $\lim_{t \to \infty} \theta_i(t)/\theta_j(t) = 1$.

Conversely, if $\lim_{t \to \infty} \theta_i(t)/\theta_j(t) = 1$, Ω must be a multiple of $\mathbf{1}$. Since W is orthogonal to $\mathbf{1}$, this implies that $W\mu = 0$. Since W is invertible, it follows that $\mu = 0$. □

As an immediate consequence of Lemma 2.1, if the solution of (2.3) tends to an equilibrium, then the corresponding solution of (1.1) is coherent. For example, if the interaction function f that satisfies (C1) and (C2), is a gradient vector field, i.e., $f(\theta) = -\nabla V_0(\theta)$ for some potential function $V_0 : \mathbb{T}^N \to \mathbb{R}$, then the u-equation (2.3) is also a gradient system:

$$\dot{u} = -\nabla \left[-\omega^T Wu + \varepsilon V(u) \right], \tag{2.4}$$

where $V(u) = V_0(Wu)$. A minimum u^* of the potential function $-\omega^T Wu + \varepsilon V(u)$ then corresponds to the vector of phase deviations for a coherent solution of the original oscillator system (1.1). As in the proof of Lemma 2.1, we have $\Omega = (\mathbf{1}^T \omega)\mathbf{1}$ in this case, meaning that the output frequency of every oscillator tends to the mean natural frequency $\bar{\omega} \equiv \sum_i \omega_i / N$ of the oscillators. For the interaction of the form (1.2), the potential function takes the form

$$V(u) = -\frac{1}{2N} \sum_{i=1}^N \sum_{j=1}^N H_{ij}(\theta_i - \theta_j) = -\frac{1}{2N} \sum_{i=1}^N \sum_{j=1}^N H_{ij} \left(\sum_{k=1}^{N-1} [W_{ik} - W_{jk}]u_k \right),$$

where $H_{ij}(x) \equiv \int_0^x h_{ij}(y)dy$.

3. Diagonalizable interaction. Assuming that the interactions among the oscillators are diagonalizable enables us to carry out a rigorous analysis of the system.

DEFINITION 3.1. *We say that the system* (1.1) *(or the interaction function f) is* diagonalizable *if there exists a $N \times (N-1)$ matrix W and real, continuous, periodic functions p_j such that*

(i) $\mathbf{1}^T W = \mathbf{0}$,

(ii) $W^T W = I_{N-1}$, *and*

(iii) $f(Wu) = Wp(u)$ *with* $p(u) = (p_1(u_1), \ldots, p_{N-1}(u_{N-1}))^T$.

For example, $W = W^{(N)}$ defined by

$$W_{jk}^{(N)} = \frac{1}{\sqrt{N}} \left(\sin \frac{2\pi jk}{N} + \cos \frac{2\pi jk}{N} \right)$$
$$= \frac{2}{\sqrt{N}} \sin \left(\frac{2\pi jk}{N} + \frac{\pi}{4} \right) \tag{3.1}$$

satisfies these conditions.

When the system (1.1) is diagonalizable, the equations for the components of u become independent of other components:

$$\dot{u}_j = a_j + \varepsilon p_j(u_j), \quad j = 1, \ldots, N-1, \tag{3.2}$$

where we set $a_j = W_j^T \omega$. Thus, the problem is reduced to solving a scalar differential equation for each j. The following lemma applies to each equation in (3.2).

LEMMA 3.2. *Let $\varepsilon > 0$, and let a be a real number. Let $p(u)$ be a real, continuous, periodic function with period $L > 0$. Define $m = \min\limits_{0 \le u < L} p(u)$, $M = \max\limits_{0 \le u < L} p(u)$. For any solution $u(t)$ of $\dot{u} = a + \varepsilon p(u)$, the limit $\mu_p(a, \varepsilon) \equiv \lim_{t \to \infty} u(t)/t$ exists, and*

$$\mu_p(a, \varepsilon) = \begin{cases} L/T(a, \varepsilon), & a < -\varepsilon M, a > -\varepsilon m \\ 0, & -\varepsilon M \le a \le -\varepsilon m, \end{cases} \tag{3.3}$$

where

$$T(a, \varepsilon) \equiv \int_0^L \frac{du}{a + \varepsilon p(u)}$$

is the "period" of the solution in the case of $a < -\varepsilon M$ or $a > -\varepsilon m$, in the sense that $u(t + T(a, \varepsilon)) = u(t) + L$.

Proof. If $-\varepsilon M \le a \le -\varepsilon m$, then any solution $u(t)$ tends to a zero of the function $a + \varepsilon p(u)$. Hence, $\mu_p(a, \varepsilon) = 0$.

For notational simplicity, let us drop the dependence of $T(a, \varepsilon)$ on a and ε below. Suppose $a > -\varepsilon m$, so that $a + \varepsilon p(u) > 0$ for all u. It is straightforward to show that the function $u(t)$ defined implicitly by the formula

$$\int_{u_0}^{u(t)} \frac{du}{a + \varepsilon p(u)} = t$$

is the unique solution of $\dot{u} = a + \varepsilon p(u)$ with the initial condition $u(0) = u_0$, and that

it satisfies $u(t + T) = u(t) + L$. We have

$$\mu_p(a, \varepsilon) = \lim_{t \to \infty} u(t)/t$$

$$= \lim_{t \to \infty} \frac{1}{t} \left(u_0 + \int_0^t [a + \varepsilon p(u(s))] ds \right)$$

$$= a + \varepsilon \lim_{t \to \infty} \frac{1}{t} \int_0^t p(u(s)) ds$$

Let n be the largest integer for which $nT \le t$. Then, by changing the variables in each integral using the translation by multiples of T, we see that

$$\int_0^t p(u(s)) ds = \sum_{k=1}^n \int_{(k-1)T}^{kT} p(u(s)) ds + \int_{nT}^t p(u(s)) ds$$

$$= n \int_0^T p(u(s)) ds + \int_0^{t-nT} p(u(s)) ds.$$

Consequently,

$$\left| \frac{1}{t} \int_0^t p(u(s)) ds - \frac{1}{T} \int_0^T p(u(s)) ds \right|$$

$$= \left| \left(\frac{n}{t} - \frac{1}{T} \right) \int_0^T p(u(s)) ds + \frac{1}{t} \int_0^{t-nT} p(u(s)) ds \right|$$

$$\le \frac{|nT - t|}{tT} \int_0^T |p(u(s))| ds + \frac{1}{t} \int_0^{t-nT} |p(u(s))| ds$$

$$\le \frac{T}{tT} T \max\{|m|, |M|\} + \frac{1}{t} T \max\{|m|, |M|\}$$

$$\to 0,$$

as $t \to \infty$, showing that the limit $\lim_{t \to \infty} \frac{1}{t} \int_0^t p(u(s)) ds$ exists and is equal to $\frac{1}{T} \int_0^T p(u(s)) ds$. Thus, by changing variables from s to u and translating by u_0, we see that $\mu_p(a, \varepsilon)$ exists, and

$$\mu_p(a, \varepsilon) = a + \frac{\varepsilon}{T} \int_0^T p(u(s)) ds$$

$$= a + \frac{\varepsilon}{T} \int_0^L \frac{p(u) du}{a + \varepsilon p(u)}$$

$$= L/T.$$

If $a < -\varepsilon M$, then by replacing u with $-u$, a with $-a$, ε with $-\varepsilon$ and m with M the problem reduces to the previous case. The lemma is proved. $\square$

The function $\mu_p(\cdot, \varepsilon)$ in Lemma 3.2, which can easily be shown to be differentiable with positive derivative outside the interval $[-\varepsilon M, \varepsilon M]$, determines the relationship between the input frequency a and the output frequency $\mu_p(a, \varepsilon)$. If we take $p(u) = \sin(u)$, for example, the integration in the expression of μ_p can be carried out, and we get

$$\mu_p(a, \varepsilon) = \begin{cases} -\sqrt{a^2 - \varepsilon^2}, & a < -\varepsilon, \\ 0, & -\varepsilon \le a < \varepsilon, \\ \sqrt{a^2 - \varepsilon^2}, & a \ge \varepsilon, \end{cases}$$

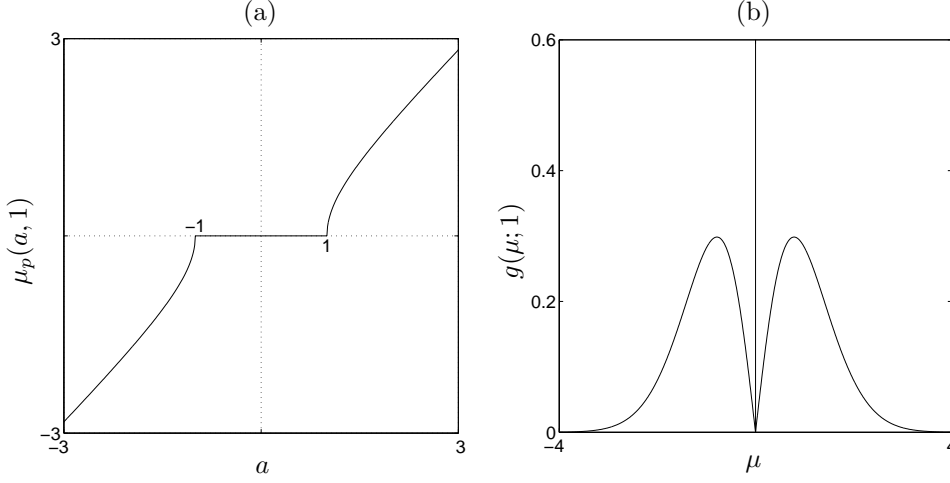(a)   (b)

FIG. 3.1. *(a) The graph of the input-output frequency function $\mu_p(a, \varepsilon)$ vs a for $\varepsilon = 1$ and $p(u) = \sin(u)$. (b) The corresponding density $g(\mu; 1)$ when a is the standard Gaussian random variable.*

whose graph is depicted in Fig. 3.1(a) for $\varepsilon = 1$. With this function, the output frequency vector μ of the u-equation (2.3) can be written as

$$\mu = \mu(a) = (\mu_{p_1}(a_1, \varepsilon), \ldots, \mu_{p_{N-1}}(a_{N-1}, \varepsilon))^T.$$

It is important to note here that μ does *not* depend on the initial condition $u(0) = u_0$, which implies that it is also independent of the initial condition for θ. In other words, the initial condition for the system (1.1) does not affect the behavior of its solution, as far as its coherence properties are concerned. Therefore, in this sense, coherence, partial coherence and incoherence are properties of the system rather than of individual solutions for a diagonalizable system.

4. Randomly Distributed Frequencies. In this section we consider ω to be a random vector in $\mathbb{R}^N$. We take the components $\omega_1, \ldots, \omega_N$ of ω to be independent and identically distributed (i.i.d.) random variables with mean 0 and variance $\sigma^2 > 0$. In the general case of mean $\omega_0 \neq 0$, the problem can always be reduced to the zero-mean case by the translation of θ by $-\omega_0 t$.

Since ω is random, the vectors a and μ are also random vectors in $\mathbb{R}^{N-1}$. Lemma 3.2 along with the relation $a = W^T \omega$ can be used to determine the distribution of μ from the distribution of ω. For example, if each ω_j is standard Gaussian, then so is each a_j, in which case the density $g(\mu; \varepsilon)$ for the random variable $\mu_p(a_j, 1)$ when $p(u) = \sin(u)$ can be computed. The result is

$$g(\mu; \varepsilon) = \frac{|\mu| e^{-(\mu^2 + \varepsilon^2)/2}}{\sqrt{2\pi(\mu^2 + \varepsilon^2)}} + \delta(\mu) \operatorname{erf}\left(\frac{\varepsilon}{\sqrt{2}}\right),$$

where $\delta(\mu)$ is Dirac's delta function. The graph of this density is shown in Fig. 3.1 for $\varepsilon = 1$.

Our main goal is this section is to compute the probabilities that the system (1.1) is coherent, partially coherent, or incoherent. The following theorem reveals a curious

property of a generic diagonalizable system of phase oscillators.

THEOREM 4.1. *Let the natural frequency vector ω be a random vector in $\mathbb{R}^N$, whose components are i.i.d. with a common continuous distribution. Suppose (1.1) is a diagonalizable system of N phase oscillators such that W satisfies the condition that $W_{ki} \neq W_{kj}$ for all $k = 1, 2, \ldots, N$ and for all $i, j = 1, 2, \ldots, N-1$ such that $i \neq j$. Then, the partial coherence of the system almost surely implies coherence, i.e., given that the system is partially coherent, the probability that it is coherent is one.*

Proof. Once again, let $\Omega(\omega) = \lim_{t \to \infty} \theta(t)/t = (\mathbf{1}^T \omega)\mathbf{1} + W\mu(\omega)$. Let S_c be the set of ω in $\mathbb{R}^N$ that corresponds to coherent systems, i.e., $S_c = \left\{\omega \in R^N : \Omega(\omega) = \mathbf{0}\right\}$. By Lemma 2.1, we may also write $S_c = \left\{\omega \in R^N : \mu(\omega) = \mathbf{0}\right\}$. Let S_{pc} be the set corresponding to partially coherent systems, that is, $S_{pc} = \left\{\omega \in R^N : \Omega_i(\omega) = \Omega_j(\omega)\right.$ for some $\left. i \neq j\right\}$. It is easy to see that we can also rewrite this in terms of μ as

$$S_{pc} = \left\{\omega \in R^N : \text{ There are } i \neq j \text{ s.t. } \sum_{k=1}^{N-1}(W_{ki} - W_{kj})\mu_k(\omega) = 0\right\}$$

$$= \bigcup_{i \neq j}\left\{\omega \in R^N : \sum_{k=1}^{N-1}(W_{ki} - W_{kj})\mu_k(\omega) = 0\right\} \equiv \bigcup_{i \neq j} S_{pc}^{(i,j)}$$

The probability that the system is coherent, given that the system is partially coherent, is $P(S_c)/P(S_{pc})$ since $S_c \subset S_{pc}$. This probability is one if and only if $P(S_{pc} \setminus S_c) = 0$, which would be satisfied if $P(S_{pc}^{(i,j)} \setminus S_c) = 0$ for every pair $i \neq j$. We shall show this next.

Let us fix i and j. For any $A \subset \{1, 2, \ldots, N-1\}$, denote by Z_k the subspace $\left\{\mu \in \mathbb{R}^{N-1} : \mu_k = 0\right\}$, and let $Z_A = \bigcup_{k \in A} Z_k$ and $Z'_A = \bigcap_{k \notin A} Z_k$. Let R_A denote the subspace $\left\{\mu \in \mathbb{R}^{N-1} : \sum_{k \in A}(W_{ki} - W_{kj})\mu_k = 0\right\}$. Define $Q_A = R_A \cap Z'_A \setminus \{\mathbf{0}\}$. We will show that $P(Q_A) = 0$ for any choice of A. $P(S_{pc}^{(i,j)} \setminus S_c) = 0$ follows from this by taking $A = \{1, 2, \ldots, N-1\}$.

We shall prove $P(Q_A) = 0$ by induction on $n = |A|$, the cardinality of A. Suppose first that $n = 1$ and say, $A = \{1\}$. Since $W_{1i} - W_{1j} \neq 0$, we have $R_A = \left\{\mu \in \mathbb{R}^{N-1} : \mu_1 = 0\right\} = Z_1$ and $Z'_A = \bigcap_{k=2}^{N-1} Z_k$. So, $Q_A = \bigcap_{k=1}^{N-1} Z_k \setminus \{\mathbf{0}\} = \varnothing$, which implies $P(Q_A) = 0$. The same holds for any other A with $|A| = 1$.

Suppose $P(Q_A) = 0$ for any A with $|A| = n - 1$ and consider the case $|A| = n$. We have

$$P(Q_A) = P(Q_A \cap Z_A) + P(Q_A \setminus Z_A)$$

$$= P\left(\bigcup_{k \in A} Q_A \cap Z_k\right) + P(Q_A \setminus Z_A)$$

$$\leq \sum_{k \in A} P(Q_A \cap Z_k) + P(Q_A \setminus Z_A)$$

We see that $P(Q_A \cap Z_k) = 0$ for each $k \in A$, by the induction hypothesis, since we can write $Q_A \cap Z_k = R_A \cap Z'_A \cap Z_k \setminus \{\mathbf{0}\} = R_{A_k} \cap Z'_{A_k} \setminus \{\mathbf{0}\} = Q_{A_k}$ with $A_k = A \setminus \{k\}$, for which we have $|A_k| = n - 1$. Thus, if we can show $P(Q_A \setminus Z_A) = 0$, then we are done.

We show $P(Q_A \setminus Z_A) = 0$ in three steps. First, since Z'_A is an n-dimensional subspace, and $Q_A \subset R_A \cap Z'_A$ is an $(n-1)$-dimensional subspace, the n-dimensional Lebesgue measure of Q_A in Z'_A must be zero.

Next, note that the conditional probability distribution of μ, given $\mu \in Z'_A$, is continuous with respect to Lebesgue measure outside the set Z_A. This can be seen by noting the following: (1) each component can be written as $\mu_j = \mu_{p_j}(a_j)$ by Lemma 3.2, (2) $\mu_{p_j}^{-1}$ exists and is differentiable except at the origin, again by Lemma 3.2, and (3) the conditional distribution of $a_j = \sum_k W_{kj}\omega_k$, given that $\mu(\omega) \in Z'_A$ (which is equivalent to $-\varepsilon M_k \leq \sum_l W_{lk}\omega_l \leq -\varepsilon m_k$ for all $k \notin A$), is continuous everywhere.

Finally, combining these two observations, we see that $P(Q_A \setminus Z_A \mid Z'_A) = 0$, which implies that $P(Q_A \setminus Z_A) = P(Z'_A)P(Q_A \setminus Z_A \mid Z'_A) = 0$. This completes the proof of the theorem. $\square$

We next describe the behavior of a generic diagonalizable system in the limit of $N \to \infty$. In order to formalize the process of taking the limit, we need to choose a sequence of systems of the form (1.1). Such a sequence can be characterized by the following sequences.

1. Consider a sequence $\{W^{(N)}\}_{N=1,2,...}$ of matrices with the following properties.
   (a) Each $W^{(N)}$ is an $N \times (N-1)$ matrix with orthonormal columns.
   (b) $\mathbf{1}_N^T W^{(N)} = \mathbf{0}$ for all N.
   (c) Each $W^{(N)}$ satisfies the condition for W in Theorem 4.1.
   (d) $||W^{(N)}||_\infty \to 0$ as $N \to \infty$. ($|| \cdot ||_\infty$ denotes the maximum matrix norm defined by $||A||_\infty = \max |A_{ij}|$, where the maximum is taken over all elements of A.) This is like a mixing condition that will be necessary later in order to apply Proposition 4.3.
2. Consider a sequence $\{p_j\}_{j=1,2,...}$ of real, continuous, periodic functions such that the corresponding sequence of norms $||p_j|| \equiv \max_j |p_j(u)|$ is bounded.
3. Consider a sequence $\{\omega_j\}_{j=1,2,...}$ of i.i.d. random variables with mean ω_0 and variance σ^2.

The sequence of matrices $W^{(N)}$ defined by (3.1) satisfies the conditions above. Given such sequences, for each $\varepsilon > 0$ and N, we define $\mathcal{S}_{N,\varepsilon}$ to be the diagonalizable system (1.1) of phase oscillators using the natural frequency vector $\omega = (\omega_1, \ldots, \omega_N)^T$, the functions $\{p_1, \ldots, p_{N-1}\}$ and the matrix $W^{(N)}$. We are now ready to state and prove our main theorem.

THEOREM 4.2. *Let $\mathcal{S}_{N,\varepsilon}$ be defined as above. Then, for any fixed $\varepsilon > 0$, $\mathcal{S}_{N,\varepsilon}$ is almost surely incoherent as $N \to \infty$, i.e., the probability that $\mathcal{S}_{N,\varepsilon}$ is incoherent tends to one in the limit of $N \to \infty$.*

Proof. As mentioned before, we may assume $\omega_0 = 0$ without loss of generality, since $\omega_0 \neq 0$ case can always be reduced to $\omega_0 = 0$ case.

From Theorem 4.1, we know that the probability that $\mathcal{S}_{N,\varepsilon}$ is *not* incoherent is equal to the probability q_c that it is coherent. We need to show that $q_c \to 0$ as $N \to \infty$.

Let $N_0 < N$ be fixed. From Lemma 2.1, it follows that $q_c = P(\mu = \mathbf{0})$. Since the sequence $\{||p_j||\}_{j=1,2,...}$ is bounded, we can define $M = \sup M_j$ and $m = \inf m_j$, where $m_j = \min_{0 \leq u < L} p_j(u)$, $M_j = \max_{0 \leq u < L} p_j(u)$ for each j. Then, Lemma 3.2 implies

$$
\begin{aligned}
q_c &= P(\mu = \mathbf{0}) \\
&= P(-\varepsilon M_j \leq a_j^{(N)} \leq -\varepsilon m_j, \ j = 1, \ldots, N-1) \\
&\leq P(-\varepsilon M \leq a_j^{(N)} \leq -\varepsilon m, \ j = 1, \ldots, N_0),
\end{aligned}
$$

where $a_j^{(N)} = \left(W_j^{(N)} \right)^T \omega$.

The following Proposition shows that for each j, $a_j^{(N)}$ converges to a Gaussian random variable with mean 0 and variance σ^2.

PROPOSITION 4.3. *Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $EX_j = 0$ and $Var(X_j) = E(X_j^2) = \sigma^2$. Suppose that for each N, real numbers $b_{N,1}, \ldots, b_{N,N}$ satisfy $\sum_{j=1}^{N} b_{N,j}^2 = 1$. Also, suppose that*

$$\lim_{N \to \infty} \max_{1 \leq j \leq N} |b_{N,j}| = 0.$$

Then, we have

$$S_N = \sum_{j=1}^{N} b_{N,j} X_j \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

as $N \to \infty$.

Proof. Let $Y_{N,j} = b_{N,j} X_j$. We will apply the Lindeberg-Feller Central Limit Theorem (see, for example, [3]) to $Y_{N,j}$. For this we need to check three conditions. The first is $EY_{N,j} = b_{N,j} EX_j = 0$. The second condition is satisfied because $\sum_{j=1}^{N} EY_{N,j}^2 = \sum_{j=1}^{N} b_{N,j}^2 EX_j^2 = \sigma^2 > 0$. To show that the third condition is satisfied, let $\varepsilon > 0$ be fixed. We have

$$\sum_{j=1}^{N} E\left(|Y_{N,j}|^2 \,\Big|\, |Y_{N,j}| > \varepsilon\right) = \sum_{j=1}^{N} b_{N,j}^2 E\left(|X_j|^2 \,\Big|\, |X_j| > \frac{\varepsilon}{|b_{N,j}|}\right)$$

where $E(X|A)$ denotes the conditional expectation of X given A. Let j be fixed. For each N, set $Z_N = |X_j|^2$ if $|X_j| > \varepsilon/|b_{N,j}|$, and 0 otherwise. Since $|b_{N,j}| \to 0$, $Z_N \leq |X_j|^2$ for each N, and $Z_N \to 0$ almost surely, we may use the Dominated Convergence Theorem to show that for each $j = 1, 2, \ldots$, $EZ_N = E\left(|X_j|^2 \,\big|\, |X_j| > \varepsilon/|b_{N,j}|\right) \to 0$ as $N \to \infty$. Thus, the third condition $\sum_{j=1}^{N} E\left(|Y_{N,j}|^2 \,\big|\, |Y_{N,j}| > \varepsilon\right) \to 0$ is satisfied. The conclusion now follows directly from application of the Lindeberg-Feller Theorem. $\square$

For each $j = 1, \ldots, N-1$, we take $b_{N,i} = W_{ij}^{(N)}$ and $X_i = \omega_i$ in Proposition 4.3, and we see that $a_j^{(N)}$ converges in distribution to $a_j^{(\infty)}$ as $N \to \infty$, where $a_j^{(\infty)}$ is a Gaussian random variable. Moreover, due to the orthogonality of $W^{(N)}$, $a_1^{(N)}, \ldots, a_{N-1}^{(N)}$, in some sense, become independent in the limit.

LEMMA 4.4. *The random variables $a_1^{(\infty)}, a_2^{(\infty)}, \ldots$ are independent.*

Proof. We need to show that for any finite $A \subset \mathbb{N}$, the collection $\{a_k^{(\infty)}\}_{k \in A}$ is a set of independent random variables. For simplicity, we only prove this for $A = \{1, 2\}$, but a similar argument works for a general case.

Let t_1 and t_2 be given. Set

$$b_{N,j} = \frac{t_1 W_{1j}^{(N)} + t_2 W_{2j}^{(N)}}{\sqrt{t_1^2 + t_2^2}}.$$

Then, as $N \to \infty$, $\max_j |b_{N,j}|$ approaches zero because $\max_j |W_{1j}|$ and $\max_j |W_{2j}|$ go to zero. Also, it is easy to check that $\sum_{j=1}^{N} b_{N,j}^2 = 1$ for all N. Applying Proposition 4.3, we see that

$$\frac{t_1 a_1^{(N)} + t_2 a_2^{(N)}}{\sqrt{t_1^2 + t_2^2}} = \sum_{j=1}^{N} b_{N,j} \omega_j \xrightarrow{d} \mathcal{N}(0, \sigma^2),$$

which implies that $t_1 a_1^{(N)} + t_2 a_2^{(N)} \xrightarrow{d} \mathcal{N}(0, \sigma^2(t_1^2 + t_2^2))$, which in turn implies the convergence of the joint characteristic function of $a_1^{(N)}$ and $a_2^{(N)}$ as $N \to \infty$. Specifically,

$$Ee^{it_1 a_1^{(\infty)} + it_2 a_2^{(\infty)}} = \lim_{N \to \infty} Ee^{i(t_1 a_1^{(N)} + t_2 a_2^{(N)})} = e^{-\sigma^2(t_1^2 + t_2^2)/2} = e^{-\sigma^2 t_1^2/2} e^{-\sigma^2 t_2^2/2}.$$

Therefore, $a_1^{(\infty)}$ and $a_2^{(\infty)}$ are independent. □

Let us come back to the proof of Theorem 4.2. As a consequence of $a_1^{(\infty)}, a_2^{(\infty)}, \dots$ being independent Gaussian random variables, we have

$$q_c \leq P(\{-\varepsilon M \leq a_j^{(N)} \leq -\varepsilon m, \ j = 1, \dots, N_0\})$$

$$\to \left[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\varepsilon M}^{-\varepsilon m} e^{-x^2/2\sigma^2} dx\right]^{N_0},$$

as $N \to \infty$. Since this holds for any fixed N_0, and the right hand side goes to zero as $N_0 \to \infty$, we conclude that $q_c \to 0$. This completes the proof of Theorem 4.2. □

5. Example. A network of voltage-controlled oscillator (VCO) devices can be built as an example of systems with diagonalizable interaction. The behavior of the jth VCO in the network is described by its phase variable θ_j, which satisfies [5, 7]

$$\dot{\theta}_j = \omega_j + I_j(t)$$

where ω_j is the center frequency and $I_j(t)$ is the input signal from other VCO's. The system is diagonalizable if, for example, $I_j(t)$ has the form

$$I_j(t) = \varepsilon \sum_{k=1}^{N-1} W_{jk}^{(N)} \sin\left(\sum_{\ell=1}^{N} W_{\ell k}^{(N)} \theta_\ell\right),$$

with $W^{(N)}$ defined by (3.1).

This type of interaction can be implemented using commercially available circuit elements, as follows. The sine terms on the right hand side can be constructed as sums and products of output voltages:

$$\sin\left(\sum_{\ell=1}^{N} W_{\ell k}^{(N)} \theta_\ell\right) = \sum_b \prod_{\ell=1}^{N} \cos\left(W_{\ell k}^{(N)} \theta_\ell - \frac{b_\ell \pi}{2}\right),$$

where the sum is taken over all (ordered) binary N-tuples $b = (b_1, \dots, b_N)$, $b_\ell = 0, 1$, such that $\sum_\ell b_\ell$ is odd. This means that $I_j(t)$ is a sum of terms that are products of $\sin(W_{\ell k}^{(N)} \theta_\ell)$ and $\cos(W_{\ell k}^{(N)} \theta_\ell)$. Signals of the form $\sin(W_{\ell k}^{(N)} \theta_\ell)$ can be obtained by using the amplified version of the input $W_{\ell k}^{(N)} I_\ell(t)$ as the controlling voltage in a separate VCO with center frequency $W_{\ell k}^{(N)} \omega_\ell$. From these we can get $\cos(W_{\ell k}^{(N)} \theta_\ell)$ by the phase shift of $\pi/2$. Finally, $I_j(t)$ is obtained by putting these signals through multipliers and adding the outputs.

6. Discussion. In order to gain additional insights, let us consider the case when the functions $p_j = p$ do not depend on j, and have the range $[-1, 1]$. The arguments in the proof of Theorem 4.2 suggest that for a diagonalizable system (1.1) with large N, the probability q_c that it is coherent is approximately

$$q_c \approx \left[\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\varepsilon}^{\varepsilon} e^{-x^2/2\sigma^2} dx\right]^{N-1} = \left[\mathrm{erf}\left(\frac{\varepsilon}{\sigma\sqrt{2}}\right)\right]^{N-1} \equiv \tilde{q}_c(\varepsilon; N),$$

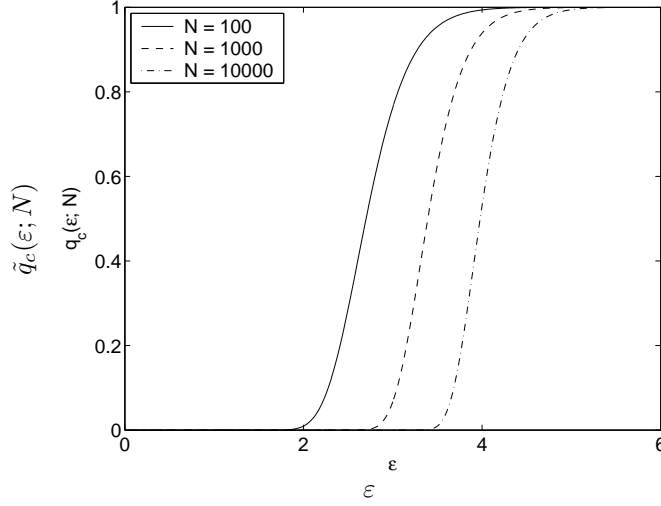FIG. 6.1. *Probability of coherence $\tilde{q}_c(\varepsilon; N)$ as a function of ε for $\sigma = 1$ and $N = 100, 1000, 10000$.*

where $\mathrm{erf}(x)$ is the error function. Typical graphs of $\tilde{q}_c(\varepsilon; N)$ are plotted in Fig. 6.1. One can see that for any finite value of N, there seems to be a sharp transition point through which $\tilde{q}_c(\varepsilon; N)$ changes from 0 to 1. However, unlike the mean-field model of Kuramoto, this point keeps shifting to the right as N increases, and tends to ∞ in the limit of $N \to \infty$, although it can be shown that this increase is at most $O(\sqrt{\ln N})$:

LEMMA 6.1. *Let $\sigma > 0$ and $0 < q < 1$ be fixed. Define $\varepsilon_{q,\sigma}(N)$ implicitly by $q = \tilde{q}_c(\varepsilon_{q,\sigma}(N); N)$. Then, $\varepsilon_{q,\sigma}(N) = O(\sqrt{\ln N})$ as $N \to \infty$, i.e., $\varepsilon_{q,\sigma}(N)/\sqrt{\ln N}$ is bounded as $N \to \infty$.*

Proof. Let $x = (\sigma\sqrt{2})^{-1}\varepsilon_{q,\sigma}(N)$. Then, using a known estimate for the error function, we have for $x \geq 1$,

$$\mathrm{erf}(x) \geq 1 - \frac{2e^{-x^2}}{\sqrt{\pi}x + \sqrt{\pi x^2 + 4}} \geq 1 - \frac{2e^{-x^2}}{\sqrt{\pi} + \sqrt{\pi + 4}}$$

Hence, we have the estimate

$$q \geq (1 - C_0 e^{-x^2})^{N-1} \geq 1 - (N-1)C_0 e^{-x^2},$$

where

$$C_0 = \frac{2}{\sqrt{\pi} + \sqrt{\pi + 4}}.$$

Here we used the relation $(1 - x)^n \geq 1 - nx$ which is valid for $n \geq 0$ and $0 \leq x \leq 1$. The estimate for $\varepsilon_{q,\sigma}(N)$ can be obtained by rearranging:

$$\varepsilon_{q,\sigma}(N) \leq \sigma\sqrt{C_1 + 2\ln(N-1)},$$

where $C_1 = 2\ln C_0 - \ln(1 - q)$. This implies $\varepsilon_{q,\sigma}(N) = O(\sqrt{\ln N})$. $\square$

7. Conclusions. In this paper, we have defined a class of systems of phase oscillators characterized by having diagonalizable interactions. For a system in this class, complete separation of variables through appropriate changes of variable is possible, which enables us to draw rigorous conclusions about the probabilistic properties of the system. In particular, we have shown that partial coherence of the system almost surely implies coherence, and in the limit of large system size, the system is almost surely incoherent. A major implication of our result is that, unlike mean-field model of Kuramoto, diagonalizable systems cannot exhibit a sharp transition from incoherence to coherence. This provides some insight into what is necessary to see such a transition in a system of phase oscillators.

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