

EXISTENCE OF TRAVELLING WAVES IN DISCRETE SINE-GORDON RINGS

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ABSTRACT. We prove existence results for travelling waves in discrete, damped, dc-driven sine-Gordon equations with periodic boundary conditions. Methods of nonlinear functional analysis are employed. Some unresolved questions are raised.

1. INTRODUCTION

The damped, dc-driven discrete sine-Gordon equation, known also as the driven Frenkel-Kontorova model, with periodic boundary conditions, arises as a model of many physical systems, including circular arrays of Josephson junctions, the motions of dislocations in a crystal, the adsorbate layer on the surface of a crystal, ionic conductors, glassy materials, charge-density wave transport, sliding friction, as well as the mechanical interpretation as a model for a ring of pendula coupled by torsional springs (we refer to [8, 10, 11] and references therein). This model has thus become a fundamental one for nonlinear physics, and has been the subject of many theoretical, numerical and experimental studies. The system of equations is

$$(1) \quad \phi_j'' + \Gamma \phi_j' + \sin(\phi_j) = F + K[\phi_{j+1} - 2\phi_j + \phi_{j-1}], \quad \forall j \in \mathbb{Z}$$

with the parameters $\Gamma > 0, K > 0, F > 0$, with the periodic boundary-condition

$$(2) \quad \phi_{j+n}(t) = \phi_j(t) + 2\pi m \quad \forall j \in \mathbb{Z}$$

where $m \geq 1$ (we note that in view of the boundary conditions we are really dealing with an n -dimensional system of ODE's rather than an infinite-dimensional one). In numerical simulations, as well as in experimental work on systems modelled by (1),(2), it is observed that solutions often converge to a travelling wave: a solution satisfying

$$(3) \quad \phi_j(t) = f\left(t + j\frac{m}{n}T\right),$$

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where the waveform $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying

$$(4) \quad f(t + T) = f(t) + 2\pi \quad \forall t \in \mathbb{R}.$$

The velocity of the travelling wave is given by

$$(5) \quad v = \frac{2\pi}{T}.$$

However, as has been pointed out in [10], even the *existence* of such a solution, has not been proven, except for the case of small K in which existence of a travelling wave for some values of F had been proven in [4].

In the ‘super-damped’ case, in which the second-derivative term in (1) is removed, there are very satisfactory results about existence and also global stability of travelling waves ([1], theorem 2). Such results rely strongly on monotonicity arguments. Recently Baesens and Mackay [2] have managed to extend these arguments to the ‘overdamped’ case of (1): their results apply when

$$(6) \quad \Gamma > 2\sqrt{2K + 1},$$

and say that there exists a travelling-wave solution which is globally stable if and only if (1),(2) does not have stationary solutions. We do not know whether in general the non-existence of stationary solutions implies the existence of a travelling wave.

We note that a function f is a waveform if and only if it satisfies (4) and

$$(7) \quad f''(t) + \Gamma f'(t) + \sin(f(t)) = F + K \left[f\left(t + \frac{m}{n}T\right) - 2f(t) + f\left(t - \frac{m}{n}T\right) \right].$$

Here we obtain several existence results for travelling-waves under conditions not covered by the existing work, described above.

Theorem 1. *Fixing any $\Gamma > 0$, $K > 0$, and given any velocity $v > 0$, there exists a travelling-wave solution of (1), (2) with velocity v for an appropriate $F > 0$.*

Theorem 2. *For any $F > 1$ there exists a travelling-wave solution of (1), (2).*

Theorem 3. *Assume that n does not divide m . Fixing any $\tilde{F} > 0$ and $\tilde{\Gamma} > 0$, for all K sufficiently large there exists a travelling-wave solution of (1), (2) for any $F \geq \tilde{F}$, $\Gamma \geq \tilde{\Gamma}$.*

Theorem 4. *Fixing any $\tilde{F} > 0$ and $\tilde{K} > 0$, for all $\Gamma > 0$ sufficiently small there exists a travelling-wave solution of (1), (2) for any $F \geq \tilde{F}$, $0 < K \leq \tilde{K}$.*

We remark that the assumption that n does not divide m cannot be removed from theorem 3, since if n divides m the coupling term vanishes and (7),(4) reduce to the equation of a running solution of a dc-forced pendulum, which, fixing $\Gamma > 0$, is known to have a solution only when F exceeds a positive critical value [5].

It is interesting to note that theorem 4 demonstrates that for some parameter ranges there is coexistence of stationary solutions and travelling waves of (1), (2). Indeed, it is well known [3] that, fixing K , for F sufficiently small there exist stationary solutions of (1), (2), and these obviously do not depend on Γ . Hence we can take $\Gamma > 0$ sufficiently small so that theorem 4 ensures also the existence of travelling waves. This phenomenon cannot happen in the super-damped case, nor in the overdamped case in which (6) holds, since in these cases existence of stationary solutions implies that the ω -limit set of every orbit is contained in the set of stationary solutions [1, 2].

Along the way we will prove that

Proposition 5. *An upper bound for the velocity v of any travelling wave is given by*

$$(8) \quad v < \frac{F}{\Gamma}.$$

and a lower bound, in the case $F > 1$, is given by

$$(9) \quad v > \frac{F-1}{\Gamma}.$$

In the next section we prove the results stated above. In section 3 we discuss the meaning of our results in connection with existing numerical studies of the discrete sine-Gordon equation, and point out some further mathematical questions which arise from our results and remain open.

2. PROOFS OF THE RESULTS

Our method of proof involves re-formulating the problem as a fixed point problem in a Banach space, and applying results of nonlinear functional analysis. Our approach is thus close in spirit to [6], which deals with travelling waves in globally coupled Josephson junctions.

We transform the problem (4),(7) by setting

$$f(t) = u(vt) + vt,$$

where the wave-velocity v is defined by (5) and u satisfies:

$$(10) \quad u(z + 2\pi) = u(z) \quad \forall z \in \mathbb{R}.$$

(7) can then be written as

$$(11) \quad \begin{aligned} v^2 u''(z) &+ \Gamma v u'(z) + \sin(z + u(z)) \\ &= F - \Gamma v + K \left[u\left(z + 2\pi \frac{m}{n}\right) - 2u(z) + u\left(z - 2\pi \frac{m}{n}\right) \right] \end{aligned}$$

Dividing by v^2 and setting

$$\lambda = \frac{1}{v}$$

we re-write (11) in the form

$$(12) \quad \begin{aligned} u''(z) &+ \lambda \Gamma u'(z) + \lambda^2 \sin(z + u(z)) \\ &= \lambda^2 F - \lambda \Gamma + \lambda^2 K \left[u\left(z + 2\pi \frac{m}{n}\right) - 2u(z) + u\left(z - 2\pi \frac{m}{n}\right) \right]. \end{aligned}$$

We note that if $u(z)$ satisfies (10),(12) then so does $\tilde{u}(z) = u(z + c) + c$, for any $c \in \mathbb{R}$. Thus by adjusting c we may assume that u satisfies

$$(13) \quad \int_0^{2\pi} u(s) ds = 0.$$

We note now that if u satisfies (10),(12), then by integrating both sides of (12) over $[0, 2\pi]$ we obtain

$$(14) \quad F = \frac{\Gamma}{\lambda} + \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds.$$

We can thus re-write (12) as

$$(15) \quad \begin{aligned} u''(z) + \lambda \Gamma u'(z) &+ \lambda^2 \sin(z + u(z)) = \lambda^2 \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds \\ &+ \lambda^2 K \left[u\left(z + 2\pi \frac{m}{n}\right) - 2u(z) + u\left(z - 2\pi \frac{m}{n}\right) \right]. \end{aligned}$$

Conversely, if u satisfies (14) and (15) then it satisfies (12). We have thus reformulated our problem as: find solutions (λ, u) of (10),(13),(14),(15). The idea now is to consider λ as a *parameter* in (15) and try to find solutions u satisfying (10),(13),(15) and then substitute λ and u into (14) to obtain the corresponding value of F . This is the same idea as used in the numerical method presented in [8], but here it is used as part of existence proofs. We claim that

Proposition 6. *For any value λ , there exists a solution u of (15) satisfying (10),(13).*

We note that this proposition immediately implies theorem 1, since given any $v > 0$ it shows that we can solve (15) with $\lambda = \frac{1}{v}$, hence obtain a travelling wave with velocity v , for the value of F given by (14).

To prove proposition 6 we will use the Schauder fixed-point theorem. We denote by X, Y the Banach spaces of real-valued functions

$$X = \{u \in H^2[0, 2\pi] \mid u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad \int_0^{2\pi} u(s) ds = 0\},$$

$$Y = \{u \in L^2[0, 2\pi] \mid \int_0^{2\pi} u(s) ds = 0\},$$

with the norm

$$\|u\|_Y = \left(\frac{1}{2\pi} \int_0^{2\pi} (u(s))^2 ds \right)^{\frac{1}{2}},$$

and by $L_\lambda : X \rightarrow Y$ the linear mapping

$$L_\lambda(u)(z) = u''(z) + \lambda \Gamma u'(z) - \lambda^2 K \left[u\left(z + 2\pi \frac{m}{n}\right) - 2u(z) + u\left(z - 2\pi \frac{m}{n}\right) \right].$$

We want to show that this mapping is invertible and derive an upper bound for the norm of its inverse. Noting that any $u \in X$ can be decomposed in a Fourier series $u(z) = \sum_{l \neq 0} a_l e^{ilz}$ (with $a_{-l} = \overline{a_l}$), we apply L_λ to the Fourier elements, obtaining,

$$L_\lambda(e^{ilz}) = \mu_l e^{ilz},$$

where

$$\mu_l = -l^2 - 2K\lambda^2 \left(\cos\left(\frac{2\pi ml}{n}\right) - 1 \right) + \lambda \Gamma i,$$

so that

$$(16) \quad |\mu_l| = \left[\left(l^2 + 2K\lambda^2 \left(\cos\left(\frac{2\pi ml}{n}\right) - 1 \right) \right)^2 + \lambda^2 l^2 \Gamma^2 \right]^{\frac{1}{2}},$$

which does not vanish if $\Gamma > 0$. Thus the mapping L_λ has an inverse satisfying $L_\lambda^{-1}(e^{ilz}) = \frac{1}{\mu_l} e^{ilz}$. Since L_λ^{-1} takes Y onto X , and since X is compactly embedded in Y , we may consider L_λ^{-1} as a mapping from Y to itself, in which case it is a *compact* mapping. We also note using (16) that

$$(17) \quad \|L_\lambda^{-1}\|_{Y,Y} \leq \max_{l \geq 1} \frac{1}{|\mu_l|} \leq \frac{1}{\lambda \Gamma}.$$

We also define the nonlinear operator $N : Y \rightarrow Y$ by

$$N(u)(z) = -\sin(z + u(z)) + \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds.$$

It is easy to see that N is continuous, and that the range of N is contained in a bounded ball in Y , indeed we have

$$\|\sin(z + u(z))\|_{L_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} (\sin(s + u(s)))^2 ds \right)^{\frac{1}{2}} \leq 1,$$

and since $N(u)$ is the orthogonal projection of $-\sin(z + u(z))$ into Y , we have

$$(18) \quad \|N(u)\|_Y \leq 1 \quad \forall u \in Y.$$

We can now rewrite the problem (10),(13),(15) as the fixed-point problem:

$$(19) \quad u = \lambda^2 L_\lambda^{-1} \circ N(u).$$

The operator on the right-hand side is compact by the compactness of L_λ^{-1} , and has a bounded range by (17),(18), so that Schauder's fixed-point theorem implies that (19) has a solution, proving proposition 6 (we note that by a simple bootstrap argument a solution in Y is in fact smooth). Moreover, defining

$$\Sigma = \{(\lambda, u) \in [0, \infty) \times Y \mid u = \lambda^2 L_\lambda^{-1} \circ N(u)\},$$

Rabinowitz's continuation theorem [7] implies that the connected component of Σ containing $(\lambda, u) = (0, 0)$, which we denote by C , is unbounded in $[0, \infty) \times Y$. Since for any $\lambda_0 > 0$ we have, from (18), (19), the bound $\|u\|_Y \leq \frac{\lambda_0}{\Gamma}$ for solutions (λ, u) of (19) with $\lambda \in [0, \lambda_0]$, the unboundedness of the set C must be in the λ -direction, that is, there exist $(\lambda, u) \in C$ with arbitrarily large values of λ .

We can now consider the right-hand side of (14) as a functional on $[0, \infty) \times Y$:

$$(20) \quad \Phi(\lambda, u) = \frac{\Gamma}{\lambda} + \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds,$$

and our strategy in proving theorems 2-4 is to prove solvability of the equation

$$(21) \quad \Phi(\lambda, u) = F, \quad (\lambda, u) \in \Sigma$$

(in fact we shall prove solvability of (21) with Σ replaced by $C \subset \Sigma$). We note that by the boundedness of the sine function we have

$$(22) \quad \lim_{\lambda \rightarrow 0+, (\lambda, u) \in C} \Phi(\lambda, u) = +\infty,$$

$$(23) \quad \limsup_{\lambda \rightarrow +\infty, (\lambda, u) \in C} \Phi(\lambda, u) \leq 1.$$

Since C is a connected set and Φ is continuous, (22) implies that

Proposition 7. *For any F satisfying*

$$(24) \quad F > \underline{F} \equiv \inf_{(\lambda, u) \in C} \Phi(\lambda, u),$$

there exists a travelling wave.

Since (23) implies that $\underline{F} \leq 1$, this proves theorem 2.

We now prove the lower and upper bounds for the velocities of travelling waves given in proposition 5. These follow from (5), (14) and

Proposition 8. *For any $(\lambda, u) \in \Sigma$ with $\lambda > 0$ we have*

$$0 < \frac{\Gamma}{\lambda} < \Phi(\lambda, u) < \frac{\Gamma}{\lambda} + 1.$$

The upper bound follows immediately from the definition (20) of $\Phi(\lambda, u)$ since $\frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds < 1$. The lower bound follows from the claim that

$$(25) \quad (\lambda, u) \in \Sigma \Rightarrow \int_0^{2\pi} \sin(s + u(s)) ds > 0.$$

To prove this claim we multiply (15) by $1 + u'(z)$ and integrate over $[0, 2\pi]$, noting that

$$\begin{aligned} \int_0^{2\pi} u\left(s + 2\pi\frac{m}{n}\right)u'(s)ds &= \int_0^{2\pi} u(s)u'\left(s - 2\pi\frac{m}{n}\right)ds \\ &= - \int_0^{2\pi} u'(s)u\left(s - 2\pi\frac{m}{n}\right)ds, \end{aligned}$$

so that we obtain

$$(\lambda, u) \in \Sigma \Rightarrow \Gamma \frac{1}{2\pi} \int_0^{2\pi} (u'(s))^2 ds = \lambda \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds.$$

This proves (25) since the left-hand side is non-negative and cannot vanish unless $u \equiv 0$, but $(\lambda, 0) \notin \Sigma$ for $\lambda > 0$.

We now turn to the proof of theorem 3.

Proposition 9. *Assume n does not divide m and $\tilde{\Gamma} > 0$. Given any $\lambda_0 > 0$ and $\epsilon > 0$, there exists K_0 such that for $K \geq K_0$, $\Gamma \geq \tilde{\Gamma}$ we have that*

$$(26) \quad \left| \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds \right| < \epsilon \quad \text{if } (\lambda_0, u) \in \Sigma.$$

To see that proposition 9 implies theorem 3, we fix some $\tilde{F} > 0$, $\tilde{\Gamma} > 0$, and assume $\Gamma \geq \tilde{\Gamma}$. We choose $\lambda_0 > \frac{\Gamma}{\tilde{F}}$ and set $\epsilon = \tilde{F} - \frac{\Gamma}{\lambda_0}$. We then choose K_0 according to proposition 9, so that (26) holds, which implies that when $K \geq K_0$ we have $\Phi(\lambda_0, u) < \tilde{F}$ for any u with $(\lambda_0, u) \in C$. Thus $\underline{F} < \tilde{F}$, where \underline{F} is defined by (24), so proposition 7 implies the existence of a travelling wave for any $F \geq \tilde{F}$.

We now prove proposition 9. Let $\lambda_0 > 0$ and $\epsilon > 0$ be given. Assume $(\lambda_0, u) \in \Sigma$, so that (19) holds with $\lambda = \lambda_0$. Let (m, n) denote the greatest common divisor of m, n and let

$$p = \frac{m}{(m, n)}, \quad q = \frac{n}{(m, n)}.$$

Since we assume n does not divide m we have $q \geq 2$. Let Y_0 be the subspace of Y consisting of $\frac{2\pi}{q}$ -periodic functions, and let Y_1 be its orthogonal complement in Y . We denote by P the orthogonal projection of Y to Y_0 . Setting

$$u_0 = P(u), \quad u_1 = (I - P)(u),$$

we have $u = u_0 + u_1$ with $u_0 \in Y_0$, $u_1 \in Y_1$. Applying P and $I - P$ to (19), and noting that L_λ commutes with P , we have

$$(27) \quad u_0 = \lambda_0^2 L_{\lambda_0}^{-1} \circ P \circ N(u_0 + u_1)$$

$$(28) \quad u_1 = \lambda_0^2 L_{\lambda_0}^{-1} \circ (I - P) \circ N(u_0 + u_1)$$

We will now use (16) to derive a bound for $\|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1, Y_1}$ which goes to 0 as $K \rightarrow \infty$. We note that

$$(29) \quad \|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1, Y_1} \leq \max_{l \geq 1, q \nmid l} \frac{1}{|\mu_l|},$$

so we need to find lower bounds for the $|\mu_l|$'s for which q does not divide l . We define

$$\rho = \max_{l \geq 1, q \nmid l} \cos\left(\frac{2\pi pl}{q}\right)$$

and note that since p, q are coprime we have $\rho < 1$.

We define

$$\alpha = 2K\lambda_0^2(1 - \rho) - \sqrt{K},$$

and we shall henceforth assume that K is sufficiently large so that $\alpha > 0$. For each $l \geq 1$ we have either $l^2 < \alpha$ or $l^2 \geq \alpha$, and we treat each of these cases separately.

(1) In case $l^2 < \alpha$, we have

$$l^2 + 2K\lambda_0^2(\rho - 1) < -\sqrt{K},$$

and by the definition of ρ

$$\cos\left(\frac{2\pi ml}{n}\right) = \cos\left(\frac{2\pi pl}{q}\right) \leq \rho,$$

so that

$$l^2 + 2K\lambda_0^2\left(\cos\left(\frac{2\pi ml}{n}\right) - 1\right) < -\sqrt{K},$$

which by (16) implies

$$(30) \quad |\mu_l| > \sqrt{K}.$$

(2) In case $l^2 \geq \alpha$, we have, since (16) implies $|\mu_l| > \lambda_0 \Gamma l$:

$$(31) \quad |\mu_l| \geq \lambda_0 \Gamma \sqrt{\alpha} \geq \lambda_0 \tilde{\Gamma} \sqrt{\alpha} = \lambda_0 \tilde{\Gamma} \left[2K\lambda_0^2(1 - \rho) - \sqrt{K}\right]^{\frac{1}{2}}.$$

From (30),(31) we obtain that $\lim_{K \rightarrow \infty} |\mu_l| = +\infty$ uniformly with respect to $l \geq 1$ which are not multiples of q , hence by (29)

$$\lim_{K \rightarrow \infty} \|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1, Y_1} = 0.$$

In particular we may choose K_0 such that for $K \geq K_0$ we will have

$$\|L_{\lambda_0}^{-1}|_{Y_1}\|_{Y_1, Y_1} < \frac{\epsilon}{\lambda_0^2}.$$

By (28) and (18) this implies

$$(32) \quad \|u_1\|_Y \leq \epsilon.$$

Thus

$$\begin{aligned}
 \left| \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds \right| &\leq \left| \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u_0(s)) ds \right| \\
 &+ \left| \frac{1}{2\pi} \int_0^{2\pi} [\sin(s + u(s)) - \sin(s + u_0(s))] ds \right| \\
 (33) \qquad \qquad \qquad &\leq \left| \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u_0(s)) ds \right| + \frac{1}{2\pi} \int_0^{2\pi} |u_1(s)| ds
 \end{aligned}$$

From (32) and the Cauchy-Schwartz inequality we have

$$(34) \qquad \frac{1}{2\pi} \int_0^{2\pi} |u_1(s)| ds \leq \frac{1}{\sqrt{2\pi}} \left(\int_0^{2\pi} (u_1(s))^2 ds \right)^{\frac{1}{2}} \leq \epsilon.$$

From trigonometry we have

$$\int_0^{2\pi} \sin(s + u_0(s)) ds = \int_0^{2\pi} \sin(s) \cos(u_0(s)) ds + \int_0^{2\pi} \cos(s) \sin(u_0(s)) ds,$$

but the functions $\cos(u_0(s))$, $\sin(u_0(s))$ are $\frac{2\pi}{q}$ -periodic with $q \geq 2$, which implies that they are orthogonal to $\cos(s)$, $\sin(s)$, so that we have

$$\int_0^{2\pi} \sin(s + u_0(s)) ds = 0,$$

which together with (33) and (34) implies (26), concluding the proof of proposition 9.

We now turn to the proof of theorem 4. We first note that from (16) we have

$$|\mu_l| \geq \left| l^2 + 2K\lambda^2 \left(\cos\left(\frac{2\pi ml}{n}\right) - 1 \right) \right|,$$

so that if we assume

$$0 < \lambda < \lambda_0 = \frac{1}{\sqrt{8K}} \leq \frac{1}{\sqrt{8K}}$$

then we have $|\mu_l| > \frac{1}{2}$ for all $l \geq 1$, hence $\|L_\lambda^{-1}\|_{Y,Y} < 2$, independently of Γ . From (19) we thus have

$$(\lambda, u) \in \Sigma, \ 0 < \lambda < \lambda_0 \Rightarrow \|u\|_Y < 2\lambda^2.$$

We now choose $\lambda_1 \leq \lambda_0$ so that $2\lambda_1^2 \leq \frac{1}{2}\tilde{F}$. Thus $(\lambda_1, u) \in \Sigma$ implies that

$$\begin{aligned}
 (35) \qquad \left| \frac{1}{2\pi} \int_0^{2\pi} \sin(s + u(s)) ds \right| &= \left| \frac{1}{2\pi} \int_0^{2\pi} [\sin(s + u(s)) - \sin(s)] ds \right| \\
 &\leq \frac{1}{2\pi} \int_0^{2\pi} |u(s)| ds \leq \|u\|_Y < 2\lambda_1^2 \leq \frac{1}{2}\tilde{F}.
 \end{aligned}$$

Finally, we choose Γ_0 so that

$$(36) \quad \frac{\Gamma_0}{\lambda_1} < \frac{1}{2} \tilde{F}.$$

(35), (36) thus imply that when $0 < \Gamma < \Gamma_0$

$$(\lambda_1, u) \in \Sigma \Rightarrow \Phi(\lambda_1, u) < \tilde{F},$$

so that we have $\underline{F} < \tilde{F}$, where \underline{F} is defined by (24), hence proposition 7 implies the existence of a travelling wave for any $F \geq \tilde{F}$.

3. DISCUSSION AND FURTHER QUESTIONS

In the numerical and experimental explorations of the dynamics of sine-Gordon rings [8, 10, 11], a useful method of representation consists in displaying the velocity-force characteristic. In the case of the travelling waves studied here, since the velocity of the waves is given by $v = \frac{1}{\lambda}$, the velocity-force characteristic is the subset of the (F, v) -plane given by

$$\left\{ \left(\Phi\left(\frac{1}{v}, u\right), v \right) \mid \left(\frac{1}{v}, u\right) \in \Sigma \right\},$$

where the set Σ and the functional Φ are as defined in the previous section. Examining the velocity-force characteristic as numerically computed in [8] (fig.2), we see that F is a non-monotone function of v . This means that for some values of F equation (21) has more than one solution, or in other words that there exist multiple travelling waves with different velocities for the same value of F . On the other hand, the fact that according to the available numerical evidence F is a function of v leads us to conjecture that, fixing $\Gamma > 0, K > 0$, for each given velocity $v > 0$ there is a unique travelling wave with velocity v , for an appropriate F . Thus, our conjecture is that uniqueness holds in theorem 1, or in other words that the fixed point problem (19) always has a unique solution. Let us note that for $0 < \lambda < \Gamma$ it is easy to show, using (17), that the right-hand side of (19) is a contraction from Y to itself, hence we may replace the use of the Schauder fixed-point theorem by the Banach contraction-mapping principle, which implies uniqueness. Thus, at least for velocities $v > \frac{1}{\Gamma}$, we have uniqueness in theorem 1. However for lower velocities this argument does not work so a proof of the above conjecture will require some new idea.

The non-uniqueness of travelling waves for some values of the parameters Γ, K, F , mentioned above, implies important consequences for the dynamics of the discrete sine-Gordon ring, such as instability of some of the travelling waves, and bistability of travelling waves leading to hysteresis as the force F is varied. It would be interesting to determine whether these phenomena can occur in the large K and the small Γ regimes for which existence of a travelling wave has been proved in theorems 3,4. Moreover, stationary solutions and travelling waves do not exhaust the dynamical repertoire of the

discrete sine-Gordon equations in the under-damped case: quasi-periodic and chaotic behavior is reported [8, 9, 11]. An interesting question is to determine conditions on the parameters which ensure that there exists at least one *locally asymptotically stable* travelling wave.

Returning to the issue of existence of travelling waves, which has been the focus of our investigation, we note an intriguing question which arises from our results, and remains unanswered.

Let us define, for fixed $\Gamma > 0$, $K > 0$

$$F_0(\Gamma, K) = \inf\{F \geq 0 \mid \text{a travelling wave of (1), (2) exists}\}.$$

Theorem 2 implies that $F_0(\Gamma, K) \leq 1$ for all $\Gamma > 0$, $K > 0$. Theorem 3 implies that, when n does not divide m , $\lim_{K \rightarrow \infty} F_0(\Gamma, K) = 0$. Theorem 4 implies that $\lim_{\Gamma \rightarrow 0} F_0(\Gamma, K) = 0$. Is it true, though, that for each $\Gamma > 0$, $K > 0$ we have $F_0(\Gamma, K) > 0$? In other words, is it always true that (fixing $\Gamma > 0$, $K > 0$) for sufficiently small $F > 0$ a travelling wave does not exist? We have not been able to prove or disprove this conjecture, and can only offer the following remarks:

- (i) If Γ, K satisfy (6), then indeed $F_0(\Gamma, K) > 0$, since for sufficiently small F there exists a stationary solution of (1),(2), so the results of [2] imply that no travelling wave exists for small F . However, as we have remarked, theorem 4 shows that in general travelling waves and stationary solutions may coexist.
- (ii) If n divides m then, as was noted in the introduction, the existence of travelling waves reduces to that of running solutions of the forced pendulum, hence it is well known that $F_0(\Gamma, K) > 0$ for all Γ, K . However, in the case that n divides m we also have noted also that the result of theorem 3 does not hold, so that the case that n divides m is rather special and may not be indicative of the general case.
- (iii) The conjecture that $F_0(\Gamma, K) > 0$ is supported by the notion of ‘pinning’ - the phenomenon whereby travelling waves are unable to propagate in discrete systems when the applied force is small. However, whether this effect indeed holds in general in underdamped systems (as opposed to the overdamped case - see (i) above) is unclear to the best of our knowledge. Moreover, it is conceivable that $F_0(\Gamma, K) = 0$ but pinning still occurs - if for small F a travelling wave exists but is unstable.
- (iv) Since, by proposition 8, we have $\Phi(\lambda, u) > 0$ for all $(\lambda, u) \in \Sigma$, $\lambda > 0$, we have that $F_0(\Gamma, K) = 0$ if and only if

$$\liminf_{\lambda \rightarrow +\infty, (\lambda, u) \in \Sigma} \Phi(\lambda, u) = 0.$$

Thus, determining whether the above equality can hold could be a route to resolving our question, but we have not been able to do so.

We conclude with one more question: clarify the connection, if any, between the travelling waves obtained in [4] for small values of $K > 0$ and those obtained by us for large values of K in theorem 3.

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