ON THE DYNAMICS OF A CLOSED THERMOSYPHON

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1. INTRODUCTION

In this work we shall study the motion of an incompressible fluid at the interior of a closed loop under the effect of convective forces arising from differences of temperature. We will assume that the section of the pipe is constant and small, so that the motion of the fluid can be supposed to be one-dimensional. Then the equations describing conservation of mass and momentum are given respectively by

$$\frac{\partial \rho}{\partial \tau} + \frac{\partial}{\partial s}(\rho u) = 0 \tag{1.1}$$

$$\frac{\partial u}{\partial \tau} + \frac{1}{2}\frac{\partial}{\partial s}u^2 + \frac{1}{\rho}\frac{\partial p}{\partial s} + \frac{\partial U}{\partial s} = -\frac{\lambda \mid u \mid u}{2D} \tag{1.2}$$

where s is the length coordinate along the loop and τ is the time coordinate: $\rho(s,\tau), p(s,\tau)$, and $u(s,\tau)$ denote respectively the density, pressure and velocity of the fluid. $U(s)$ represents the potential corresponding to gravitational forces, so that $f_m = -\frac{\partial U}{\partial s}$, $f_m(s)$ being the component of gravitational forces along the loop by unit of mass. D stands for the diameter of the pipe, whereas λ is Darcy-Weissbach coefficient. Using standard thermodynamical relations, we also obtain an equation for the enthropy $S(s,\tau)$, namely

$$\theta\left(\frac{\partial S}{\partial \tau} + v\frac{\partial S}{\partial s}\right) = \frac{\lambda u^3}{2D} + \frac{4q_s}{\rho D} \tag{1.3}$$

where $\theta = \theta(s,\tau)$ is the temperature of the fluid and q_s is the heat transfer coefficient at the wall. As it is often done when dealing with convective motions in liquids, we shall make use of the Boussinesq approximation here. We therefore will assume that density is constant in the continuity equation (1.1) (which yields $u = u(\tau)$), and consider ρ in the forms $\rho = \rho_0(1-(\theta-\theta_0))$ in (1.2), where ρ_0 is constant, θ_0 is some mean temperature, and $(\theta-\theta_0) << 1$, so that variations of temperature are assumed to be small. (Here and henceforth we shall make use of the customary asymptotic notations $O, o, <<$, etc.). Furthermore, one then supposes that variations of pressure are very small when compared with the hydrostatic pressure at equilibrium, p_h. If we then write $p = p_h + \tilde{p}$, where $\tilde{p} << p_h$, and notice that $\frac{1}{\rho_0}\,\frac{\partial p_h}{\partial s} + \frac{\partial U}{\partial s} = 0$, it readily follows that, retaining only lower order terms,

$$\frac{1}{\rho}\frac{\partial \tilde{p}}{\partial s} + \frac{\partial U}{\partial s} = \frac{1}{\rho_0}\frac{\partial \tilde{p}}{\partial s} + (\theta - \theta_0) f_m$$

We substitute this equation in (1.2), and assume that the loop geometry is described by a function $\tilde{z} = \tilde{z}(s)$, so that $f_m = -g\,\frac{d\tilde{z}}{ds}$. Integrating then the resulting equation along the loop, whose total length is denoted by L, we arrive at

$$L\frac{du}{d\tau} = g\int_0^L (\theta - \theta_0)\tilde{z}'(s)ds - \frac{\lambda L}{\tau D} \mid u \mid u \tag{1.4}$$

In the enthropy equation, we suppose that $q_s = \frac{\lambda \rho v}{8} c_\ell(\theta_w - \theta)$ where c_ℓ is the specific heat of the liquid and θ_w is the temperature at the wall (Reynolds analogy). Assuming that $u^2 << c_\ell(\theta_w - \theta)$, we obtain, after setting $S = c_\ell \log\theta + s_0$ for some constant s_0

$$\frac{\partial \theta}{\partial \tau} + u(\tau)\frac{\partial \theta}{\partial s} = \frac{\lambda \mid u \mid}{2D}(\theta_w - \theta) \tag{1.5}$$

We now turn our attention to the Darcy-Weissbach coefficient λ, which is a function of the Reynolds number

$$Re = \frac{v_c D}{\nu},$$

where v_c is a characteristic velocity of the problem and ν is the viscosity coefficient. Thus $\lambda = \lambda(\tilde{Re})$, where function $\lambda(\xi)$ is usually in the form

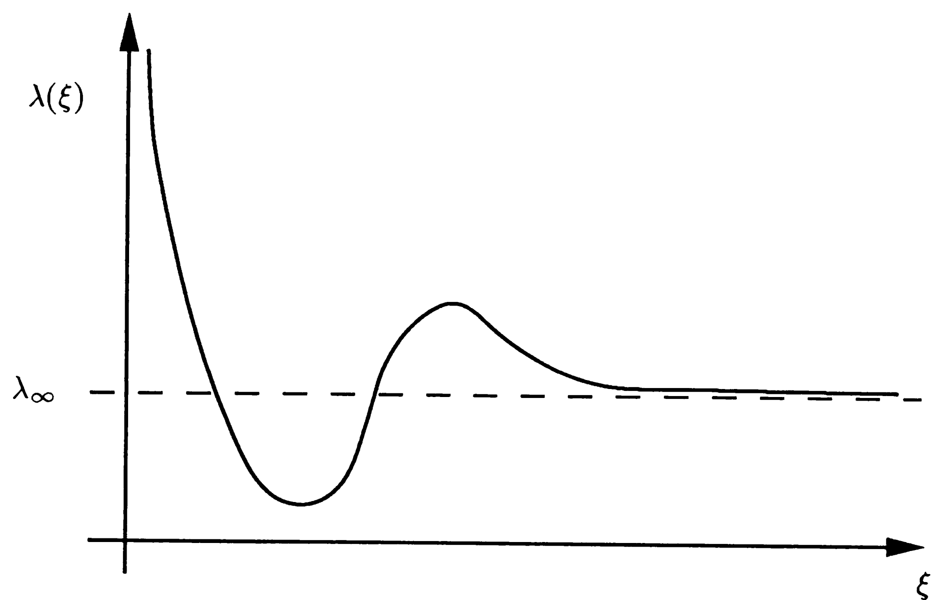

Figure 1: The dependence of λ on the Reynolds number

(cf. for instance [PT], Chapter III, Section B). More precisely, we shall assume that

$$\lambda(\xi) \approx \frac{A_1}{\xi} \text{ as } \xi \downarrow 0 \text{ for some } A_1 > 0$$

$$\lambda(\xi) \to \lambda_\infty \text{ as } \xi \to \infty \text{ for some } \lambda_\infty > 0$$

Additional requirements on $\lambda(\xi)$ will be made explicit later when required. We now define a new set of non-dimensional variables by setting

$$t = \frac{\tau}{t_0}, \quad x = \frac{s}{L}, \quad v = \frac{u}{v_c}, \quad z = \frac{\tilde{z}}{L}, \quad T = \frac{\theta - \theta_0}{\theta_c}$$

where θ_c is some characteristic temperature, $v_c = \left\{\frac{2g\theta_c D}{g}\right\}^{1/2}$, and $t_0 = \frac{L}{v_c}$. Putting $\epsilon = \frac{2D}{\lambda_\infty L}$ and $T_w = \frac{\theta_w - \theta_0}{\theta_c}$, we obtain that (1.4) and (1.5) give

(1.6a)
$$\epsilon \frac{dv}{dt} = \int_0^1 T(x,t) z'(x) dx - \frac{\lambda}{\lambda_\infty} \mid v \mid v,$$

(1.6b)
$$\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial x} = \frac{\lambda \mid v \mid}{\epsilon \lambda_\infty} (T_w - T)$$

We finally define

(1.7a)
$$g(\xi) = \frac{\lambda(\xi)}{\lambda_\infty}$$

so that $\frac{\lambda}{\lambda_\infty} \frac{|u| D}{\nu} = g\left(\frac{v_c D}{\nu} \mid v \mid\right)$. Therefore, g is such that

(1.7b)
$$g(\xi) \approx \frac{A}{\xi} \text{ as } \xi \downarrow 0 \text{ for some } A > 0,$$

(1.7c)
$$\lim_{\xi \to \infty} g(\xi) = 1$$

(compare with Figure 1). We shall denote by Re the associated Reynolds number given by

$$Re = \frac{v_c D}{\nu}$$

We shall refer henceforth to Re as the Reynolds number for short. If we now write $f(x) = z'(x)$, we are thus led to the study of the following

Problem.- To find $v(t)$ and $T(x,t)$ such that

(1.8a)
$$\epsilon \frac{dv}{dt} = \oint T(x,t) f(x) dx - g(Re \mid v \mid) \mid v \mid v \text{ for } t > 0,$$

(1.8b)
$$\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial x} = \frac{1}{\epsilon} g(Re \mid v \mid) \mid v \mid (T_w(x) - T)$$

for $t > 0$, and $x \in (0,1)$

(1.8c)
$$v(0) = v_0, \quad T(x,0) = T_0(x)$$

(1.8d)
$$T(0,t) = T(1,t) \text{ for any } t > 0$$

where for $\xi > 0, f(\xi)$ is a continuous function which satisfies (1.7). Here f, T_0 and T_w are periodic given functions with period one (cf. Section 2 below for precise functional assumptions on them), and so will be $T(x,t)$ with respect to the space coordinate x. To call attention to this fact, we have replaced the integral symbol in (1.6a) by $\oint$ in (1.8a) above. This notation will be retained henceforth.

As to the precedings of this work, there is a large literature devoted to the asymptotics of models alike to (1.8), which are usually referred to as thermosyphons. We should first mention the pioneering papers by Keller [K] and Welander [W], and refer to Chen [C], Hart [H], Sen, Ramos and Treviño [SRT] and Liñán [L] (among others) for recent interesting work on this topic. Of course, this bibliographical relation is far from being complete. In particular, further related work can be found in the references included in the previous papers. We should point out that most of these works used formal methods (basically, singular perturbation techniques) in their analysis, and quite often specialized to particular geometries (mainly toroidal) of the circuit considered.

Recently, Herrero and Velázquez considered in [HV] a simplified version of (1.8). Namely, they replaced the right hand side in (1.8) by a prescribed heat flux $q(x)$, and function $g(Re \mid v \mid)$ by a constant (which essentially amounts, to substitute g by its asymptotic value as $Re \mid v \mid \rightarrow \infty$). Moreover, they dropped the term $\epsilon \frac{dv}{dt}$ in the left hand side of (1.8a). This last assumption was motivated by consideration of the intermediate asymptotics of the model, a stage in which such hypothesis is suggested by formal analysis in [L].A somewhat surprising result which was proved in [HV] is that stationary solutions of such system are generically linearly unstable under small perturbations of the geometry of the pipe or the heating applied there. We refer to [HV] for details and related results.

The paper is concerned with the rigorous analysis of (1.8) for large values of the Reynolds number Re. Our plan here is as follows. Global existence and uniqueness are briefly discussed in a suitable functional frame in Section 2 below. Section 3 is then devoted to the study and characterization of stationary solutions of (1.8) when $Re >> 1$. We introduce there a meromorphic function $L(z)$(cf. (3.8)) which plays a central role in our analysis. In particular, we shall show later in Section 4 that sharp asymptotic estimates on the zeroes of $L(z)$ provide the key to establish a linear stability analysis of the stationary solutions for large Reynolds numbers. Such stability analysis makes the content of Section 4, which is the last in the paper.

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2. GLOBAL EXISTENCE AND UNIQUENESS.

Consider the following problem

$$\epsilon\frac{dv}{dt} = \oint T(x,t)f(x)dx - g(Re \mid v \mid) \mid v \mid v \quad \text{for } t > 0, \tag{2.1}$$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \frac{1}{\epsilon}g(Re \mid v \mid) \mid v \mid (T_w(x) - T) \tag{2.2}$$

for $t > 0$ and $x \in (0,1)$

$$v(0) = v_0, \quad T(x,0) = T_0(x) \tag{2.3}$$

$$T(0,t) = T(1,t) \text{ for any } t > 0 \tag{2.4}$$

where $\epsilon > 0$ and $Re > 0$. Define $H^1(0,1) = \{h \in L^2(0,1) : h^1 \in L^2(0,1)\}$ and denote by H^1_p the set consisting on the periodic extension to the whole line of those functions $h \in H^1(0,1)$ such that $h(0) = h(1)$. We shall assume that f, T_w and T_0 belong to H^1_p and $f(x)$ is such that $\int_0^1 f(x)dx = 0$. As to $g(\xi)$, in addition to (1.7) we shall require the following

(2.5a) $$g(\xi) \geq \eta > 0 \text{ for any } \xi \in (0,\infty),$$

(2.5b) $$\xi g(\xi) \in C^1 \text{ for } \xi > 0.$$

These assumptions will be retained henceforth.

We shall say that a pair of functions $(v(t), T(x,t))$ is a solution of (2.1)-(2.4) in a time interval $I = [0,\tau_0)$ if:

i) $v(t) \in C^1[0,\tau_0)$,

ii) $T(\cdot,t) \in C^1([0,\tau_0) : H_p^1)$,

iii) (2.1)–(2.4) are satisfied for $t \in [0,\tau_0)$.

If these conditions hold true for any $\tau_0 > 0$, the solution is said to be global. We now proceed to prove the following result

THEOREM 2.1. *Let f, T_0, T_w and g be given functions satisfying our previous assumptions. Then there exists a unique global solution of (2.1)-(2.4).*

Proof. a) Local existence follows from a classical fixed point argument. By standard results, any local solution of (2.1)-(2.4) will satisfy

(2.6) $$T(x,t) = \exp\left(-\frac{1}{\epsilon}\int_0^1 g(Re \mid v(s) \mid) \mid v(s) \mid ds\right) T_0\left(x - \int_0^t v(s)ds\right) +$$
$$+\frac{1}{\epsilon}\int_0^t \exp\left(-\frac{1}{\epsilon}\int_s^t g(Re \mid v(\lambda) \mid) \mid v(\lambda) \mid d\lambda\right) g(Re \mid v(s) \mid) \mid v(s) \mid T_w\left(-\int_s^t v(\lambda)d\lambda\right) ds,$$

and

(2.7) $$v(t) = v_0 - \frac{1}{\epsilon}\int_0^t g(Re \mid v(s) \mid) \mid v(s) \mid v(s)ds +$$
$$+\frac{1}{\epsilon}\int_0^t \exp\left(-\frac{1}{\epsilon}\int_s^t g(Re \mid v(\lambda) \mid) \mid v(\lambda) \mid d\lambda\right)\left(\int_0^1 f(x)T^0 x - \int_0^s v(\lambda)d\lambda\right) dxds$$
$$+\frac{1}{\epsilon^2}\int_0^t ds \int_0^s \exp\left(-\int_\lambda^s g(Re \mid v(r) \mid) \mid v(r) \mid dr\right) g(Re \mid v(\lambda) \mid) \mid v(\lambda) \mid$$
$$\left(\int_0^1 f(x) T_w\left(x - \int_\lambda^s v(r)dr\right) dx\right) d\lambda$$
$$\equiv Sv(t).$$

whenever t lies in the existence interval of (v,T). For a given constant $M>0$, we now consider the space $X \equiv X_{\delta,M} = \{u \in C[0,\delta] : \mid u(t)-v_0 \mid \leq M \text{ for } t \leq \delta\}$, endowed with the supremum norm. A straightforward computation reveals then that, for $\delta>0$ small enough, S is a contractive operator from X into itself. This yields the existence of a unique solution (v,T) of (2.1)-(2.4) for small times.

$b)$ To obtain global existence we recall that, by our assumptions on g, there exist positive constants σ, A and B such that

$$\frac{\sigma}{Re} \leq g(Re \mid v(t) \mid) \mid v(t) \mid \leq A + B \mid v(t) \mid$$

as for as $v(t)$ is well defined. For any $h \in L^2$, let us denote by $\|h\|$ the L^2-norm of h. Substituting (2.6) into (2.1) and multiplying both sides there by (sgnv), we obtain

$$\epsilon \frac{d \mid v \mid}{dt} \leq -g(Re \mid v \mid) \mid v \mid^2 + \|f\| \, \|T_0\|$$

$$+ \|f\| \, \|T_w\| \left(\frac{1}{\epsilon} \int_0^t e^{-\frac{\sigma(t-s)}{Re}} g(Re \mid v(s) \mid) \mid v(s) \mid ds \right)$$

and, by (2.5a)

$$\epsilon \frac{d \mid v \mid}{dt} \leq \eta^2 \mid v \mid^2 + \|f\| \, \|T_0\| + \frac{\|f\| \, \|T_w\|}{\epsilon Re} \int_0^t e^{-\frac{\sigma(t-s)}{Re}} (A + \text{BRe} \mid v(s) \mid) ds \tag{2.8}$$

$$\leq \eta^2 \mid v \mid^2 + \|f\| \, \|T_0\| + \|f\| \, \|T_w\| \frac{A}{\sigma\epsilon} + \frac{B}{\epsilon} \int_0^t e^{-\frac{\sigma(t-s)}{Re}} \mid v(s) \mid ds$$

Suppose now that $\mid v(s) \mid \leq C$ for $s \in [0,t]$, where $C > 2 \mid v_0 \mid$ will be precised later. Then (2.8) yields

$$\epsilon \frac{d \mid v \mid}{dt} \leq \eta^2 \mid v \mid^2 + \|f\| \, \|T_0\| + \|f\| \, \|T_w\| \frac{A}{\sigma\epsilon} + \frac{\text{BRe}}{\sigma\epsilon}$$

In particular, if $\mid v(t) \mid = C$, we would have $\frac{d|v(t)|}{dt} < 0$ provided that C is selected large enough. We have obtained a bound for $\mid v(t) \mid$ which is independent of its existence interval. Using (2.6), we obtain a similar bound for $T(x,t)$, and we now conclude by means of a standard continuation argument. □

3. STATIONARY SOLUTIONS FOR LARGE REYNOLDS NUMBERS.

In this Section we shall analyze the set of stationary solutions of (2.1)-(2.4) in the case where $Re >> 1$. We will show that, according to their velocity, there can be classified as fast solutions (when $\mid v \mid \approx 1$) and slow solutions, which are such that $\mid v \mid \approx \frac{1}{Re}$.

To begin with, stationary solutions satisfy

$$\oint f(x)T(x)dx = g(Re \mid v \mid) \mid v \mid v \tag{3.1}$$

$$v\frac{\partial T}{\partial x} = \frac{1}{\epsilon} g(Re \mid v \mid) \mid v \mid (T_w - T) \tag{3.2}$$

$$T(0) = T(1) \tag{3.3}$$

where f, T_w, ϵ and f are as in the previous Section. We shall often use Fourier expansions for f, T_w and T, which will be written as follows

$$f(x) = \sum_{-\infty}^{\infty} a_k e^{2\pi ikx} \text{ where } a_0 = 0 \text{ (since } \int_0^1 f(x)dx = 0) \tag{3.4a}$$

$$T_w(x) = \sum_{-\infty}^{\infty} b_k e^{2\pi \mathrm{ikx}}, \tag{3.4b}$$

$$T(x) = \sum_{-\infty}^{\infty} c_k e^{2\pi \mathrm{ikx}}. \tag{3.4c}$$

Furthermore, since all functions considered are real, one has that $a_k = \bar{a}_{-k}$ for any k, and a similar result holds for b_k, c_k. Clearly, one has that

$$\|f\|_{H^1}^2 = \sum_{-\infty}^{\infty}(1+k^2)a_k^2 < +\infty \tag{3.5a}$$

$$\|T_w\|_{H^1}^2 = \sum_{-\infty}^{\infty}(1+k^2)b_k^2 < +\infty \tag{3.5b}$$

The following quantity will play an important role in what follows

$$\chi_0 = \int_0^1 f(x)T_w(x)dx = \sum_{-\infty}^{\infty} a_k \bar{b}_k \tag{3.6}$$

From (3.1) and (3.4), we obtain

$$c_k = \frac{|v|}{\epsilon} g(Re \mid v \mid) b_k \left(\frac{1}{\epsilon} g(Re \mid v \mid) \mid v \mid +2\pi k i v \right)^{-1} \tag{3.7}$$

for any integer $k, -\infty < k < +\infty$

Taking into account (3.7), it turns out that (3.1) can be recast as follows

$$\epsilon v^2 = \sum_{-\infty}^{\infty} \frac{a_k \bar{b}_k}{z - 2\pi k_i} \equiv L(z) \tag{3.8a}$$

where

$$z = \frac{1}{\epsilon} \operatorname{sgn}(v) g(Re \mid v \mid) \tag{3.8b}$$

Analysis of function $L(z)$ in (3.8a) will be instrumental in deriving the main results in this Section (cf. Theorem 3.2 below). One readily sees that $L(z)$ is a meromorphic function with poles at points $\tilde{z}_j = 2\pi i j$ for which $\overline{a_j b_j} \neq 0$. To avoid a separate study of the cases where $\overline{a_j b_j} = 0$, we now introduce some definitions. We shall say that $L(z)$ has generalized poles at points $\tilde{z}_j = 2\pi i j$ for $j = \pm 1, \pm 2, \ldots$ A point z_0 will be said to be a generalized zero of $L(z)$ if either z_0 is a zero of $L(z)$ or $z_0 = 2\pi i j$ and $a_j \bar{b}_j = 0$ for some j. Notice that this last case can be viewed as a collapse of a zero and a pole, which takes place when $a_j \bar{b}_j = 0$. It is readily seen that the classical arguments principle in complex variable as well as Rouche's Theorem (which follows from it) still hold true when zeroes and poles are replaced by generalized zeroes and generalized poles respectively. We shall keep to this terminology from now on, although the term generalized will be dropped for convenience. We thus shall refer henceforth to zeroes and poles without any further specification.

Some useful properties of function $L(z)$ are gathered in the following

LEMMA 3.1. *Assume that $\chi_0 \neq 0$ in (3.6). Then the zeroes of $L(z)$ can be labelled as a sequence $\{z_j\} : j = 0, \pm 1, \pm 2, \ldots$. Moreover, there holds*

$$\lim_{|j| \to \infty} \mid z_j - 2\pi i j \mid = 0. \tag{3.9}$$

Remark. It follows from (3.9) that $L(z)$ has at most a finite number of real zeroes.

Proof of Lemma 3.1. It will follow from a suitable application of the argument principle, a technique which will be repeatedly used in the sequel. Consider the contour

$\gamma_N = \{z \in \mathbb{C} : z = (2N+1)\pi e^{i\theta}$ with $\theta \in [0, 2\pi]\}$, where $N = 0, 1, 2, \ldots$ For any $z \in \gamma_N$, we have that

$$| L(z) - \frac{\chi_0}{z} | \leq \sum_{|k| > \frac{N}{2}} \frac{2\pi \mid k \mid\mid a_k \bar{b}_k \mid}{\mid z \mid\mid z - 2\pi i k \mid} + \sum_{|k| \leq \frac{N}{2}} \frac{2\pi \mid k \mid\mid a_k \bar{b}_k \mid}{\mid z \mid\mid z - 2\pi i k \mid}$$
$$\equiv S_1 + S_2$$

Since $\mid z - 2\pi i k \mid \geq \pi$ whenever $z \in \gamma_N$, it follows from (3.5) that

$$S_1 \leq \frac{2}{\mid z \mid} \left(\sum_{|k| > \frac{N}{2}} k^2 \mid a_k \mid^2 \right)^{1/2} \left(\sum_{|k| > \frac{N}{2}} k^2 \mid b_k \mid^2 \right)^{1/2}$$
$$\equiv o\left(\frac{1}{\mid z \mid}\right) \quad \text{as } N \to \infty.$$

If $\mid k \mid \leq \frac{N}{2}$, then $\mid z - 2\pi i k \mid \geq \mid z \mid - 2\pi \mid k \mid \geq \pi N$ whenever $z \in \gamma_N$, whence

$$S_2 \leq \frac{2}{N \mid z \mid} \|f\|_{H^1} \|T_w\|_{H^1} = o\left(\frac{1}{\mid z \mid}\right) \quad \text{as } N \to \infty$$

for $z \in \gamma_N$. We thus have that

$$\mid L(z) - \frac{\chi_0}{z} \mid = o\left(\frac{1}{\mid z \mid}\right) \text{ whenever } \mid z \mid = (2N+1)\pi \text{ and } N \to \infty \tag{3.10}$$

It then follows from the argument principle that the number of zeroes of $L(z)$ exceeds in one to that to their poles at the interior of γ_N when $N >> 1$. Recalling (3.3a), we conclude that zeroes of $L(z)$ can be labelled as $z_0, z_{\pm 1}, z_{\pm 2}, \ldots$ Moreover, for large enough N, there are exactly two such zeroes in the annulus $A_N = \{ z : (2N+1)\pi < \mid z \mid < (2N+3)\pi \}$. We now proceed to derive (3.9). To this end, we notice that

$$\mid L(z) - \frac{a_j \bar{b}_j}{z - 2\pi i j} - \frac{\chi_0}{z} \mid = o\left(\frac{1}{\mid z \mid}\right) \quad \text{as } \mid j \mid \to \infty, \tag{3.11}$$

uniformly on sets $\mid z - 2\pi i j \mid \leq \frac{1}{10}$.

The proof of (3.11) is similar to that of (3.10) and will therefore be omitted. Consider now the contours $C_j = \{z \in \mathbb{C} : | z - 2\pi i j | = \frac{4\pi|j|a_j\bar{b}_j}{|\chi_0|}\}$. Recalling (3.5), one obtains after a routine computation that

$$\left|\frac{a_j\bar{b}_j}{z-2\pi ij} + \frac{\chi_0}{z}\right| \geq \frac{|\chi_0|}{8|z|} \text{ for } z \in C_j, |j| >> 1 \tag{3.12}$$

From (3.11), (3.12) and the argument principle, we deduce that $L(z)$ and $L_1(z) = \left(\frac{a_j\bar{b}_j}{z-2\pi ij} + \frac{\chi_0}{z}\right)$ have the same number of zeroes within the discs $B_j = \left\{z : | z - 2\pi ij | < \frac{4\pi|j||a_j\bar{b}_j|}{|\chi_0|}\right\}$. We now conclude by observing that zeroes of $L_1(z)$ are located at points

$$\tilde{z}_j = (2\pi ij)\left(1 + \frac{a_j\bar{b}_j}{\chi_0}\right)^{-1}$$

so that, by Taylors expansion

$$|\tilde{z}_j - 2\pi ij | < 4\pi |j|| a_j\bar{b}_j || \chi_0|^{-1} \text{ if } |j| >> 1,$$

and recalling (3.5), (3.9) follows. □

We now impose further conditions on the function $g(\xi)$, namely

(3.13a) There exists χ_0 and ξ_1 with $0 < \xi_0 < \xi_1 < +\infty$ such that

$$g'(\xi) < 0 \text{ for } 0 < \xi < \xi_0, g'(\xi) > 0 \text{ for } \xi_0 < \xi < \xi_1 \text{ and}$$

$$g'(\xi) < 0 \text{ for } \xi > \xi_1. \text{ At } \xi_0, \xi_1 \text{ we have that } g(\xi_1) = M, g(\xi_2) = m.$$

(3.13b) There exist $B > 0$ such that

$$g(\xi) = 1 + \frac{B}{\xi} + o\left(\frac{1}{\xi}\right) \text{ as } \xi \to \infty,$$

$$g'(\xi) = -\frac{B}{\xi^2} + o\left(\frac{1}{\xi^2}\right) \text{ as } \xi \to \infty,$$

In view of (3.13a), it is clear that if z and v are related by (3.8b), we should have $|z| \geq \frac{m}{\epsilon}$. Consider now the set

(3.14a) $$S = \{z \in \mathbb{C} : L(z) = 0 \text{ and } |z| \geq \frac{m}{\epsilon}\}.$$

For any $z_j \in S$, we define a set of complex numbers $\{\xi_j^\ell\}, \ell = 1, \ldots, j(z_j)$ given by the roots of the equation

(3.14b) $$z_j = \frac{1}{\epsilon} \operatorname{sgn}(\xi) g(\xi_j^\ell)$$

The main result of this Section is

THEOREM 3.2. *Assume that hypotheses (3.13) hold, and*

(3.15a) $$\chi_0 \neq 0, L(\pm\frac{1}{\epsilon}) \neq 0, L(\pm\frac{m}{\epsilon}) \neq 0, L(\pm\frac{M}{\epsilon}) \neq 0.$$

(3.15b) *The roots $z_j \in S$ are simple.*

Then for large enough Reynolds numbers there exists a one-to-one correspondence between the possible values of the velocity for solutions of (3.1)-(3.3) , and the elements of the following set

$$I = \Big\{\{\xi_j^\ell\} : \ell = 1, \ldots, j(z_j), \text{ where } \xi_j^\ell \text{ and } z_j \text{ are related by (3.14) } \Big\} \bigcup \left\{ z : z = \pm\frac{1}{\epsilon} \text{ and } L(z) > 0 \right\}.$$

Moreover, the dependence of the velocity values on Reynolds numbers in such case is described by the following formulae

(3.16) $$v_j^\ell = \frac{\xi_j^\ell}{Re} + o\left(\frac{1}{Re}\right) \quad \text{as } Re \to \infty$$
$$\text{for } \ell = 1, \ldots, j(z_j)$$

(3.17a) $$v_+ = \left(\frac{1}{\epsilon} L\left\{\frac{1}{\epsilon}\right\}\right)^{1/2} + o(1) \text{ as } Re \to \infty$$
$$\text{if } L\left(\frac{1}{\epsilon}\right) > 0$$

(3.17b) $$v_- = -\left(\frac{1}{\epsilon}L\left\{-\frac{1}{\epsilon}\right\}\right)^{1/2} + o(1) \text{ as } Re \to \infty$$
$$\text{if } L\left(-\frac{1}{\epsilon}\right) > 0.$$

Finally, solutions satisfying (3.16) actually exist.

Remark. Theorem 3.2 states that, under suitable transversality conditions, stationary solutions corresponding to large Reynolds numbers may be of two types: slow solutions, for which velocities are of order $\frac{1}{Re}$, and fast solutions, for which velocities are of order unity. Notice that assumptions (3.14), (3.15) are generic, in the sense of being preserved under small perturbations of f, T_w, ϵ and g.

Proof of Theorem 3.2. We shall consider separately equation (3.8a) in the cases where $| v | \geq (Re)^{-1/2}$ and $| v |$ that $| v | \geq (Re)^{-1/2}$. Then $g(Re \mid v \mid) = 1 + o(1)$, uniformly as $Re \to \infty$. This in turns implies that

$$z = \frac{\text{sgn}(v)}{\epsilon} + o(1), L(z) = L\left(\frac{\text{sgnv}}{\epsilon}\right) + o(1)$$

uniformly as $Re \to \infty$ where z and v are related through (3.8b). Since $L\left(\pm\frac{1}{\epsilon}\right) \neq 0$ by assumption, we deduce that for such solutions to exist it is necessary to have $L\left(\frac{1}{\epsilon}\right) > 0$ or $L\left(-\frac{1}{\epsilon}\right) > 0$, in which cases (3.17a) or (3.17b) respectively hold. To show uniqueness of such solutions, we argue as follows. Let us define

(3.18) $$H(v) = L\left(\frac{1}{\epsilon}\text{sgn}(v)\right) g(Re \mid v \mid)$$

Recalling condition (3.13b), we see that for $| v | \geq (Re)^{-1/2}$ and large enough (Re)

(3.19) $$| H'(v) | \leq C(Re)^{-1/2}$$

where here and henceforth C will denote a generic constant, depending possibly on $\|f\|, \|T_w\|$, ϵ and g. Suppose now that there exist two different solutions with velocity values v_1 and v_2, and that (3.17a) holds. Then $v_i \geq \mu > 0$ $(i = 1, 2)$ for some μ and $Re >> 1$. By (3.8a), one then has

(3.20a) $$\frac{H(v_1) - H(v_2)}{v_1 - v_2} = \epsilon(v_1 + v_2) \geq 2\mu\epsilon$$

whereas on the other hand, by (3.19)

$$(3.20b) \qquad \frac{H(v_1) - H(v_2)}{v_1 - v_2} = \frac{1}{v_1 - v_2} \int_{v_2}^{v_1} H'(s)ds \leq C(Re)^{-1/2}$$

From (3.20) one derives a contradiction when $Re >> 1$. The case where (3.17b) holds is similar.

Assume now that $| v |< (Re)^{-1/2}$. By (3.18) and (3.8a), it then follows that $| H(v) | \leq \epsilon(Re)^{-1/2}$. If $Re >> 1$, this can only happen if one of the following cases holds

$$(3.21a) \qquad \frac{1}{\epsilon} \operatorname{sgn}(v) g(Re \mid v \mid) \approx z_j, \text{ where } z_j \in S \text{ (cf. (3.13a))}$$

or

$$(3.21b) \qquad Re \mid v \mid \to 0 \text{ as } Re \to \infty$$

We next proceed to rule out the second alternative above. Indeed, if (3.21b) occurs, by (3.13b), we should have

$$\frac{A}{2\xi} \leq g(\xi) \leq \frac{3A}{2\xi} \text{ if } \xi \text{ is small enough}$$
$$\text{(independently of } Re)$$

so that, by (3.8b)

$$\frac{A}{2\epsilon Re|v|} \leq | z | \leq \frac{3A}{2\epsilon Re|v|} \text{ for } Re >> 1$$

We then can use (3.10) and (3.13*b*), to obtain that for $Re >> 1$,

$$\epsilon v^2 = | L(z) | \geq \frac{\epsilon \mid \chi_0 \mid}{3A} Re \mid v \mid$$

whence

$$(3.22a) \qquad v = 0$$

or

$$\text{(3.22b)} \qquad |\, v \,| \geq \frac{|\, \chi_0 \,|\, Re}{3A}$$

If (3.22b) holds, we get at once a contradiction with the assumption $|\, v \,| \leq (Re)^{-1/2}$. If on the other hand $v = 0$, we deduce from (3.2) that $T(x) \equiv T_w(x)$, in which case (3.1) yields $\chi_0 = 0$, which contradicts (3.14). Then (3.21a) must be satisfied. Taking into account (3.13b), it follows that such solutions (if any) should verify (3.16). It remains yet to prove existence and uniqueness for this last type of solutions. To this end, we use the fact that

$$L\left(\pm\frac{m}{\epsilon}\right) \neq 0, L\left(\pm\frac{M}{\epsilon}\right) \neq 0, L\left(\pm\frac{1}{\epsilon}\right) \neq 0.$$

(cf. (3.15a)). Together wit (3.15b), this implies that $g'(\xi_j^\ell) \neq 0$ for any j and ℓ as before. Using Taylor's expansion, we then obtain that

$$H(v) = \frac{Re\ \mathrm{sgn}(v)}{\epsilon}\left[L'\left(\frac{sgn(v)g(\xi_j^\ell)}{\epsilon}\right) g'(\xi^{\ell_j})\left(v - \frac{\xi_j^\ell}{Re}\right) + o\left(v - \frac{\xi_j^\ell}{Re}\right)\right]$$

$$\text{for } v - \frac{\xi_j^\ell}{Re} = o\left(\frac{1}{Re}\right), \text{ uniformly as } Re \to \infty.$$

It is then readily seen that, for $\chi > 0$ small enough , $H\left(\frac{\xi_j^\ell}{Re} - \frac{\delta}{Re}\right)$ and $H\left(\frac{\xi_j^\ell}{Re} + \frac{\delta}{Re}\right)$ have opposite signs, whence existence follows. Uniqueness is obtained by noting that

$$\frac{d}{dr}(H(r) - \epsilon r^2) = \frac{Re\ \mathrm{sgn}(r)}{\epsilon} L'\left(\frac{\mathrm{sgn}(r) g(Re\,|\, r\,|)}{\epsilon}\right) - 2\epsilon r$$

so that $\frac{d}{dv}(H(v) - \epsilon v^2)$ has a definite sign for $v \in \left(\frac{\xi_j^\ell}{Re} - \frac{\delta}{Re}, \frac{\xi_j^\ell}{Re} + \frac{\delta}{Re}\right)$ when $\delta > 0$ is small enough. □

4. LINEAR STABILITY OF STATIONARY SOLUTIONS FOR LARGE REYNOLDS NUMBERS.

4.1. Preliminaries. Linear stability of fast solutions.

In this Section we shall perform a linear stability analysis of the stationary solutions considered in Section 3. To this end, it will be useful to introduce a new velocity related variable Φ and a new time s, defined as follows

$$\Phi = \frac{|v|v}{2}, \tag{4.1a}$$

$$ds = \frac{1}{\epsilon} g(Re \mid v \mid) \mid v \mid dt \tag{4.1b}$$

It is then readily seen that (2.1), (2.2) can be rewritten in the form

$$\frac{d\Phi}{ds} + 2\Phi = \frac{1}{g(Re|v|)} \oint f(x)T(x,s)ds, \tag{4.2}$$

$$\frac{\partial T}{\partial s} + \frac{\epsilon \text{ sgn } \Phi}{g(Re \mid v \mid)} \cdot \frac{\partial T}{\partial x} = (T_w(x) - T(x,s)) \tag{4.3}$$

Classical linear stability theory proceeds by setting

$$\begin{aligned} \Phi &= \Phi_s + \delta\Phi = \Phi_s + ce^{-\lambda s} + \dots, \\ T &= T_s + \delta T = T_s + e^{-\lambda s}\psi + \dots, \end{aligned}$$

where it is assumed that $\delta\Phi << 1, \delta T << 1$, as $s \to \infty, \Phi_s$ and T_s being stationary solutions of (4.2), (4.3). Retaining only first order terms, standard computations yield then

$$(2-\lambda)c = -Q\int_0^1 f(x)T_s(x)dx + (g(Re \mid 2\Phi_s \mid)^{1/2})^{-1}\int_0^1 f(x)\psi(x)dx \tag{4.4a}$$

$$-\lambda\psi + \frac{\psi'}{2} - \epsilon c \text{ sgn } \Phi_s Q T_s = -\psi \tag{4.4b}$$

where

$$Q = \frac{g'(Re \mid 2\Phi_s \mid^{1/2}) Re \text{ sgn}(\Phi_s)}{g(Re \mid 2\Phi_s \mid^{1/2})(2 \mid \Phi_s \mid)^{1/2}} \tag{4.4c}$$

and z is given in (3.8b). If we now write

$$(4.5) \qquad \psi(x) = \sum_{\infty}^{\infty} d_k e^{2\pi \mathrm{ikx}}.$$

Then, recalling (3.4) and (3.7), we see that (4.4) can be written in the form

$$(4.6a) \qquad (2-\lambda)c = -Q(\sum \bar{a}_k c_k)c + g(Re \mid 2\Phi_s \mid^{1/2})^{-1} \sum \bar{a}_k d_k,$$

$$(4.6b) \qquad ((1-\lambda) + \frac{2\pi i k}{z})d_k = cQ\epsilon \operatorname{sgn}(v_s) 2\pi \mathrm{ikc}_k : k = \pm 1, \pm 2, \ldots$$

Set now

$$(4.7) \qquad \mu = (1-\lambda)z$$

Then straightforward (but tedious) calculations show that there exists nontrivial solutions of (4.6) if and only if

$$(4.8) \qquad \mu \neq 2\pi k i : \text{ and } S(\mu) = 1 + \frac{\mu}{z} + \frac{Q\mu z}{\mu - z} L(z) - L(\mu) =$$

$$v_s^2 - L(\mu) = 0,$$

or $\mu = 2\pi k i$ and $a_k \bar{b}_k = 0$, where $k = \pm 1, \pm 2, \ldots, v_s$ is a stationary velocity value (related to Φ_s via (4.1b)), and Q, L are respectively given in (4.4c) and (3.8a). Notice that without loss of generality, the eigenvalues μ in (4.7) are given by the roots of

$$(4.9) \qquad (1 - e^{\mu})S(\mu) = 0$$

Consider now S as a function of the complex variable μ. Then the poles of $S(\mu)$ are located at $\mu = 2\pi i k, k = \pm 1, \pm 2, \ldots$ In particular, $\mu = z$ is not a pole for the meromorphic function S. By means of a slight modification of the arguments leading to the proof of Lemma 3.1, we then see that

$$S(\mu) \approx \frac{\mu}{z} \text{ as } N \to \infty \text{ if } \mid \mu \mid = (2N+1)\pi$$

and

$$(4.10) \qquad \begin{array}{c}\text{The set of generalized zeroes of } S(\mu) \text{ can be labelled} \\ \text{as a sequence } \{\mu_j\} \text{ where } j = 0, \pm 1, \pm 2, \pm 3, ..\end{array}$$

Furthermore

$$(2 \mid j \mid -1)\pi <\mid \mu_j \mid< (2 \mid j \mid +1)\pi \text{ for } j = \pm 1, \pm 2, \pm 3, \ldots$$

We next set out to estimate the roots of $(1 - e^{\mu})S(\mu)$. As a first result, we shall prove:

LEMMA 4.1. *Assume that the hypotheses in Theorem 3.2 are satisfied and v_s behaves as in (3.17). Then there holds*

$$(4.11) \qquad \mid Q \mid = O\left(\frac{1}{Re}\right) \text{ as } Re \to \infty$$

Moreover, there exists a constant C independent of Re and j, such that

(4.12a) *There is one root μ_0 of $S(\mu)$ satisfying*

$$\mid \mu_0 + z \mid \leq C \mid Q \mid \quad as\ Re \to \infty$$

(4.12b) *For $j = \pm 1, \pm 2, \pm 3, \ldots$ there is a unique root of $(1 - e^{\mu})S(\mu)$ satisfying*

$$\mid \mu_j - 2\pi i j \mid \leq \frac{C|Q| \mid \bar{a}_j b_j \mid}{\mid j \mid} \quad as\ Re \to \infty$$

Proof. Estimate (4.11) follows at once from (4.4c) and (3.13b). Furthermore, under our current assumptions $z \to \pm\frac{1}{\epsilon}$ as $Re \to \infty$. If we now fix $\delta > 0$, we have that

$$\mid S(\mu) - \left(1 + \frac{\mu}{2}\right) \mid = \mid Q\mu z \frac{L(z) - L(\mu)}{z - \mu} \mid \leq K \mid Q \mid$$

for some $K > 0$, provided that $\mid \mu + z \mid \leq \delta$.
Set now $\mid \mu + z \mid = 2K \mid z \mid\mid Q \mid$. By (4.11), we see that $\mid \mu + z \mid \leq \delta$ if $Re >> 1$. One then has that

$$\mid S(\mu) - \left(1 + \frac{\mu}{2}\right) \mid \leq K \mid Q \mid < 2K \mid Q \mid = \mid 1 + \frac{\mu}{z} \mid$$

Then by Rouche's Theorem we obtain the existence of a root of $S(\mu)$ at the interior of the ball $B_1 = (\mu :\mid \mu + z \mid < 2K \mid Q \mid\mid z \mid)$ if $Re >> 1$, whence (4.12a) follows. As to (4.12b) it suffices to consider the case $a_j \bar{b}_j \neq 0$, since otherwise the result holds in view of (4.8) and (4.9). A first estimate on the location of the zeroes $\{\mu_j\}$ is then provided by (4.10). Assume now that

$$\mid \mu - 2\pi i j \mid = \frac{2 \mid Q \mid\mid j \mid z^2 \mid a_j \bar{b}_j \mid}{|2\pi i j - z||2\pi i j + z|} \tag{4.13}$$

Standard calculations yield then

$$\mid S(\mu) - 1 + \frac{2\pi i j}{z} + \frac{Q\mu z}{2\pi i j - z} \cdot \frac{a_j \bar{b}_j}{\mu - 2\pi i j} \mid \tag{4.14a}$$

$$\leq \frac{\mid \mu - 2\pi i j \mid}{|z|} + \frac{\mid Q \mid\mid \mu \mid\mid z \mid}{|\mu - z|} \left(\mid L(z) \mid + \mid \sum_{k \neq j} \frac{a_k \bar{b}_k}{\mu - 2\pi i k} \mid \right)$$

$$\leq C \mid Q \mid$$

where C depends on $\|f\|$, $\|T_w\|$ and z, but not on $\mid j \mid$. In a similar way, we obtain that, if (4.13) holds

$$\mid 1 + \frac{2\pi i j}{z} - \frac{Q\mu z}{2\pi i j - z} \cdot \frac{a_j \bar{b}_j}{\mu - 2\pi i j} \mid \geq \frac{1}{2} \mid 1 + \frac{2\pi i j}{z} \mid \tag{4.14b}$$

Taking into account (4.14), the conclusion follows by Rouche's Theorem, since there are only a finite number of such roots in $\mid j \mid$ is bounded. □

An immediate consequence of Lemma 4.1, is the following

COROLLARY 4.2. *Assume that the hypotheses in Theorem 3.2 hold. Then stationary solutions whose velocities satisfy (3.17) are linearly stable for large Reynolds numbers.*

Proof. Recalling (4.7) and (4.12), we readily see that as $Re \to \infty$, the eigenvalues of the linearized problem satisfy

$$\lambda_0 = 2 + O\left(\frac{1}{Re}\right) \tag{4.15a}$$

$$(4.15b) \qquad \mid \lambda_k - 1 + \frac{2\pi i k}{z} \mid \leq \frac{C}{Re} \cdot \frac{\mid a_k \bar{b}_k \mid}{|k|} \text{ for } k = \pm 1, \pm 2, \ldots$$

where C depends on z, f and T_w. $\square$

4.2. Linear stability of slow solutions. Statement of the main result. Refined estimates on L(z).

We shall now consider the case where (3.16) holds. Then the function Q given in (4.4c) is such that

$$(4.16) \qquad Q = \frac{g'(Re \mid v_s \mid) Re}{(g(Re \mid v_s \mid))^2} \sim \frac{g'(\xi_j^\ell) Re^2}{g(\xi_j^\ell)^2 \xi_j^\ell} \text{ as } Re \to \infty$$

where v_s and ξ_j^ℓ are related as explained in the statement of Theorem 3.2. Notice that $g'(\xi_j^\ell) \neq 0$ since $L\left(\pm\frac{m}{\epsilon}\right) \neq 0$, $L\left(\pm\frac{1}{\epsilon}\right) \neq 0$ and $L\left(\pm\frac{M}{\epsilon}\right) \neq 0$. In what follows, we shall require that

$$(4.17) \qquad \frac{M_0}{\chi_0} + \frac{g(\xi_j^\ell)^2}{g'(\xi_j^\ell)\xi_j^\ell z_j^2} + \epsilon \neq 0, \text{ where}$$

$M_0 = \sum_{-\infty}^{\infty} 2\pi i k a_k \bar{b}_k, \chi_0$ is given in (3.6), z_j is one of the zeroes of $L(z)$ and z_j, ξ_j^ℓ are related through (3.14b) (cf. also (3.16)).

Let us consider now a stationary solution whose velocity $v \equiv v_s$ satisfies (3.16). Our goal consists in proving.

THEOREM 4.3. *Assume that the hypotheses in Theorem 3.2 are fulfilled. Suppose also that (3.16) and (4.17) hold. Then the roots of* $(1 - e^\mu)S(\mu)$ *can be labelled as a sequence* $\{\mu_k\} : k = 0, \pm 2, \pm 3, \ldots$ *whose elements have the following asymptotic behaviours.*

$$(4.18) \qquad \mid \mu_k - z_k \mid = o(1) \text{ as } Re \to \infty \text{ for } k = \pm 2, \pm 3, .. \text{ and } k \neq j,$$

$$(4.19) \qquad \mu_0 = o(1) \text{ as } Re \to \infty,$$

$$(4.20) \qquad \mu_{\pm 1} = \pm (Q z^2 \chi_0)^{1/2} + \frac{q \epsilon z^2 (\xi_j^\ell)^2}{2} + \frac{M_0}{2\chi_0} + o(1),$$

$$\text{as } Re \to \infty \text{ , where } q = \lim_{Re \to \infty} \frac{Q}{(Re)^2} \text{ (cf. (4.16))}$$

Remark. Notice that all the assumptions in Theorem 4.3 are generic in the sense of being preserved under small perturbations on f, T_w, g and ϵ. Notice also that $\mu_k = 2\pi i k$ is one root of $(1-\epsilon^{\mu})S(\mu)$ if $a_k\overline{b}_k = 0$.

For convenience, the proof of Theorem 4.3 will be divided into several steps. We first need a number of refined estimates on $L(z)$, and its sequence of zeroes $\{z_k\}$. We begin by

LEMMA 4.4. *Under the assumptions of Lemma 3.1, there holds*

$$\lim_{|k|\to\infty} \frac{|\, z_k - 2\pi i k(1 - a_k\overline{b}_k\chi_0^{-1})\,|}{|\, k \,||\, a_k\overline{b}_k \,|} = 0 \tag{4.21}$$

Proof. It follows from a suitable application of Rouche's Theorem. Take $\sigma > 0$ fixed, and consider the disks

$$B_j = \left\{ z \in \mathbb{C} : z = 2\pi i k \left(1 - \frac{a_k\overline{b}_k}{\chi_0}\right) + \sigma \mid k \mid\mid a_k\overline{b}_k \mid e^{i\theta}, \theta \in [0, 2\pi] \right\}$$

Since for $\delta > 0$ small enough we have that $(1+\delta)^{-1} = 1 - \delta + O(\delta^2)$, standard calculations show that, if $|\, k \,| >> 1$ and $\sigma > 0$ is sufficiently small,

$$\left|\frac{\chi_0}{z} + \frac{a_k\overline{b}_k}{z - 2\pi i k}\right| = \tag{4.22}$$

$$= \left| \frac{\chi_0}{2\pi i k}\left(1 + \frac{a_k\overline{b}_k}{\chi_0} - \frac{\sigma}{2\pi i}\frac{|\, a_k\overline{b}_k |k\,|\, e^{i\theta}}{k} + O|\, a_k\overline{b}_k \,|^2\right) - \right.$$

$$\left. - \frac{\chi_0}{2\pi i k}\left(1 - \frac{i\sigma\chi_0}{2\pi}\cdot\frac{|\, a_k\overline{b}_k |k\,|\, e^{i\theta}}{k a_k\overline{b}_k} + O(\sigma^2)\right) \right|$$

$$\geq \frac{\sigma\chi_0^2}{8\pi^2|k|}$$

Putting together (4.22) and (3.11), we deduce that $L(z)$ and $\left(\frac{\chi_0}{z} + \frac{a_k\overline{b}_k}{z-2\pi i k}\right)$ have the same number of zeroes at the interior of B_k, whence

$$|\, z_k - 2\pi i k\left(1 - \frac{a_k\overline{b}_k}{\chi_0}\right) | < \sigma \mid k \mid\mid a_k\overline{b}_k \mid$$

and since $\sigma > 0$ can be taken arbitrarily small, (4.21) follows. □

We next show

LEMMA 4.5. *Assume that $L(z)$ has no multiple roots. Then*

$$L'(z_k) = \frac{\chi_0^2}{4\pi^2 k^2 (a_k \bar{b}_k)} + o\left(\frac{1}{k^2(a_k\bar{b}_k)}\right) \quad as \quad |k| \to \infty \tag{4.23a}$$

in particular

$$| L'(z_k) | \geq \frac{C}{k^2 \mid a_k \bar{b}_k \mid} \; for \; k = \pm 2, \pm 3, \pm 4, \ldots \tag{4.23b}$$

and some $C > 0$.
Furthermore, there holds

$$L''(\mu) = \frac{2 a_k \bar{b}_k}{(\mu - 2\pi i k)^3} + o\left(\frac{1}{|\mu|^2}\right) \quad as \; k \to \infty \tag{4.24}$$

uniformly on sets $| \mu - 2\pi i k | \leq \frac{1}{10}$

Proof. It follows from $(3.8a)$ that

$$\begin{aligned} L'(z_k) &= -\frac{a_k \bar{b}_k}{(z_k - 2\pi i k)^2} - \sum_{k \neq \ell} \frac{a_\ell \bar{b}_\ell}{(z_\ell - 2\pi i \ell)^2} \\ &\equiv L_1 + L_2 \end{aligned}$$

We now claim that

$$\left| \sum_{k \neq \ell} \frac{a_\ell \bar{b}_\ell}{(z - 2\pi i \ell)^2} - \frac{1}{z^2} \sum_{-\infty}^{\infty} a_\ell \bar{b}_\ell \right| = o\left(\frac{1}{|z|^2}\right) \tag{4.25}$$

$$\text{as } \; |k| \to \infty \text{ uniformly on sets } \; |z - 2\pi i k| \leq \frac{1}{10}$$

In order to derive (4.25), we observe that

$$\begin{aligned} &\left| \sum_{k \neq \ell} \frac{a_\ell \bar{b}_\ell}{(z - 2\pi i \ell)^2} - \frac{1}{z^2} \sum_{-\infty}^{\infty} a_\ell \bar{b}_\ell \right| \\ &\leq \frac{| a_k \bar{b}_k |}{|z|^2} + \left| \sum_{k \neq \ell} a_\ell \bar{b}_\ell \left(\frac{1}{(z - 2\pi i \ell)^2} \right) - \frac{1}{z^2} \sum_{-\infty}^{\infty} a_\ell \bar{b}_\ell \right| \\ &\leq \frac{| a_k \bar{b}_k |}{|z|^2} + \frac{1}{|z|^2} \sum_{k \neq \ell} \frac{(4\pi \mid \ell \mid\mid z \mid + 4\pi^2 \ell^2}{(z - 2\pi i \ell)^2} \mid a_\ell \bar{b}_\ell \mid \\ &\equiv \frac{| a_k \bar{b}_k |}{|z|^2} + S \end{aligned}$$

Therefore , if $| z - 2\pi ik |\leq \frac{1}{10}$ and $| k |\to \infty$, we can bound the last term as follows

$$S \leq C \sum_{k\neq \ell} (| k || \ell | + \ell^2) \frac{| a_\ell \bar{b}_\ell |}{|k-\ell|^2} \leq C \sum_{k\neq\ell} (| k - \ell || \ell | + \ell^2) \frac{| a_\ell \bar{b}_\ell |}{|k-\ell|^2}$$

for some $C > 0$ independent of k. Recalling (4.21), we then have that, as $| k |\to \infty$

$$\begin{aligned} L'(z_k) &= -\frac{a_k \bar{b}_k}{(z_k - 2\pi ik)^2} - \frac{1}{z_k} \sum a_\ell \bar{b}_\ell = o\left(\frac{1}{| z_k |^2}\right) \\ = -a_k \bar{b}_k & \left(\frac{-2\pi ik \mid a_k \bar{b}_k \mid}{\chi_0} + o| k || a_k \bar{b}_k |^{-1} - \frac{\chi_0}{(2\pi ik)^2} + o\left(\frac{1}{| z_k |^2}\right)\right) \\ &= \frac{\chi_0^2}{4\pi^2 k^2 a_k \bar{b}_k}(1 + o(1)) + \frac{\chi_0}{4\pi^2 k^2} + o\left(\frac{1}{|k|^2}\right) \\ &= \frac{\chi_0^2}{4\pi^2 k^2 a_k \bar{b}_k} + o\left(\frac{1}{k^2 a_k \bar{b}_k}\right) \end{aligned}$$

where use has been made of the fact that $k^2 \mid a_k \bar{b}_k \mid = o(1)$ as $| k |\to \infty$ (since $\sum \ell^2 \mid a_\ell \bar{b}_\ell \mid < +\infty$). This yields (4.23a) whence (4.23b) follows. To derive (4.24), we argue in a similar way. We start with

$$L"(z) = \frac{2a_k \bar{b}_k}{(z - 2\pi ik)^3} + \sum_{k\neq\ell} \frac{2a_\ell \bar{b}_\ell}{(z - 2\pi i\ell)^3}$$

Instead of (4.25), we now obtain

$$(4.26) \qquad \sum_{k\neq\ell} \frac{a_\ell \bar{b}_\ell}{(z - 2\pi ik)^3} = o\left(\frac{1}{k^2}\right) \quad \text{as} \quad | k |\to \infty$$

uniformly on sets $| z - 2\pi ik |\leq \frac{1}{10}$ which in turn gives at once (4.24). We shall deduce (4.26) after some algebra. Namely, we notice that, as $| k |\to \infty$,

$$\begin{aligned} (4.27) \qquad & \left| \sum_{k\neq\ell} \frac{a_\ell \bar{b}_\ell}{(z - 2\pi i\ell)^3} - \frac{1}{z^3} \sum_{-\infty}^{\infty} a_\ell \bar{b}_\ell \right| \\ &\leq \frac{| a_k \bar{b}_k |}{|z|^3} + \sum_{k\neq\ell} | a_\ell \bar{b}_\ell | \left| \frac{1}{(z - 2\pi i\ell)^3} - \frac{1}{z^3} \right| \\ &\leq C \sum_{k\neq\ell} \frac{(k^2 | \ell | + \ell^2 | k | + | \ell |^3) | a_\ell \bar{b}_\ell |}{|k|^3 |\ell - k|^3} + o\left(\frac{1}{|z|^3}\right) \\ &\equiv S_1 + o\left(\frac{1}{|z|^3}\right) \end{aligned}$$

for some generic constant $C > 0$. To proceed further, we split S_1 as follows

$$S_1 \;=\; S_1^1 + S_1^2 + S_1^3$$

where in S_1^1 (resp. in S_1^2, S_1^3) summation is extended to indexes ℓ with $\mid \ell \mid \geq 2 \mid k \mid$ (resp. $\frac{|k|}{2} \leq \mid \ell \mid < 2 \mid k \mid, \mid \ell \mid < \frac{|k|}{2}$). A careful analysis of these terms reveals that

$$S_1 = o\left(\frac{1}{k^2}\right) \text{ as } \mid k \mid \to \infty \tag{4.28}$$

Indeed, consider S_1^2, and denote by $\sum_{\ell,k}$ the corresponding summation therein. We then have that

$$\begin{aligned} S_1^2 &= C \sum_{k,\ell} \frac{(k|\ell| + \ell^2|k| + |\ell|^3) \mid a_\ell \bar{b}_\ell \mid}{|k|^3 |\ell - k|^3} \\ &\leq \frac{C}{|k|^3} \sum_{k,\ell} \mid k \mid \ell^2 \mid a_\ell \bar{b}_\ell \mid = o\left(\frac{1}{k^2}\right) \text{ as } \mid k \mid \to \infty \end{aligned}$$

Putting together (4.27) and (4.28), (4.26) follows and the proof is concluded. □
We next observe that, by (4.21)

$$\begin{aligned} &\text{There exists } \theta > 0 \text{ such that, if } z_k \neq 2\pi i k, \\ &\mid z_k - 2\pi i k \mid \geq \theta \mid k \mid\mid a_k \bar{b}_k \mid \text{ for } k \;=\; 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{4.29}$$

Suppose now that

$$\mid \mu - z_k \mid \leq \frac{\theta}{2} \mid k \mid\mid a_k \bar{b}_k \mid \tag{4.30}$$

Then $\mid \mu - 2\pi i k \mid \geq \frac{\theta}{2} \mid k \mid\mid a_k \bar{b}_k \mid$, and (4.24) yields at once.

COROLLARY 4.6. *If (4.30) holds, then*

$$\sup_{\Lambda} \mid L''(\mu) \mid \leq \frac{C}{|k|^3 \mid a_k \bar{b}_k \mid^2} + o\left(\frac{1}{k^2}\right) \;\; as \;\; \mid k \mid \to \infty \tag{4.31}$$

for some $C > 0$, where $\Lambda = \left\{\mu \neq 2\pi i k : \mid \mu - z_k \mid \leq \frac{\theta}{2} \mid k \mid\mid a_k \bar{b}_k \mid, \theta \text{ being as in } (3.29)\right\}$

Our next step is

LEMMA 4.7. *Assume that $L(0) \neq 0$ and $L(z)$ has no multiple roots. Suppose also that $z_k \neq 2\pi i k$, and define*

$$\rho_k = \frac{2}{\mid L'(z_k) \mid} \frac{\mid z_k - z \mid}{\mid Q z_k z \mid} \left(\left| 1 + \frac{z_k}{z} \right| + A \right) \tag{4.32}$$

where A is a fixed positive constant. Then there exists $C > 0$ (independent of Re, k) such that

$$\mid L(\mu) - L'(z_k)(\mu - z_k) \mid \leq \frac{C|k|^3}{|Q|^2} \text{ for } k = \pm 1, \pm 2, \ldots \tag{4.33}$$
$$\text{provided that } \mid \mu - z_k \mid = \rho_k \text{ and } \mid k \mid \leq CRe$$

Proof. Since $L(0) \neq 0$, one has that $z_k \neq 0$ for any k. By (4.23b)

$$\rho_k \leq CK^2 \mid a_k \bar{b}_k \mid \frac{\mid z_k - z \mid}{\mid Q z_k z \mid} \left(\left| 1 + \frac{z_k}{z} \right| + A \right) \equiv B_k \tag{4.34a}$$

where here and henceforth C will denote a generic constant which is independent of Re and k. We want to have

$$B_k \leq \frac{\theta}{2} \mid k \mid\mid a_k \bar{b}_k \mid \tag{4.34b}$$

where θ is the constant in (4.29). This last inequality holds provided that

$$\frac{2C}{\theta} \cdot \frac{\mid z_k - z \mid\mid k \mid}{|Q| \mid z_k z \mid} \left(\left| 1 + \frac{z_k}{z} \right| + A \right) \leq 1$$

Taking into account (4.16) as well as the fact that $z = O(1)$ for $Re >> 1$, (4.34b) will hold if $\mid k \mid \leq \sigma Re$ for some $\sigma > 0$ and Re is large enough. Then, if $\mid \mu - z_k \mid \leq \rho_k$ and Λ is the set defined in Corollary 4.6, we will have that, as $\mid j \mid \to \infty$

$$\begin{aligned}
\mid L(\mu) - L'(z_k)(\mu - z_k) \mid &\leq \frac{1}{2} |\sup_{\Lambda} L''(\mu)| \mid \mu - z_k \mid^2 \\
&\leq \frac{\rho_k^2}{2} \left(\frac{C}{|k|^3 \mid a_k \bar{b}_k \mid^2} + o\left(\frac{1}{k^2}\right) \right) \\
\leq \frac{1}{2} \left(\frac{C^2 k^4 \mid a_k \bar{b}_k \mid^2 \mid z_k - z \mid^2}{\mid Q z_k z \mid^2} \mid 1 + \frac{z_k}{z} \mid^2 \right) &\left(\frac{C}{|k|^3 \mid a_k \bar{b}_k \mid^2} + o\left(\frac{1}{k^2}\right) \right) \\
&\leq \frac{C \mid k \mid^3}{|Q|^2 |z|^2} + o\left(\frac{k^4 \mid a_k \bar{b}_k \mid^2}{|Q|^2 |z|^2} \right),
\end{aligned}$$

so that (4.33) follows. □

4.3. The proof of Theorem 4.3.

Lemmata 4.4-4.7 have proved the ground to tackle the proof of our result on the stability of slow solutions. To begin with, we are now in a position to show a particular case of (4.18), (4.19), namely

LEMMA 4.8. *Assume that the hypotheses in Theorem 3.2 hold. Suppose also that (3.16) and (4.17) are satisfied. Then the roots $\{\mu_k\}$ of $(1-e^{\mu})S(\mu)$ with $k=\pm1,\pm2,\pm3,..k\neq j$, are such that*

$$\mid \mu_k - z_k \mid = o(1) \ for \ \mid k \mid \leq (Re)^{1/3} \ as \ Re \to \infty \tag{4.35a}$$

On the other hand, there is one root μ_0 such that

$$\mid \mu_0 \mid = o(1), \ as \ Re \to \infty \tag{4.35b}$$

Proof. It clearly suffices to consider the case where $a_k\bar{b}_k \neq 0$. Assume first that $L(0)\neq 0$, and let ρ_k be as in (4.32). Then (4.35a) will follow as soon as we prove that

$$\text{If } \mid \mu - z_k \mid = \rho_k, \text{ then}$$
$$\left| S(\mu) - \left(\frac{\mathrm{Qzz}_k}{z_k - z}\right) L'(z_k)(\mu - z_k)\right| < 2\left(\left|1 + \frac{z_k}{z}\right| + A\right) \tag{4.36}$$

To derive (4.36), we just observe that , if $\mid \mu - z_k \mid = \rho_k$,

$$\begin{aligned}
&\left| S(\mu) - \left(\frac{\mathrm{Qzz}_k}{z_k - z}\right) L'(z_k)(\mu - z_k)\right| \\
&\leq 1 + \frac{\mid \mu \mid}{\mid z \mid} + \epsilon \mid Qv^2 \mid \left|\frac{\mu z}{\mu - z}\right| + C\frac{\mid Q\mu z \mid}{|\mu - z|}\frac{\mid k \mid^3}{|Q|^2} \\
&+ \mid L'(z_k) \mid\mid \mu - z_k \mid\mid Q \mid\mid z \mid \left|\frac{\mu}{\mu - z} - \frac{z_k}{z_k - z}\right| \\
&\equiv I_1 + I_2 + I_3 + I_4
\end{aligned}$$

for some $C>0$. Taking into account that $\mid v \mid \approx Re$ as $Re \to \infty$ we readily see that

$$\sum_{i=1}^{4} I_i \leq 1 + \frac{|\mu|}{|z|} + B + \frac{C |k|^3}{|Q|} + \frac{|L'(z_k)| |Q||z|^2 |\mu - z_k|^2}{|\mu - z| |z - z_k|}$$

We now relate k and Re as follows

$$|k| \leq (Re)^{1/3} \tag{4.37}$$

Notice that this relation is compatible with the one in (4.33). Since $\rho_k \to 0$ as $Re \to \infty$, one then has

$$\sum_{i=1}^{4} I_i \leq M + 1 + \frac{|\mu|}{|z|} \text{ if } Re >> 1$$

for some $M > 0$ large enough. Taking then $A = 2M$, (4.36) follows. As a matter of fact, by our previous estimates we have shown that

$$\begin{aligned} \rho_k &\leq Ck^2 |a_k \overline{b}_k| \frac{|z_k - z|}{|Q| |z| |z_k|} \left(|1 + \frac{z_k}{z}| + A \right) \\ &\leq C_1 \frac{(1 + |k|^3) |a_k \overline{b}_k|}{|Q|} \leq C_1 \frac{(1 + k^2) |a_k \overline{b}_k| Re^{1/3}}{|Q|} \\ &\leq C_1 (1 + k^2) |a_k \overline{b}_k| (Re)^{-5/3} \end{aligned}$$

for some C_1 depending on A and z. Hence

$$|\mu_k - z_k| \leq C(1 + k^2) |a_k \overline{b}_k| (Re)^{-5/3}, \text{ whenever} \tag{4.38}$$

$|k| \leq (Re)^{1/3}$, $k = \pm 1, \pm 2, \pm 3, \ldots$ and Re is large enough.

To derive (4.35b) when $L(0) \neq 0$, we modify our previous argument as follows. We consider the curves $\Gamma = \left\{ \mu : |\mu| = \frac{3}{|Q| \, |L(0)|} \right\}$, and obtain after standard computations that

$$|S(\mu) - (1 - QL(0)\mu)| < |1 - QL(0)\mu| \quad \text{on } \gamma$$

whereupon we conclude as before. Finally, if $L(0) = 0$, there is a root $z_0 = 0$ of $L(z)$ which cannot be dealt with as before. For those roots approaching z_k with $k \neq 0$, the previous argument still yields the result. The roots $\mu = o(1)$ are kept track of just by replacing $(1 - QL(0)\mu)$ by $(1 - QL'(0)\mu^2)$ above. $\square$

We now proceed to remove the assumption $|k| \leq (Re)^{1/3}$ in (4.35a). To this end, we prove

LEMMA 4.9. *Let* $D_k = \left\{\mu \in \mathbb{C} : 2\pi\left(|k| - \frac{1}{2}\right) < |\mu| < 2\pi\left(|k| + \frac{1}{2}\right)\right\}$. *Then there holds*

$$\sup_{\mu \in D_k} \mu^2 \left| L(\mu) - \frac{a_k \bar{b}_k}{\mu - 2\pi i k} - \frac{\chi_0}{\mu} - \frac{M_0}{\mu^2} \right| \to 0 \tag{4.39}$$
$$\text{as } \mid k \mid \to \infty.$$

where M_0 *and* χ_0 *are as in the statement of Theorem 4.8.*

Proof. We start from

$$L(\mu) \;=\; \frac{a_k \bar{b}_k}{\mu - 2\pi i k} + \sum_{k \neq \ell} \frac{a_\ell \bar{b}_\ell}{\mu - 2\pi i \ell}$$

and compute

$$\sum_{k \neq \ell} \frac{a_\ell \bar{b}_\ell}{\mu - 2\pi i \ell} \;=\; \frac{\chi_0}{\mu} + \frac{1}{\mu} \sum_{k \neq \ell} \frac{2\pi i \ell a_k \bar{b}_k}{\mu - 2\pi i \ell} - \frac{a_k \bar{b}_k}{\mu},$$

$$\left| \sum_{k \neq \ell} \frac{2\pi i \ell a_k \bar{b}_k}{\mu - 2\pi i \ell} - \frac{M_0}{\mu} \right| \leq \frac{4\pi |k| \mid a_k \bar{b}_k \mid}{\mid k \mid} + \frac{4\pi^2}{\mid k \mid} \sum_{k \neq \ell} \frac{\ell^2 \mid a_\ell \bar{b}_\ell \mid}{\mid \mu - 2\pi i \ell \mid},$$

since $\mid \mu - 2\pi i \ell \mid \geq \theta > 0$ for some θ whenever $\mu \in \left\{\mathbb{C} : 2\pi(\mid k \mid - \frac{1}{2}) < \mid \mu \mid < 2\pi(\mid k \mid + \frac{1}{2})\right\}$ we also have that

$$\lim_{|k| \to \infty} \left(\sum_{k \neq \ell} \frac{\ell^2 \mid a_\ell \bar{b}_\ell \mid}{\mid \mu - 2\pi i \ell \mid} \right) = 0.$$

Putting all these estimates together (4.39) follows. □

We are now prepared to show

LEMMA 4.10. *Under the assumptions of Lemma 4.8, there holds*

$$\mid \mu_k - z_k \mid \;=\; o(1) \text{ as } Re \to \infty \text{ for } k \;=\; 0, \pm 1, \pm 2, \ldots \tag{4.40}$$

Proof. It involves a number of tedious (but otherwise routine) computations. We shall therefore sketch the main lines in the argument and leave most of the details to the reader. Clearly, it suffices to examine the case where $\mid k \mid \geq (Re)^{1/3}$. Consider the function

$$G_k(\mu) = \frac{\mu}{z} - \frac{Qz\chi_0}{\mu} + Qz\left(\epsilon v^2 - \frac{M_0}{\mu^2}\right) - \frac{\mathrm{Qza}_k\overline{b}_k}{\mu - 2\pi ik} \tag{4.41}$$

Using (4.39), we then have that

$$|S(\mu) - G_k(\mu)| \leq \frac{C}{|\mu|}\left(1 + \frac{|Q| \mid a_k\overline{b}_k \mid}{|\mu - 2\pi ik|}\right) + \left|1 + \frac{Qz^2\chi_0}{|\mu|^2}\right| \tag{4.42}$$
$$+ o\left(\frac{|Q|}{|\mu|^2}\right) \; as \mid j \mid \to \infty,$$
$$\text{uniformly for } \mid \mu - 2\pi ik \mid \leq \frac{1}{10}$$

Set now

$$\rho_k = 2 \mid Qa_k\overline{b}_k \mid \left(\left|Qz\left(\epsilon v^2 + \frac{M_0}{4\pi^2k^2}\right) + i\left(\frac{2\pi k}{z} + \frac{Qz\chi_0}{2\pi k}\right)\right|\right)^{-1} \tag{4.43}$$
$$\equiv 2 \mid \mathrm{Qza}_k\overline{b}_k \mid H^{-1}$$

Notice that

$$\text{If } \left|\frac{2\pi k}{z} + \frac{Qz\chi_0}{2\pi k}\right| \geq \frac{1}{2}, \text{ then } \mid H \mid \geq \frac{1}{2}, \tag{4.44a}$$

$$\text{If } \left|\frac{2\pi k}{z} + \frac{Qz\chi_0}{2\pi k}\right| \leq \frac{1}{2}, \text{ then } C_1k^2 \leq \mid Q \mid \leq C_2k^2 \tag{4.44b}$$

for some constants C_1, C_2 independent of Re and k

On the other hand, we readily see that

$$|H| \geq \left|Qz\left(\epsilon v^2 + \frac{M_0}{4\pi^2k^2}\right)\right|$$
$$\geq \left|Qz\left(\epsilon v^2 + M_0(Qz^2\chi_0 + 4\pi^2k^2 - Qz^2\chi_0)^{-1}\right)\right|$$
$$\geq \left|Qz\epsilon v^2 + M_0(Qz^2\chi_0)^{-1}\right|$$
$$- \left|\mathrm{QzM}_0\left((Qz^2\chi_0 + 4\pi^2k^2 - Qz^2\chi_0)^{-1} - (Qz^2\chi_0)^{-1}\right)\right|$$
$$\geq \left|Q\epsilon v^2z + \frac{M_0}{\chi_0 z}\right| - C\frac{M_0 \mid k \mid}{|Q|\chi_0^2}$$

for some $C > 0$. Since the second term above behaves asymptotically as $| k |^{-1}$ for large $| k |$, we obtain that

(4.45) $$| H | \geq C_1 > 0 \text{ as } Re \to \infty, \text{ where } C_1 \text{ is}$$

independent of k.

We next claim that

(4.46) $$\begin{gathered} \rho_k \leq C \mid k \mid\mid a_k \bar{b}_k \mid \text{ for some } C > 0, \text{ so that} \\ \rho_k \to 0 \text{ if } \mid k \mid \geq Re^{1/3} \text{ and } Re \to \infty. \end{gathered}$$

Estimate (4.46) is obtained by arguments alike to those in (4.44). Our next goal is to show that

(4.47) $$|G_k(\mu)| >> |S(\mu) - G_k(\mu)| \text{ as } Re \to \infty, \text{ uniformly for}$$

$$| \mu - 2\pi i k | = \rho_k \text{ with } | k | \geq Re^{1/3}$$

To derive (4.47), we set $\xi = \mu - 2\pi i k$, so that $\frac{Qz\chi_0}{\mu} = \frac{Qz\chi_0}{2\pi i k}\left(1 - \frac{\xi}{2\pi i k} + \dots\right)$ in the region under consideration. We have there that

$$\begin{aligned}
|G_k(\mu)| = &\left| \frac{2\pi i k}{z} - \frac{Qz\chi_0}{2\pi i k} + \frac{\xi}{z} + O\left(\frac{| Qz\chi_0 || \xi |}{k^2}\right) \right. \\
&\left. + \frac{\mathrm{QzM_0}}{4\pi^2 k^2} + Q\epsilon v^2 z + O\left(\frac{| \mathrm{QzM_0} || \xi |}{|k|^3}\right) - \frac{\mathrm{Qza}_k \bar{b}_k}{\xi} \right| \\
\geq &\left| Qz \left(\epsilon v^2 \frac{M_0}{4\pi^2 k^2}\right) + i\left(\frac{2\pi k}{z} + \frac{Qz\chi_0}{2\pi k}\right) \right| - \frac{\mathrm{Qza}_k \bar{b}_k}{| \xi |} \\
&- | \xi | \left(\frac{1}{|z|} + O\left(\frac{| Qz\chi_0 |}{k^2}\right)\right) \\
\geq &\frac{1}{4} \left| Qz\left(\epsilon v^2 + \frac{M_0}{4\pi^2 k^2}\right) + i\left(\frac{2\pi k}{z} + \frac{Qz\chi_0}{2\pi k}\right) \right| \\
&>> | S(\mu) - G_k(\mu) |
\end{aligned}$$

where we have used (4.42) to derive the last statement above. By the argument principle, we then deduce that

(4.48)

As $Re \to \infty$, the difference between the number of zeroes and poles in the region $\mid \mu - 2\pi ik \mid \leq \rho_k$ ($\mid k \mid \leq Re^{1/3}$), is the same for functions $S(\mu)$ and $G_k(\mu)$.

As a further step, we now consider function $\tilde{G}_k(\mu)$ given by

$$\tilde{G}_k(\mu) \;=\; \frac{2\pi k}{z} - \frac{Qz\chi_0}{2\pi ik} + Qz\left(\epsilon v^2 + \frac{M_0}{4\pi^2k^2}\right) - \frac{\mathrm{Qza}_k\overline{b}_k}{\mu 2\pi ik}$$

It is readily seen that

$$\left|\tilde{G}_k(\mu)\right| \geq \left|Qz\left(\epsilon v^2 + \frac{M_0}{4\pi^2k^2}\right) + i\left(\frac{2\pi k}{z} + \frac{Qz\chi_0}{2\pi k}\right)\right| - \\ - \frac{\mid \mathrm{Qza}_k\overline{b}_k \mid}{\mid \rho_k \mid}$$

whereas, if $\xi \;=\; \mu - 2\pi ik$,

$$\mid G_k(\mu) - \tilde{G}_k(\mu) \mid \leq C\left(\frac{1}{|z|} + |Q|k^2 + \frac{|Q|M_0}{|k|^3}\right) \mid \xi \mid$$

for some $C > 0$. We then deduce that, if $\mid \mu - 2\pi ik \mid$ with $Re >> 1$, $\mid G_k(\mu) - \tilde{G}_k(\mu) \mid <<$ $\mid G_k(\mu) \mid$ whence

(4.49)

As $Re \to \infty$, the difference between the number of zeroes and poles in the region $\mid \mu - 2\pi ik \mid \leq \rho_k$ ($\mid k \mid \geq Re^{1/3}$) is the same for functions $G_k(\mu)$ and $\tilde{G}_k(\mu)$.

From (4.48) and (4.49), we deduce that $S(\mu)$ has one zero μ_k such that $\mid \mu_k - 2\pi ik \mid \leq \rho_k$. From this and (3.9), (4.40) follows. □

The last result in this Section is

LEMMA 4.11. *There exists two roots $\mu_{\pm 1}$ of $S(\mu)$ such that*

$$\mu_{\pm 1} = \pm(Qz^2\chi_0)^{1/2} - \frac{\epsilon z^2 q}{2}(\xi_j^\ell)^2 + \frac{M_0}{2\chi_0} \\ +o(1) \textit{ as } Re \to \infty, \tag{4.50}$$

where

$$q = \lim_{Re\to\infty} \frac{Q}{Re^2}.$$

Proof. Let us define

$$G \;=\; \pm(Qz^2\chi_0)^{1/2}$$

Clearly, $G \approx$ CRe as $Re \to \infty$ for some $C > 0$. Consider now the region

$$\sum = \left\{\mu :\mid \mu - G \mid = \left| \frac{z}{2}\left(\frac{M_0}{\chi_0 z} + \epsilon \mathrm{Qzv}^2\right) + \rho e^{i\theta}\right|\right\}$$

where $\rho > 0$ will be selected presently. Once again, we shall make use of Rouche's Theorem to derive the result. To this end, we consider the auxiliary function

$$F(\mu) = \frac{2}{z}(\mu - G) - \left(\frac{M_0}{\chi_0 z} - \epsilon \mathrm{Qzv}^2\right)$$

Arguing as in Lemma 4.9, we readily see that

$$L(\mu) = \frac{\chi_0}{\mu} + \frac{M_0}{\mu^2} + g(\mu), \text{ where } g(\mu) = o(\mid \mu \mid^{-2}) \text{ as } \mid \mu \mid \to \infty. \tag{4.51}$$

We now claim that

$$\mid S(\mu) - F(\mu) \mid < \mid F(\mu) \mid \text{ in } \sum \tag{4.52}$$

whereupon (4.50) follows. The above inequality is obtained much in the same way as many of our previous estimates. We shall therefore omit most of the details involved. We merely point out that in view of (4.51), we have that

$$\begin{aligned} S(\mu) - F(\mu) = &-Qz\left(\frac{\chi_0}{\mu} + \frac{M_0}{\mu^2} + g(\mu)\right) - \frac{Qz^2\mu_0}{\mu^2} \\ &+O\left(\frac{1}{\mid \mu \mid}\right) + 1 - \frac{\mu}{z} + \frac{2G}{z} + \frac{M_0}{\chi_0 z} \end{aligned}$$

as $Re \to \infty$, whence

$$|S(\mu) - F(\mu)| \leq \mid Q \mid\mid z \mid\mid g(\mu) \mid + O\left(\frac{1}{\mid \mu \mid}\right) \text{ on } \sum$$

as $Re \to \infty$, whereas on the other hand

$$| F(\mu) | = \frac{2}{|z|}\rho \text{ on } \sum$$

so that selecting now $\rho = C_1 \max\left\{| \mu |^2 g(\mu), | \mu |^{-1/2}\right\}$ for some suitable $C_1 > 0$, (4.52) follow. □

Putting together Lemmata 4.8-4.11, the proof of Theorem 4.3 is now complete.

4.4. On the existence of linearly stable slow solutions

We have already shown that fast solutions (if any) are always linearly stable (cf. Corollary 4.2). On the other hand, Theorem 4.3 provides a description of the asymptotics of the eigenvalues corresponding to linearizing around a given stationary solution. It is remarkable that, for most of such eigenvalues, their asymptotic behaviour is encoded in the zeroes of the function $L(z)$(cf. (4.18)). A question which naturally arises is whether functions f and T_w can be selected so that linearly stable solutions satisfying (3.16) exist. An affirmative answer is provided by the following result

THEOREM 4.12. *There exists a set of functions f and T_w for which there are some linearly stable slow solutions and two stable fast solutions.*

For convenience, we shall deduce Theorem 4.12 from two technical results. Set $\Gamma_k = a_k\overline{b}_k$, where a_k and b_k are given in (3.4). We then have

LEMMA 4.13. *Let $L(z)$ be the function given in (3.8) and let $\tilde{z}_j; j = 0, \pm 2, \pm 3, \pm 4, \dots$ the sequence of its zeroes.* Fix $k = 0, \pm 2, \pm 3, \pm 4, \dots$ *and a set of autoconjugate numbers $\tilde{z}_j$ $j = 0, \pm 2, \pm 3, \pm 4, \dots \pm k$. Then there exists a function $\tilde{L}(z)$ such that $\tilde{L}(z) = \sum_{\infty}^{\infty} \frac{\tilde{\Gamma}_k}{z - 2\pi k i}$, where $\overline{\Gamma}_k = \tilde{\Gamma}_{-k}, \sum_{-\infty}^{\infty} k^2 \mid \tilde{\Gamma}_k \mid < +\infty$, and the set of zeroes of $\tilde{L}(z)$ is given by*

$$\tilde{S} = \{\tilde{z}_j; j = 0, \pm 2, \dots, \pm k\} \bigcup \{z_j : j = \pm k + 1, \dots\}$$

Proof. Set

$$\tilde{L}(z) = \prod_{j=-k}^{j=+k} \left(\frac{z - \tilde{z}_j}{z - z_j}\right) L(z) = \sum_{\ell=-\infty}^{\ell=+\infty} \prod_{j=-k}^{j=+k} \left(\frac{z - \tilde{z}_j}{z - z_j}\right) \frac{\Gamma_\ell}{z - 2\pi i \ell} \tag{4.53}$$

Clearly, $\tilde{L}(z)$ is a meromorphic function having poles at most at points $z = 2\pi i \ell$. Define now

$$\tilde{\Gamma}_\ell = \lim_{z-2\pi i\ell}\left(\tilde{L}(z)(z-2\pi i\ell)\right) = \prod_{j=-k}^{j=+k}\left(\frac{2\pi ij-\tilde{z}_j}{2\pi ij - z_j}\right)\Gamma_\ell$$

It then follows that $R(z) = \tilde{L}(z) - \sum_{-\infty}^{\infty}\frac{\tilde{\Gamma}_\ell}{z-2\pi i\ell}$ is an entire function, and since $R(z) = o(1)$ when $|z| = (2N+1)\pi$ and $N\to\infty$, we then deduce that $R(z)\equiv 0$. $\square$

We deduce from Lemma 4.13 that, for any given pair of functions f and T_w, we can modify their Fourier coefficients (thus changing the geometry of the pipe and the temperature at the wall) so that

i) For the new data $\tilde{f}$ and $\tilde{T}_w$, the corresponding function $\tilde{L}(z)$ has exactly $2J+1$ zeroes $\{z_{-J}, z_{-J+1}, \dots, z_0, \dots, z_{J-1}, z_J\}$ lying on the real axis.

ii) When linearization is performed around velocity values satisfying (3.16) with $j = J$, the eigenvalues corresponding to (4.18) and (4.19) are positive.

iii) $L\left(\pm\frac{1}{\epsilon}\right) > 0$ and $|z| \pm J \in \left[\frac{m}{\epsilon}, \frac{M}{\epsilon}\right]$.

To discuss the situation corresponding to eigenvalues satisfying (4.20), we need yet to evaluate how parameters χ_0 and M_0 change under transformations of type (4.53). The information required is contained in the following result.

LEMMA 4.14. *Let N be a positive integer, $\alpha > 1, \nu > 0$, and let $\{r_j\}, -\infty < j < +\infty$, be a sequence of real numbers such that $r_j = -r_{-j}$ for any j. For any integer k, we set*

$$\tilde{\Gamma}_k^N = \left(\prod_{N\le j\le \alpha N}\left(\frac{2\pi i(k-j)-\epsilon_j}{2\pi ik - z_j}\right)\right)\Gamma_k \equiv R_{N,k}\cdot\Gamma_k \tag{4.54a}$$

where Γ_k as in our previous Lemma, and

$$\epsilon_j = \frac{\nu}{|j|} + r_j \tag{4.54b}$$

Assume now that $\chi_0 \neq 0$, and define $\tilde{\chi}_0^N, \tilde{M}_0^N$ as follows

$$\tilde{\chi}_0^N = \sum_{\infty}^{\infty}\tilde{\Gamma}_k^N, \tilde{M}_0^N = \sum_{\infty}^{\infty} 2\pi ik\tilde{\Gamma}_k$$

We then have that

$$\lim_{n\to\infty}\tilde{\chi}_0^N = \chi_0, \tag{4.55}$$

$$\lim_{n\to\infty} \tilde{M}_0^N = M_0 - \chi_0 \nu \log \alpha. \tag{4.56}$$

Proof. We shall prove first (4.55). To this end, we set $\delta_j = z_j - 2\pi i j$. By (4.25), we then have that

$$\delta_j = -\frac{2\pi i j}{\chi_0}\Gamma_j + o(| j || \Gamma_j |) \text{ as } | j |\to \infty$$

We now split $\tilde{\chi}_0^N$ as follows

$$\begin{aligned}\tilde{\chi}_0^N &= \sum_{\infty}^{\infty} R_{N,k}\Gamma_k = \sum_{k|<N} R_{N,k}\Gamma_k + \sum_{N\le|k|\le\alpha N} R_{N,K}\Gamma_k \\ &+ \sum_{\alpha N<|k|} R_{N,k}\Gamma_k \equiv J_1^N + J_2^N + J_3^N\end{aligned} \tag{4.57}$$

Assume now that $| k |< \frac{N}{2}$ or $| k |> 2\alpha$N. Then $| k-j |\ge \frac{N}{2}$ and $| k-j |\ge C | j |$, where here and henceforth C will denote a generic constant (changing possibly from line to line), which is independent of k and j. We then have that

$$\begin{aligned}\log R_{N,k} &= \sum_{N\le|j|\le\alpha N} \log\left(1+\frac{\delta_j-\epsilon_j}{2\pi i(k-j)-\delta_j}\right) \\ &\le C\left(\sum_{N\le|j|\le\alpha N} | \delta_j | + \sum_{N\le|j|\le\alpha N}\frac{| \epsilon_j |}{| j |}\right)\end{aligned}$$

Recalling the well known formula

$$\sum_{j=1}^{n}\frac{1}{j} = \log n + \gamma + o(1) \text{ as } n \to \infty$$

where γ is Euler's constant, we obtain

$$\log R_{N,k} \le 2\log\alpha + o(1) \text{ as } N \to \infty \tag{4.58}$$

uniformly when $| k |< \frac{N}{2}$ or $| k |> 2\alpha N$.

Suppose now that $\frac{N}{2} \le| k |<$ N. Arguing as before, we obtain

$$\log R_{N,k} \leq C\delta(\log(\alpha N - N + 1) - \gamma) + o(1)$$

as $N \to \infty$, where $\delta > 0$ can be taken so small that,

$$\text{say, } C\delta < \frac{1}{5}$$

A similar argument applies to the case $N \leq \mid k \mid \leq 2\alpha$N. We thus arrive at

$$(4.59) \qquad \mid R_{N,k} \mid \leq C \mid k \mid^{1/5} \text{ as } N \to \infty, \text{ uniformly}$$

$$\text{when } \frac{N}{2} \leq \mid k \mid \leq 2\alpha N$$

Define now $\mu_{N,k} = R_{N,k}$ if $\mid k \mid < N$ or $\mid k \mid > \alpha N, \mu_{N,k} = 0$ otherwise. Clearly,

$$J_1^N + J_3^N = \sum_{-\infty}^{\infty} \mu_{N,k}\Gamma_k$$

Taking into account (4.58) and (4.59), it follows from Lebesgue's Theorem that

$$\lim_{N\to\infty}(J_1^N + J_3^N) = \sum_{-\infty}^{\infty}(\lim_{n\to\infty} \mu_{n,k})\Gamma_k = \sum_{-\infty}^{\infty}(\lim_{n\to\infty} R_{n,k})\Gamma_k$$

where limits within the summation terms above are to be considered for fixed k. We now claim that

$$(4.60) \qquad \text{For any fixed k }, \lim_{n\to\infty} R_{n,k} = 1$$

To check (4.59), we notice that if we set $\sum_1 = \sum_{N\leq|j|\leq\alpha N}$, for fixed k and large enough N there holds

$$\begin{aligned}\log R_{N,k} &= \sum_1 \left(\frac{\delta_j - \epsilon_j}{2\pi i(k-j) - \delta_j} + O\left(\frac{(\delta_j - \epsilon_j)^2}{(k-j)^2}\right)\right) \\ &= \sum_1 O\left(\frac{\mid \delta_j \mid + \mid \epsilon_j \mid}{|j|} + \left(\delta_j^2 + \mid \epsilon_j \mid\mid \delta_j \mid + \frac{\epsilon_j^2}{j^2}\right)\right) \\ &= o(1) \text{ as } n \to \infty\end{aligned}$$

whence (4.60). We have therefore proved that

$$\lim_{n\to\infty} J_1^N + J_3^N = \sum_{-\infty}^{\infty} \Gamma_k = \chi_0 \tag{4.61}$$

The next step consists in showing that

$$\lim_{n\to\infty} J_2^N = 0. \tag{4.62}$$

As soon as this had been done, (4.55) will follow from (4.57), (4.61) and (4.62). To prove (4.62), let us write

$$R_{N,k} = \frac{\epsilon_k}{\delta_k}\left(\prod_2 \frac{2\pi i(k-j)-\epsilon_j}{2\pi i(k-j)\bar{j}\delta_j}\right) \equiv \frac{\epsilon_k}{\delta_k} S_{N,k}$$

where the symbol $\prod_2$ denotes that the product there is extended to indexes j such that $j \neq k, N \leq | j | \leq \alpha N$. Clearly, $\overline{R}_{N,k} = R_{N,k}$ hence

$$J_2^N = \sum_1 R_{N,k}\Gamma_k + \left(\overline{\sum_1 R_{N,k}\Gamma_k}\right) \equiv L_N + \overline{L}_N$$

Let us define $\sum_2$ in a similar way to $\sum_1$ above. Then, for fixed $k > 0$ and large N,

$$\begin{aligned}\log(S_{N,k}) &= \sum_2 \log\left(1 + \frac{\delta_j - \epsilon_j}{2\pi i(k-j) - \delta_j}\right)\\ &= \sum_2 \log\left(1 + \frac{\epsilon_j}{2\pi i(k-j)}\right) + o(1)\end{aligned}$$

Let us denote by $[\sigma]$ the entire part of the real positive number σ, and assume that $k \neq N, k \neq [\alpha N]$. We now split $\sum_2$ in the form $\sum_2 = \sum_{2,1} + \sum_{2,2} + \sum_{2,3}$, where summation in $\sum_{2,1}$ (resp. $\sum_{2,2}, \sum_{2,3}$) is extended to the indexes $N \leq | j | \leq k-1$ (resp. $k+1 \leq j \leq [\alpha N], -[\alpha N] \leq j < N$). This yields

$$\begin{aligned}\log(S_{N,k}) &= \sum_{\ell=1}^{k-N} \log\left(1 - \frac{\epsilon_{k-\ell}}{2\pi i\ell}\right) \sum_{\ell=1}^{[\alpha N]-k} \log\left(1 + \frac{\epsilon_{\ell+k}}{2\pi i\ell}\right)\\ &\quad + \sum_{\ell=1}^{[\alpha N]-k} \log\left(1 + \frac{\epsilon_{\ell-k}}{2\pi i\ell}\right) + o(1)\\ &\equiv K_1^N + K_2^N + K_3^N + o(1)\end{aligned} \tag{4.63}$$

It is readily seen that, as $N \to \infty$

$$K_3^N = \sum_{\ell=N+k}^{[\alpha N]+k} \left(-\frac{\epsilon_{k-\ell}}{2\pi i \ell} + O\left(\left(\frac{\epsilon_{k-\ell}}{\ell}\right)^2\right)\right) \tag{4.64a}$$

$$= o(1)\left(\sum_{\ell=N+k}^{[\alpha N]+k}\left(\frac{1}{\ell}+\frac{1}{\ell^2}\right)\right) = o(1)1 + \log\left(\frac{[\alpha N]+k}{N+k}\right)$$

In a similar way, we obtain that, as $N \to \infty$

$$K_1^N + K_2^N = \sum_1^{k-N}\left(-\frac{\epsilon_{k-\ell}}{2\pi i \ell}\right) + \sum_1^{[\alpha N]-k} \frac{\epsilon_{k+\ell}}{2\pi i \ell} + o(1)$$

If we now suppose that $k-1 \neq N, k+1 \neq [\sigma N]$, and set $\beta_{N,k} = \min\,\{k-N, [\alpha N]-k\}$, $\gamma_{N,k} = \max\,\{k-N,[\alpha N]-k\}$, we see that as $N \to \infty$

$$\begin{aligned} K_1^N + K_2^N &= \sum_{\ell=1}^{\beta_{N,k}} \frac{1}{2\pi i \ell}\left(\epsilon_{k+\ell} - \epsilon_{k-\ell}\right) + \\ &\quad + o(1)\left(\sum_{\beta_{N,k}+1}^{k-N}\frac{1}{\ell} + \sum_{\beta_{n,k}+1}^{[\alpha N]-k}\frac{1}{\ell}\right) + o(1) \\ &\leq \frac{|\,\nu\,|}{\pi}\sum_{\ell=1,\ell\neq k}^{\beta_{N,k}} \frac{1}{|\,k+\ell\,||\,k-\ell\,|} + \frac{1}{2\pi}\sum_1^{\beta_{N,k}} |\,r_{k+\ell}\,| + |\,r_{k-\ell}\,| \\ &\quad + o(1)\left(1+\log\left(\frac{k-N}{N,k}\right) + \log\left(\frac{[\alpha N]-k}{\beta_{N,k}+1}\right)\right) \end{aligned} \tag{4.64b}$$

uniformly for $k>0, N<k<[\alpha N]-1$, as $N \to \infty$. In view of (4.63) and (4.64), for $\delta_1 > 0$ arbitrarily small and N large enough, we have that

$$|\,S_{N,K}\,| \leq C\left(\frac{k-N}{\beta_{N,k}+1}\right)^{\delta_1}\left(\frac{[\alpha N]-k}{\beta_{N,k}+1}\right)^{\delta_1} \leq C\,|\,k\,|^{2\delta_1} \tag{4.65}$$

whereupon $|\,L_N\,| = o(1)$ as $N \to \infty$. Finally, minor modifications of the previous argument enable us to remove the assumptions made on the relations between k and N, and (4.62) follows.

We next set out to derive (4.56). To this end we notice that, keeping to our previous notation

$$(4.66) \qquad \tilde{M}_0^N = \sum_{\infty}^{\infty} 2\pi i k \mu_{N,k} \Gamma_k + \sum_{N \le |k| \le \alpha N} 2\pi \mathrm{ik} \mathrm{R}_{N,k} \Gamma_k \equiv B_1^N + B_2^N$$

Arguing as in the previous case, we readily see that

$$(4.67a) \qquad \lim_{N \to \infty} B_1^N = M_0$$

Therefore the proof will be finished as soon as can show that

$$(4.67b) \qquad \lim_{N \to \infty} B_2^N = -\chi_0 \nu \log \alpha$$

To derive (4.67b), we write B_2^N in the form

$$(4.68a) \qquad \begin{aligned} B_2^N &= \left(\sum_{N}^{[\alpha,N]} 2\pi i k R_{N,k} \Gamma_k \right) + \overline{\left(\sum_{N}^{[\alpha,N]} 2\pi i k R_{N,k} \Gamma_k \right)} \\ &\equiv Q_N + \overline{Q}_N \end{aligned}$$

We know have that, as $N \to \infty$,

$$(4.68b) \qquad \begin{aligned} Q_N &= -\chi_0 \sum_{N+1}^{[\alpha N]-1} \epsilon_k \delta_{N,k} + o(1) \\ &= -\chi_0 \nu \sum_{N+1}^{[\alpha N]-1} \frac{\delta_{N,k}}{k} - \chi_0 \sum_{N+1}^{[\alpha N]-1} r_k S_{N,k} + o(1) \\ &\equiv -\chi_0 \nu S_1 + o(1). \end{aligned}$$

On the other hand, for $1 < \theta_1 < \theta_2 < \alpha$ with θ_1, θ_2 close enough to 1 and α respectively, we see that

$$(4.69) \qquad \begin{aligned} | S_1 | &\le \sum_{N+1}^{[\theta_1 N]-1} \frac{| S_{N,k} |}{k} + \sum_{N+1}^{[\theta_2 N]} \frac{| S_{N,k} |}{k} + \\ &+ \sum_{N+1}^{[\alpha N]-1} \frac{| S_{N,k} |}{k} \equiv \\ &\equiv S_{1,1} + S_{1,2} + S_{1,3} \end{aligned}$$

Recalling (4.65), it follows at once that, for some $C > 0$

(4.70a)
$$S_{1,1} \leq \frac{C}{N} \sum_{N+1}^{[\theta_1 N]-1} \left(\frac{k}{N}\right)^{-1} \left(\frac{k}{N} - 1\right)^{-\delta_1} \leq$$
$$\leq C \int_1^{\theta_1} \frac{dr}{r(r-1)^{\delta_1}}$$

In a similar way, we obtain that

(4.70b)
$$S_{1,2} \leq C \int_{\theta_2}^{\alpha} \frac{dr}{r(r-1)^{\delta_1}}$$

Finally, one readily checks that $S_{N,k} = 1 + o(1)$ as $N \to \infty$, uniformly for $k \in (\theta_1 N, \theta_2 N)$. Hence

(4.70c)
$$S_{1,3} = \sum_{[\theta_1, N]}^{[\theta_2, N]} \frac{1}{k}(1 + o(1)) = \log\left(\frac{\theta_2}{\theta_1}\right) + o(1) \text{ as } N \to \infty.$$

Therefore, letting $\theta_1 \to 1$ and $\theta_2 \to \infty$ in (4.70), we deduce that $S_1 \to \log\ \alpha$ as $N \to \infty$. From this and (4.68), (4.67), follows. $\square$

End of the proof of Theorem 4.12.

By Lemma 4.13 we can select f ant T_w so that an arbitrary (but finite) number of zeroes of the corresponding function $L(z)$ is prescribed at will. Let $S_R = \{\ z_1, .., \tilde{z}_N\}$ be the real roots of $L(z)$ and assume that $\tilde{z}_N > \frac{1}{\epsilon}$ and $\tilde{z}_N > \mid \tilde{z}_j \mid$ for any $\tilde{z}_j \in S_R$ with $h \neq$N. Let ξ_N^ℓ be one of the roots of (3.14b) with z_j replaced by $\tilde{z}_N$ there. Clearly, we may assume that $\xi_N^\ell > 0$ and $g'(\xi_N^\ell) < 0$. Furthermore, it may also be supposed that $L\left(\pm\frac{1}{\epsilon}\right) > 0$ if roots $\tilde{z}_j$ are located in a suitable way.

It is easy to see that we can have that $\chi_0 \neq 0$ and $M_0 \neq 0$, after possibly a slight perturbation in the location of the zeroes above. Indeed, suppose that we change the coefficient Γ_k, Γ_{-k} in $L(z)$, where $\Gamma_k = a_k \bar{b}_k$, into $\Gamma_k + \Delta\Gamma_k, \Gamma_{-k} + \Delta\Gamma_{-k}$. Then χ_0 would go into $\tilde{\chi}_0 = \chi_0 + \Delta\chi_0 = \chi_0 + 2Re(\Gamma_k)$, and M_0 would be transformed into $\tilde{M}_0 = M_0 - 4\pi k Im(\Gamma_k)$, whence the result. Indeed this change can be done so that it also yields (4.17). On the other hand, by (4.16) we see that, as $Re \to \infty$

$$Qz^2\chi_0 \approx \left(\frac{z^2 Re^2}{g(\xi_N^\ell)^2}\right) \cdot \frac{g'(\xi_N^\ell)\chi_0}{\xi_N^\ell}$$

Therefore, by selecting coefficients so that $\chi_0 > 0$, the first term in the right of (4.20) yields purely imaginary values, whereas the second one there behaves as

$$\frac{1}{2}\left(\frac{\epsilon z^2 \xi_N^\ell g'(\xi_N^\ell)}{(g(\xi_N^\ell))^2} + \frac{M_0}{\chi_0}\right)$$

which by (4.55), (4.56) can approach any arbitrary real value if coefficients Γ_k with $N \leq\mid k \mid\leq \alpha N$ ($\alpha > 1$, and N large enough) are suitably selected.

5. STABILITY OF LINEARLY STABLE SOLUTIONS

In this Section we analyze the local stability of the stationary solutions for the complete nonlinear problem (4.2), (4.3). Assume that (Φ_s, T_s) is a stationary solution as in Section 4. We then define

$$w = \Phi(s) - \Phi_s \tag{5.1}$$

$$\psi(x,s) = T(x,s) - T_s(x) \tag{5.2}$$

Assume $\Phi_s \not\equiv 0$ that is generically true. By (4.2), (4.3) we obtain that as long as $|w|$ is small enough,

$$\begin{aligned} \frac{\partial\psi}{\partial s} + \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)}\frac{\partial\psi}{\partial x} &= -\psi + \\ &+ \varepsilon \operatorname{sgn}(\Phi_s)\frac{\partial T_s}{\partial x}(x)\left\{\frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(s)|)}\right\} + \\ &+ \varepsilon \operatorname{sgn}(\Phi_s)\left\{\frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(s)|)}\right\}\frac{\partial\psi}{\partial x}(x,s) \end{aligned} \tag{5.3}$$

$$\begin{aligned} \frac{dw}{ds} + 2w &= \frac{1}{g(Re|v_s|)}\oint f(x)\psi(x,s)dx + \\ &+ \left(\frac{1}{g(Re|v(s)|)} - \frac{1}{g(Re|v_s|)}\right)\oint f(x)T_s(x)dx + \\ &+ \left(\frac{1}{g(Re|v(s)|)} - \frac{1}{g(Re|v_s|)}\right)\oint f(x)\psi(x,s)dx \end{aligned} \tag{5.4}$$

where by (4.1a) $v_s = \operatorname{sgn}(\Phi_s)(2|\Phi_s|)^{1/2}$, $v(s) = \operatorname{sgn}(\Phi(s))(2|\Phi(s)|)^{1/2}$. By (5.3) we have that

$$\psi(x,s) = e^{-s}\psi_0\left(x - \int_0^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)d\lambda}{g(Re|v(\lambda)|)}\right) + \tag{5.5}$$

$$+\int_0^s e^{-(s-\lambda)}\varepsilon\,\mathrm{sgn}(\Phi_s)\left\{\frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(\lambda)|)}\right\}$$

$$\frac{\partial T_s}{\partial x}\left(x - \int_\lambda^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v(\Phi)|)}d\xi\right)d\lambda$$

Plugging (5.5) into (5.4) we obtain the differential equation

$$\frac{dw}{ds} + 2w = \frac{e^{-s}}{g(Re|v(s)|)}\oint f(x)\psi_0\left(x - \int_0^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v(\lambda)|)}d\lambda\right)$$

$$+\int_0^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v(\lambda)|)}e^{-(s-\lambda)}\oint f(x)\frac{\partial T_s}{\partial_x}\left(x - \int_\lambda^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v(s)|)}\lambda\xi\right)dxdx$$

$$+\left(\frac{1}{g(Re|v(s)|)} - \frac{1}{g(Re|v_0|)}\right)\oint f(x)T_0(x)dx$$

We make the change at variables $w(s) = e^{-s}g(s)$. Then $g(s)$ satisfies the equation

$$\frac{dq}{ds} + q = \frac{1}{g(Re|v(s)|)}\oint f(x)\psi_0\left(x - \int_0^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g|Re|x(\lambda)|)}d\lambda\right) \tag{5.6}$$

$$+\int_0^s e^{\lambda}\frac{\varepsilon\,\mathrm{sgn}(\Phi_e)}{g(Re|v(\lambda)|)}\left(\oint f(x)\frac{\partial T_s}{\partial x}\left(x - \int_\lambda^s \frac{\varepsilon\,\mathrm{sgn}(\Phi_\varepsilon)}{g|Re|v(s)|)}d\xi\right)dx\right)d\lambda$$

$$+e^{s}\left(\frac{1}{g|Re|v(e))} - \frac{1}{g(Re|v(s)|)}\right)\oint f(x)T_s(x)dx$$

We now define a new time scale as:

$$\eta \equiv \int_0^{Z(\eta)} \frac{g(Re|v_s|)}{g|Re|v(\eta)|)}d\lambda, \quad Z(\eta) = s \tag{5.7}$$

It is readily seen that $\eta \to \infty$ as $s \to \infty$.

On the other hand, (5.6) may be written as

$$\frac{dq}{d\eta} + \frac{g(Re|v(s)|)}{g(Re|v_s|)} q = \tag{5.8}$$

$$= \frac{1}{g(Re|v_s|)} \oint f(x)\psi_0 \left(x - \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \right) dx +$$

$$+ \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \left(\frac{g(Re|v(s)|)}{g(Re|v_s|)} \right) \int_0^\eta e^{Z(\gamma)} Q\omega |Z(\gamma)| \tilde{K}(\eta - \gamma) d\gamma$$

$$+ \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \left(\frac{g(Re|v(s)|)}{g(Re|v_s|)} \right) \int_0^\eta e^{Z(\gamma)} \left\{ \frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(\lambda)|)} - Qw(Z(\gamma)) \right\} \tilde{K}(\eta - \gamma) d\gamma +$$

$$+ e^{Z(\eta)} \left(\frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|s(s)|)} \right) \oint f(x) T_s(x) dx$$

where

$$\tilde{K}(\xi) = \oint f(x) \frac{\partial T_s}{\partial x} \left(x - \frac{\xi}{g(Re|v_s|)} \right) dx \tag{5.9}$$

and Q has been defined in (4.16).

Set

$$\Gamma \equiv -Q \oint f(x) T_s(x) dx \tag{5.10}$$

We can transform (5.8) in

$$\frac{dq}{d\eta} + (1+\Gamma)q = \frac{1}{g(Re|v_s|)} \oint f(x)\psi_0 \left(x - \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \eta \right) dx +$$

$$+ \frac{Q\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \int_0^\eta \tilde{K}(\eta - \gamma) q(Z(\gamma)) d\gamma +$$

$$+ \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \left(\frac{g(Re|v(s)|)}{g(Re|v_s|)} \right) \int_0^\eta e^{Z(\gamma)} \left\{ \frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(Z(\gamma))|)} - Qw(Z(\gamma)) \right\} \tilde{K}(\eta - \gamma) d\gamma +$$

$$+ \frac{Q\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \left(\frac{g(Re|v(s)|) - g(Re|v_s|)}{g(Re|v_s|)} \right) \int_0^\eta q(Z(\gamma)) \tilde{K}(\eta - \gamma) d\gamma -$$

$$- \left(\frac{g(Re|v(s)|) - g(Re|v_s|)}{g(Re|v_s|)} \right) q(s) +$$

$$+ e^{Z(q)} \left\{ \frac{1}{g(Re|v_0|)} - \frac{1}{g(Re|v(s)|)} - Qw(s) \right\} \oint f(x) T_s(x) dx$$

Define

$$\Omega(\xi) \equiv e^{-\eta} q(Z(\eta)) \tag{5.11a}$$

$$K(\xi) = e^{-\xi} \tilde{K}(\xi) \tag{5.11b}$$

Then, we have

(5.12)

$$\begin{aligned}
\frac{d\Omega}{d\eta} + (2+\Gamma)\Omega = & \frac{e^{-\eta}}{g(Re(v_s))} \oint f(x)\psi_0\left(x - \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re(v_s))}\eta\right) dx \\
& + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \int_0^\eta K(\eta-\gamma)\Omega(\gamma)d\gamma + \\
& + \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \left(\frac{g(Re|v(s)|)}{g(Re|v_s|)}\right) \int_0^\eta e^{-\eta} e^{Z(\gamma)} \left\{ \frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(\gamma)|)} - \right. \\
& \qquad \left. - Q\omega(Z(\gamma)) \right\} \tilde{K}(\eta-\gamma)d\gamma + \\
& + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re(v_s))} \left(\frac{g(Re|v(s)|) - g(Re|v_s|)}{g(Re|v_s|)}\right) \int_0^\eta K(\eta-\gamma)\Omega(\gamma)d\gamma \\
& - \left(\frac{g(Re|v(s)|) - g(Re|v_s|)}{g(Re|v_s|)}\right) \Omega(\eta) + \\
& + e^{-\eta} e^{Z(\eta)} \left\{ \frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(s)|)} - Q\omega(s) \right\} \oint f(x) T_s(x) dx
\end{aligned}$$

Our goal is to analyze equation (5.10) for $|\Omega(0)|\ \|\psi_0(\ )\|_{H^1_p(C)}$ small. To this end, we first study the linear integrodifferential equation

$$\frac{d\Omega}{d\eta} + (2+\Gamma)\Omega = \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \int_0^\eta K(\eta-\gamma)\Omega(\gamma)d\gamma \tag{5.13}$$

Suppose that the roots of the linearized problem, that have been analyzed in Section 4, are placed at $J = \{\mu_k \in \mathbb{C} : k = 0, \pm 1, \pm 2 \dots\}$. Then

LEMMA 5.1. *Assume that $-1 < \tau < 0$, $Re\{\mu_k\} < \tau$ for $\mu_k \in J$. The problem (5.13) with the initial condition $\Omega(0) = \Omega_0$ has a unique solution $\Omega(t) \in C[0+\infty) \cap C^1(0+\infty)$. This solution satisfies*

$$|\Omega(\eta)| \leq C(\Omega_0)e^{\tau\eta} \tag{5.14}$$

where $C > 0$ is some constant depending only on Re, $\|f\|_{H'_p}, \|T_w\|_{H^1_p}, g$.

Proof. Local existence and uniqueness of solutions for the problem (5.12), $\Omega(0) = \Omega_0$ follows from a standard fixed point argument. On the other hand by the variation of constants formula

$$\Omega(\eta) = \Omega_0 e^{-(2+\Gamma)\eta} + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \int_0^\eta d\xi e^{-(2+\Gamma)(\eta-\xi)} - \int_0^\xi K(\xi-\gamma)\Omega(\gamma)d\gamma$$

whence taking into account (3.9), (5.11)

$$|\Omega(\eta)| \le |\Omega_0| e^{-(2+\Gamma)\eta} + C \int_0^\eta d\xi e^{-(2+\Gamma)(\eta-\xi)} \int_0^\xi e^{-(\xi-\gamma)} |\Omega(\gamma)| d\gamma$$

where $C > 0$ depends on f, T_w, Re, and g. A straightforward calculation gives

$$|\Omega(\eta)| \le |\Omega_0| e^{-(2+\Gamma)\eta} + C \int_0^\eta e^{\tilde{L}(\eta-\gamma)} |\Omega(\gamma)| d\gamma$$

where $\tilde{L} > 0$ and $C > 0$ may change from line to line. A continuation argument implies that for $L > 0$ large enough

$$|\Omega(\eta)| \le C e^{L\eta} \tag{5.15}$$

and global existence follows.

By (5.13) we can apply Laplace's transform to (5.13). Set

$$\hat{f}(z) = \int_0^{+\infty} e^{-zt} f(t) dt$$

to obtain $z\widehat{\Omega}(z) - \Omega_0 + (2+\Gamma)\widehat{\Omega}(z) = \frac{Q\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \widehat{K}(z)\widehat{\Omega}(z)$. Notice that $\widehat{\Omega}(z)$ is analytic for $Re(z) > L$. Then

$$\widehat{\Omega}(z) = \frac{\Omega_0}{\left(z + (2+\Gamma) - \frac{Q\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} \widehat{K}(z)\right)} \tag{5.16}$$

By the inversion's formula for the Laplace transform

$$\Omega(\eta) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-Ri+2L}^{Ri+2L} \frac{\Omega_0 e^{z\eta}}{\left(z + (2+\Gamma) - \frac{Q\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re(v_s))} \widehat{K}(z)\right)} dz \tag{5.17}$$

On the other hand, it is readily seen that the zeroes of $z + (2+\Gamma) - \frac{Q\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re(v_s))} \widehat{K}(z)$ are placed at the same points that the zeroes of $(1 - e^z) S(z)$ (cf. (4.9)).

Notice that

$$\begin{aligned} \widehat{K}(z) &= \int_0^{+\infty} dt e^{-zt} K(t) dt = \\ &= \int_0^{+\infty} dt e^{-zt} e^{-t} \oint \frac{\partial T_s}{\partial x} \left(x - \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)} t \right) f(x) dx = \\ &= \sum_{k=-\infty}^{k=+\infty} \frac{2\pi k i \bar{a}_k c_k}{z + 1 + 2\pi k i \frac{\varepsilon \operatorname{sgn}(\Phi_s)}{g(Re|v_s|)}}, \end{aligned}$$

where c_k is given in (3.7).

It is readily seen that uniformly in

(5.18)
$$\left| z + 1 + 2\pi k i \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \right| \geq \frac{1}{2} |\pi k| \frac{\varepsilon}{|g(Re|v_s|)|}$$
$$\widehat{K}(z) \simeq + \frac{\sum_{k=-\infty}^{k=+\infty} 2\pi k i \bar{a}_k c_k}{z} \quad \text{as } |z| \to \infty$$

We now define a sequence of contours C_n as in the figure

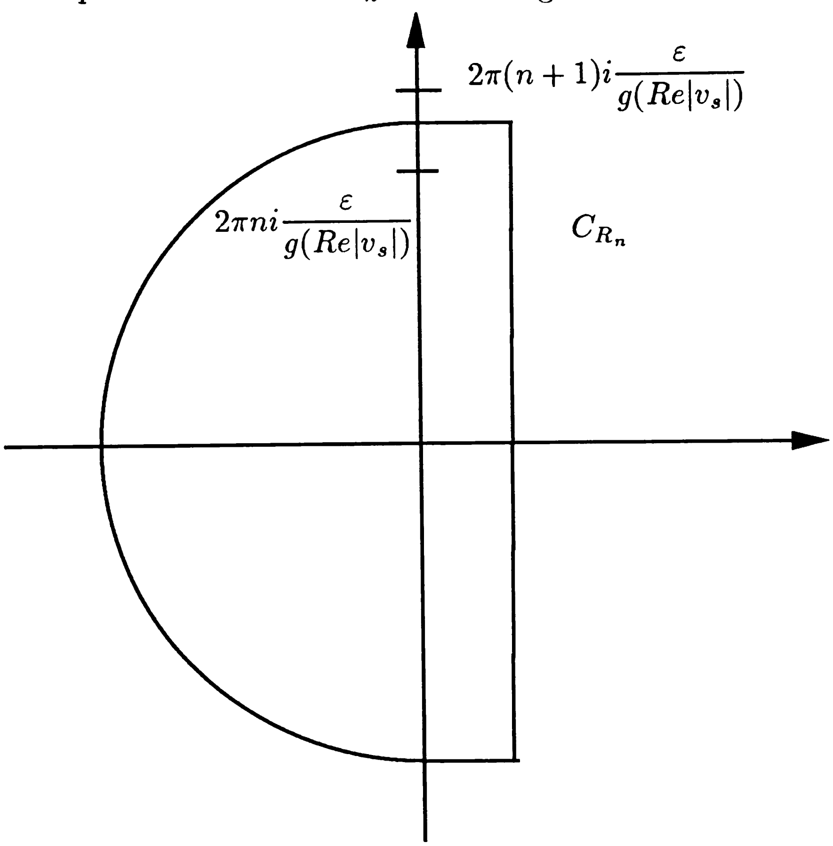

Figure 2:

Then it may be easily seen from (5.16), (5.17) that:

$$\Omega(\eta) = \frac{\Omega_0}{2\pi i} \lim_{n\to\infty} \int_{C_n} \frac{e^{z\eta}}{\left(z + (2+\Gamma) - \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \widehat{K}(z) \right)} dz$$

and by residue theorem we finally arrive to the representation formula

(5.19)
$$\Omega(\eta) = \lim_{N\to\infty} \sum_{|\ell|\leq N} Re\left\{ \widehat{\Omega}(z) e^{z\eta}, \mu_\ell \right\}$$

where μ_ℓ are the zeroes of $(1-e^z)S(z)$ that have been analyzed in Section 4.

In order to obtain bounds of $\Omega(\eta)$ we need precise estimates on the zeroes of

$$W(z) = \left(z + (2+\Gamma) - \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\widehat{K}(z)\right)$$

Notice that

$$W(z) = z + (2+\Gamma) + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re(v_s))}\sum_{k=-\infty}^{k=+\infty}\frac{2\pi k i \bar{a}_k c_k}{\left(z+1+2\pi k i\frac{\varepsilon\,\mathrm{sgn}(\Phi_i)}{g(Re(v_s))}\right)}$$

Arguing as in previous sections, we can obtain an asymptotic formula for the roots of W. Notice that the equation $W(z)=0$ may be approximated as:

$$z = \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\frac{2\pi j i \bar{a}_j c_j}{\left(z+1+2\pi j i\frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\right)}$$

Then,

$$\mu_j \simeq -1 - 2\pi j i\frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \quad \text{as } |j|\to\infty$$

whence:

$$(2+\Gamma) - 1 - 2\pi j i\frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \simeq \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} - \frac{2\pi j i \bar{a}_j c_j}{\left(2+1+\frac{+2\pi j i\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\right)}$$

Then:

$$\mu_j \simeq -\left(1+\frac{2\pi j i\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\right) - \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_\Gamma)}{g(Re|v_s|)}\frac{2\pi j i \bar{a}_j c_j}{2\pi j i\frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}}$$

as $|j|\to\infty$. Then

$$\mu_j \simeq -\left(1+\frac{2\pi j i\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\right) - \bar{a}_j c_j, \qquad \text{as } |j|\to\infty.$$

Generically we can assume that the roots of $W(z)$ are simple.

Then

$$Res\left\{\widehat{\Omega}(z)e^{z\eta}, \mu_\ell\right\} \simeq \frac{e^{\mu_\ell \tau}}{W'(\mu_\ell)} \tag{5.20}$$

Notice that

$$W'(z) = 1 + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \sum_{k=-\infty}^{k=+\infty} \frac{2\pi k i \bar{a}_k c_k}{\left(z + 1 + \frac{2\pi k i \varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\right)^2}$$

$$\begin{aligned} W'(\mu_j) &\simeq 1 + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \cdot \frac{2\pi j i \bar{a}_j c_j}{\left(\mu_j + 1 + \frac{2\pi j_i \varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\right)^2} \simeq \\ &\simeq 1 + \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \frac{2\pi j i \bar{a}_j c_j}{(\bar{a}_j c_j)^2} \simeq \\ &\simeq \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)} \frac{2\pi j i}{\bar{a}_j c_j} \qquad \text{as } |j| \to \infty \end{aligned}$$

Then, by (5.20)

$$Res\{\widehat{\Omega}(z) e^{z\eta} : \mu_l\} \simeq \frac{\Omega_0 g(Re(v_s))}{Q\varepsilon\,\mathrm{sgn}(\Phi_s)} \frac{\bar{a}_l c_l}{2\pi l i} e^{\mu_l \eta} \tag{5.21}$$

Then, if we use (5.18) we obtain that $(\Omega(\eta)) \leq C e^{\tau\eta} |\Omega_0|$, where C depends on f, $T_w(x)$, and $-1 < \tau < 0$. Notice that similar bounds may be obtained if $\bar{a}_l c_l = 0$, because $\tau > -1$. □

Set $T(\eta) = \Omega(\eta)$ the solution of (5.13) with $\Omega(0) = 1$.

We now consider the problem

$$\frac{d\Omega}{d\eta} + (2 + \Gamma)\Omega = \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re(v_s))} \int_0^\eta K(\eta - \gamma)\Omega(\gamma) d\gamma + M(\eta) \tag{5.22a}$$

$$\Omega(0) = \Omega_0 \tag{5.22b}$$

where $M(\eta)$ is a continuous function for $\eta \geq 0$. Then, we have

LEMMA 5.2. *The problem (5.22) has a unique solution $\Omega(\eta) \in C[0, +\infty)$ that admits the representation formula*

$$\Omega(\eta) = T(\eta)\Omega_0 + \int_0^\eta T(\eta - \gamma) M(\eta) d\gamma \tag{5.23}$$

Proof. Local existence and uniqueness follows from a standard fixed point argument. By Lemma 5.1 we have that $T(\cdot) \in C[0 + \infty) \cap C^1(0 + \infty)$. Then, a straightforward calculation shows that $\Omega(\eta)$ solves (5.22a) and (5.22b). □

Finally we can state the main result of this section.

THEOREM 5.3. *Assume that there exists* $-1 < \tau < 0$ *such that* $Re\{\mu_k\} < \tau$ *for* $\mu_k \in J$. *There exists* $\delta > 0$ *such that for* $|\Omega_0| + \|\psi_0\|_{H_p^1(C)} \leq \delta$ *the solution of (5.12)* $(\Omega(\eta), \psi(\eta))$ *satisfies*

$$|\Omega(\eta)| + \|\psi(\cdot,\eta)\|_{H_p^1(C)} \leq C(|\Omega_0| + \|\psi_0\|_{H_p^1(C)})e^{\tau\eta} \tag{5.24}$$

where $C > 0$ *depends on* f, T_w, g, Re, τ.

Proof. Problem (4.2)-(4.3) is equivalent to (5.5)-(5.12) if $|\Omega(\eta)|$ is small enough. Then global existence and uniqueness of (5.12) follows from Theorem 2.1 and estimate (5.24) if $\delta > 0$ is small enough.

Assume that $|\Omega(\eta)| \leq \bar{\delta}$, where $\bar{\delta}$ is small enough. Then, by (5.7) we have that $(1-\beta)\eta \leq Z(\eta) \leq (1+\beta)\eta$, where $\beta > 0$ is arbitrarily small. We apply the representation formula (5.23) in (5.17) to obtain

$$\begin{aligned}
\Omega(\eta) = T(\eta)\Omega_0 + &\\
&+ \frac{1}{g(Re|v_s|)}\int_0^\eta d\gamma T(\eta-\gamma)e^{-\gamma}\oint f(x)\psi_0\left(x - \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\gamma\right)d_x \\
&+ \frac{\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re|v_s|)}\int_0^\eta d\gamma T(\eta-\gamma)\left(\frac{g(Re|v(Z(\gamma))|)}{g(Re|v_s|)}\right) \\
&\qquad - \int_0^\gamma e^{-\gamma}e^{Z(\xi)}\left\{\frac{1}{g(Re|v_s|)} - \frac{1}{g(Re|v(\xi)|)} - Qw(Z(\xi))\right\}\widetilde{K}(\gamma-\xi)d\xi \\
&+ \frac{Q\varepsilon\,\mathrm{sgn}(\Phi_s)}{g(Re(v_s))}\int_0^\eta d\gamma T(\eta-\gamma)\left(\frac{g(Re|v(Z(\gamma))|) - g(Re(v_s))}{g(Re(v_s))}\right) \\
&\qquad - \int_0^\gamma K(\gamma-\xi)\Omega(\xi)d\xi - \\
&\qquad - \int_0^\eta d\gamma T(\eta-\gamma)\left(\frac{g(Re|v(Z(\gamma))|) - g(Re(v_s))}{g(Re(v_s))}\right)\Omega(\gamma) + \\
&+ \int_0^\eta d\gamma T(\eta-\gamma)e^{-\gamma}e^{Z(\gamma)}\left\{\frac{1}{g(Re(v_s))} - \frac{1}{g(Re|v(Z(\gamma))|)} - Qw(Z(0))\right\} \\
&\qquad\qquad \oint f(x)T_s(x)dx
\end{aligned}$$

Standard estimates that use Lemma 5.1 imply that

$$
\begin{aligned}
|\Omega(\eta)| \le\ & C(\Omega_0)e^{\tau\eta} + C\|\psi_0\|_{L^2(C)}e^{-\eta} + \\
& + C\int_0^\eta d\gamma e^{\tau(\eta-\gamma)}e^{-\gamma}\int_0^\gamma e^{(1+\beta)\xi}(\Omega(Z(\xi)))^2 \\
& + C\int_0^\eta d\gamma e^{\tau(\eta-\gamma)}(\Omega(Z(\gamma)))\int_0^\gamma e^{-(\gamma-\xi)}(\Omega(\xi))d\xi + \\
& + C\int_0^\eta d\gamma e^{\tau(\eta-\gamma)}(\Omega(Z(\gamma)))(\Omega(\gamma)) + \\
& + C\int_0^\eta d\gamma e^{\tau(\eta-\gamma)}e^{-\gamma}e^{(1+\beta)\gamma}(\Omega(Z(\gamma))).
\end{aligned}
$$

Then, if we take into account that as long as (5.24) holds $Z(\eta) \le (1+\beta)\eta$, we can use a standard continuation argument to obtain that if β is small enough

$$|\Omega(\eta)| \le C\left(|\Omega_0| + \|\psi_0\|_{L^2(C)}\right)e^{\tau\eta}$$

We now can use (5.5) to obtain

$$\|\psi(\cdot,\tau)\|_{H^1_p(C)} \le C\left(|\Omega_0| + \|\psi_0\|_{H^1_p(C)}\right)e^{\tau\eta}$$

This concludes the proof of (5.24). □

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