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RECONSTRUCTION OF A SPHERICALLY SYMMETRIC SPEED OF SOUND*

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Abstract. Consider the inverse acoustic scattering problem for a spherically symmetric inhomogeneity of compact support that arises, among other places, in nondestructive testing. Define the corresponding homogeneous and inhomogeneous interior transmission problems, see, e.g., [D. Colton and P. Monk, *Quart. J. Mech. Math.*, 41 (1988), pp. 97–125]. Here the authors study the subset of transmission eigenvalues corresponding to spherically symmetric eigenfunctions of the homogeneous interior transmission problem. It is shown in McLaughlin and Polyakov [*J. Differential Equations*, to appear] that these eigenvalues are the zeros of an average of the scattering amplitude, and a uniqueness theorem for the inverse acoustic scattering problem is presented where these eigenvalues are the given data. In the present paper an algorithm for finding the solution of the inverse acoustic scattering problem from this subset of transmission eigenvalues is developed and implemented. The method given here completely determines the sound speed when the size, measured by an integral, satisfies a particular bound. The algorithm is based on the Gel'fand–Levitan integral equation method [I. M. Gelfand and B. M. Levitan, *Amer. Math. Soc. Trans.*, 1 (1951), pp. 253–304], [W. Rundell and P. E. Sacks, *Inverse Problems*, 8 (1992), pp. 457–482].

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1. Introduction. We consider the inverse acoustic scattering problem for a spherically symmetric inhomogeneous medium of compact support in a ball Ω_b of a radius b in $\mathbf{R}^3$. We seek to recover the local speed of sound $c(\vec{r})$, supposed to have the following properties:

$$c(\vec{r}\,) = c(r),$$
$$c(r) = c_0 > 0, \quad \text{for} \quad r > r_0,$$

where $r = |\vec{r}|$ and $c_0, r_0 < b$ are given constants. Denoting $n(r) = [c_0/c(r)]^2$, we assume that $n(r) - 1 = [c_0/c(r)]^2 - 1 \in H_0^2(\mathbf{R}^3)$ and that $n(r) \not\equiv 1$. Letting ω, α be the frequency and direction of an incident planewave and letting $k = \omega/c_0$ be the wavenumber, the scattered wave, $w^s(\vec{r}, t)$, satisfies

$$w(\vec{r}, t) = e^{-i\omega t + ik(\vec{r}\cdot\alpha)} + w^s(\vec{r}, t),$$

where

$$\frac{1}{[c(r)]^2} w_{tt} = \Delta w,$$

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the acoustic wave equation. Assuming that $w(\vec{r},t) = e^{-i\omega t}u(\vec{r}), w^s(\vec{r},t) = e^{-i\omega t}u^s(\vec{r})$, then the velocity potential $u(\vec{r})$ satisfies

$$\begin{aligned} \Delta u + k^2 n(r) u &= 0, \\ u(\vec{r}\,) &= e^{ik(\vec{r}\cdot\alpha)} + u^s(\vec{r}\,), \end{aligned} \tag{1}$$

and the scattered field, $u^s(\vec{r})$, satisfies the Sommerfeld (outgoing) radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0$$

and the asymptotic form

$$u^s(\vec{r}\,) = \frac{e^{ikr}}{r} F(\hat{r}, k, \alpha) + O\left(\frac{1}{r^2}\right).$$

Here $F(\hat{r}, k, \alpha)$ is the scattering amplitude and $\hat{r}$ are the spherical coordinates on the unit sphere $\partial\Omega$ in $\mathbf{R}^3$. When c is allowed to be a function of all three variables, one is interested in recovering $c(\vec{r})$ from the five variable function, $F(\hat{r}, k; \alpha)$. A uniqueness result and further references can be found in [CK].

In this paper we are concerned with recovering $c = c(r)$ from the zeros of certain integral averages of F that have been shown to be eigenvalues of a related boundary value problem; for completeness we repeat that argument in this paper. We then reconstruct $c(r)$ from this data. In this paper we reconstruct $c(r)$ when $a = b\{\int_0^b [c_0/c(r)]\,dr\}^{-1} \geq 3$; the reconstruction for the cases $0 < a < 1$ and $1 < a < 3$ will be presented in a subsequent paper. The reconstruction presented here is then applicable when the "size" of $c(r)$, as measured by $a = b\{\int_0^b [c_0/c(r)]\,dr\}^{-1}$, is sufficiently large. We show that even for a small number, say 10, of the zeros of the integral average, the reconstruction is reasonably good; for more eigenvalues, say 30, the reconstruction can be quite accurate.

Our method is new. We have extracted functionals, that is the eigenvalues, from the scattering amplitude; we use this data to determine a related overposed Goursat problem (see also [RS1], [RS2]) leading to the recovery of $n(r)$. Other known numerical methods for recovering n from the scattering amplitude fall roughly into two groups. In one type the problem is reformulated as a nonlinear optimization problem subject to a priori constraints. See, for example, [JT1], [JT2], [KvdB], [T], [WC], [W1], [W2]. In the second type, a dual space method is used. One simultaneously seeks $n(r)$ and the solution of the Lippmann–Schwinger equation subject to a carefully chosen constraint, see [CM2]. See also [CK, pp. 10–11] for a discussion comparing the two types of numerical methods.

The boundary value problem that we use for determining the eigenvalues is the so-called *homogeneous interior transmission* problem. It was formulated in [CM1] and consists of finding a solution to the system of equations and boundary conditions:

$$\begin{aligned} \Delta v + k^2 n(r) v &= 0, \\ \Delta w + k^2 w &= 0, \\ v(\vec{r}\,) - w(\vec{r}\,) &= 0 \quad \text{on} \quad \partial\Omega_b, \\ \frac{\partial}{\partial r}(v(\vec{r}\,) - w(\vec{r}\,)) &= 0 \quad \text{on} \quad \partial\Omega_b. \end{aligned} \tag{2}$$

The k^2 for which (2) has a solution are called transmission eigenvalues. We will confine ourselves to a subset of the set of transmission eigenvalues, namely to the eigenvalues of (2) with spherically symmetric eigenfunctions $\{v, w\}$. These eigenvalues can be considered as the eigenvalues of an associated one-dimensional eigenvalue problem with the eigenvalue parameter in the boundary condition. This will not be a standard Sturm–Liouville problem; it will not be a self-adjoint boundary value problem, nor will it have a well-defined adjoint problem. Still, we have adapted some of the known techniques (see [PT], [MP], [RS1], [RS2]) to establish asymptotics of the eigenvalues, uniqueness of the inverse eigenvalue problem, and the reconstruction procedure. Please note that a new feature of this inverse eigenvalue problem will be that the size of $c(r)$, as measured by $a = b\{\int_0^b [c_0/c(r)]\,dr\}^{-1}$, is exceptionally important. The reduction is made as follows [CM1], [MP].

Because n is radially symmetric, the solution of (2) has the form:

$$v(\vec{r}\,) = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} y_l(r) Y_l^m(\vec{r}\,),$$

$$w(\vec{r}\,) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} b_{lm} j_l(kr) Y_l^m(\hat{r}),$$

where y_l is a solution of

$$y'' + \left(k^2 n(r) - \frac{l(l+1)}{r^2}\right) y = 0$$

with initial condition

$$\lim_{r\to 0} \left(\frac{1}{r} y(r) - j_l(kr)\right) = 0$$

for $\hat{r}$, spherical coordinates; Y_l^m, spherical harmonics; and j_l, Bessel functions.

If we consider only radially symmetric solutions of (2), then we must have $l = m = 0$ or

$$v(x) = \frac{1}{r} a_{00} y_0(r),$$
$$w(x) = b_{00} j_0(kr).$$

Relabeling y_0 as y, we can rewrite the problem (2) for radially symmetric solutions in a form

$$y'' + k^2 n(r) y = 0, \qquad 0 \le r \le b,$$
$$\lim_{r\to 0} \left(\frac{1}{r} y(r) - 1\right) = 0,$$
$$\det \begin{pmatrix} \frac{1}{r} y(r) & -j_0(kr) \\ \frac{d}{dr}\left(\frac{1}{r} y(r)\right) & -\frac{d}{dr} j_0(kr) \end{pmatrix} = 0 \quad \text{for } r = b. \tag{3}$$

Then, taking into account that $n'(b)=0$, $n(b)=1$, $j_0(kr)=\sin kr/kr$ and making the change of variables:

$$\begin{aligned} x &= \frac{1}{B}\int_0^r n^{1/2}(t)\,dt, \\ y(x) &= y(r(x)), \\ \tilde{n}(x) &= n(r(x)), \\ \lambda &= B^2k^2, \\ a &= \frac{b}{B}, \end{aligned}$$

where $B=\int_0^b n^{1/2}(t)\,dt$, we arrive at the following eigenvalue problem in impedance form [CM1], [MP], [RS2]:

$$\begin{aligned} (\tilde{n}^{1/2}\cdot y_x)_x + \lambda\tilde{n}^{1/2}\cdot y &= 0 \quad \text{for } 0<x<1, \\ y(0) &= 0, \\ y(1)\cdot\cos\sqrt{\lambda}\,a - y_x(1)\cdot\frac{\sin\sqrt{\lambda}\,a}{\sqrt{\lambda}} &= 0. \end{aligned} \tag{4}$$

To characterize the same subset of transmission eigenvalues from the point of view of $F(\hat{x},k,\alpha)$, let us suppose that k^2 is not an eigenvalue of (3). Then there exists a solution $\{r^{-1}a_{00}y(r), b_{00}j_0(kr)\}$ of the *nonhomogeneous interior transmission problem* [CM1]:

$$\begin{aligned} \Delta v + k^2 n(r) v &= 0, \\ \Delta w + k^2 w &= 0, \\ v(\vec{r}\,) - w(\vec{r}\,) &= \frac{e^{ikr}}{r} \quad \text{on } \partial\Omega_b, \\ \frac{\partial}{\partial r}(v(\vec{r}\,) - w(\vec{r}\,)) &= \frac{d}{dr}\frac{e^{ikr}}{r} \quad \text{on } \partial\Omega_b, \end{aligned}$$

with

$$b_{00} = \frac{\det\begin{pmatrix} \frac{1}{r}y(r) & \frac{e^{ikr}}{r} \\ \frac{d}{dr}\left(\frac{1}{r}y(r)\right) & \frac{d}{dr}\frac{e^{ikr}}{r} \end{pmatrix}\Bigg|_{r=b}}{\det\begin{pmatrix} \frac{1}{r}y(r) & -j_0(kr) \\ \frac{d}{dr}\left(\frac{1}{r}y(r)\right) & -\frac{d}{dr}j_0(kr) \end{pmatrix}\Bigg|_{r=b}}.$$

Using then the equality

$$\int_{\partial\Omega} e^{-ik(\hat{t}\cdot\vec{r})}\,d\sigma(\hat{t}) = 4\pi j_0(k|\vec{r}|)$$

we conclude that w is a *Herglotz wave function* [CM1], [HW]

$$w(\vec{r}\,) = \int_{\partial\Omega} g(\hat{t}) e^{-ik(\hat{t}\cdot\vec{r})}\, d\sigma(\hat{t}),$$

with *Herglotz kernel* $g(\hat{t}) = b_{00}/4\pi$, where we note that $g(\hat{t})$ and b_{00} both depend on k. Applying the equality from [CM1],

$$1 = \int_{\partial\Omega} g(\hat{t}) F(\hat{t}, k, \alpha)\, d\sigma(\hat{t})$$

and we obtain,

$$\frac{1}{b_{00}} = \frac{1}{4\pi} \int_{\partial\Omega} F(\hat{t}, k, \alpha)\, d\sigma(\hat{t}).$$

Both sides of the last equation are meromorphic functions of k, and the zeros of $1/b_{00}$ are the eigenvalues of (3). Thus, we obtain the characterization of the selected subset of transmission eigenvalues as the set of zeros of averages of the scattering amplitude.

Our method is as follows. We reconstruct $\tilde{n}(x)$ from the spectrum of (4). This spectrum is almost the same as the spectrum of (3). The only difference is that each eigenvalue of (4) is B^2 times an eigenvalue of (3). From the knowledge of $\tilde{n}(x)$, where $x = B^{-1}\int_0^r n^{1/2}(t)\, dt$, we can reconstruct $r(x)$ as the unique solution of the differential equation,

$$\frac{dr}{dx} = \frac{B}{\sqrt{\tilde{n}(x)}},$$

with initial condition,

$$r(0) = 0.$$

Using then the positivity of $\tilde{n}^{1/2}(x)$, we can uniquely reconstruct the dependency $x(r)$ and consequently the function $c(r) = c_0 \tilde{n}^{-1/2}(x(r)) = c_0 n^{-1/2}(r)$.

From the above arguments it follows that the reconstruction of $c(r)$ can be reduced to the problem of reconstruction of $\tilde{n}(x)$ from the spectrum of (4).

The spectra of (4) was investigated in [MP]. Making the change of variables

$$z(x) = \tilde{n}^{1/4}(x) \cdot y(x),$$

we transform (4) into a canonical form of the Sturm–Liouville problem:

$$\begin{aligned} &z_{xx} + (\lambda - q(x)) z = 0 \quad \text{for } 0 < x < 1, \\ &z(0) = 0, \\ &z(1)\cos\sqrt{\lambda}\, a - z_x(1) \frac{\sin\sqrt{\lambda}\, a}{\sqrt{\lambda}} = 0, \end{aligned} \tag{5}$$

where

$$q(x) = \frac{(\tilde{n}^{1/4}(x))''}{\tilde{n}^{1/4}(x)} \in L^2[0,1].$$

The following theorems were established in [MP].

THEOREM 1. *Let $q = (\tilde{n}^{1/4})_{xx} \tilde{n}^{-1/4} \in L^2_R[0,1]$, be a real valued function. There exists an integer i_0 that has the properties*:

(1) *the number $N(i)$ of all such eigenvalues λ of* (4) *such that $|\lambda| < (i + \frac{1}{2})^2 \pi^2 / (a-1)^2$ satisfies the condition $N(i) \geq i$ for $i \geq i_0$*;

(2) *there are infinitely many real eigenvalues of* (4), *satisfying the condition* $\mu > i_0^2\pi^2/(a-1)^2$, *all of them are isolated and have the following form*:

$$\left| \mu_i(q) - \frac{\pi^2 i^2}{(a-1)^2} + \frac{1}{a-1}\int_0^1 q(t)\,dt - \frac{1}{a-1}\int_0^1 q(t)\cdot\cos\left(\frac{2\pi i t}{a-1}\right)dt \right| < \frac{C}{i} \tag{6}$$

for

$$i > \max\left\{1, A, \frac{25A|a-1|}{\pi^2}\right\},$$

$$A = \max\{e^{\|q\|}, (\|q\|+1)e^{\|q\|}+1\},$$

$$C = 1.6A + \frac{0.5A}{|a-1|}(1 + A\times|a-1|)$$

$$+ \|q\|\left(1.5A + 2.4\frac{A}{|a-1|} + 0.51\frac{1}{|a-1|} + 1.2\frac{A}{|a-1|^2}\right),$$

$$\|q\| = \left(\int_0^1 q^2(x)\,dx\right)^{1/2}.$$

THEOREM 2. *Let* $q_1 = (\tilde{n}_1^{1/4})_{xx}\tilde{n}_1^{-1/4}, q_2 = (\tilde{n}_2^{1/4})_{xx}\tilde{n}_2^{-1/4} \in L_R^2[0,1]$ *be real valued functions. Let* $a \geq 3$ *and let there exist a common sequence of eigenvalues* $\lambda_j (j = 1, \ldots, \infty)$ *of* (4), *for* $\tilde{n}_1$ *and* $\tilde{n}_2$ *satisfying the following properties*:

(1) *there exists* $i_0 \in \mathbf{Z}^+$ *such that* $|\lambda_j| < (i+\frac{1}{2})^2\pi^2/(a-1)^2$ *for* $j = 1, \ldots, i$ *and* $i \geq i_0$;

(2) *all* λ_j *such that* $|\lambda_j| > (i_0+\frac{1}{2})^2\pi^2/(a-1)^2$ *are real.*

Then $\tilde{n}_1 = \tilde{n}_2$.

Note that the asymptotic form (6) holds because $\lambda_j = \mu_j(q_i)$ for

$$|\lambda_j| > \left(i_0 + \frac{1}{2}\right)^2 \pi^2/(a-1)^2$$

and $q_i = (\tilde{n}_i^{1/4})_{xx}\tilde{n}_i^{-1/4}, i = 1, 2$.

Theorem 2 makes it reasonable to try to find a procedure of reconstruction of an $\tilde{n}(x)$, with $q(x) = (\tilde{n}^{1/4})_{xx}\tilde{n}^{-1/4} \in L_R^2[0,1]$ and $a \geq 3$, from the spectrum of (4). In this paper we discuss such a procedure.

The paper is organized as follows. In §1 we consider the Goursat problem [GL], [RS2] associated with the boundary value problem (4). Then an intermediate problem of finding the boundary values for this Goursat problem is formulated. This problem is formulated in terms of a certain infinite dimensional system of linear equations. The main theorem on the solvability of the infinite dimensional system is stated. In §2 the proof of this theorem is given. We then have the required theory to implement the numerical method. Section 3 contains the description of this numerical method and the computed results.

1. Derivation of the reconstruction procedure. Following [RS1], [RS2] we consider for $q = (\tilde{n}^{1/4})_{xx}\tilde{n}^{-1/4} \in L_R^2[0,1]$ the Gelfand–Levitan kernel $K(x,t;q)$ defined on the triangle $0 \leq |t| \leq x \leq 1$ and satisfying the following conditions [GL]:

$$\begin{aligned} K_{tt} - K_{xx} + q(x)\cdot K &= 0 \quad \text{for } 0 \leq |t| \leq x \leq 1, \\ K(x,0) &= 0 \quad \text{for } 0 \leq x \leq 1, \\ K(x, \pm x) &= \pm\frac{1}{2}\int_0^x q(u)\,du \quad \text{for } 0 \leq x \leq 1. \end{aligned} \tag{7}$$

Defining

$$M(x,t) = \frac{1 + \int_t^x K(x,u)\,du}{\tilde{n}^{1/4}(x)}, \tag{8}$$

we conclude from (7) that $M(x,t)$ satisfies the conditions

$$\begin{aligned} \tilde{n}^{1/2}(x)\cdot M_{tt} - (\tilde{n}^{1/2}(x)\cdot M_x)_x &= 0 \quad \text{for } 0 \le t \le x \le 1, \\ M_t(x,0) &= 0 \quad \text{for } 0 \le x \le 1, \\ M(x,x) &= \frac{1}{\tilde{n}^{1/4}(x)} \quad \text{for } 0 \le x \le 1. \end{aligned} \tag{9}$$

The boundary values $M_t(1,t)$ and $M_x(1,t)$ for $t \in [0,1]$ together with (9) provide the overdetermined boundary Goursat problem for M from which $\tilde{n}(x)$ can be reconstructed. The method of the reconstruction of $\tilde{n}(x)$ from this overdetermined problem was developed in [RS2]. The method consists of two parts:

(i) reconstruction of the boundary values $\{M_t(1,t), M_x(1,t)\}$ from the spectral data, and

(ii) reconstruction of $\tilde{n}(x)$ as a fixed point of a nonlinear mapping defined by (9) and boundary values $\{M_t(1,t), M_x(1,t)\}$.

We deal in this and the next section only with part (i) of the reconstruction. Namely, we discuss the numerical procedure of the reconstruction of $M_t(1,t)$ and $M_x(1,t)$ from the spectral data for (4) with a large enough.

The possibility of such a reconstruction arises from the integral equation [GL]:

$$z(x,\lambda,q) = \frac{\sin\sqrt{\lambda}\,x}{\sqrt{\lambda}} + \int_0^x K(x,t;q)\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}\,dt, \tag{10}$$

where $z(x,\lambda,q)$ is the solution of the differential equation (1) with initial conditions $z(0)=0$, $z'(0)=1$. Using (10) in the boundary condition at $x=1$, we get

$$\begin{aligned} \cos\sqrt{\lambda}\,\alpha\int_0^1 K(1,t)\sin\sqrt{\lambda}\,t\,dt - \frac{\sin\sqrt{\lambda}\,a}{\sqrt{\lambda}}\int_0^1 K_x(1,t)\sin\sqrt{\lambda}\,t\,dt \\ = \sin\sqrt{\lambda}\,(a-1) + \frac{K(1,1)}{\sqrt{\lambda}}\cdot\sin\sqrt{\lambda}\cdot\sin\sqrt{\lambda}\,a \end{aligned} \tag{11}$$

for $\lambda = \lambda_j(q)$ $(j = 1,\dots,\infty)$.

Substituting then into (11) $K(1,t) = -M_t(1,t)$ and $K_x(1,t) = -M_{tx}(1,t)$, integrating by parts in the second integral, and using $M_x(1,1) = K(1,1)$ we get

$$\cos\sqrt{\lambda}\,a\int_0^1 M_t(1,t)\sin\sqrt{\lambda}\,t\,dt + \sin\sqrt{\lambda}\,a\int_0^1 M_x(1,t)\cos\sqrt{\lambda}\,t\,dt = \sin\sqrt{\lambda}\,(1-a) \tag{12}$$

for $\lambda = \lambda_j(q)$ $(j = 1,\dots,\infty)$.

We want to consider (12) as the system of equations for the coefficients of Fourier series for $M_t(1,t)$ and $M_x(1,t)$. To do this we construct special bases in $L_R^2[0,1]$ and show the procedure of the reconstruction of $K(1,1) = M_x(1,1)$.

The latter is a consequence of Theorem 1 and of (7), because from (7) we have

$$K(1,1) = \frac{1}{2}\int_0^1 q(t)\,dt,$$

and using this in (6) we obtain

$$\lambda_i - \frac{\pi^2 i^2}{(a-1)^2} + \frac{2}{a-1} M_x(1,1) = \mu_i(q) - \frac{\pi^2 i^2}{(a-1)^2} + \frac{2}{a-1} K(1,1) \in l_2, \tag{13}$$

which makes it possible to reconstruct $M_x(1,1)$ from the real spectrum of (4), or $K(1,1)$ from the real spectrum of (5).

We consider then two bases in $L_R^2[0,1]$,

$$\{\sin i\pi t\}_{i=1}^{\infty} \quad \text{and} \quad \left\{\cos\left(i - \frac{1}{2}\right)\pi t\right\}_{i=1}^{\infty},$$

and Fourier series in these two bases,

$$M_t(1,t) = -tM_x(1,1) + \sum_{i=1}^{\infty} x_{2i} \sin i\pi t, \tag{14}$$

$$M_x(1,t) = M_x(1,1) + \sum_{i=1}^{\infty} x_{2i-1} \cos\left(i - \frac{1}{2}\right)\pi t. \tag{15}$$

Let us denote $j(s) = [((a-1)/2)s] \equiv$ (integer part of $((a-1)/2)s$) for $s \in Z^+$. We assume further on that $\tilde{n}(x)$ has the additional property; namely, that there exists s_0 such that for $s \geq s_0$ we have

$$N_R(j(2s)) > 2s, \tag{16}$$

where $N_R(i)$ is the number of real eigenvalues of (5) satisfying the condition $|\lambda| < (1 + \frac{1}{2})^2\pi^2/(a-1)^2$.

Together with the asymptotics (6) the property (16) allows us to choose a special sequence of real eigenvalues of (5). Assuming that s_0 from (16) is such that $j(2s_0)$ is bigger than i_0 from (6), we choose any $2s_0$ real eigenvalues from the eigenvalues satisfying $|\lambda| < (j(2s_0 + \frac{1}{2}))^2\pi^2/(a-1)^2$, and then we choose eigenvalues $\lambda_{j(s)}$ for $s > 2s_0$. We denote this sequence by $\{\lambda_{j(s)}\}$.

Substituting now (14) and (15) into (12) evaluated at $\lambda_{j(s)}(q)$ we obtain the infinite system of linear equations:

$$\begin{aligned} &\sum_{i=1}^{\infty} x_{2i} \cdot \left(\cos\sqrt{\lambda_{j(s)}}a\right) \int_0^1 \sin i\pi t \cdot \sin\sqrt{\lambda_{j(s)}}t\, dt \\ &\quad + \sum_{i=1}^{\infty} x_{2i-1}\left(\sin\sqrt{\lambda_{j(s)}}a\right) \int_0^1 \cos\left(i - \frac{1}{2}\right)\pi t \cdot \cos\sqrt{\lambda_{j(s)}}t\, dt \\ &= \sin\sqrt{\lambda_{j(s)}}(1-a) + M_x(1,1)\frac{\cos\sqrt{\lambda_{j(s)}}a \cdot \sin\sqrt{\lambda_{j(s)}}}{\lambda_{j(s)}} - M_x(1,1)\frac{\cos\sqrt{\lambda_{j(s)}}(1-a)}{\sqrt{\lambda_{j(s)}}} \end{aligned} \tag{17}$$

for $s = 1, \ldots, \infty$.

Let l_1^2 be the Hilbert space of real sequences $\{x_s\}_{s=1}^{\infty}$ with the norm $\|x\|_{l_1^2} = (\sum_{s=1}^{\infty} s^2 x_s^2)^{1/2}$.

Our main goal is to prove the following theorem.

THEOREM 3. *Let us denote by α_0 the minimal positive solution of the equation*

$$(1 - \cos\pi\alpha + \sin\pi\alpha)^2 + (2 - \cos\pi\alpha + \sin\pi\alpha)^2 \cdot (\tan\pi\alpha)^2 = \tfrac{1}{2},$$

and let us denote $a_0 = ((1 + \alpha_0)/\alpha_0) \sim 9.4$.

If $a > a_0$, then the system (17) *is a Fredholm type system in l_1^2, namely it is equivalent to a system*

$$(I + N + K)x = y,$$

where I is the identity operator, N is bounded operator with $\|N\|_{l_1^2} < 1$, K is a finite-dimensional operator, and $y \in l_1^2$.

2. Proof of Theorem 3. Before proving Theorem 3 we will prove several lemmas.

LEMMA 1. *For $a > 3$ and $s > (1/a - 1)\cdot \max\{1, A, 25A(a-1)/\pi^2\}$*

$$\left|\sqrt{\lambda}_{j(2s)}a - (2m+1)\frac{\pi}{2}\right| \geq \left(\frac{\pi}{2} - \frac{\pi}{a-1} - \frac{B}{s}\left(1 + \frac{2}{a-2}\right)\right) \tag{18}$$

and

$$|\sqrt{\lambda}_{j(2s+1)}a - m\pi| \geq \left(\frac{\pi}{2} - \frac{\pi}{a-1} - \frac{B}{s}\left(1 + \frac{2}{a-2}\right)\right), \tag{19}$$

for any $m \in Z^+$, where $B = 2\|q\|/\pi + C(a-1)/\pi$, and C from (6).

Proof. Let us start with the proof of (18).

Denote $p = j(2s) = [s(a-1)]$. Then from the asymptotic estimate (6) we conclude that

$$\left|\sqrt{\lambda}_{j(2s)} - \frac{p\pi}{a-1}\right| < \frac{B}{p},$$

and subsequently

$$\left|\sqrt{\lambda}_{j(2s)}a - \frac{p\pi a}{a-1}\right| < \frac{Ba}{s(a-1)-1} < \frac{B}{s}\left(1 + \frac{2}{a-2}\right). \tag{20}$$

It follows from (20) that to prove (18) it suffices to prove that

$$\left|\frac{p\pi a}{a-1} - (2m+1)\frac{\pi}{2}\right| \geq \left(\frac{1}{2} - \frac{1}{a-1}\right)\pi \tag{21}$$

for any $m = 1, \dots, \infty$. To prove (21) we represent $(p\pi a/(a-1)) = (2m+1)(\pi/2) + \alpha\pi$. Then

$$a = \frac{2m + 1 + 2\alpha}{2(m-p) + 1 + 2\alpha}$$

and

$$a - 1 = \frac{2p}{2(m-p) + 1 + 2\alpha}. \tag{22}$$

From the equality $[s(a-1)] = p$ we deduce that

$$\left[p\frac{2s}{2(m-p) + 1 + 2\alpha}\right] = p$$

and subsequently

$$1 \leq \frac{2s}{2(m-p) + 1 + 2\alpha} < 1 + \frac{1}{p},$$

or

$$(m-p)+\frac{1}{2}+\alpha \leq s<(m-p)+\frac{1}{2}+\alpha-1+\frac{2m+1+2\alpha}{2p}. \tag{23}$$

From (22) we get

$$2ma-2pa+a+2\alpha a=2m+1+2\alpha$$

and subsequently

$$\frac{2m+1+2\alpha}{2p}=\frac{a}{a-1}=1+\frac{1}{a-1}. \tag{24}$$

Applying now (24) in (23) we obtain

$$(m-p)+\frac{1}{2}+\alpha \leq s<(m-p)+\frac{1}{2}+\alpha+\frac{1}{a-1}. \tag{25}$$

From (25) we conclude that if $|\alpha|<\frac{1}{2}-(1/a-1)$, then there is no $s \in Z^{+}$ satisfying (25). This proves (21) and (18).

To prove (19) we use the same approach. Denoting $p=j(2s+1)=[(s+\frac{1}{2})(a-1)]$ and using the asymptotic estimate (6) we conclude that to prove (19) it suffices to prove that

$$\left|\frac{p\pi a}{a-1}-m\pi\right| \geq\left(\frac{1}{2}-\frac{1}{a-1}\right)\pi \tag{26}$$

for any $m=1,\ldots,\infty$. Representing $(p\pi a/(a-1))=m\pi+\alpha\pi$ we obtain from the equality $[(s+\frac{1}{2})(a-1)]=p$ that

$$2(m-p)+2\alpha \leq 2s+1<2(m-p)+2\alpha-2+\frac{2(m+\alpha)}{p}. \tag{27}$$

Using then in (27) the equality

$$\frac{2(m+\alpha)}{p}=\frac{2a}{a-1}=2+\frac{2}{a-1},$$

we conclude that (27) has no solutions s for $|\alpha|<\frac{1}{2}-(1/(a-1))$. This proves (26) and (19). □

LEMMA 2. *For $a>3$ and $s>\dfrac{1}{a-1}\max\{1, A, 25A(a-1)/\pi^{2}\}$ the following estimates are valid*:

$$\left|\sqrt{\lambda}_{j(2s)}-\pi s\right| \leq \frac{\pi}{a-1}+\frac{B}{sa}\left(1+\frac{2}{a-2}\right), \tag{28}$$

$$\left|\sqrt{\lambda}_{j(2s+1)}-\pi\left(s+\frac{1}{2}\right)\right| \leq \frac{\pi}{a-1}+\frac{B}{sa}\left(1+\frac{2}{a-2}\right). \tag{29}$$

Proof. From the estimate (20) we conclude that to prove (28) it suffices to prove the following estimate:

$$\left|\frac{[s(a-1)]\pi}{a-1}-s\pi\right|<\frac{\pi}{a-1}. \tag{30}$$

Representing for any s

$$a-1=\frac{l+\alpha}{s},$$

with $l \in Z$ and $0 \leq \alpha < 1$, we deduce that

$$\frac{[s(a-1)]}{a-1}=\frac{l \cdot s}{l+\alpha}=s-\frac{s\alpha}{l+\alpha}=s-\frac{\alpha}{a-1}$$

and subsequently the inequality (30). The estimate (29) follows analogously from the inequality

$$\left|\frac{\left[\frac{(2s+1)}{2}(a-1)\right]\pi}{a-1}-\left(s+\frac{1}{2}\right)\pi\right|<\frac{\pi}{a-1}. \qquad \square \tag{31}$$

In the next lemma we prove that the righthand side of (17) belongs to l_1^2.

LEMMA 3. *Let*

$$u_s=\sin\sqrt{\lambda}_{j(s)}(1-a)-\frac{M_x(1,1)}{\sqrt{\lambda}_{j(s)}}\cdot\cos\sqrt{\lambda}_{j(s)}(1-a)$$
$$+\frac{M_x(1,1)}{\sqrt{\lambda}_{j(s)}}\cdot\left(\cos\sqrt{\lambda}_{j(s)}a\right)\cdot\sin\sqrt{\lambda}_{j(s)}.$$

Then $(u_1,\dots,u_n,\dots)\in l_1^2$.

Proof. From the asymptotic estimate (6) we have:

$$\lambda_{j(s)}=\frac{\pi^2 j^2(s)}{(a-1)^2}-\frac{2M_x(1,1)}{a-1}+\alpha_s, \tag{32}$$

where $\{\alpha_s\}\in l_2$.

Then from (32) we estimate

$$\begin{aligned}\sqrt{\lambda}_{j(s)}&=\sqrt{\frac{\pi^2 j^2(s)}{(a-1)^2}-\frac{2M_x(1,1)}{a-1}+\alpha_s}\\&=\frac{\pi j(s)}{a-1}\sqrt{1-\frac{2(a-1)M_x(1,1)}{\pi^2 j^2(s)}+\frac{(a-1)^2\alpha_s}{\pi^2 j^2(s)}}\\&=\frac{\pi j(s)}{a-1}\left(1-\frac{(a-1)M_x(1,1)}{\pi^2 j^2(s)}+\frac{\beta_s}{j^2(s)}\right),\end{aligned} \tag{33}$$

where $\{\beta_s\}\in l_2$. From (33) we have the following estimates:

$$\begin{aligned}\sin\sqrt{\lambda}_{j(s)}(1-a)&=-\sin\left(\pi j(s)-\frac{(a-1)M_x(1,1)}{\pi j(s)}+\frac{\pi\beta_s}{j(s)}\right)\\&=(-1)^{j(s)+1}\frac{(a-1)M_x(1,1)}{\pi j(s)}+\frac{\gamma_s}{j(s)},\end{aligned} \tag{34}$$

where $\{\gamma_s\} \in l_2$, and

$$\cos\sqrt{\lambda_{j(s)}}(1-a) = \cos\left(\pi j(s) - \frac{(a-1)M_x(1,1)}{\pi j(s)} + \frac{\pi\beta_s}{j(s)}\right) = (-1)^{j(s)} + \delta_s, \tag{35}$$

where $\{\delta_s\} \in l_2$. Substituting now the estimates (33), (34), and (35) into the formula for u_s and noting that

$$\left\{\sigma_s = \frac{M_x(1,1)}{\lambda_{j(s)}} \cdot \cos\left(\sqrt{\lambda_{j(s)}}a\right) \cdot \sin\sqrt{\lambda_{j(s)}}\right\} \in l_1^2$$

we obtain

$$\left\{u_s = (-1)^{j(s)}\frac{(a-1)M_x(1,1)}{\pi \cdot j(s)} + \frac{\gamma_s}{j(s)} - \frac{M_x(1,1)}{\sqrt{\lambda_{j(s)}}} \cdot \left((-1)^{j(s)} + \delta_s\right) + \sigma_s\right\} \in l_1^2.$$

This concludes the proof of Lemma 3. □

Proof of Theorem 3. Let us denote

$$\begin{aligned} a_{si} &= \int_0^1 \sin i\pi t \cdot \sin\sqrt{\lambda_{j(2s)}}t\,dt, \\ b_{si} &= \int_0^1 \cos\left(i - \frac{1}{2}\right)\pi t \cdot \cos\sqrt{\lambda_{j(2s-1)}}t\,dt, \\ c_{si} &= \int_0^1 \cos\left(i - \frac{1}{2}\right)\pi t \cdot \cos\sqrt{\lambda_{j(2s)}}t\,dt, \\ d_{si} &= \int_0^1 \sin i\pi t \cdot \sin\sqrt{\lambda_{j(2s-1)}}t\,dt, \end{aligned} \tag{36}$$

and by $A = \{a_{si}\}$, $B = \{b_{si}\}$, $C = \{c_{si}\}$, $D = \{d_{si}\}$ the corresponding matrices. Then the system (17) becomes

$$\begin{aligned} \sum_{i=1}^{\infty} a_{si}\left(\cos\sqrt{\lambda_{j(2s)}}a\right)x_{2i} - \sum_{i=1}^{\infty} c_{si}\left(\sin\sqrt{\lambda_{j(2s)}}a\right)\cdot x_{2i-1} &= u_{2s}, \\ \sum_{i=1}^{\infty} d_{si}\left(\cos\sqrt{\lambda_{j(2s-1)}}a\right)x_{2i} - \sum_{i=1}^{\infty} b_{si}\left(\sin\sqrt{\lambda_{j(2s-1)}}a\right)\cdot x_{2i-1} &= u_{2s-1}. \end{aligned} \tag{37}$$

To make further transformation of the system (37) we will use the estimates (18) and (19) from Lemma 1.

From the definition of a_0 we have $a_0 > 7$. Therefore, there exists s_1 such that for $s > s_1$ we will have from (18) and (19), respectively,

$$\left|\cos\sqrt{\lambda_{j(2s)}}a\right| > \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

and

$$\left|\sin\sqrt{\lambda_{j(2s-1)}}a\right| > \frac{\sqrt{3}}{2}. \tag{38}$$

Thus, making the change of variables

$$\begin{aligned} y_l &= u_l & 1 < l < 2s_1, \\ y_{2r} &= u_{2r}\left(\cos\sqrt{\lambda}_{j(2r)}a\right)^{-1} & r > s_1, \\ y_{2r-1} &= u_{2r-1}\left(\sin\sqrt{\lambda}_{j(2r-1)}a\right)^{-1} & r > s_1, \end{aligned}$$

and grouping even and odd coordinates of x and y we transform (37) into a system, which up to a finite dimensional transformation is equivalent to

$$\begin{pmatrix} A & E \\ F & B \end{pmatrix}\begin{pmatrix} x^e \\ x^o \end{pmatrix} = \begin{pmatrix} y^e \\ y^o \end{pmatrix}, \tag{39}$$

where $y \in l_1^2$, matrices A and B are defined in (36) and

$$e_{si} = c_{si}\left(\tan\sqrt{\lambda}_{j(2s)}a\right), \qquad f_{si} = d_{si}\left(\cot\sqrt{\lambda}_{j(2s-1)}a\right). \tag{40}$$

Let us consider the operator corresponding to the matrix A', transpose of the matrix A, in the Hilbert space $L^2[0,1]$—the space of functions on $[0,1]$ with the scalar product

$$(f,g) = \left(\int_0^1 f(t)\,dt\right)\cdot\left(\int_0^1 g(t)\,dt\right) + \int_0^1 f'(t)\cdot g'(t)\,dt. \tag{41}$$

At first let us notice that in the basis $\{\sin s\pi t\}$ in $L^2[0,1]A'$ can be interpreted as a matrix of the linear transformation

$$\mathscr{A}' : \{\sin s\pi t\} \to \left\{\sin\sqrt{\lambda}_{j(2s)}t\right\}.$$

Using then the canonical isomorphism $\tau: L_1^2[0,1] \to \mathbf{C} \oplus L^2[0,1]$ defined by the formulas

$$\tau(f) = \int_0^1 f(t)\,dt \oplus f',$$

$$\tau^{-1}(c \oplus g) = \int_0^x g(t)\,dt + c - \int_0^1 dx \int_0^x g(t)\,dt$$

and a diagram

$$\begin{array}{ccc} L_1^2[0,1] & \xrightarrow{A'} & L_1^2[0,1] \\ {\scriptstyle\tau}\downarrow & & \uparrow{\scriptstyle\tau^{-1}} \\ \mathbf{C} \oplus L^2[0,1] & \xrightarrow{I \oplus \mathscr{A}'} & \mathbf{C} \oplus L^2[0,1] \end{array}$$

we can interpret A' as a matrix of the linear transformation

$$A' : \left\{\frac{\cos s\pi t}{s\pi}\right\} \to \left\{\frac{\cos\sqrt{\lambda}_{j(2s)}t}{\sqrt{\lambda}_{j(2s)}}\right\}$$

in $L_1^2[0,1]$.

From the estimate (28) and from the inequality $a_0 > 7$, we conclude that there exists s_2 such that for $s > s_2$

$$\left|\sqrt{\lambda}_{j(2s)} - s\pi\right| < \frac{\pi}{4}. \tag{42}$$

Therefore, we can apply the "Kadec $\frac{1}{4}$-Theorem" [Y] and obtain

$$A' = I + M' + S', \tag{43}$$

where S' is a finite dimensional operator, and

$$\|M'\|_{l_1^2} \le 1 - \cos \pi\beta + \sin \pi\beta \tag{44}$$

with $\beta = (1/\pi)\sup_{s>s_2}|\sqrt{\lambda}_{j(2s)} - s\pi|$.

The same arguments apply to B'. That is, B' is the matrix that represents the operator $\mathscr{B}'$: $\{\cos(s-\frac{1}{2})\pi t\} \to \{\cos\sqrt{\lambda}_{j(2s-1)}t\}$ in $L^2[0,1]$, and it can be shown that

$$B' = I + P' + R', \tag{45}$$

where R' is a finite dimensional operator, and

$$\|P'\|_{l_1^2} \le 1 - \cos \pi\delta + \sin \pi\delta \tag{46}$$

with $\delta = (1/\pi)\sup_{s>s_2}|\sqrt{\lambda}_{j(2s-1)} - (s-\frac{1}{2})\pi|$.

To estimate the norm of the operator E we note that C' is the matrix that represents the operator $\mathscr{C}'$: $\{\cos(s-\frac{1}{2})\pi t\} \to \{\cos\sqrt{\lambda}_{j(2s)}t\}$ in $L^2[0,1]$ and that $\mathscr{C}'$ is a composition of two operators

$$\mathscr{C}_1 : \{\cos(s-\tfrac{1}{2})\pi t\} \to \{\cos s\pi t\},$$
$$\mathscr{C}_2 : \{\cos s\pi t\} \to \left\{\cos\sqrt{\lambda}_{j(2s)}t\right\}.$$

$\mathscr{C}_1$ and the corresponding matrix C_1 are unitary operators. The norm of $\mathscr{C}_2$, or equivalently C_2, can be estimated with the use of the Kadec $\frac{1}{4}$-Theorem

$$\|C_2\|_{l_1^2} \le 1 + (1 - \cos \pi\beta + \sin \pi\beta), \tag{47}$$

with β from (44). Therefore we have the following estimate for E

$$\|E\|_{l_1^2} \le (2 - \cos \pi\beta + \sin \pi\beta)\cdot\gamma, \tag{48}$$

where $\gamma = \sup_{s>s_2}|\tan\sqrt{\lambda}_{j(2s)}a|$.

Analogous arguments lead to the following estimate for F

$$\|F\|_{l_1^2} \le (2 - \cos \pi\delta + \sin \pi\delta)\cdot\sigma, \tag{49}$$

where $\sigma = \sup_{s>s_2}|\cot\sqrt{\lambda}_{j(2s-1)}a|$. From (39), (43), and (45) we conclude now that the system (17) is equivalent to a system

$$(I + T + V)x = y,$$

where I is the identity operator, V is a finite dimensional operator, and

$$T = \begin{pmatrix} M & E \\ F & P \end{pmatrix}.$$

Using the estimates (44), (46), (48), and (49) we obtain

$$\begin{aligned} \|T\|_{l_1^2} &\le \sqrt{\|M\|_{l_1^2}^2 + \|P\|_{l_1^2}^2 + \|E\|_{l_1^2}^2 + \|F\|_{l_1^2}^2} \\ &\le \big((1 - \cos \pi\beta + \sin \pi\beta)^2 + (1 - \cos \pi\delta + \sin \pi\delta)^2 \\ &\quad + (2 - \cos \pi\beta + \sin \pi\beta)^2\cdot\gamma^2 + (2 - \cos \pi\delta + \sin \pi\delta)^2\cdot\sigma^2\big)^{1/2}. \end{aligned} \tag{50}$$

We modify now the operator T by a finite dimensional operator to make its norm small.

From the definition of β and from (28) using the inequality $a > a_0$, we conclude that there exists s_3 such that for $s > s_3$

$$|\sqrt{\lambda}_{j(2s)} - \pi s| < \frac{\pi}{a_0 - 1}. \tag{51}$$

Thus, if we change s_2 to s_3 in the definition of β, then we will have

$$\beta < \frac{1}{a_0 - 1}. \tag{52}$$

Analogous consideration shows that after another possible change of s_3 to s_4 we will have

$$\delta < \frac{1}{a_0 - 1}. \tag{53}$$

From the inequalities (18) and (19) we conclude that there exists s_5 such that for $s > s_5$ we will have

$$\inf_{\left\{ \substack{s > s_5 \\ m \in Z} \right\}} \left| \sqrt{\lambda}_{j(2s)} a - (2m+1)\frac{\pi}{2} \right| > \frac{\pi}{2} - \frac{\pi}{a_0 - 1},$$

$$\inf_{\left\{ \substack{s > s_5 \\ m \in Z} \right\}} |\sqrt{\lambda}_{j(2s-1)} a - m\pi| > \frac{\pi}{2} - \frac{\pi}{a_0 - 1},$$

and thus

$$\begin{aligned} \gamma &= \sup_{s > s_5} |\tan \sqrt{\lambda}_{j(2s)} a| < \tan \frac{\pi}{a_0 - 1}, \\ \sigma &= \sup_{s > s_5} |\cot \sqrt{\lambda}_{j(2s-1)} a| < \tan \frac{\pi}{a_0 - 1}. \end{aligned} \tag{54}$$

Using now the inequalities (52), (53), and (54), we choose $s_6 = \sup\{s_1, s_2, s_3, s_4, s_5\}$ and come to the decomposition

$$T = N + U, \tag{55}$$

where U is a finite dimensional operator and the norm of N can be estimated as follows:

$$\begin{aligned} \|N\|_{l_1^2} &\le \big((1 - \cos \pi\beta + \sin \pi\beta)^2 + (1 - \cos \pi\delta + \sin \pi\delta)^2 \\ &\quad + (2 - \cos \pi\beta + \sin \pi\beta)^2 \cdot \gamma^2 + (2 - \cos \pi\delta + \sin \pi\delta)^2 \cdot \sigma^2 \big)^{1/2} \\ &< \left(2\left(1 - \cos\left(\frac{\pi}{a_0 - 1}\right) + \sin\left(\frac{\pi}{a_0 - 1}\right)\right)^2 \right. \\ &\quad \left. + 2\left(2 - \cos\left(\frac{\pi}{a_0 - 1}\right) + \sin\left(\frac{\pi}{a_0 - 1}\right)\right)^2 \cdot \tan^2\left(\frac{\pi}{a_0 - 1}\right) \right)^{1/2}. \end{aligned} \tag{56}$$

Applying now in (56) the definition of a_0 and α_0 and monotonicity of the functions

$$\phi(\alpha) = (1 - \cos \pi\alpha + \sin \pi\alpha)^2 + (2 - \cos \pi\alpha + \sin \pi\alpha)^2 \cdot \tan^2 \pi\alpha,$$
$$\alpha(a) = \frac{1}{a-1},$$

we obtain

$$\|N\|_{l_1^2} < 1.$$

This concludes the proof of Theorem 3. □

We state again that the theorem proved here shows that up to a finite dimensional subspace in l_1^2 functions $M_t(1,t)$ and $M_x(1,t)$ can be uniquely reconstructed from the spectrum of the boundary value problem (4) in the case $a > a_0$.

3. Numerical results. In this section we discuss a numerical solution procedure that is suggested by the earlier analysis. As was mentioned in the introduction, we can obtain the unknown index of refraction $\tilde{n}(x)$ by the iteration procedure of [RS2] provided that the functions $M_t(1,t), M_x(1,t)$ may be found from the spectral data, where $M(x,t)$ is the kernel defined in (8). Using the expansions (14)–(15), we may expect to determine these functions by solving the system (17), which we write, with the obvious notation as

$$\sum_{i=1}^{\infty} A_{si}x_{2i} + B_{si}x_{2i-1} = U_s.$$

The given data for the inversion procedure consists of the sequence $\{k_j^2\}$, the real positive eigenvalues of (3) (equivalently the zeros of the integral averages of the scattering amplitude $F(\hat{r}, k, \alpha)$), together with the values of b and c_0. To evaluate the constants A_{si}, B_{si}, and U_s we need also the values of B (equivalently a) and $M_x(1,1)$. In principle both of these numbers are determined by the asymptotics (6) of the eigenvalues:

$$\frac{k_j}{(j\pi)} \mapsto \frac{1}{b-B}, \tag{57}$$

$$k_j^2 - \left(\frac{(\pi j)}{(b-B)}\right)^2 \mapsto \frac{1}{B(B-b)}\int_0^1 q(s)\,ds = \frac{2}{B(B-b)}M_x(1,1), \tag{58}$$

at least if $\tilde{n} \in H^2(0,1)$. If $\tilde{n}$ has less regularity, then (58) may still hold in the sense of Cesaro $(C,1)$ convergence [RS2].

There are various strategies one might use to numerically estimate the two constants $M_x(1,1)$ and B based on (57)–(58) and a finite number of eigenvalues $\{k_j^2\}_{j=1}^n$. Because (6) implies that

$$Pj^2 + Q = k_j^2 + o(1)$$

with $P = [\pi/(b-B)]^2$ and $Q = 2M_x(1,1)/B(B-b)$, we used a least-square approach as follows. For $m = 1, \ldots, n-2$ solve the overdetermined linear system,

$$Pj^2 + Q = k_j^2 \qquad j = m, \ldots, n \tag{59}$$

in the least-square sense, to get approximate values $P^{(m)}, Q^{(m)}$, and hence corresponding approximate values $M_x(1,1)^{(m)}, B^{(m)}$. Our estimate for B is the average of the

$B^{(m)}$'s. We could of course estimate $M_x(1,1)$ as the average of the $M_x(1,1)^{(m)}$'s, and that works well enough for smooth $\tilde{n}(x)$, but in general we got a better result by using the approximation to the Cesaro $(C,1)$ limit of (58), namely

$$\frac{2M_x(1,1)}{B(B-b)} \approx \frac{1}{n}\sum_{j=1}^{n}\left(k_j^2 - \left(\frac{\pi j}{b-B}\right)^2\right) \tag{60}$$

substituting the value for B just obtained.

At this point we may regard the numbers $\lambda_j = (Bk_j)^2$, $1 \le j \le n$, $a = b/B$, and $M_x(1,1)$ as (approximately) known so that A_{si}, B_{si}, U_s are known for $j(s) \le n$.

Next, we must choose the subsequence of eigenvalues $\lambda_{j(s)}$ appearing in (17), and the statement of Theorem 3 allows us considerable freedom in doing this. We recall that for some sufficiently large s_0 we are to choose any $2s_0$ eigenvalues λ satisfying $\sqrt{\lambda} < (j(2s_0 + \frac{1}{2}))\pi/(a-1)$ and then use $\lambda = \lambda_{j(s)}$ for $s > 2s_0$, with $j(s) =$ integer part of $\frac{1}{2}(a-1)s$. Because s_0 could always be increased, and because computationally we only work with a finite number of eigenvalues, we have in effect complete freedom to choose which eigenvalues we use, as long as the "spacing" of them is correct. Now of course Theorem 3 doesn't guarantee uniqueness (or say anything at all when $a < a_0$) so it is still up to us to select the subset of eigenvalues to use in an "intelligent" manner, namely to obtain a system that is as well conditioned as possible. It seems clear that to achieve this one should have the eigenvalues as evenly distributed as possible in the interval $[0, j(2s_0 + \frac{1}{2})^2\pi^2/(a-1)^2]$. With a set of eigenvalues in hand, such a selection could always be made. The automatic choice $j(s) = [\frac{1}{2}(a-1)s]$ will generally tend to achieve this goal. One can also imagine carrying out the reconstruction using several different choices of subsets of the eigenvalues, and then averaging the results in some manner.

Let us now summarize the complete procedure for numerical solution of the inverse eigenvalue problem. Assume that we are given k_j^2, $j = 1, \ldots, n$ real, nonnegative eigenvalues of (4), and the value of b.

(i) Estimate the values of $M_x(1,1)$ and $a = b/B$ as described above.

(ii) Select a suitable subset of the eigenvalues for use in the system (17). We used $\{k_{j(s)}^2\}_{s=1}^{2N}$ with $j(s) =$ integer part of $\frac{1}{2}(a-1)s$, and $2N$ the largest even integer less or equal to $2n/(a-1)$.

(iii) Solve the approximate system

$$\sum_{i=1}^{N}(A_{si}x_{2i} + B_{si}x_{2i-1}) = U_s \qquad s = 1, \ldots, 2N \tag{61}$$

for $\{x_i\}_{i=1}^{2N}$.

(iv) Obtain the approximate Cauchy data

$$\begin{aligned} \tilde{M}_t(1,t) &= -tM_x(1,1) + \sum_{i=1}^{N} x_{2i}\sin i\pi t \\ \tilde{M}_x(1,t) &= M_x(1,1) + \sum_{i=1}^{N} x_{2i-1}\cos\left(i - \frac{1}{2}\right)\pi t \end{aligned} \tag{62}$$

(v) Use the iterative procedure from [RS2] to obtain the impedance $A(x) = \sqrt{\tilde{n}(x)}$: For a given $A(x)$ let $u = u(x,t;A)$ solve the "sideways" Cauchy problem

$$\begin{aligned} &A(x)u_{tt} - (A(x)u_x)_x = 0 \qquad 0 \le |t| \le x \le 1 \\ &u_t(1,t) = \tilde{M}(1,t) \qquad u_x(1,t) = \tilde{M}_x(1,t) \qquad -1 \le t \le 1 \end{aligned} \tag{63}$$

with $\tilde{M}_t, \tilde{M}_x$ defined for $t < 0$ by odd and even extension, respectively. Let $A_0 \equiv 1$ and for $m = 1, 2, \ldots,$ set

$$A_{m+1}(x) = \frac{1}{M^2(x,x)}.$$

See [RS2] for details about convergence of this iteration scheme.

The impedance $A(x)$ could also be found by so-called layer-stripping methods (see, for example, [BB]).

(vi) Recover $c(r)$ from $\tilde{n}(x)$. We have the pair of equations

$$\sqrt{n(r(x))} = A(x), \qquad x = \frac{1}{B}\int_0^r \sqrt{n(u)}\, du, \tag{64}$$

where $A(x)$ is the function determined in step (v) and $B = b/a$ is known from step (i). We thus obtain

$$r(x) = B\int_0^x \frac{dx}{A(x)}$$

and hence $x(r)$ and subsequently

$$c(r) = \frac{c_0}{\sqrt{n(r)}} = \frac{c_0}{A(x(r))}$$

may be found. Computationally, this may be done by straightforward interpolation and quadrature techniques.

In Fig. 1 a sample velocity $c(r)$ is shown, for which we may take $b = 1$, along with its corresponding impedance. The exact value of $a = 1/B \approx 3.366$. We have carried out the above reconstruction procedure using $n = 10$ eigenvalues and then with $n = 30$ eigenvalues. The spectral data was generated numerically using the NAG subroutine D02HBF.

For the reconstruction using $n = 10$ eigenvalues we obtained estimated constants $a = 3.371$ and $M_x(1,1) = 1.230$, whereas for $n = 30$ the numbers were $a = 3.366$ and $M_x(1,1) = 2.327$. Although the a values are consistent with each other and quite accurate, the values for $M_x(1,1)$ evidently were not. The $n = 30$ value is naturally more accurate, and most of the error in the $n = 10$ estimate is attributable to the poorer estimate for a. That is to say, if the exact value of a happened to be available, a much better estimate for $M_x(1,1)$ would have resulted. We remark, however, that in [RS2] the effect of error in $M_x(1,1)$ was discussed, and found not to be so crucial. It tends to affect the reconstruction only for x very close to 0.

A subset of the eigenvalues was selected as described in step (ii) above, leaving us with $2N = 8$ and $2N = 24$ for the cases $n = 10, 30$ respectively. Reconstruction of $A(x) = \sqrt{\tilde{n}(x)}$ was carried out as in steps (iii)–(v) above, and the results are shown along with the exact $A(x)$ in Fig. 2. The condition number for the linear system solved in step iii is about 11 in either case. The iteration process for $n = 30$ is shown in Fig. 3; effective convergence has taken place after three iterations. Finally the inversion of the $x \to r$ transformation leads to the final reconstructions of $c(r)$ shown in Fig. 4.

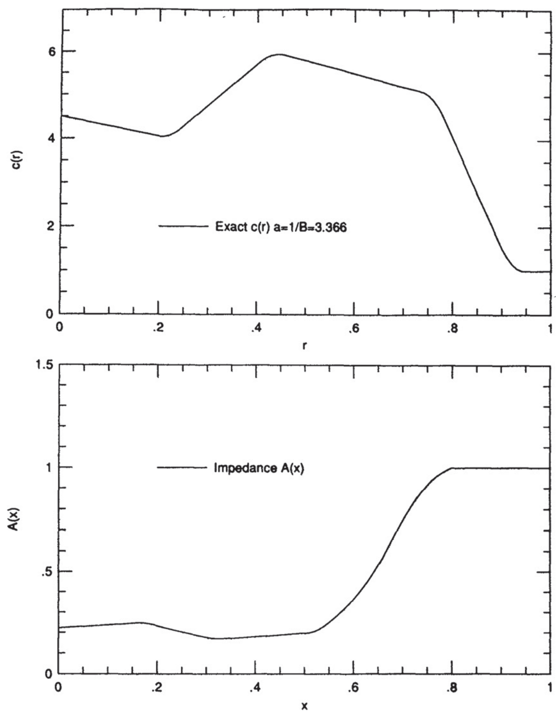

FIG. 1. *(Top) Velocity profile $c(r)$ to be found. (Bottom) Corresponding impedance* $A(x) = \sqrt{n(r(x))}$.

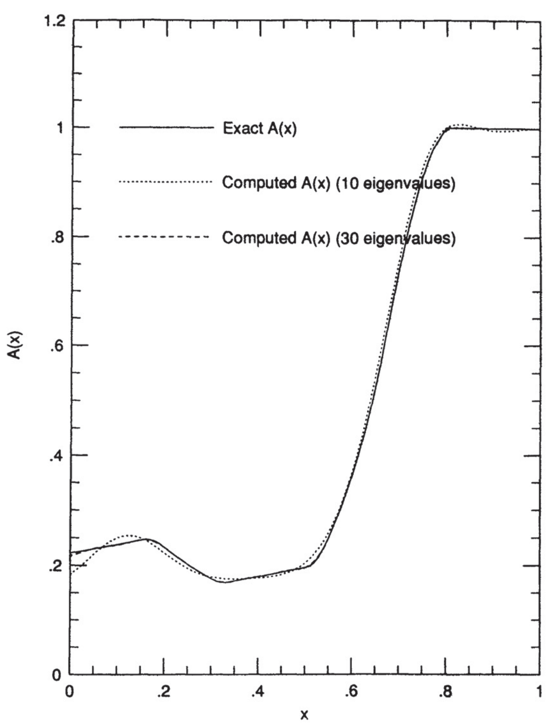

FIG. 2. *Reconstructions of $A(x)$ for $n = 10$ and $n = 30$.*

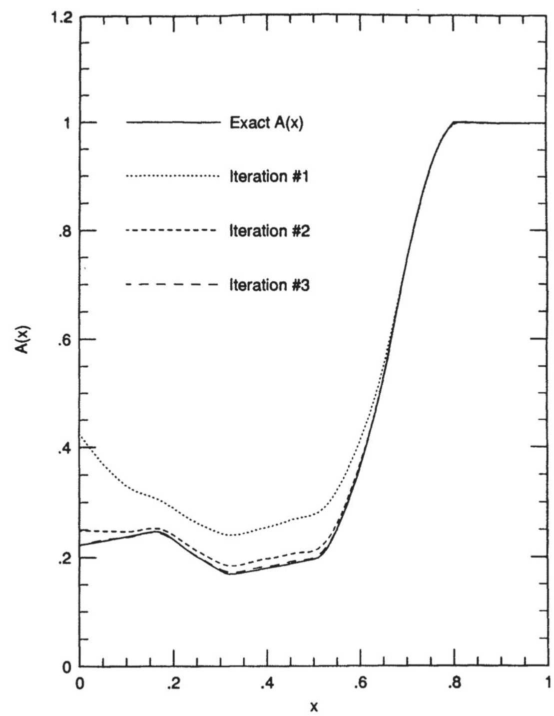

FIG. 3. *The iterates* $A_i(x)$ *for* $i = 1, 2, 3$.

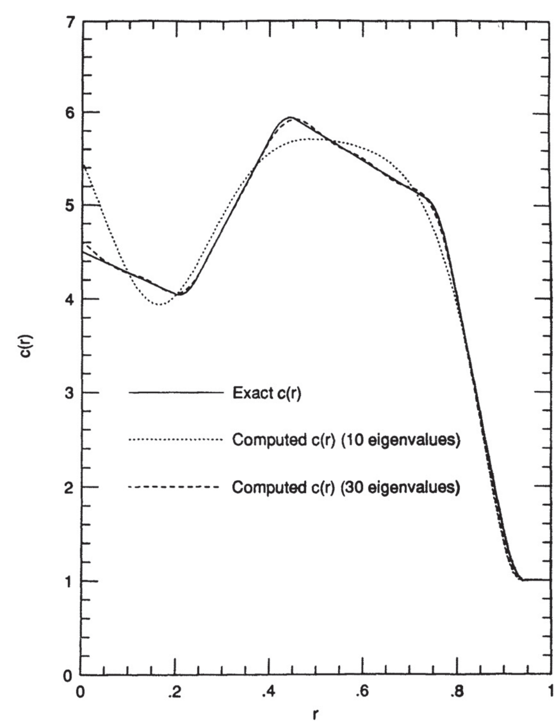

FIG. 4. *Reconstructions of* $c(r)$ *for* $n = 10$ *and* $n = 30$.

We remark that the velocity $c(r)$ in this example belongs to $H^2(0, b)$, but we have obtained reconstructions nearly as accurate for c's that are only Lipschitz continuous.

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