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Harald Garcke, Kazuo Ito and Yoshihito Kohsaka

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Abstract. The linearized stability of stationary solutions to the surface diffusion flow with angle conditions and no-flux conditions as boundary conditions is studied. We perform a linearized stability analysis in which the H^{-1}-gradient flow structure plays a key role. As a byproduct our analysis also gives a criterion for the stability of critical points of the length functional of curves which come into contact with the outer boundary. Finally, we study the linearized stability of several examples.

Key words. surface diffusion, gradient flow, stability of stationary solutions, eigenvalues, isoperimetric problems

AMS subject classifications. 35G30, 35B35, 35R35, 80Axx

1 Introduction

The geometrical evolution law

$$V = -\Delta\kappa$$

was derived by Mullins [17] to model the motion of interfaces in the case that the motion of interfaces is governed purely by mass diffusion within the interfaces (for simplicity we set the diffusion constant to 1). Here V is the normal velocity of the evolving interface, Δ is the Laplace-Beltrami operator and κ is the mean curvature of the interface where we use the sign convention that a sphere with the normal pointing to the inside has positive curvature. We also refer to work by Davi and Gurtin [5] who derived the above law from balance laws in conjunction with an appropriate version of the second law of thermodynamics and to work by Cahn, Elliott and Novick-Cohen [2] who derived this evolution law as the sharp interface limit of a Cahn-Hilliard equation with degenerate mobility. This evolution law has the property that for closed embedded hypersurfaces the enclosed volume is preserved and the surface area decreases in time (see e.g. [8], [9]). An existence result for curves in the plane and stability of spheres - which are stationary under the flow - has been shown by Elliott and Garcke [8]. This result was generalized to the higher dimensional case by Escher, Mayer and Simonett [9].

In general interfaces will meet an outer boundary or they might intersect at triple or multiple junctions. In this case boundary conditions have to hold and they were derived by Garcke and Novick-Cohen [11] as the asymptotic limit of a Cahn-Hilliard system with a degenerate mobility matrix. At the outer boundary and at triple junctions angle conditions and a balance condition for the mass fluxes have to hold. At triple junction in addition a continuity condition for chemical potentials has to hold. An existence result for surface diffusion of curves that intersect the outer boundary and meet at triple junctions has been given by Garcke and Novick-Cohen [11]. The stability problem for stationary solutions for the surface diffusion flow with triple junctions has been addressed by Ito and Kohsaka [15] and by Escher, Garcke and Ito [10] in the case of a geometry with a mirror symmetry and by Ito and Kohsaka [16] in a triangular domain. The general case is still open. This is partly due to the fact that the stability depends in a nontrivial way on the geometry of the boundary.

For motion by mean curvature which is given by the law

$$V = \kappa \tag{1.1}$$

the stability of stationary interfaces with boundaries was studied by Rubinstein, Sternberg and Keller [18] in the case where the evolving curves intersect an outer boundary with a $90°$ angle. For stability results for the stationary solutions of (1.1) in the presence of triple junctions we refer to Sternberg and Ziemer [19] and Ikota and Yanagida [13]. The last authors developed a linear stability criterion that is based on ideas of Ei, Sato and Yanagida [7], [6].

One main difference between motion by mean curvature and motion by surface diffusion is that the former does not preserve volume and whereas the latter does. This implies that the stationary solutions are different. For motion by surface diffusion spherical arcs that intersect the outer boundary perpendicular are stationary. It is the goal of this paper to study the stability of such stationary solutions under surface diffusion. More precisely we study the following problem. Given an open bounded domain Ω we look for evolving curves $\Gamma = (\Gamma_t)_{t>0}$ (for a definition see Gurtin [12]) lying in Ω with the following properties (for a precise definition of the flow see Section 2):

$$\begin{cases} V = -\kappa_{ss} & \text{for all points on the curve,} \\ \partial\Gamma_t \subset \partial\Omega & \text{at all times,} \\ \sphericalangle(\partial\Omega, \Gamma) = \pi/2 & \text{at the boundary,} \\ \kappa_s = 0 & \text{at the boundary} \end{cases} \tag{1.2}$$

here a subscript s denotes differentiation with respect to arc-length. The second and third condition imply that the boundary of the curves at all times intersect the outer boundary perpendicularly. The last condition says that there is no mass flux at the outer boundary (see [11]). It is not difficult to show that (see [11])

$$\frac{d}{dt}\text{Area}_\Gamma(t) = 0, \qquad \frac{d}{dt}\text{Length}_\Gamma(t) \leq 0$$

under surface diffusion with the above boundary conditions. Here we denote by $\text{Area}_\Gamma(t)$ the area enclosed by the curve and $\partial\Omega$ at time t (for definiteness we take the side of Γ in which the normal points) and by $\text{Length}_\Gamma(t)$ the length of Γ at time t.

We will introduce a linear stability criterion based on the work of [7], [6], [13] which deal with mean curvature flow. The analysis in the case of surface diffusion is more difficult because the surface diffusion flow is the gradient flow with respect to the H^{-1}-inner product (see [20]) in contrast to the case of motion by mean curvature which is a gradient flow with respect to the L^2-inner product. We want to emphasize that the observation that also the linearized problem is an H^{-1} gradient flow of the bilinearized area functional is an important ingredient of our analysis (see Section 4). Indeed, the zero solution is an asymptotically stable solution of the linearized equation $\rho_t = \mathcal{A}\rho$ ($\mathcal{A}$ being the linearized operator) if and only if all eigenvalues of $\mathcal{A}$ are negative and it will turn out that this is equivalent to the fact that the bilinearized area functional is positive definite.

The stability of stationary arcs that are attached perpendicular to the outer boundary depend on their curvature, their length and the curvature of the outer boundary in a nontrivial way. The reader is advised to have a look at Section 7 where we illustrate the stability behaviour with the help of several examples. Taking advantage of the gradient flow property of the evolution and using variational arguments we are able to analyze the linear stability behaviour, i.e. the stability of the zero solution of the linearized operator (see Section 6). It would remain to show that the principle of linearized stability holds, that means except for the critical (or neutral) case of stability the zero solution of the linearized problem has the same stability as the stationary solution of the nonlinear problem around which we linearized the equation. Due to the highly nonlinear boundary condition this is a nontrivial task. We refer to [10] for a result in this direction for a mirror-symmetric situation.

Finally, we remark that our results also have some relevance for isoperimetric problems as they give stability results for critical points of the length functional which is restricted to curves that enclose a fixed area. Since the surface diffusion flow reduces length conserving area at the same time, the study of critical points of the length functional (given an area constraint) is what the stability analysis for the evolution problem can be reduced to.

2 Parameterization

In this section we give a precise definition of the flow (1.2) and in particular we introduce a parameterization of an evolving curve that will be convenient for our analysis. For a smooth function $\psi : \mathbb{R}^2 \to \mathbb{R}$ with $\nabla\psi(x) \neq 0$ if $\psi(x) = 0$, set

$$\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) < 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^2 \mid \psi(x) = 0\}.$$

Let Γ_* be a stationary solution and σ be the arc-length parameter of Γ_*. Then we denote an arc-length parameterization of Γ_* as

$$\Gamma_* = \{\Phi_*(\sigma) \mid \sigma \in [-l, l]\}.$$

Note that we can extend Γ_* naturally either to the full circle when Γ_* is a part of circle or to the straight line when Γ_* is a line segment. Also note that the curvature κ_* of Γ_* is a constant. We denote

$$\bar{l} := \begin{cases} \pi/|\kappa_*|, & \kappa_* \neq 0, \\ +\infty, & \kappa_* = 0. \end{cases}$$

i.e. $\bar{l}$ is the length of the extension of Γ_* to a full circle (if $\kappa_* \neq 0$). Define

$$\begin{cases} \xi_+(q) = \max\{\sigma \in (-\bar{l}, \bar{l}) \mid \Phi_*(\sigma) + qN_*(\sigma) \in \Omega\}, \\ \xi_-(q) = \min\{\sigma \in (-\bar{l}, \bar{l}) \mid \Phi_*(\sigma) + qN_*(\sigma) \in \Omega\}. \end{cases}$$

where $q \in [-d, d]$ for a small $d > 0$, and $N_*(\sigma)$ is a unit normal vector of Γ_* at σ and is obtained by rotating the unit tangent vector $T_*(\sigma)$ of Γ_* with $\pi/2$. Then it holds $\psi(\Phi_*(\xi_\pm(q)) + qN_*(\xi_\pm(q))) = 0$. In addition, we have $\xi_\pm(0) = \pm l$. Using the implicit function theorem, we see that $\xi_+(q)$ and $\xi_-(q)$ are smooth. Let

$$\Psi(\sigma, q) := \Phi_*(\xi(\sigma, q)) + qN_*(\xi(\sigma, q))$$

with

$$\xi(\sigma, q) := \xi_-(q) + \frac{\sigma + l}{2l}(\xi_+(q) - \xi_-(q)).$$

Note that $\xi(\pm l, q) = \xi_\pm(q)$ and $\xi(\sigma, 0) = \sigma$.

Let Γ be curves in the neighbourhood of Γ_*, which touch the boundary $\partial\Omega$ and are contained in Ω. For some functions $\rho : [-l, l] \to [-d, d]$, we define $\Phi(\sigma) := \Psi(\sigma, \rho(\sigma))$ for $\sigma \in [-l, l]$, which denotes a parameterization of such curves Γ. Thus we set

$$\Gamma(t) := \{\Phi(\sigma, t) \mid \sigma \in [-l, l]\} \tag{2.1}$$

with $\Phi(\sigma, t) := \Psi(\sigma, \rho(\sigma, t))$ for a function ρ depending on σ and t. We remark that $\rho \equiv 0$ means that curves Γ coincide with a stationary curve Γ_*.

Let us derive the representation of (1.2) to the parameterization (2.1). For the arc-length parameter s of Γ, we have

$$\frac{ds}{d\sigma} = |\Phi_\sigma| = \sqrt{|\Psi_\sigma|^2 + 2(\Psi_\sigma, \Psi_q)_{\mathbb{R}^2}\rho_\sigma + |\Psi_q|^2\rho_\sigma^2}\ (=: J(\rho)). \tag{2.2}$$

Here and hereafter $(\cdot, \cdot)_{\mathbb{R}^2}$ denotes the inner product in $\mathbb{R}^2$. Then we find

$$T = \frac{1}{J(\rho)}\Phi_\sigma, \quad N = \frac{1}{J(\rho)}R\Phi_\sigma,$$

where T and N are the unit tangent and normal vector of Γ respectively, and R is the rotation matrix with $\pi/2$. The normal velocity V of $\Gamma(t)$ is denoted by

$$V = (\Phi_t, N)_{\mathbb{R}^2} = \frac{1}{J(\rho)}(\Phi_t, R\Phi_\sigma)_{\mathbb{R}^2} = \frac{1}{J(\rho)}(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}\rho_t.$$

Moreover, since (2.2) gives

$$\partial_s^2 = \frac{1}{J(\rho)}\partial_\sigma\left(\frac{1}{J(\rho)}\partial_\sigma\right) = \frac{1}{(J(\rho))^2}\partial_\sigma^2 + \frac{1}{J(\rho)}\left(\partial_\sigma\frac{1}{J(\rho)}\right)\partial_\sigma\ (=: \Delta(\rho)), \tag{2.3}$$

the curvature κ of $\Gamma(t)$ is written by

$$\begin{aligned}
\kappa(\rho) &= (\Delta(\rho)\Phi, N)_{\mathbb{R}^2} \\
&= \frac{1}{(J(\rho))^3}(\Phi_{\sigma\sigma}, R\Phi_\sigma)_{\mathbb{R}^2} \\
&= \frac{1}{(J(\rho))^3}\Big[(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}\rho_{\sigma\sigma} + \big\{2(\Psi_{\sigma q}, R\Psi_\sigma)_{\mathbb{R}^2} + (\Psi_{\sigma\sigma}, R\Psi_q)_{\mathbb{R}^2}\big\}\rho_\sigma \\
&\quad + \big\{(\Psi_{qq}, R\Psi_\sigma)_{\mathbb{R}^2} + 2(\Psi_{\sigma q}, R\Psi_q)_{\mathbb{R}^2} + (\Psi_{qq}, R\Psi_q)_{\mathbb{R}^2}\rho_\sigma\big\}\rho_\sigma^2 \\
&\quad + (\Psi_{\sigma\sigma}, R\Psi_\sigma)_{\mathbb{R}^2}\Big].
\end{aligned} \tag{2.4}$$

Thus the surface diffusion flow equation is described by

$$\rho_t = -L(\rho)\Delta(\rho)\kappa(\rho), \tag{2.5}$$

where

$$L(\rho) := \frac{1}{(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2}} J(\rho). \tag{2.6}$$

Let us derive the representation of the boundary conditions which are the Neumann boundary condition and the no-flux condition $\kappa_s = 0$ on $\partial\Omega$ (the second condition in (1.2) is automatically fulfilled). Since the Neumann boundary condition $(\Phi_\sigma, T_{\partial\Omega})_{\mathbb{R}^2} = 0$ is equivalent to $(R\Phi_\sigma, \nabla\psi(\Phi))_{\mathbb{R}^2} = 0$, we have

$$(R\Psi_\sigma + R\Psi_q\rho_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} = 0.$$

By (2.2) and (2.4) the no-flux condition $\kappa_s = 0$ is denoted by

$$\partial_\sigma\kappa(\rho) = 0.$$

Consequently we have the following proposition.

Proposition 2.1 *For a parameterization (2.1), the problem (1.2) is represented by*

$$\begin{cases} \rho_t = -L(\rho)\Delta(\rho)\kappa(\rho) & \textit{for} \quad \sigma \in (-l, l), \ t > 0, \\ (R\Psi_\sigma + R\Psi_q\rho_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} = 0 & \textit{at} \quad \sigma = \pm l, \\ \partial_\sigma\kappa(\rho) = 0 & \textit{at} \quad \sigma = \pm l, \end{cases} \tag{2.7}$$

where $L(\rho)$, $\Delta(\rho)$ and $\kappa(\rho)$ are defined by (2.6), (2.3) and (2.4) respectively.

3 Linearization

To study the linearized stability of a stationary solution Γ_*, the curvature κ_* of which is a constant, we linearize (2.7) around $\rho \equiv 0$. For this purpose we need the following properties of Ψ at $q = 0$.

Lemma 3.1 *For the parameterization of Section 2 it holds*
i) $\Psi(\sigma, 0) = \Phi_*(\sigma)$.
ii) $\Psi_\sigma(\sigma, 0) = T_*(\sigma)$ *and* $\Psi_q(\sigma, 0) = N_*(\sigma)$.
iii) $\Psi_{\sigma\sigma}(\sigma, 0) = \kappa_* N_*(\sigma)$ *and* $\Psi_{\sigma q}(\sigma, 0) = -\kappa_* T_*(\sigma)$.
iv) $\Psi_{\sigma\sigma q}(\sigma, 0) = -\kappa_*^2 N_*(\sigma)$.

Proof. By the definition of Ψ, i) is obvious. Using i), we readily derive $\Psi_\sigma(\sigma, 0) = T_*(\sigma)$. To derive $\Psi_q(\sigma, 0) = N_*(\sigma)$, we first prove $\xi'_+(0) = \xi'_-(0) = 0$. Note that $\xi(\pm l, q) = \xi_\pm(q)$ and $\xi_q(\pm l, q) = \xi'_\pm(q)$. Since it follows from Frenet-Serret formula that

$$\Psi_q(\sigma, q) = \xi_q(\sigma, q)(1 - q\kappa_*)T_*(\xi(\sigma, q)) + N_*(\xi(\sigma, q)), \tag{3.1}$$

we are led to

$$\begin{aligned} 0 &= \frac{d}{dq}\psi(\Psi(\pm l, q)) \\ &= (1 - q\kappa_*)(\nabla\psi(\Psi(\pm l, q)), T_*(\xi_\pm(q)))_{\mathbb{R}^2}\xi'_\pm(q) + (\nabla\psi(\Psi(\pm l, q)), N_*(\xi_\pm(q)))_{\mathbb{R}^2}. \end{aligned}$$

Putting $q=0$, we have $(\nabla\psi(\Phi_*(\pm l)), T_*(\pm l))_{\mathbb{R}^2}\xi'_\pm(0)=0$, so that $\xi'_+(0)=\xi'_-(0)=0$. Then this implies

$$\xi_q(\sigma,0)=\xi'_-(0)+\frac{\sigma+l}{2l}(\xi'_+(0)-\xi'_-(0))=0.$$

Putting $q=0$ in (3.1), we derive $\Psi_q(\sigma,0)=N_*(\sigma)$. By virtue of ii) and Frenet-Serret formula, we readily derive iii). Finally, by differentiating $\Psi_{\sigma q}(\sigma,0)=-\kappa_* T_*(\sigma)$ with respect to σ and applying Frenet-Serret formula, we are led to iv). $\square$

Set $G(\rho):=-L(\rho)\Delta(\rho)\kappa(\rho)$ and $\mathcal{A}:=\partial G(0)$, where $\partial G(0)$ is the Fréchet derivative of G at 0. Then we have the following representation of $\mathcal{A}$.

Lemma 3.2 *For the Fréchet derivative $\mathcal{A}$ of the right hand side of (2.5) we obtain*

$$\mathcal{A}=-\partial_\sigma^2(\partial_\sigma^2+\kappa_*^2).$$

Proof. Since $G(\rho)=-L(\rho)\Delta(\rho)\kappa(\rho)$, we have

$$\mathcal{A}\rho=\partial G(0)\rho=-(\partial L(0)\rho)\Delta(0)\kappa(0)-L(0)(\partial\Delta(0)\rho)\kappa(0)-L(0)\Delta(0)\partial\kappa(0)\rho. \tag{3.2}$$

By virtue of Lemma 3.1 and the definition of $L(\rho)$, $\Delta(\rho)$ and $\kappa(\rho)$, we observe

$$L(0)\equiv 1,\quad \Delta(0)=\partial_\sigma^2,\quad \kappa(0)\equiv\kappa_*. \tag{3.3}$$

Then, since κ_* is a constant, we have $\Delta(0)\kappa(0)=0$ and

$$\begin{aligned}&(\partial\Delta(0)\rho)\kappa(0)\\ &\quad=\left(\frac{d}{d\varepsilon}(J(\varepsilon\rho))^{-2}\Big|_{\varepsilon=0}\right)\partial_\sigma^2\kappa(0)+\left(\frac{d}{d\varepsilon}(J(\varepsilon\rho))^{-1}\partial_\sigma(J(\varepsilon\rho))^{-1}\Big|_{\varepsilon=0}\right)\partial_\sigma\kappa(0)=0.\end{aligned}$$

Let us derive $\partial\kappa(0)\rho$. Set

$$\begin{cases} a_1(\rho)=(\Psi_q, R\Psi_\sigma)_{\mathbb{R}^2},\\ a_2(\rho)=2(\Psi_{\sigma q}, R\Psi_\sigma)_{\mathbb{R}^2}+(\Psi_{\sigma\sigma}, R\Psi_q)_{\mathbb{R}^2},\\ a_3(\rho)=(\Psi_{qq}, R\Psi_\sigma)_{\mathbb{R}^2}+2(\Psi_{\sigma q}, R\Psi_q)_{\mathbb{R}^2}+(\Psi_{qq}, R\Psi_q)_{\mathbb{R}^2}\rho_\sigma,\\ a_4(\rho)=(\Psi_{\sigma\sigma}, R\Psi_\sigma)_{\mathbb{R}^2}.\end{cases}$$

Then $\kappa(\rho)$ is written by

$$\kappa(\rho)=(J(\rho))^{-3}a(\rho),$$

where

$$a(\rho):=a_1(\rho)\rho_{\sigma\sigma}+a_2(\rho)\rho_\sigma+a_3(\rho)\rho_\sigma^2+a_4(\rho).$$

Thus we have

$$\begin{aligned}\partial\kappa(0)\rho &= \frac{d}{d\varepsilon}\kappa(\varepsilon\rho)\Big|_{\varepsilon=0}\\ &= (J(0))^{-3}\frac{d}{d\varepsilon}a(\varepsilon\rho)\Big|_{\varepsilon=0}+\left(\frac{d}{d\varepsilon}(J(\varepsilon\rho))^{-3}\Big|_{\varepsilon=0}\right)a(0).\end{aligned}$$

By virtue of Lemma 3.1, we observe $J(0) = 1$ and $a(0) = a_4(0) = \kappa_*$. In addition, it holds

$$\frac{d}{d\varepsilon} a(\varepsilon\rho)\Big|_{\varepsilon=0} = a_1(0)\rho_{\sigma\sigma} + a_2(0)\rho_\sigma + \partial a_4(0)\rho = \partial_\sigma^2 \rho - 2\kappa_*^2 \rho,$$
$$\frac{d}{d\varepsilon}(J(\varepsilon\rho))^{-3}\Big|_{\varepsilon=0} = -3(J(0))^{-4} \frac{d}{d\varepsilon} J(\varepsilon\rho)\Big|_{\varepsilon=0} = 3\kappa_* \rho.$$

Consequently, we are led to

$$\partial\kappa(0)\rho = (\partial_\sigma^2 + \kappa_*^2)\rho. \tag{3.4}$$

The assertion follows from (3.2)-(3.4). $\square$

Let us consider the boundary condition. Set

$$\begin{cases} B_1(\rho) := (R\Psi_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2} + (R\Psi_q, \nabla\psi(\Psi))_{\mathbb{R}^2}\rho_\sigma, \\ B_2(\rho) := \partial_\sigma \kappa(\rho), \end{cases}$$

and denote $x_*^\pm := \Phi_*(\pm l)$. Define

$$\mathcal{B} := \begin{pmatrix} \partial B_1(0)/(\mp|\nabla\psi(x_*^\pm)|) \\ \partial B_2(0) \end{pmatrix} \quad \text{at} \quad \sigma = \pm l$$

Then we have the following representation of $\mathcal{B}$.

Lemma 3.3 *Let $h_\pm$ be the curvatures of $\partial\Omega$ at $x_*^\pm \in \Gamma_* \cap \partial\Omega$ respectively (where we use the sign convention that $h_\pm < 0$ if Ω is convex). Then*

$$\mathcal{B} = \begin{pmatrix} \partial_\sigma \pm h_\pm \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2) \end{pmatrix} \quad \textit{at} \quad \sigma = \pm l.$$

Proof. First we derive $\partial B_1(0)$. Set

$$b_1(\rho) = (R\Psi_\sigma, \nabla\psi(\Psi))_{\mathbb{R}^2}, \quad b_2(\rho) = (R\Psi_q, \nabla\psi(\Psi))_{\mathbb{R}^2}.$$

Then we have $B_1(\rho) = b_1(\rho) + b_2(\rho)\rho_\sigma$, so that

$$\partial B_1(0)\rho = \frac{d}{d\varepsilon} B_1(\varepsilon\rho)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} b_1(\varepsilon\rho)\Big|_{\varepsilon=0} + b_2(0)\rho_\sigma.$$

It follows from Lemma 3.1 that

$$\frac{d}{d\varepsilon} R\Psi_\sigma(\sigma, \varepsilon\rho)\Big|_{\varepsilon=0} = -\kappa_* N_*(\sigma)\rho,$$
$$\frac{d}{d\varepsilon} \nabla\psi(\Psi(\sigma, \varepsilon\rho))\Big|_{\varepsilon=0} = [D^2\psi(\Phi_*(\sigma))] N_*(\sigma)\rho,$$

where $D^2\psi$ is the Hessian matrix of ψ. Since $(N_*(\sigma), \nabla\psi(\Phi_*(\sigma)))_{\mathbb{R}^2} = 0$ at $\sigma = \pm l$, we are led to

$$\frac{d}{d\varepsilon} b_1(\varepsilon\rho)\Big|_{\varepsilon=0} = (N_*(\sigma), [D^2\psi(\Phi_*(\sigma))] N_*(\sigma))_{\mathbb{R}^2}\rho \quad \text{at} \quad \sigma = \pm l.$$

This implies that for $\sigma = \pm l$

$$\partial B_1(0)\rho = -(T_*(\sigma), \nabla\psi(\Phi_*(\sigma)))_{\mathbb{R}^2}\rho_\sigma + (N_*(\sigma), [D^2\psi(\Phi_*(\sigma))]N_*(\sigma))_{\mathbb{R}^2}\rho.$$

Let the arc-length parameter of $\partial\Omega$ run clockwise. Here we have

$$\kappa_{\partial\Omega} = -\frac{1}{|\nabla\psi|}([D^2\psi]T_{\partial\Omega}, T_{\partial\Omega})_{\mathbb{R}^2},$$

where $T_{\partial\Omega}$ is the unit tangent vector of $\partial\Omega$ and $\kappa_{\partial\Omega}$ is computed in the direction of the unit normal vector $N_{\partial\Omega}$ of $\partial\Omega$, which is obtained by rotating $T_{\partial\Omega}$ with $\pi/2$. Note that $h_\pm = \kappa_{\partial\Omega}(x_*^\pm)$, and denote $T_{\partial\Omega}^\pm := T_{\partial\Omega}(x_*^\pm)$ and $N_{\partial\Omega}^\pm := N_{\partial\Omega}(x_*^\pm)$. At $\sigma = \pm l$, we observe $T_*(\pm l) = \pm N_{\partial\Omega}^\pm$ and $N_*(\pm l) = \mp T_{\partial\Omega}^\pm$. This implies that for $\sigma = l$

$$\begin{aligned}
\partial B_1(0)\rho &= -(N_{\partial\Omega}^+, \nabla\psi(x_*^+))_{\mathbb{R}^2}\rho_\sigma + (-T_{\partial\Omega}^+, [D^2\psi(x_*^+)](-T_{\partial\Omega}^+))_{\mathbb{R}^2}\rho \\
&= -|\nabla\psi(x_*^+)|\Big\{\rho_\sigma + \Big(-\frac{1}{|\nabla\psi(x_*^+)|}(T_{\partial\Omega}^+, [D^2\psi(x_*^+)]T_{\partial\Omega}^+)_{\mathbb{R}^2}\Big)\rho\Big\} \\
&= -|\nabla\psi(x_*^+)|(\rho_\sigma + h_+\rho),
\end{aligned}$$

and that for $\sigma = -l$

$$\begin{aligned}
\partial B_1(0)\rho &= -(-N_{\partial\Omega}^-, \nabla\psi(x_*^-))_{\mathbb{R}^2}\rho_\sigma + (T_{\partial\Omega}^-, [D^2\psi(x_*^-)]T_{\partial\Omega}^-)_{\mathbb{R}^2}\rho \\
&= |\nabla\psi(x_*^-)|\Big\{\rho_\sigma - \Big(-\frac{1}{|\nabla\psi(x_*^-)|}(T_{\partial\Omega}^-, [D^2\psi(x_*^-)]T_{\partial\Omega}^-)_{\mathbb{R}^2}\Big)\rho\Big\} \\
&= |\nabla\psi(x_*^-)|(\rho_\sigma - h_-\rho).
\end{aligned}$$

Consequently, we have

$$\mp\frac{1}{|\nabla\psi(x_*^\pm)|}\partial B_1(0)\rho = (\partial_\sigma \pm h_\pm)\rho \quad \text{at} \ \ \sigma = \pm l.$$

Let us also derive $\partial B_2(0)$. From (3.4) we have

$$\partial B_2(0)\rho = \partial_\sigma[\partial\kappa(0)\rho] = \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho \quad \text{at} \ \ \sigma = \pm l.$$

This completes the proof. $\square$

By the Lemmas 3.2 and 3.3 we have derived the linearization of (2.7) around $\rho \equiv 0$.

Theorem 3.4 *The linearization of (2.7) around $\rho \equiv 0$ is as follows:*

$$\begin{cases}
\rho_t = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho & \textit{for} \quad \sigma \in (-l, l), \ t > 0, \\
(\partial_\sigma \pm h_\pm)\rho = 0 & \textit{at} \quad \sigma = \pm l, \\
\partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0 & \textit{at} \quad \sigma = \pm l.
\end{cases} \tag{3.5}$$

4 Gradient flow structure

The surface diffusion flow can be interpreted as the H^{-1}-gradient flow of the area functional (see [20]). In this section we demonstrate that the linearization (3.5) derived in Section 3 can also be interpreted as a gradient flow. This observation will be important for our stability analysis.

In what follows we need the duality pairing $\langle\cdot,\cdot\rangle$ between $(H^1(-l,l))'$ and $(H^1(-l,l))$; and the following weak formulation. We denote by $\|\cdot\|_s$ the norm on $H^s(-l,l)$ where $H^0(-l,l) = L^2(-l,l)$.

Definition 4.1 *We say that* $u_v \in H^1(-l,l)$ *for a given* $v \in (H^1(-l,l))'$ *with* $\langle v, 1\rangle = 0$ *is a weak solution of*

$$\begin{cases} -\partial_\sigma^2 u_v = v & \text{for} \quad \sigma \in (-l,l)\,, \\ \partial_\sigma u_v = 0 & \text{at} \quad \sigma = \pm l \end{cases} \tag{4.1}$$

if u_v *satisfies*

$$\langle v, \xi\rangle = \int_{-l}^{l} \partial_\sigma u_v \partial_\sigma \xi$$

for all $\xi \in H^1(-l,l)$.

Definition 4.2 *For a given* $v \in (H^1(-l,l))'$ *with* $\langle v, 1\rangle = 0$, *we say that* $\rho \in H^3(-l,l)$ *with* $\int_{-l}^{l} \rho = 0$ *is a weak solution of the boundary value problem*

$$\begin{cases} v = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho & \text{for} \quad \sigma \in (-l,l)\,, \\ (\partial_\sigma \pm h_\pm)\rho = 0 & \text{at} \quad \sigma = \pm l\,, \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0 & \text{at} \quad \sigma = \pm l \end{cases} \tag{4.2}$$

if ρ *satisfies*

$$\langle v, \xi\rangle = \int_{-l}^{l} \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho\, \partial_\sigma \xi, \quad \text{and} \quad (\partial_\sigma \pm h_\pm)\rho = 0 \quad \text{at} \quad \sigma = \pm l$$

for all $\xi \in H^1(-l,l)$.

In the case that $v \in L^2(-l,l)$ we obtain that $v = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho$ is fulfilled almost everywhere in $(-l,l)$ and $\partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0$ is fulfilled for $\sigma = \pm l$.

In addition we also need the symmetric bilinear form on $H^1(-l,l)$

$$I(\rho_1, \rho_2) := \int_{-l}^{l} \{\partial_\sigma \rho_1 \partial_\sigma \rho_2 - \kappa_*^2 \rho_1 \rho_2\} d\sigma + h_+ \rho_1(l)\rho_2(l) + h_- \rho_1(-l)\rho_2(-l)$$

and the inner product

$$(\rho_1, \rho_2)_{-1} := \int_{-l}^{l} \partial_\sigma u_{\rho_1} \partial_\sigma u_{\rho_2}$$

where $u_{\rho_i} \in H^1(-l,l)$ for a given $\rho_i \in (H^1(-l,l))'$ with $\langle \rho_i, 1\rangle = 0$ is defined as the weak solution of (4.1). The bilinear form I is defined on $H^1(-l,l)$ and the inner product $(\cdot\,,\cdot)_{-1}$ is defined for all pairs of elements in $(H^1(-l,l))'$ with $\langle \rho_i, 1\rangle = 0$. We remark that by Definition 4.1

$$(\rho_1, \rho_2)_{-1} = \langle \rho_1, u_{\rho_2}\rangle \tag{4.3}$$

holds for $\rho_i \in (H^1(-l,l))'$ with $\langle \rho_i, 1\rangle = 0$.

Now we are going to show that the linearized problem (3.5) is the gradient flow of $E(\rho) := I(\rho,\rho)/2$ with respect to the H^{-1}-inner product $(\cdot,\cdot)_{-1}$. Let us review the concept of gradient flows. For a given functional E on a linear space X and an inner product $(\cdot,\cdot)_X$ on X we say that a time dependent function ρ with values in X is a solution of the gradient flow equation to E and $(\cdot,\cdot)_X$ if and only if

$$(\rho_t(t),\xi)_X = -\partial E(\rho(t))(\xi)$$

holds for all $\xi \in X$ and all t. Here $\partial E(\rho(t))(\xi)$ denotes the derivative of E at the point $\rho(t)$ in the direction ξ. The fact that the linearized problem (3.5) is the gradient flow of $I(\rho,\rho)/2$ with respect to the $(\cdot,\cdot)_{-1}$ inner product follows from the following lemma. This is true since the derivative of $E(\rho) = I(\rho,\rho)/2$ in a direction ξ is given by $I(\rho,\xi)$.

Lemma 4.3 *Let $v \in (H^1(-l,l))'$ with $\langle v,1\rangle = 0$ be given. Then a function $\rho \in H^3(-l,l)$ with $\int_{-l}^{l}\rho = 0$ is a weak solution of (4.2) if and only if*

$$(v,\xi)_{-1} = -I(\rho,\xi)$$

holds for all $\xi \in H^1(-l,l)$ with $\int_{-l}^{l}\xi = 0$.

Proof. Let $\rho \in H^3(-l,l)$ be a weak solution of (4.2). By (4.3) and Definition 4.2, we have

$$(v,\xi)_{-1} = \langle v,u_\xi\rangle = \int_{-l}^{l} \partial_\sigma(\partial_\sigma^2+\kappa_*^2)\rho\,\partial_\sigma u_\xi.$$

for all $\xi \in H^1(-l,l)$ with $\int_{-l}^{l}\xi = 0$. Note that $u_\xi \in H^1(-l,l)$ is a weak solution of (4.1) with $\xi \in H^1(-l,l)$. Then, by virtue of $(\partial_\sigma^2+\kappa_*^2)\rho \in H^1(-l,l)$, we see

$$\int_{-l}^{l} \partial_\sigma(\partial_\sigma^2+\kappa_*^2)\rho\,\partial_\sigma u_\xi = \int_{-l}^{l}(\partial_\sigma^2+\kappa_*^2)\rho\,\xi.$$

This implies that

$$\begin{aligned}(v,\xi)_{-1} &= \int_{-l}^{l}(\partial_\sigma^2+\kappa_*^2)\rho\,\xi \\ &= -\int_{-l}^{l}(\partial_\sigma\rho\,\partial_\sigma\xi - \kappa_*^2\rho\,\xi) + [\partial_\sigma\rho\,\xi]_{\sigma=-l}^{\sigma=l} \\ &= -I(\rho,\xi)\,.\end{aligned}$$

The last equality is shown by using $(\partial_\sigma \pm h_\pm)\rho = 0$ at $\sigma = \pm l$.

Conversely, assume that $\rho \in H^1(-l,l)$ with $\int_{-l}^{l}\rho = 0$ satisfies

$$(v,\xi)_{-1} = -I(\rho,\xi) \tag{4.4}$$

for all $\xi \in H^1(-l,l)$ with $\int_{-l}^{l}\xi = 0$. Choose $\xi = -\partial_\sigma^2\eta$ in (4.4) for a given function $\eta \in H^3(-l,l)$ with $\partial_\sigma\eta = 0$ at $\sigma = \pm l$. Then it holds

$$\begin{aligned}\langle v,\eta\rangle &= (v,\xi)_{-1} \\ &= -I(\rho,\xi) \\ &= -\int_{-l}^{l}(\partial_\sigma\rho\,\partial_\sigma\xi - \kappa_*^2\rho\,\xi) - \{h_+\rho(l)\xi(l) + h_-\rho(-l)\xi(-l)\} \\ &= -\int_{-l}^{l}(-\partial_\sigma\rho\,\partial_\sigma^3\eta + \kappa_*^2\rho\,\partial_\sigma^2\eta) + \{h_+\rho(l)\partial_\sigma^2\eta(l) + h_-\rho(-l)\partial_\sigma^2\eta(-l)\}\,.\end{aligned}$$

Since $v \in (H^1(-l,l))'$, we deduce from the above identity that $\rho \in H^3(-l,l)$. Integration by parts gives

$$\begin{aligned} \langle v, \eta \rangle &= \int_{-l}^{l} (-\partial_\sigma^2 \rho \, \partial_\sigma^2 \eta + \kappa_*^2 \partial_\sigma \rho \, \partial_\sigma \eta) + [(\partial_\sigma \rho \pm h_\pm \rho) \partial_\sigma^2 \eta]_{\sigma=-l}^{\sigma=l} \\ &= \int_{-l}^{l} \partial_\sigma (\partial_\sigma^2 + \kappa_*^2) \rho \, \partial_\sigma \eta + [(\partial_\sigma \rho \pm h_\pm \rho) \partial_\sigma^2 \eta]_{\sigma=-l}^{\sigma=l} , \end{aligned} \tag{4.5}$$

where $[(\partial_\sigma \rho \pm h_\pm \rho) \partial_\sigma^2 \eta]_{\sigma=-l}^{\sigma=l} = (\partial_\sigma \rho + h_+ \rho) \partial_\sigma^2 \eta|_{\sigma=l} - (\partial_\sigma \rho - h_- \rho) \partial_\sigma^2 \eta|_{\sigma=-l}$. Since $\partial_\sigma^2 \eta$ can be chosen arbitrarily at $\sigma = \pm l$ and v is a bounded linear functional on $H^1(-l,l)$, we can deduce that the first boundary condition $(\partial_\sigma \pm h_\pm)\rho = 0$ at $\sigma = \pm l$ holds. The remaining identity in (4.5) then is a weak formulation of $v = -\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho$ for $\sigma \in (-l,l)$ together with $\partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0$ at $\sigma = \pm l$ (see Definition 4.2). □

5 Self-adjointness of the linearized operator

It is the aim of this section to show that the linearized operator is self-adjoint and to study its spectrum. By choosing an appropriate domain of definition, the linearized operator of (3.5) is given by

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to H$$

with

$$\begin{cases} \mathcal{D}(\mathcal{A}) = \{\rho \in H^3(-l,l) \mid (\partial_\sigma \pm h_\pm)\rho = 0 \ \text{ at } \ \sigma = \pm l \ \text{ and } \ \int_{-l}^{l} \rho = 0\}, \\ H = \{\rho \in (H^1(-l,l))' \mid \langle \rho, 1 \rangle = 0\} \end{cases}$$

by

$$\langle \mathcal{A}\rho, \xi \rangle := \int_{-l}^{l} \partial_\sigma (\partial_\sigma^2 + \kappa_*^2) \rho \, \partial_\sigma \xi \, . \tag{5.1}$$

Then the boundary value problem (4.2) corresponds to the problem in finding a $\rho \in \mathcal{D}(\mathcal{A})$ with

$$\mathcal{A}\rho = v \, .$$

We also remark that this definition gives for all $\xi \in H^1(-l,l)$ with $\int_{-l}^{l} \xi = 0$

$$(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi).$$

For this operator $\mathcal{A}$, we have the following lemma.

Lemma 5.1 *The operator $\mathcal{A}$ is symmetric with respect to the inner product $(\cdot,\cdot)_{-1}$.*

Proof. For all $\rho, \xi \in \mathcal{D}(\mathcal{A})$ we have

$$(\mathcal{A}\rho, \xi)_{-1} = -I(\rho, \xi) = -I(\xi, \rho) = (\mathcal{A}\xi, \rho)_{-1} = (\rho, \mathcal{A}\xi)_{-1},$$

so that $\mathcal{A}$ is symmetric. □

We need to analyze the spectrum of $\mathcal{A}$ in order to decide on the stability behaviour of the linearized problem (3.5). Using classical principles of the variational calculus, we can describe the spectrum of $\mathcal{A}$ with the help of the inner-product $(\cdot\,,\cdot)_{-1}$ and I. In fact, if ρ is an eigenfunction to the eigenvalue λ, it holds

$$\lambda(\rho,\xi)_{-1} = (\mathcal{A}\rho,\xi)_{-1} = -I(\rho,\xi)\,.$$

We remark that eigenvalues $\lambda \neq 0$ always correspond to eigenfunctions that have the mean value zero. This follows by integrating the identity

$$-\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho = \lambda\rho$$

and using the boundary conditions. In what follows we will only study eigenvalues which have eigenfunctions with mean value zero. This is a natural request for the linearized problem. It follows when we take the mass constraint in the nonlinear problem into account. This makes sense because the surface diffusion flow is mass preserving (cp. [11]).

Therefore we define $V = \{\rho \in H^1(-l,l) \mid \int_{-l}^{l} \rho = 0\}$. The following two lemmas will be needed to show the boundness of the eigenvalue from above.

Lemma 5.2 *For all $\delta > 0$ there exists a C_δ such that for all functions $\rho \in V$ the inequality*

$$\rho(l)^2 \leq \delta\|\partial_\sigma\rho\|_0^2 + C_\delta\|\rho\|_{-1}^2$$

holds. The same inequality holds for $\rho(-l)^2$ instead of $\rho(l)^2$.

Proof. We prove the assertion by contradiction. Assume that there exists a $\delta > 0$ such that for all $n \in \mathbb{N}$, $\rho_n \in V$ with $\rho_n(l)^2 = 1$ satisfy

$$1 = \rho_n(l)^2 > \delta\|\partial_\sigma\rho_n\|_0^2 + n\|\rho_n\|_{-1}^2\,.$$

This implies

$$\|\rho_n\|_{-1}^2 < \frac{1}{n} \longrightarrow 0 \qquad \text{as } n \to \infty$$

and

$$\|\partial_\sigma\rho_n\|_0^2 < \frac{1}{\delta}\,.$$

Since $\int_{-l}^{l} \rho_n = 0$, we conclude from Poincaré's inequality that ρ_n is bounded uniformly in $H^1(-l,l)$. This gives

$$\rho_n \longrightarrow 0 \quad \text{weakly in } H^1(-l,l)$$

and therefore (since the embedding $H^1(-l,l)$ into $C^0([-l,l])$ is compact)

$$\rho_n(l) \longrightarrow 0\,.$$

This is a contradiction and therefore the lemma is shown. $\square$

Lemma 5.3 *There exist positive constants c_1 and c_2 such that*

$$\|\rho\|_1^2 \leq c_1\|\rho\|_{-1}^2 + c_2 I(\rho,\rho) \ \textit{ for all } u \in V\,.$$

Proof. Since the embedding $H^1(-l,l) \hookrightarrow L^2(-l,l)$ is compact, we obtain that for all $\delta > 0$ there exists a $\hat{C}_\delta > 0$ such that

$$\|\rho\|_0^2 \le \delta \|\partial_\sigma \rho\|_0^2 + \hat{C}_\delta \|\rho\|_{-1}^2 \,.$$

This can, for example, be shown in exactly the same manner as in the proof of the preceeding lemma. Therefore we obtain with the help of Lemma 5.2 and the above inequality

$$\begin{aligned}
I(u,u) &= \int_{-l}^{l} |\partial_\sigma \rho|^2 - \kappa_*^2 \int_{-l}^{l} \rho^2 + h_+ \rho(l)^2 + h_- \rho(-l)^2 \\
&\ge \int_{-l}^{l} |\partial_\sigma \rho|^2 - \kappa_*^2 \int_{-l}^{l} \rho^2 - |h_+|\rho(l)^2 - |h_-|\rho(-l)^2 \\
&\ge (1-\varepsilon) \int_{-l}^{l} |\partial_\sigma \rho|^2 - C_\varepsilon \|\rho\|_{-1},
\end{aligned}$$

which holds for suitable ε and C_ε. The above inequality proves the lemma. $\square$

Corollary 5.4 *The largest eigenvalue of $\mathcal{A}$ is bounded from above by c_1/c_2.*

Proof. Let λ be an eigenvalue of $\mathcal{A}$ then there exists a $\rho \neq 0$ such that

$$\lambda(\rho,\rho)_{-1} = -I(\rho,\rho) \,.$$

Assume $\lambda > c_1/c_2$. This implies

$$0 = I(\rho,\rho) + \lambda(\rho,\rho)_{-1} > I(\rho,\rho) + c_1/c_2(\rho,\rho)_{-1} \ge 1/c_2 \|\rho\|_1^2 > 0,$$

which is a contradiction. $\square$

By virtue of Lemma 5.1 and Corollary 5.4, we have following theorem.

Theorem 5.5 *(i) The operator $\mathcal{A}$ is self-adjoint with respect to the inner product $(\cdot,\cdot)_{-1}$.*
(ii) The spectrum of $\mathcal{A}$ contains a countable system of real eigenvalues.
(iii) The initial value problem (3.5) is solvable for initial data in H.
(iv) The zero solution is an asymptotically stable solution of (3.5) if and only if the largest eigenvalue of $\mathcal{A}$ is negative.

Proof. First we show that the resolvent $(\mathcal{A} - \omega)^{-1}$ exists for some $\omega \in \mathbb{R}$. Choosing $\omega > c_1/c_2$ and using Corollary 5.4, we know that $\mathcal{A} - \omega$ is injective. It remains to show that $\mathcal{A} - \omega$ is surjective. For a given $f \in H$ we need to prove that there exists a weak solution ρ of the boundary value problem

$$\begin{cases} -\partial_\sigma^2\{-(\partial_\sigma^2 + \kappa_*^2)\}\rho + \omega\rho = f & \text{for} \quad \sigma \in (-l,l), \\ (\partial_\sigma \pm h_\pm)\rho = 0 & \text{at} \quad \sigma = \pm l, \\ \partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0 & \text{at} \quad \sigma = \pm l. \end{cases} \tag{5.2}$$

To obtain a solution to (5.2) we use the fact that the minimizing problem

$$F(\rho) := \int_{-l}^{l} \Big(\frac{1}{2}|\partial_\sigma \rho|^2 - \frac{1}{2}\kappa_*^2 \rho^2\Big) + \frac{1}{2} h_+ \rho^2(l) + \frac{1}{2} h_- \rho^2(-l) + \frac{\omega}{2}\|\rho\|_{-1}^2 - \int_{-l}^{l} u_f \rho \to \min$$

under all $\rho \in H^1(-l,l)$ with $\int_{-l}^{l} \rho = 0$ admits as solutions $\tilde{\rho}$. This holds since F is coercive which follows from Lemmas 5.2 and 5.3. Taking the first variation of F, we observe that

$$-(\partial_\sigma^2 + \kappa_*^2)\tilde{\rho} + \omega u_{\tilde{\rho}} = u_f \quad \text{for} \quad \sigma \in (-l,l), \tag{5.3}$$

with $(\partial_\sigma \pm h_\pm)\tilde{\rho} = 0$ at $\sigma = \pm l$ holds in a weak sense. Since $u_{\tilde{\rho}}$, $u_f \in H^1(-l,l)$, we have $\tilde{\rho} \in H^3(-l,l)$. Furthermore, it follows from $\partial_\sigma u_{\tilde{\rho}} = \partial_\sigma u_f = 0$ at $\sigma = \pm l$ that $\partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\bar{\rho} = 0$ at $\sigma = \pm l$. Taking second derivatives of (5.3) in a weak sense, we derive that $\tilde{\rho}$ solves (5.2). This shows that $\mathcal{A} - \omega$ is surjective and hence $(\mathcal{A} - \omega)^{-1}$ exists.

Let us prove (i). We already know from Lemma 5.1 that $\mathcal{A}$ is symmetric. Since the self-adjointness of $\mathcal{A}$ follows from the self-adjointness of $\mathcal{A} - \omega$ for some $\omega \in \mathbb{R}$, we show the self-adjointness of $\mathcal{A} - \omega$. Suppose that there are v, $w \in H$ such that

$$((\mathcal{A} - \omega)\rho, v)_{-1} = (\rho, w)_{-1} \tag{5.4}$$

for all $\rho \in D(\mathcal{A} - \omega)$. By the above argument $\mathcal{A} - \omega$ is invertible if ω is large enough. Then there exists a $z \in D(\mathcal{A} - \omega)$ such that

$$(\mathcal{A} - \omega)z = w \tag{5.5}$$

for sufficiently large ω. By (5.4), (5.5) and Lemma 5.1, we have

$$((\mathcal{A} - \omega)\rho, v)_{-1} = (\rho, (\mathcal{A} - \omega)z)_{-1} = ((\mathcal{A} - \omega)\rho, z)_{-1}.$$

Since $\mathcal{A} - \omega$ is surjective, we obtain $v = z$. This implies that $v \in D(\mathcal{A} - \omega)$ and

$$(\mathcal{A} - \omega)v = w,$$

so that $\mathcal{A} - \omega$ is self-adjoint.

Since $(\mathcal{A} - \omega)^{-1}$ exists and is compact, (ii) follows from Theorem 6.29, Chapter III in [14] and the fact that $\mathcal{A}$ is self-adjoint.

Using the fact that $\mathcal{A}$ is a self-adjoint operator on H, the theory of semigroups is applicable to show (iii) (see e.g. the functional calculus in 5.8-5.10 of [21]). Semigroup theory also gives (iv). $\Box$

To decide on the linearized stability, it will be important to know that the eigenvalues of $\mathcal{A}$ depend continuously on h_+, h_- and κ_*^2, and are also monotone in each of these parameters. The following lemma assures these properties.

Lemma 5.6 *Let*

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$$

be the eigenvalues of $\mathcal{A}$ (taking the multiplicity into account).
i) Then it holds for all $n \in \mathbf{N}$

$$-\lambda_n = \inf_{W \in \Sigma_n} \sup_{u \in W \setminus \{0\}} \frac{I(u,u)}{(u,u)_{-1}},$$

$$-\lambda_n = \sup_{W \in \Sigma_{n-1}} \inf_{u \in W^\perp \setminus \{0\}} \frac{I(u,u)}{(u,u)_{-1}}.$$

Here Σ_n is the collection of n-dimensional subspaces of V and $W^\perp$ is the orthogonal complement with respect to the $(\cdot\,,\cdot)_{-1}$-scalar product.
ii) The eigenvalues λ_n depend continuously on h_+, h_- and κ_*^2; and are monotone decreasing in each of the parameters h_+, h_- and $(-\kappa_*^2)$.

Proof. The lemma follows with the help of Courant's maximum-minimum principle together with the fact that I depends in a monotone and continuous way on h_+, h_- and $(-\kappa_*^2)$. The proof follows the lines of Courant and Hilbert [4], VI.2. $\square$

6 Stability analysis

To obtain a linearized stability result for stationary solutions of (2.7) it is enough to show that $I(\rho, \rho)$ is positive for all $\rho \in V \setminus \{0\}$. Then $\lambda_1 < 0$ which implies stability. This is true since λ_1 allows the characterization

$$-\lambda_1 = \inf_{\rho \in V \setminus \{0\}} \frac{I(\rho, \rho)}{(\rho, \rho)_{-1}}$$

and the infimum is in fact a minimum and therefore it is enough to show the positivity of I pointwise.

In the following arguments we only consider the case $\kappa_* > 0$ (or $\kappa_* = 0$). We remark that the same result is derived for $\kappa_* < 0$. Also note that the stationary solution is a part of a circle with radius κ_*. The length of the stationary solution is $2l$ and therefore the restriction

$$2l < \frac{2\pi}{\kappa_*},$$

which gives $\kappa_* l < \pi$, has to hold.

Now the following lemma shows that for given κ_* the stationary solution is always stable provided h_+, h_- are large enough.

Lemma 6.1 *Let $\kappa_* l < \pi$. Then there exists a constant $K > 0$, such that*

$$I(\rho, \rho) > 0 \quad \text{for all} \quad \rho \in V \setminus \{0\}$$

provided that $h_+, h_- > K$.

Proof. Using the transformation

$$u(s) = \rho\Big(\frac{2l}{\pi} s - l\Big)$$

and the fact that $\kappa_* l < \pi$ it is enough to show that there exists a constant $\overline{c} > 0$ such that

$$\|u'\|_0^2 - 4\|u\|_0^2 + \overline{c}(u(0)^2 + u(\pi)^2) \geq 0$$

for all $u \in H^1(0, \pi)$ with $\int_0^\pi u = 0$. Assume such a constant $\overline{c}$ does not exist. Then there exists a sequence u_n (without loss of generality we assume $\|u_n\|_0^2 = 1$) such that

$$\|u_n'\|_0^2 - 4\|u_n\|_0^2 + n(u_n^2(0) + u_n^2(\pi)) < 0\,.$$

This implies

$$\|u_n'\|_0^2 \leq 4$$

and we deduce the existence of a subsequence (which we also label by $\{u_n\}_{n\in\mathbf{N}}$) such that

$$\begin{array}{llll} u_n' & \to u' & \text{weakly} & \text{in } L^2(0,\pi)\,, \\ u_n & \to u & \text{strongly} & \text{in } L^2(0,\pi)\,, \\ u_n & \to u & \text{strongly} & \text{in } C^0([0,\pi])\,. \end{array}$$

Then

$$u_n^2(0) + u_n^2(\pi) \le \frac{4}{n}$$

implies

$$u(0) = u(\pi) = 0\,.$$

The lower semicontinuity of the L^2-norm under weak convergence implies

$$\|u'\|_0^2 < 4\|u\|_0^2\,,$$

which contradicts the facts that $u \in \overset{\circ}{H^1}(0,\pi)$ and $\int_0^\pi u = 0$ (see the following lemma). This proves the lemma. $\square$

Lemma 6.2 *For all $u \in \overset{\circ}{H^1}(0,\pi)$ with $\int_0^\pi u(s)ds = 0$ it holds*

$$\|u\|_0^2 \le \frac{1}{4}\|u'\|_2^2\,.$$

Proof. Each $u \in \overset{\circ}{H^1}(0,\pi)$ has a representation

$$u(s) = \sum_{k=1}^{\infty} a_k \sin ks\,.$$

Then we have

$$\|u\|_0^2 = \frac{\pi}{2}\sum_{k=1}^{\infty} a_k^2\,, \quad \|u'\|_0^2 = \frac{\pi}{2}\sum_{k=1}^{\infty} k^2 a_k^2.$$

In addition, the assumption $\int_0^\pi u(s)ds = 0$ implies

$$\sum_{\substack{k=1 \\ k\,\text{odd}}}^{\infty} \frac{2}{k} a_k = 0\,. \tag{6.1}$$

Now we readily see

$$\sum_{\substack{k=1 \\ k\,\text{even}}}^{\infty} a_k^2 \le \frac{1}{4}\sum_{\substack{k=1 \\ k\,\text{even}}}^{\infty} k^2 a_k^2\,.$$

It remains to estimate the sum over all odd k, which would follow from

$$3a_1^2 \le \sum_{\substack{k=3 \\ k\,\text{odd}}}^{\infty} (k^2-4)a_k^2\,. \tag{6.2}$$

The mean value constraint (6.1) implies

$$a_1 = -\sum_{\substack{k=3\\ k\,\mathrm{odd}}}^{\infty} \frac{1}{k} a_k,$$

which gives

$$3a_1^2 \le 3\Big(\sum_{\substack{k=3\\ k\,\mathrm{odd}}}^{\infty} \frac{1}{k} a_k\Big)^2 \le 3\Big(\sum_{\substack{k=3\\ k\,\mathrm{odd}}}^{\infty} a_k^2\Big)\Big(\sum_{\substack{k=3\\ k\,\mathrm{odd}}}^{\infty} \frac{1}{k^2}\Big) = 3\sum_{\substack{k=3\\ k\,\mathrm{odd}}}^{\infty} a_k^2 \cdot \Big(\frac{\pi^2}{8} - 1\Big).$$

Since $3(\pi^2/8 - 1) < k^2 - 4$ $(k = 3, 5, \cdots)$, the inequality (6.2) is derived. Thus the lemma follows. $\square$

The strategy now is as follows. We know that for large h_+ and h_- we have stability. In addition we know that the eigenvalues depend in a monotone and continuous way on h_+ and h_-. If we start with a stable situation $(h_+, h_- \gg 1)$ and decrease h_+ and respectively h_-, a loss of stability can therefore only occur in the case that the largest eigenvalue λ_1 passes through zero. For that reason we analyze for which values of h_+, h_-, and κ_* a zero eigenvalue is possible. To obtain a complete picture about the dimension of the unstable manifold we also determine the multiplicity of a possible zero eigenvalue.

Lemma 6.3 *i) Assume $\kappa_* \neq 0$ and $\kappa_* l < \pi$. Then the operator $\mathcal{A}$ has a zero eigenvalue if and only if*

$$\frac{a}{c} + \frac{b}{c}(h_+ + h_-) + h_+ h_- = 0 \tag{6.3}$$

where

$$\begin{aligned} a &= -2\kappa_*^2 l \sin(\kappa_* l)\cos(\kappa_* l)\,, \\ b &= \kappa_* l(\cos^2(\kappa_* l) - \sin^2(\kappa_* l)) - \sin(\kappa_* l)\cos(k_* l)\,, \\ c &= 2\Big\{-\frac{1}{\kappa_*}\sin^2(\kappa_* l) + l\sin(\kappa_* l)\cos(\kappa_* l)\Big\}\,. \end{aligned}$$

Furthermore, it holds the inequality

$$\frac{b^2}{c^2} - \frac{a}{c} > 0\,. \tag{6.4}$$

ii) If $\kappa_* = 0$ then the operator $\mathcal{A}$ has a zero eigenvalue if and only if

$$\frac{3}{l^2} + \frac{2}{l}(h_+ + h_-) + h_+ h_- = 0\,. \tag{6.5}$$

iii) If we interpret a, b, and c as functions of κ_*, we obtain

$$\frac{a}{c} \to \frac{3}{l^2} \quad \text{and} \quad \frac{b}{c} \to \frac{2}{l} \quad \text{as} \quad \kappa_* \to 0\,.$$

iv) The multiplicity of a possible zero eigenvalue is equal to one for all h_+, h_-, and κ_*.

In what follows we set

$$\mathcal{D}(h_+, h_-, \kappa_*) = \frac{a}{c} + \frac{b}{c}(h_+ + h_-) + h_+ h_-$$

for all h_+, h_-, and κ_*. The extension to $\kappa_* = 0$ is well defined by the preceeding lemma.

Remark 6.4 *a) The equations (6.3) and (6.5) define hyperbolas in the (h_-, h_+)-plane (see Figures 1-5). The hyperbolas are symmetric with respect to the $h_- = h_+$ line and the inequality (6.4) implies that the line defined by $h_+ = h_-$ always has two intersection points with the hyperbolas.*
b) From iii) in the preceeding lemma we can conclude that the hyperbolas obtained for the case $\kappa_* > 0$ tend to the one for $\kappa_* = 0$.

Proof of Lemma 6.3. i) Assume that $-\partial_\sigma^2(\partial_\sigma^2 + \kappa_*^2)\rho = 0$. Then the function ρ can be denoted by

$$\rho(\sigma) = \alpha_1 \sigma + \alpha_0 + \alpha_c \cos(\kappa_* \sigma) + \alpha_s \sin(\kappa_* \sigma)$$

for constants $(\alpha_1, \alpha_0, \alpha_c, \alpha_s)$. By the boundary conditions $\partial_\sigma(\partial_\sigma^2 + \kappa_*^2)\rho = 0$ at $\sigma = \pm l$, we have

$$\pm\alpha_c \kappa_*^3 \sin(\kappa_* l) - \alpha_s \kappa_*^3 \cos(\kappa_* l) + \kappa_*^2\{\alpha_1 \mp \alpha_c \kappa_* \sin(\kappa_* l) + \alpha_s \kappa_* \cos(\kappa_* l)\} = 0.$$

This implies that $\kappa_*^2 \alpha_1 = 0$, so that $\alpha_1 = 0$. Using the boundary conditions $(\partial_\sigma \pm h_\pm)\rho = 0$ at $\sigma = \pm l$, we derive

$$\begin{cases} h_+\alpha_0 + (-\kappa_* \sin(\kappa_* l) + h_+ \cos(\kappa_* l))\alpha_c + (\kappa_* \cos(\kappa_* l) + h_+ \sin(\kappa_* l))\alpha_s = 0, \\ -h_-\alpha_0 + (\kappa_* \sin(\kappa_* l) - h_- \cos(\kappa_* l))\alpha_c + (\kappa_* \cos(\kappa_* l) + h_- \sin(\kappa_* l))\alpha_s = 0. \end{cases}$$

Moreover, it follows from $\int_{-l}^{l} \rho = 0$ that

$$2l\alpha_0 + \Big\{\frac{2}{\kappa_*} \sin(\kappa_* l)\Big\}\alpha_c = 0.$$

Let us define the 3×3-matrix $M(h_+, h_-, \kappa_*)$ as

$$M(h_+, h_-, \kappa_*) := \begin{pmatrix} h_+ & -\kappa_* \sin(\kappa_* l) + h_+ \cos(\kappa_* l) & \kappa_* \cos(\kappa_* l) + h_+ \sin(\kappa_* l) \\ -h_- & \kappa_* \sin(\kappa_* l) - h_- \cos(\kappa_* l) & \kappa_* \cos(\kappa_* l) + h_- \sin(\kappa_* l) \\ l & \{\sin(\kappa_* l)\}/\kappa_* & 0 \end{pmatrix}.$$

Then, the operator $\mathcal{A}$ has a zero eigenvalue if and only if the equation

$$M(h_+, h_-, \kappa_*)\, {}^t(\alpha_0, \alpha_c, \alpha_s) = {}^t(0,0,0) \tag{6.6}$$

has non-zero solutions ${}^t(\alpha_0, \alpha_c, \alpha_s)$. Non-zero solutions of (6.6) are derived when

$$\det M(h_+, h_-, \kappa_*) = 0,$$

which implies (6.3). Furthermore, by the definition of a, b, c, we have

$$b^2 - ac = \{\kappa_* l - \sin(\kappa_* l)\cos(\kappa_* l)\}^2 = \frac{1}{4}\{2\kappa_* l - \sin(2\kappa_* l)\}^2 \geq 0.$$

It follows from $\kappa_* l \neq 0$ that $2\kappa_* l - \sin(2\kappa_* l) \neq 0$. This implies (6.4).

ii) Assume that $-\partial_\sigma^4 \rho = 0$. Then the function ρ can be denoted by

$$\rho(\sigma) = \alpha_3 \sigma^3 + \alpha_2 \sigma^2 + \alpha_1 \sigma + \alpha_0.$$

for constants $(\alpha_3, \alpha_2, \alpha_1, \alpha_0)$. By the boundary conditions $\partial_\sigma^3 \rho = 0$ at $\sigma = \pm l$, we have $\alpha_3 = 0$. In addition, the conditions $(\partial_\sigma \pm h_\pm)\rho = 0$ at $\sigma = \pm l$ and $\int_{-l}^{l} \rho = 0$ give the equation

$$M_0(h_+, h_-)\, {}^t(\alpha_2, \alpha_1, \alpha_0) = {}^t(0, 0, 0), \tag{6.7}$$

where the 3×3-matrix $M_0(h_+, h_-)$ is defined as

$$M_0(h_+, h_-) := \begin{pmatrix} 2l + h_+ l^2 & 1 + h_+ l & h_+ \\ -2l - h_- l^2 & 1 + h_- l & -h_- \\ l^2/3 & 0 & 1 \end{pmatrix}.$$

Applying the similar argument to the proof of i), the operator $\mathcal{A}$ with $\kappa_* = 0$ has a zero eigenvalue if and only if $\det M_0(h_+, h_-) = 0$, which implies (6.5).

iii) This follows readily from the expressions for a/c and b/c with the help of the L'Hospitals rule.

iv) In the case $\kappa_* = 0$, we needed to find non-zero solutions of (6.7) in order to derive a zero eigenvalue of $\mathcal{A}$. Each of the solutions to the linear systems (6.7) corresponds one eigenfunction to the eigenvalue zero. Assume the multiplicity of an eigenvalue zero is larger than one. This implies that the matrix $M_0(h_+, h_-)$ has rank 1 (less is not possible). This implies

$$1 + h_+ l = 1 + h_- l = 0\,.$$

Hence

$$h_+ = h_- = -\frac{1}{l}\,.$$

But then the first and third column are not linear dependent. This is a contradiction and shows the assertion for $\kappa_* = 0$. A similar argument works in the case $\kappa_* \neq 0$. $\square$

We denote by N_U and N_N the number of unstable and zero eigenvalues of $\mathcal{A}$ (counting the multiplicity). Then we obtain the following theorem.

Theorem 6.5 *Case A: If $\mathcal{D}(h_-, h_+, \kappa_*) > 0$ and if $h_- > -b/c$, then*

$$N_U = N_N = 0\,.$$

Case B: If $\mathcal{D}(h_-, h_+, \kappa_*) = 0$ and if $h_- > -b/c$, then

$$N_U = 0\,,\ N_N = 1\,.$$

Case C: If $\mathcal{D}(h_-, h_+, \kappa_*) < 0$, then

$$N_U = 1\,,\ N_N = 0\,.$$

Case D: If $\mathcal{D}(h_-, h_+, \kappa_*) = 0$ and if $h_- < -b/c$, then

$$N_U = 1\,,\ N_N = 1\,.$$

Case E: If $\mathcal{D}(h_-, h_+, \kappa_*) > 0$ and if $h_- < -b/c$, then

$$N_U = 2\,,\ N_N = 0\,.$$

Remark 6.6 *a) In the cases A,B,D and E the condition* $h_- > -b/c$ *(*$h_- < -b/c$ *respectively) can be replaced by* $h_+ > -b/c$ *(*$h_+ < -b/c$ *respectively).*

b) Theorem 6.5 says that we have stability above the upper arc of the hyperbola (see Figures 1-5). Underneath of it we have instability where the number of instable modes is one when we are above the lower arc of the hyperbola and two when we are underneath of it.

Proof of Theorem 6.5. The proof is a simple consequence of the Lemmas 5.6, 6.1 and 6.3. For large h_+ and h_- we have stability. If we decrease h_+ or h_-, the stability behaviour only changes on the curves defined by $\mathcal{D}(h_-, h_+, \kappa_*) = 0$. By virtue of iv) in Lemma 6.3, only one eigenvalue can pass through zero when crossing the curves $\mathcal{D}(h_-, h_+, \kappa_*) = 0$. The monotonicity of the eigenvalues with respect to h_+ and h_- implies that the number of unstable modes can only increase if we further decrease h_+ or h_-. This proves the theorem. $\square$

Let us discuss the signs of a, b, and c, which depends on $\kappa_* l$. It is easy to see

$$\begin{cases} a < 0 & \text{for} \quad \kappa_* l < \pi/2, \\ a = 0 & \text{for} \quad \kappa_* l = \pi/2, \\ a > 0 & \text{for} \quad \kappa_* l > \pi/2. \end{cases}$$

To derive the signs of b, we rewrite b as

$$\begin{aligned} b &= \frac{1}{2}\{2\kappa_* l \cos(2\kappa_* l) - \sin(2\kappa_* l)\} \\ &= \frac{1}{2}\cos(2\kappa_* l)\{2\kappa_* l - \tan(2\kappa_* l)\} \quad \text{if} \quad 2\kappa_* l \neq \pi/2, 3\pi/2. \end{aligned}$$

It follows from the relations between $2\kappa_* l$ and $\tan(2\kappa_* l)$ in $0 < 2\kappa_* l < 2\pi$ that

$$\begin{cases} b < 0 & \text{for} \quad \kappa_* l < \theta_0, \\ b = 0 & \text{for} \quad \kappa_* l = \theta_0, \\ b > 0 & \text{for} \quad \kappa_* l > \theta_0 \end{cases}$$

for some $\theta_0 \in (\pi/2, \pi)$. Finally, we investigate the sign of c. If $\kappa_* l \geq \pi/2$, we can easily derive $c < 0$. If $\kappa_* l < \pi/2$, we rewrite c as

$$c = \frac{2}{\kappa_*}\sin(\kappa_* l)\cos(\kappa_* l)\{\kappa_* l - \tan(\kappa_* l)\}.$$

Then $\kappa_* l < \pi/2$ implies that $\sin(\kappa_* l) > 0$, $\cos(\kappa_* l) > 0$, and $\kappa_* l - \tan(\kappa_* l) < 0$, so that $c < 0$. Thus we see $c < 0$ in any cases. Consequently Figure 1-5 follows.

7 Examples

Finally we want to discuss how the linearized stability of equilibria depends on the parameters l, κ_*, h_+ and h_-. In the following the expressions "stable" and "unstable" are to be understood in the linearized sense. If κ_* is zero and h_+ and h_- are negative then the stability depends crucially on the length of Γ_*. For fixed h_+ and h_- equilibria with a

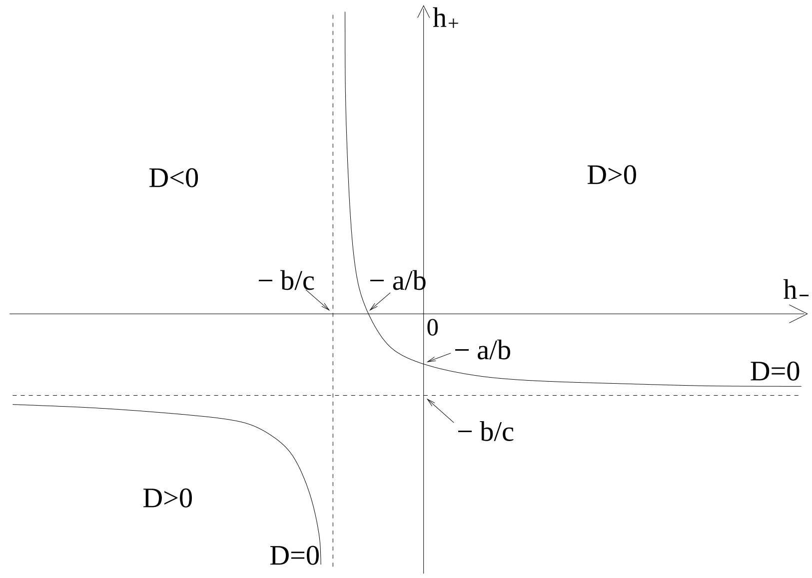

Figure 1: $\kappa_* l < \pi/2, a < 0, b < 0, c < 0$

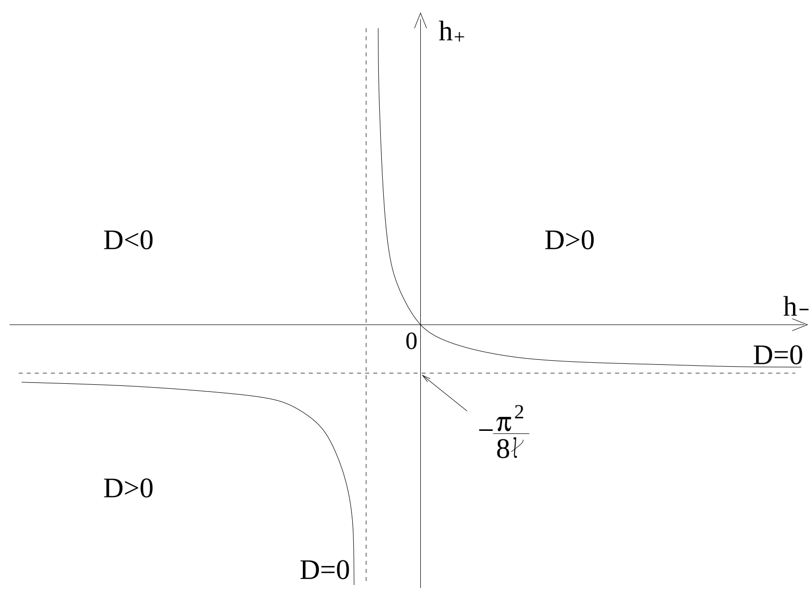

Figure 2: $\kappa_* l = \pi/2, a = 0, b = -\kappa_* l, c = -2/\kappa_*$

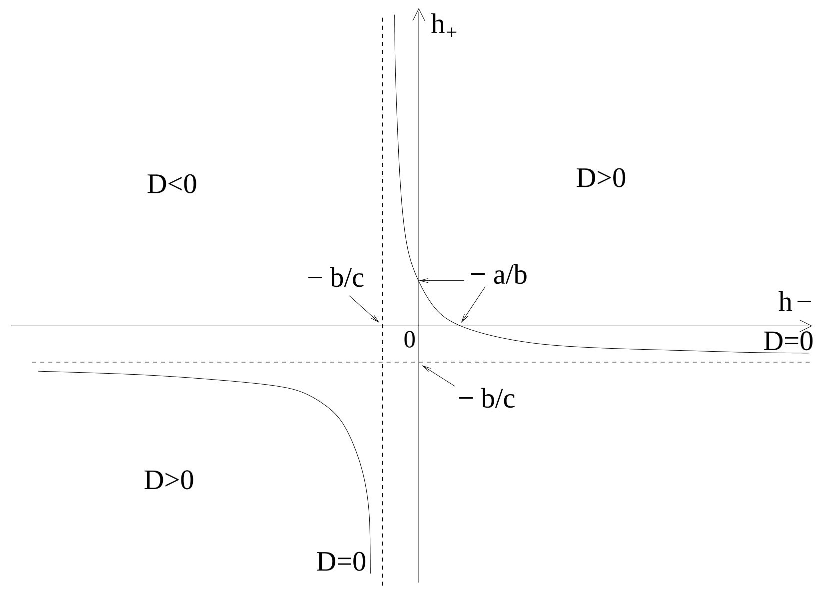

Figure 3: $\kappa_* l > \pi/2, a > 0, b < 0, c < 0$

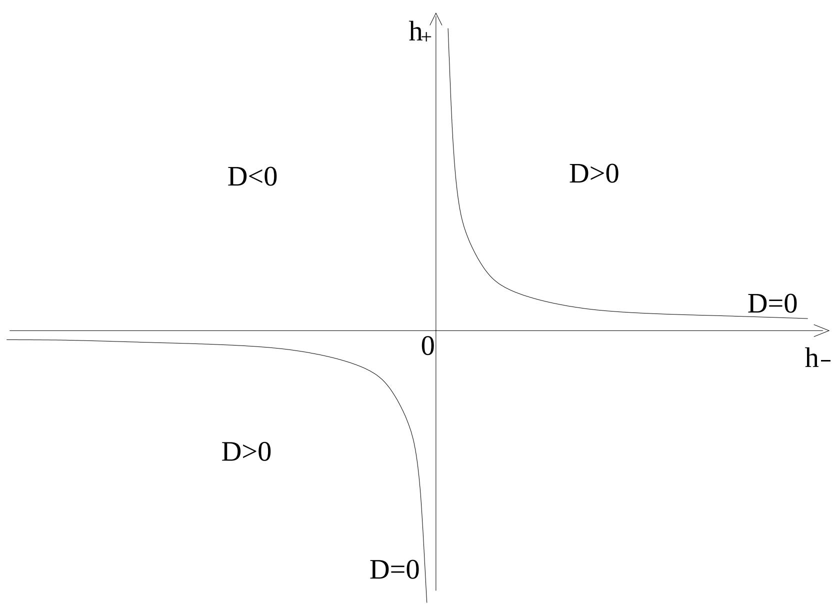

Figure 4: $\kappa_* l > \pi/2, a > 0, b = 0, c < 0$

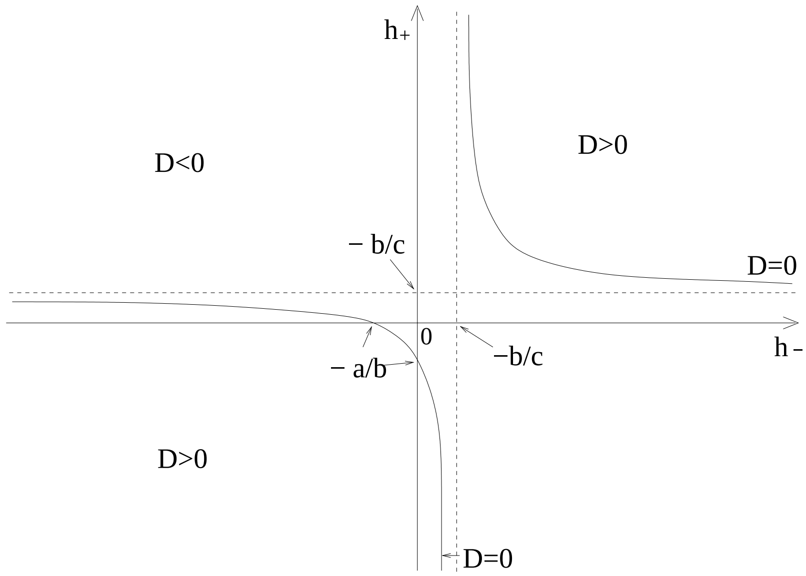

Figure 5: $\kappa_* l > \pi/2, a > 0, b > 0, c < 0$

small length are stable and equilibria with a large length are unstable. They are separated by a case which is neutral in the sense that the linearized evolution operator has besides negative eigenvalues one zero eigenvalue. This is for example the case when Ω is a ball and Γ_* is a segment intersection $\partial\Omega$ perpendicular (see Figure 5). In this case a nonlinear analysis has to decide on the stability.

If κ_* is nonzero then the linearized stability behaviour depends on the curvature of the outer boundary roughly speaking in the following sense. Cases with large positive outer curvatures h_+ and h_- are stable and cases with large negative outer curvatures are unstable. In Figure 7 we demonstrate this stability behaviour for a case where we fix κ_* and l. An equilibrium Γ_* is stable for $h > 0$ and unstable for $h < 0$. The case $h = 0$ is neutral and again a nonlinear analysis has to decide on stability. An interesting special case is when the outer boundary has constant curvature. This case is illustrated in Figure 8 and this case is always neutral. Indeed, let h be a constant curvature of the outer boundary, which implies $h_+ = h_- = h$. For the case $h = 0$, see the above explanation. If $h \neq 0$, then h is represented as

$$h = -\frac{\kappa_*}{\tan(\kappa_* l)}.$$

By the definition of a, b, c, we derive

$$\frac{a}{c} = -\frac{\kappa_*^3 l}{\kappa_* l - \tan(\kappa_* l)}, \quad \frac{b}{c} = -\frac{h}{2}\left\{1 - \frac{\kappa_* l \tan^2(\kappa_* l)}{\kappa_* l - \tan(\kappa_* l)}\right\}.$$

This implies that

$$D(h, \kappa_*) = \frac{a}{c} + \frac{b}{c} \cdot 2h + h^2 = 0.$$

In addition, $h > -b/c$ for $0 < \kappa_* l < \pi/2$ follows from

$$1 - \frac{\kappa_* l \tan^2(\kappa_* l)}{\kappa_* l - \tan(\kappa_* l)} > 2 \quad \text{for} \quad 0 < \kappa_* l < \pi/2,$$

and we also find $h > 0$ for $\pi/2 < \kappa_* l < \pi$. This means that this case is included in the line $D = 0$ on the right hand side of Figure 1 and Figure 3-5, so that this case is neutral.

Choosing for example $h_+ = h_- = 0$ we observe that $\kappa_* l$, is an important quantity (see Figure 9). As long as $\kappa_* l < \pi/2$ (i.e. Γ_* is less then a half circle) we have stability, the case $\kappa_* l = \pi/2$ is neutral (i.e. Γ_* is a half circle) and the case $\kappa_* l > \pi/2$ is unstable (i.e. Γ_* is more than a half circle).

Finally, we remark that instability for h_+, h_- positive and large is also possible. In this case $\kappa_* l$ has to be close to π, i.e. Γ_* has to be close to a full circle.

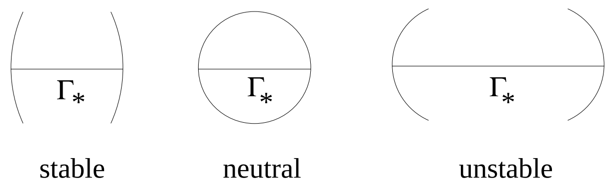

Figure 6: Three equilibria with $h_+ = h_-$ and $\kappa_* = 0$.
(The stability depends on the length of Γ_*.)

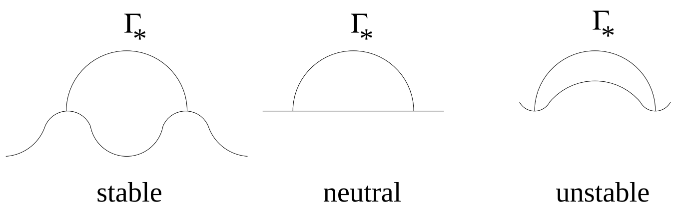

Figure 7: Three cases with κ_* and l fixed.

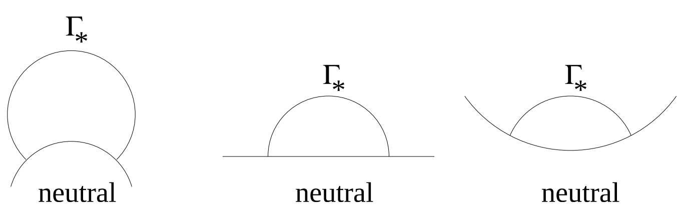

Figure 8: Three cases with the same κ_* and with constant curvature of $\partial\Omega$.

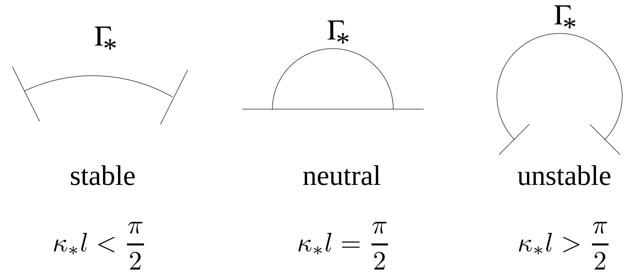

Figure 9: Three cases with $h_+ = h_- = 0$.

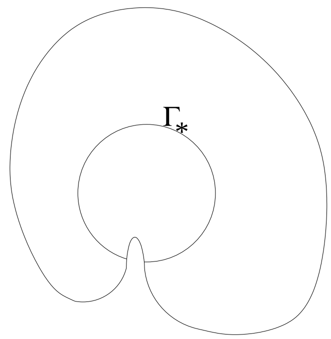

Figure 10: Instability for the case $h_+, h_- > 0$.

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