Stable determination of an inclusion by boundary measurements^{*}

G. Alessandrini[†] and M. Di Cristo[‡] Dipartimento di Scienze Matematiche Università degli Studi di Trieste, Italy

Abstract

We deal with the problem of determining an inclusion within an electrical conductor from electrical boundary measurements. Under mild a priori assumptions we establish an optimal stability estimate.

1 Introduction

In this paper we deal with an inverse boundary value problem which is a special instance of the well-known Calderón's inverse conductivity problem [C]. Given a bounded domain Ω in \mathbb{R}^n , $n \geq 2$, with reasonably smooth boundary, an open set D, compactly contained in Ω , and a constant k > 0, $k \neq 1$, consider, for any $f \in H^{1/2}(\partial\Omega)$, the weak solution $u \in H^1(\Omega)$ to the Dirichlet problem

(1.1)
$$\operatorname{div}((1+(k-1)\chi_D)\nabla u) = 0 \quad \text{in } \Omega,$$

$$(1.2) u = f on \partial \Omega$$

where χ_D denotes the characteristic function of the set D. We will denote by $\Lambda_D : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ the so called Dirichlet-to-Neumann map, that is the operator which maps the Dirichlet data onto the corresponding Neumann data $\frac{\partial u}{\partial \nu}_{|\partial\Omega}$. The inverse problem that we examine here is to determine D when Λ_D is given.

In '88 Isakov [I1] proved the uniqueness, the purpose of the present paper is to prove a result of stability. In fact we prove that, under mild a priori assumptions on the regularity and on the topology of D, there is a continuous dependence of D (in the Hausdorff metric) from Λ_D with a modulus of continuity of logarithmic type, see Theorem 2.2 below. Let us stress that, indeed, this rate of continuity is the optimal one, as it was shown by examples in the recent paper [DC-R] by the second author and Luca Rondi.

We wish to mention here a closely related, but different, problem which attracted a lot of attention starting from the papers of Friedman [F] and Friedman and Gustafsson [F-G]. That is the one of determining D when, instead of full knowledge of the Dirichlet-to-Neumann map, only one, or few, pairs of Dirichlet and Neumann data are available, see [A-I], [I2] for extended bibliographical

^{*}Work supported by Miur, grant n.2002013279.

[†]E-mail: alessang@univ.trieste.it

[‡]E-mail: dicristo@dsm.univ.trieste.it

accounts. Unfortunately, for such a problem, the uniqueness question, not to mention stability, remains a largely open issue.

Let us illustrate briefly the main steps of our arguments. We must recall that Isakov's approach to uniqueness is essentially based on two arguments

- a) the Runge approximation theorem,
- b) the use of solutions with Green's function type singularities.

Also here we shall use singular solutions, and indeed we shall need an accurate study of their asymptotic behavior when the singularity gets close to the set of discontinuity ∂D of the conductivity coefficient $1+(k-1)\chi_D$ in (1.1), see Proposition 3.2. On the other hand, it seems that Runge's theorem, which is typically based on nonconstructive arguments, (Lax, [L], Kohn and Vogelius [K-V]) is not suited for stability estimates and therefore we introduced a different approach based on quantitative estimates of unique continuation, see Proposition 3.3.

In Section 2 we formulate our main hypotheses and state the stability result, Theorem 2.2. In Section 3 we prove Theorem 2.2 on the basis of some auxiliary Propositions, whose proof is deferred to the following Section 4.

2 The main result

Let us introduce our regularity and topological assumptions on the conductor Ω and on the unknown inclusion D. To this purpose we shall need the following definitions. In places, we shall denote a point $x \in \mathbb{R}^n$ by $x = (x', x_n)$ where $x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$.

Definition 2.1. Let Ω be a bounded domain in \mathbb{R}^n . Given α , $0 < \alpha \leq 1$, we shall say that a portion S of $\partial\Omega$ is of class $C^{1,\alpha}$ with constants \overline{r} , L > 0 if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have P = 0 and

$$\Omega \cap B_{\overline{r}}(0) = \{ x \in B_{\overline{r}} : x_n > \varphi(x') \},\$$

where φ is a $C^{1,\alpha}$ function on $B_{\overline{r}}(0) \subset \mathbb{R}^{n-1}$ satisfying $\varphi(0) = |\nabla \varphi(0)| = 0$ and $\|\varphi\|_{C^{1,\alpha}(B_{\overline{r}}(0))} \leq L\overline{r}.$

Definition 2.2. We shall say that a portion S of $\partial\Omega$ is of Lipschitz class with constants \overline{r} , L > 0 if for any $P \in S$, there exists a rigid transformation of coordinates under which we have P = 0 and

$$\Omega \cap B_{\overline{r}}(0) = \{ x \in B_{\overline{r}} : x_n > \varphi(x') \},\$$

where φ is a Lipschitz continuous function on $B_{\overline{r}}(0) \subset \mathbb{R}^{n-1}$ satisfying $\varphi(0) = 0$ and $\|\varphi\|_{C^{0,1}(B_{\overline{r}}(0))} \leq L\overline{r}$.

Remark 2.1. We have chosen to scale all norms in a such a way that they are dimensionally equivalent to their argument. For instance, for any $\varphi \in C^{1,\alpha}(B_{\overline{r}}(0))$ we set

$$\|\varphi\|_{C^{1,\alpha}(B_{\overline{r}}(0))} = \|\varphi\|_{L^{\infty}(B_{\overline{r}}(0))} + \overline{r}\|\nabla\varphi\|_{L^{\infty}(B_{\overline{r}}(0))} + \overline{r}^{1+\alpha}|\nabla\varphi|_{\alpha,B_{\overline{r}}(0)}.$$

For given numbers \overline{r} , M, δ , L > 0, $0 < \alpha < 1$, we shall assume

(H1) the domain Ω satisfies the following conditions

$$(2.1) \qquad \qquad |\Omega| \le M\overline{r}^n,$$

where $|\cdot|$ denotes the Lebesgue measure of Ω ,

(2.2) $\partial \Omega$ is of class $C^{1,\alpha}$ with constants \overline{r} , L,

(H2) the inclusion D satisfies the following conditions

(2.3)
$$\Omega \smallsetminus D$$
 is connected,

(2.4)
$$\operatorname{dist}(D, \partial \Omega) \ge \delta.$$

(2.5)
$$\partial D$$
 is of class $C^{1,\alpha}$ with constants \overline{r} , L.

In the sequel we shall refer to numbers $k, n, \overline{r}, M, \delta, L, \alpha$ as to the a priori data. We shall denote by D_1 and D_2 two possible inclusions in Ω , both satisfying the properties mentioned. We shall denote by Λ_{D_i} , i = 1, 2, the Dirichlet-to-Neumann map Λ_D when $D = D_i$. We can now state the main theorem.

Theorem 2.2. Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, satisfy (H1). Let k > 0, $k \neq 1$ be given. Let D_1 and D_2 be two inclusions in Ω satisfying (H2). If, given $\varepsilon > 0$, we have

(2.6)
$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \le \varepsilon,$$

then

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \le \omega(\varepsilon),$$

where ω is an increasing function on $[0, +\infty)$, which satisfies

$$\omega(t) \le C |\log t|^{-\eta}, \qquad \qquad for \ every \quad 0 < t < 1$$

and C, η , C > 0, $0 < \eta \leq 1$, are constants only depending on the a priori data.

Here $d_{\mathcal{H}}$ denotes the Hausdorff distance between bounded closed sets of \mathbb{R}^n and $\|\cdot\|_{\mathcal{L}(H^{1/2}H^{-1/2})}$ denotes the operator norm on the space of bounded linear operators between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.

Remark 2.3. It should be emphasized that in this statement the unknown inclusion may be disconnected.

Remark 2.4. Several variations of the above results could be devised with minor adaptations on the arguments. Just to mention one, an analogous result would be obtained if the Neumann-to-Dirichlet maps N_{D_i} are available instead of the Dirichlet-to-Neumann maps Λ_{D_i} .

3 Proof of Theorem 2.2

Before proving Theorem 2.2, we shall state some auxiliary Propositions, whose proofs are collected in the next Section 4. Here and in the sequel we shall denote by \mathcal{G} the connected component of $\Omega \setminus (D_1 \cup D_2)$, whose boundary contains $\partial\Omega$, $\Omega_D = \Omega \setminus \overline{\mathcal{G}}$, $\Omega_{\overline{r}} = \{x \in \mathcal{C}\Omega : \operatorname{dist}(x, \Omega) \leq \overline{r}\}$ and $\mathcal{S}_{2\overline{r}} = \{x \in \mathbb{R}^n : \overline{r} \leq \operatorname{dist}(x, \Omega) \leq 2\overline{r}\}.$

We introduce a variation of the Hausdorff distance which we call *modified distance*.

Definition 3.1. We shall call modified distance between D_1 and D_2 the number

(3.1)
$$d_{\mu}(D_1, D_2) = \max\left\{\sup_{x \in \partial D_1 \cap \partial \Omega_D} \operatorname{dist}(x, D_2), \sup_{x \in \partial D_2 \cap \partial \Omega_D} \operatorname{dist}(x, D_1)\right\}.$$

This notion is an adaptation of the one introduced in [A-B-R-V], which was also called *modified distance*. In order to distinguish such two notions, we call d_{μ} the present one, whereas the one in [A-B-R-V] was denoted by d_m . On the other hand, we need to stress the common peculiarities: such modified distances do not satisfy the axioms of a metric and in general do not dominate the Hausdorff distance (see Section 3 in [A-B-R-V] for related arguments). The following Proposition provides sufficient conditions under which d_{μ} dominates $d_{\mathcal{H}}$. See [A-B-R-V] Proposition 3.6 for a related statement.

Proposition 3.1. Let Ω be an open set in \mathbb{R}^n satisfying (H1). Let D_1 , D_2 be two bounded open inclusions of Ω satisfying (H2). Then

(3.2)
$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \le c d_{\mu}(D_1, D_2),$$

where c depends only on the a priori assumptions.

With no loss of generality, we can assume that there exists a point O of $\partial D_1 \cap \partial \Omega_D$, where the maximum in the definition (3.1) is attained, that is

(3.3)
$$d_{\mu} = d_{\mu}(D_1, D_2) = \operatorname{dist}(O, D_2)$$

As is well-known, the Dirichlet-to-Neumann map Λ_D associated to problem (1.1), (1.2) is defined by:

(3.4)
$$\langle \Lambda_D u, v \rangle = \int_{\Omega} (1 + (k-1)\chi_D) \nabla u \cdot \nabla v$$

for every $u \in H^1(\Omega)$ solution to (1.1) and for every $v \in H^1(\Omega)$. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$. With a slight abuse of notation we shall write

$$< g, f > = \int_{\partial\Omega} gf \, d\sigma,$$

for any $f \in H^{1/2}(\partial\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. Let $\Gamma_D(x, y)$ be the fundamental solution for the operator div $((1 + (k - 1)\chi_D)\nabla \cdot))$, thus

(3.5)
$$\operatorname{div}((1+(k-1)\chi_D)\nabla\Gamma_D(\cdot,y)) = -\delta(\cdot-y),$$

where $y, w \in \mathbb{R}^n$, δ denotes the Dirac distribution. We shall denote by Γ_{D_1} , Γ_{D_2} such fundamental solutions when $D = D_1$, D_2 respectively. Recalling the well-known identity

$$\int_{\Omega} (1 + (k-1)\chi_{D_1}) \nabla u_1 \cdot \nabla u_2 - \int_{\Omega} (1 + (k-1)\chi_{D_2}) \nabla u_1 \cdot \nabla u_2 = \int_{\partial \Omega} u_1 [\Lambda_{D_1} - \Lambda_{D_2}] u_2,$$

which holds for every $u_i \in H^1(\Omega)$, i = 1, 2, solutions to (1.1) when $D = D_i$ respectively (see [I2] formula (5.0.4), Section 5.0), we have

$$\int_{\Omega} (1 + (k - 1)\chi_{D_1}) \nabla \Gamma_{D_1}(\cdot, y) \cdot \nabla \Gamma_{D_2}(\cdot, w) - \int_{\Omega} (1 + (k - 1)\chi_{D_2}) \nabla \Gamma_{D_2}(\cdot, y) \cdot \nabla \Gamma_{D_2}(\cdot, w)$$

$$= \int_{\partial\Omega} \Gamma_{D_1}(\cdot, y) [\Lambda_{D_1} - \Lambda_{D_2}] (\Gamma_{D_2}(\cdot, w)) d\sigma, \qquad \forall y, w \in \mathcal{C}\overline{\Omega}.$$

Let us define, for $y, w \in \mathcal{G} \cup \mathcal{C}\Omega$

(3.6)

(3.7)
$$S_{D_1}(y,w) = (k-1) \int_{D_1} \nabla \Gamma_{D_1}(\cdot,y) \cdot \nabla \Gamma_{D_2}(\cdot,w),$$

(3.8)
$$S_{D_2}(y,w) = (k-1) \int_{D_2} \nabla \Gamma_{D_1}(\cdot,y) \cdot \nabla \Gamma_{D_2}(\cdot,w),$$

(3.9)
$$f(y,w) = S_{D_1}(y,w) - S_{D_2}(y,w).$$

Thus (3.6) can be rewritten as

(3.10)
$$f(y,w) = \int_{\partial\Omega} \Gamma_{D_1}(\cdot,y) [\Lambda_{D_1} - \Lambda_{D_2}] (\Gamma_{D_2}(\cdot,w)) d\sigma \qquad \forall y,w \in \mathcal{C}\overline{\Omega}.$$

From now on we shall consider the dimension $n \geq 3$, since the case n = 2can be treated similarly through minor adaptations regarding the fundamental solutions. Up to a transformation of coordinates, we can assume that O, defined in (3.3), is the origin of the coordinate system. Let $\nu(O)$ be the outer unit normal vector to $\partial \Omega_D$ in the origin O. Such a normal is indeed well-defined since we are assuming that O realizes the modified distance between D_1 and D_2 , therefore, in a small neighborhood of O, $\partial \Omega_D$ is made of a part of ∂D_1 , which is known to be $C^{1,\alpha}$. We will rotate the coordinate system in such a way that $\nu(O) = (0, ..., 0, -1)$. Taking $y = w = h\nu(O)$, with h > 0, we want to evaluate f(y, y) and $S_{D_1}(y, y)$ in term of h, for h small. Then, evaluating S_{D_2} in term of d_{μ} , we will get the stability estimate for the modified distance and thus, using Proposition 3.1, for the Hausdorff distance. An important ingredient for evaluating f and S_{D_1} is the behavior of the fundamental solution. We state now a proposition that collects all the results on Γ_{D_i} , i = 1, 2, that we will need throughout the paper. For $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, we set $x^{\star} = (x', -x_n)$. We shall denote with χ^+ the characteristic function of the half-space $\{x_n > 0\}$ and with Γ_+ the fundamental solution of the operator div $((1+(k-1)\chi^+)\nabla)$. If Γ is the standard fundamental solution of the Laplace operator, we have that (see for instance [A-I-P], Theorem 4)

(3.11)
$$\Gamma_{+}(x,y) = \begin{cases} \frac{1}{k}\Gamma(x,y) + \frac{k-1}{k(k+1)}\Gamma(x,y^{\star}) & \text{for } x_{n} > 0, y_{n} > 0, \\ \frac{2}{k+1}\Gamma(x,y) & \text{for } x_{n}y_{n} < 0, , \\ \Gamma(x,y) - \frac{k-1}{k+1}\Gamma(x,y^{\star}) & \text{for } x_{n} < 0, y_{n} < 0. \end{cases}$$

The following Proposition holds.

Proposition 3.2. Let $D \subset \mathbb{R}^n$ be an open set whose boundary is of class $C^{1,\alpha}$, with constants \overline{r} , L.

(i) There exists a constant $c_1 > 0$ depending on k, n, α and L only, such that

(3.12)
$$|\nabla_x \Gamma_D(x,y)| \le c_1 |x-y|^{1-n},$$

for every $x, y \in \mathbb{R}^n$,

(ii) There exist constants c_2 , $c_3 > 0$ depending on k, n, α and L only, such that

(3.13)
$$|\Gamma_D(x,y) - \Gamma_+(x,y)| \le \frac{c_2}{\overline{r}^{\alpha}} |x-y|^{2-n+\alpha},$$

(3.14)
$$\left|\nabla_x \Gamma_D(x,y) - \nabla_x \Gamma_+(x,y)\right| \le \frac{c_3}{\overline{r}^{\alpha^2}} |x-y|^{1-n+\alpha^2},$$

for every $x \in D \cap B_r(O)$, and for every $y = h\nu(O)$, with $0 < r < \overline{r}_0$, $0 < h < \overline{r}_0$, where $\overline{r}_0 = \left(\min\left\{\frac{1}{2}(8L)^{-1/\alpha}, \frac{1}{2}\right\}\right)\frac{\overline{r}}{2}$.

The next two Propositions give us quantitative estimates on f and S_{D_1} when we move y towards O, along $\nu(O)$.

Proposition 3.3. Let Ω be an open set in \mathbb{R}^n satisfying (H1). Let D_1, D_2 be two inclusions in Ω verifying (H2) and let $y = h\nu(O)$, with O defined in (3.3). If, given $\varepsilon > 0$, we have

$$\|\Lambda_{D_1} - \Lambda_{D_2}\|_{\mathcal{L}(H^{1/2}, H^{-1/2})} \le \varepsilon,$$

then for every $h, 0 < h < \overline{c} \overline{r}$, where $0 < \overline{c} < 1$, depends on L,

$$(3.15) |f(y,y)| \le C \frac{\varepsilon^{Bh^F}}{h^A},$$

where 0 < A < 1 and C, B, F > 0 are constants that depend only on the a priori data.

Proposition 3.4. Let Ω be an open set in \mathbb{R}^n satisfying (H1). Let D_1 , D_2 be two inclusions in Ω verifying (H2) and $y = h\nu(O)$. Then for every h, $0 < h < \overline{r}_0/2$,

$$(3.16) |S_{D_1}(y,y)| \ge c_1 h^{2-n} - c_2 d_{\mu}^{2-2n} + c_3,$$

where c_1, c_2 and c_3 are positive constants only depending on the a priori data. Here \overline{r}_0 is the number introduced in Proposition 3.2.

Now we have all the tools that we need to prove Theorem 2.2.

Proof of Theorem 2.2. Let $O \in \partial D_1$ satisfying (3.3), that is

$$d_{\mu}(D_1, D_2) = \operatorname{dist}(O, D_2) = d_{\mu}.$$

Then, for $y = h\nu(O)$, with $0 < h < h_1$, where $h_1 = \min\{d_\mu, \overline{c}\overline{r}, \overline{r}_0/2\}$, using (3.12), we have

$$(3.17) |S_{D_2}(y,y)| \le c \int_{D_2} \frac{1}{(d_{\mu}-h)^{n-1}} \frac{1}{(d_{\mu}-h)^{n-1}} dx = c \frac{1}{(d_{\mu}-h)^{2n-2}} |D_2|.$$

Using Proposition 3.3, we have

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \le |S_{D_1}(y,y) - S_{D_2}(y,y)|$$

= $|f(y,y)| \le c \frac{\varepsilon^{Bh^F}}{h^A}.$

On the other hand, by Proposition 3.4 and (3.17)

$$|S_{D_1}(y,y)| - |S_{D_2}(y,y)| \ge c_1 h^{2-n} - c_2 (d_\mu - h)^{2-2n}.$$

Thus we have

$$c_3h^{2-n} - c_4(d_\mu - h)^{2-2n} \le \frac{\varepsilon^{Bh^F}}{h^A}.$$

That is

(3.18)
$$c_4(d_{\mu} - h)^{2-2n} \geq c_3 h^{2-n} - \frac{\varepsilon^{Bh^F}}{h^A} = h^{2-n}(c_3 - \varepsilon^{Bh^F} h^{\widetilde{A}})$$
$$\geq c_5 h^{2-n} (1 - \varepsilon^{Bh^F} h^{\widetilde{A}}),$$

where $\widetilde{A} = n - 2 - A$, $\widetilde{A} > 0$. Let $h = h(\varepsilon)$ where $h(\varepsilon) = \min\{|\ln \varepsilon|^{-\frac{1}{2F}}, d_{\mu}\}$, for $0 < \varepsilon \leq \varepsilon_1$, with ε_1 such that $\exp(-B|\ln \varepsilon_1|^{1/2}) = 1/2$. If $d_{\mu} \leq |\ln \varepsilon|^{-\frac{1}{2F}}$ the theorem follows using Proposition 3.1. In the other case we have

$$\varepsilon^{Bh(\varepsilon)^F} h(\varepsilon)^{\widetilde{A}} \le \varepsilon^{B|\ln \varepsilon|^{-1/2}} \le \exp\left(-B|\ln \varepsilon|^{1/2}\right).$$

Then, for any ε , $0 < \varepsilon < \varepsilon_1$,

$$(d_{\mu} - h(\varepsilon))^{2-2n} \ge c_6 h(\varepsilon)^{2-n},$$

that is

(3.19)
$$d_{\mu} \le c_7 |\ln \varepsilon|^{-\delta \frac{n-2}{2n-2}}$$

where $\delta = 1/(2F)$. When $\varepsilon \ge \varepsilon_1$, then

$$d_{\mu} \leq \operatorname{diam} \Omega \leq \operatorname{diam} \Omega \frac{|\ln \varepsilon|^{-\frac{1}{2F}}}{|\ln \varepsilon_1|^{-\frac{1}{2F}}}$$

Finally, using Proposition 3.1, the theorem follows.

4 Proofs of the auxiliary Propositions

We premise the proof of Proposition 3.1 with one lemma.

Lemma 4.1. Let Ω be an open set in \mathbb{R}^n satisfying (H1). Let D be a bounded open inclusion of Ω satisfying (H2). Then for every $P \in \partial D$, there exists a continuous path γ in $\Omega \setminus \overline{D}$ with one end-point in P and the other on $\partial\Omega$, such that for every $z \in \gamma$

$$(4.1) |z-P| \le c \operatorname{dist}(z,D),$$

where c is a positive constant depending on the a priori data only.

Proof. Using Lemma 5.2 of [A-B-R-V], (which adapted arguments due to Lieberman [Li]), we approximate dist($\cdot, \partial D$) with a regularized distance \tilde{d} such that $\tilde{d} \in C^2(\Omega \smallsetminus D) \cup C^{1,\alpha}(\overline{\Omega \smallsetminus D})$ and the following facts hold

$$\begin{split} \gamma_0 &\leq \frac{\operatorname{dist}(x, \partial D)}{\tilde{d}(x)} \leq \gamma_1, \\ |\nabla \tilde{d}(y)| &\geq c_1 \quad \text{for every } y \in \Omega \text{ s.t. } \operatorname{dist}(y, \partial D) > b\overline{r}, \\ \|\tilde{d}\|_{1,\alpha} &\leq c_2 \overline{r}, \end{split}$$

where γ_0 , γ_1 , b, c_1 and c_2 are positive constants only depending on L and α . We define for $0 < h < a\overline{r}$, with a depending on L and α only,

$$E_h = \{ x \in \Omega \setminus \overline{D} : \tilde{d}(x) > h \}.$$

Arguing as in Lemma 5.3 of [A-B-R-V], E_h is connected with boundary of class C^1 and

(4.2)
$$\widetilde{c}_1 h \leq \operatorname{dist}(x, \partial D) \leq \widetilde{c}_2 h, \quad \forall x \in \partial E_h \cap \Omega,$$

where \tilde{c}_1 , \tilde{c}_2 are positive constants depending on L and α only. Let us fix $P \in \partial D$. Let $\nu(P)$ be the outer unit normal to ∂D in P (we recall that ∂D is $C^{1,\alpha}$). Since (4.2), there exists a point $P' \in E_h$ such that $P' = \tilde{h}\nu(P)$, where \tilde{h} is a positive constant $\tilde{c}_1h < \tilde{h} < \tilde{c}_2h$. We denote by $\overline{PP'}$ the segment whose end-points are P and P'. Since E_h is connected, there exists a continuous path $\gamma' \subset E_h$ with one end-point P' and the other on $\partial \Omega$. Since $\gamma' \subset E_h$ we have that for every $x \in \gamma'$, dist $(x, \partial D) \ge ch$, where c is a positive constant. We then define $\gamma = \gamma' \cup \overline{PP'}$ and the lemma follows.

Proof of Proposition 3.1. Let us fix $P \in \partial D_1$. We distinguish the two following cases.

- i) $P \in \partial D_1 \cap \partial \mathcal{G}$,
- ii) $P \in \partial D_1 \smallsetminus \partial \mathcal{G}$.

If case i) occurs then,

$$\operatorname{dist}(P, \partial D_2) = \operatorname{dist}(P, \overline{D}_2) \le d_{\mu}.$$

Let us consider case ii). Let γ be the continuous path constructed in Lemma 4.1 from P to $\partial\Omega$. Since $P \notin \partial\mathcal{G}$, there exists $z \in \gamma \cap \partial D_2 \cap \partial\Omega_D$.

$$\operatorname{dist}(z, D_1) \leq \sup_{x \in \partial D_2 \cap \partial \Omega_D} \left\{ \operatorname{dist}(x, D_1) \right\} \leq d_\mu(D_1, D_2).$$

Thus

$$|z-P| \le cd_{\mu}(D_1, D_2),$$

where c > 0 is the constant appearing in (4.1) On the other hand

$$\operatorname{dist}(P, \partial D_2) \le |z - P|.$$

So we obtain that, for every $P \in \partial D_1$

$$\operatorname{dist}(P, \partial D_2) \le cd_{\mu}(D_1, D_2).$$

Similarly one can show that for every $Q \in \partial D_2$

$$\operatorname{dist}(Q, \partial D_1) \le cd_{\mu}(D_1, D_2).$$

Then we conclude

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \le c d_{\mu}(D_1, D_2).$$

Proof of Proposition 3.2. Let us prove (i).

Let us consider the case $x \in D$ and $y \in \partial D$. The cases in which $x, y \in D$ or $x, y \in CD$ are trivial. Let h = |x - y|. Let c be a positive number less than $\frac{1}{1+2\sqrt{n}}$. We distinguish the following two cases:

- a) dist $(x, \partial D) < ch$,
- b) dist $(x, \partial D) \ge ch$.

Let us consider the case a). Let $P \in \partial D$ be such that $|P - x| = \operatorname{dist}(x, \partial D)$. For every r > 0, let $Q_r(P)$ be the cube centered at P, with sides of length 2rand parallel to the coordinates axes. We have that the ball $B_r(P)$ is inscribed into $Q_r(P)$. In particular $x \in Q_{ch}(P)$. On the other hand

$$|P - y| \ge |y - x| - |P - x| \ge h(1 - c).$$

Then, due to our choice of c, $|P - y| > (2ch)\sqrt{n}$, that is $y \notin Q_{2ch}(P)$. Thus

$$\operatorname{div}_{z}\left((1+(k-1)\chi_{D})\nabla_{z}\Gamma_{D}(z,y)\right) = 0 \qquad \text{in } Q_{\frac{3}{2}ch}(P)$$

and for the piecewise $C^{1,\alpha}$ regularity of Γ_D , proved in [DB-E-F], see also [L-V], we have

(4.3)
$$\|\nabla\Gamma_D(\cdot, y)\|_{L^{\infty}(Q_{ch}(P))} \leq \frac{\overline{c}_1}{h} \|\Gamma_D(\cdot, y)\|_{L^{\infty}(Q_{\frac{3}{2}ch}(P))},$$

where \overline{c}_1 depends on L, k, n and α only. Using the pointwise bound of Γ_D with Γ (see [L-S-W]), we have

(4.4)
$$\|\Gamma_D(\cdot, y)\|_{L^{\infty}(Q_{\frac{3}{2}ch}(P))} \leq \overline{c}_2 \left(\frac{ch}{2}\right)^{2-n},$$

where \overline{c}_2 depends on n and k only. Hence, by (4.3) and (4.4), we get

$$(4.5) \qquad |\nabla_x \Gamma_D(x,y)| \le \|\nabla \Gamma_D(\cdot,y)\|_{L^{\infty}(Q_{ch}(P))} \le \overline{c}_3 h^{1-n} = \overline{c}_3 |x-y|^{1-n},$$

where \overline{c}_3 depends on L, k, n and α only. If case b) occurs, then $Q_{\frac{ch}{\sqrt{n}}}(x) \subset D$. Hence

$$\begin{aligned} |\nabla_x \Gamma_D(x,y)| &\leq \|\nabla \Gamma_D(\cdot,y)\|_{L^{\infty}(Q_{\frac{ch}{2\sqrt{n}}}(x))} \leq \frac{\overline{c}_4}{h} \|\Gamma_D(\cdot,y)\|_{L^{\infty}(Q_{\frac{c}{\sqrt{n}}}(P))} \\ &\leq \frac{\overline{c}_4}{h} (h(1-c))^{2-n} = \overline{c}_4' h^{1-n} = \overline{c}_4' |x-y|^{1-n}, \end{aligned}$$

where \overline{c}_4 , \overline{c}'_4 depend on L, k, n and α only. Let us prove (*ii*).

Let us fix $r_1 = \min \left\{ \frac{1}{2} (8L)^{-1/\alpha} \overline{r}, \frac{\overline{r}}{2} \right\}$. Recalling Definition 2.1, we have that

$$\partial D \cap B_{\overline{r}}(0) = \{ x \in B_{\overline{r}}(0) : x_n = \varphi(x') \}$$

where $\varphi \in C^{1,\alpha}(\mathbb{R}^{n-1})$ satisfying $\varphi(0) = |\nabla \varphi(0)| = 0$. Let $\theta \in C^{\infty}(\mathbb{R})$ be such that $0 \leq \theta \leq 1$, $\theta(t) = 1$, for |t| < 1, $\theta(t) = 0$, for |t| > 2 and $|\frac{d\theta}{dt}| \leq 2$. We consider the following change of variables $\xi = \Phi(x)$ defined by

$$\begin{cases} \xi' = x' \\ \xi_n = x_n - \varphi(x')\theta\left(\frac{|x'|}{r_1}\right)\theta\left(\frac{x_n}{r_1}\right). \end{cases}$$

It can be verified that, with the given choice of r_1 , the following properties of Φ hold

(4.6)
$$\Phi(Q_{2r_1}(0)) = Q_{2r_1}(0),$$

(4.7)
$$\Phi(Q_{r_1}(0) \cap D) = Q_{r_1}^+(0),$$

(4.8)
$$c^{-1}|x_1 - x_2| \le |\Phi(x_1) - \Phi(x_2)| \le c|x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

(4.9)
$$|\Phi(x) - x| \le \frac{c}{\overline{r}^{\alpha}} |x|^{1+\alpha}, \quad \forall x \in \mathbb{R}^n,$$

(4.10)
$$|D\Phi(x) - I| \le \frac{c}{\overline{r}^{\alpha}} |x|^{\alpha}, \quad \forall x \in \mathbb{R}^n.$$

where $Q_{r_1}^+(0) = \{x \in Q_{r_1}(0) : x_n > 0\}$ and $c \ge 1$ depends on L and α only. Φ is a $C^{1,\alpha}$ diffeomorphism from \mathbb{R}^n into itself. Let us define the cylinder C_{r_1} as

$$C_{r_1} = \{ x \in \mathbb{R}^n : |x'| < r_1, |x_n| < r_1 \}$$

For $x, y \in C_{r_1}$, we have that $\widetilde{\Gamma}_D(\xi, \eta) = \Gamma_D(x, y)$, where $\xi = \Phi(x)$, $\eta = \Phi(y)$, is solution of

(4.11)
$$\operatorname{div}_{\xi}((1+(k-1)\chi^{+})B(\xi)\nabla_{\xi}\widetilde{\Gamma}_{D}(\xi,\eta)) = -\delta(\xi-\eta),$$

where $B = \frac{JJ^T}{\det J}$, with $J = \frac{\partial \xi}{\partial x}(\Phi^{-1}(\xi))$. We observe that B is of class C^{α} and B(0) = I. Let us consider

$$\widetilde{R}(x,y) = \widetilde{\Gamma}_D(x,y) - \Gamma_+(x,y),$$

where we keep the notation x, y to indicate ξ, η . By the properties of Γ_+ and by (4.11), \widetilde{R} satisfies

$$\operatorname{div}_x((1+(k-1)\chi^+)\nabla_x\widetilde{R}(x,y)) = \operatorname{div}_x((1+(k-1)\chi^+)(I-B)\nabla_x\widetilde{\Gamma}_D(x,y)).$$

Let $\widetilde{L} > 0$, depending on the a priori data only, be such that $\overline{\Omega} \subset B_{\widetilde{L}}(0)$. Thus

using the fundamental solution Γ_+ we obtain

$$\begin{split} &-\widetilde{R}(x,y) = \int_{B_{\widetilde{L}}(0)} (1+(k-1)\chi^{+})(B-I)\nabla_{z}\Gamma_{+}(z,y)\cdot\nabla_{z}\widetilde{\Gamma}_{D}(z,x)dz \\ &+ \int_{\partial B_{\widetilde{L}}(0)} (1+(k-1)\chi^{+}) \Big[\widetilde{R}(x,z)\frac{\partial\Gamma_{+}}{\partial\nu}(z,y) - \Gamma_{+}(z,y)\frac{\partial\widetilde{R}}{\partial\nu}(x,z)\Big]d\sigma(z) \\ &= \int_{B_{\widetilde{L}}(0)\cap C_{r_{1}}} (1+(k-1)\chi^{+})(B-I)\nabla_{z}\Gamma_{+}(z,y)\cdot\nabla_{z}\widetilde{\Gamma}_{D}(z,x)dz \\ &+ \int_{B_{\widetilde{L}}(0)\smallsetminus C_{r_{1}}} (1+(k-1)\chi^{+})(B-I)\nabla_{z}\Gamma_{+}(z,y)\cdot\nabla_{z}\widetilde{\Gamma}_{D}(z,x)dz \\ &+ \int_{\partial B_{\widetilde{L}}(0)} \Big[\widetilde{R}(x,z)\frac{\partial\Gamma_{+}}{\partial\nu}(z,y) - \Gamma_{+}(z,y)\frac{\partial\widetilde{R}}{\partial\nu}(x,z)\Big]d\sigma(z). \end{split}$$

For $|x|, |y| < r_1/2$, the last two integrals are bounded. Using (3.12) we obtain

$$\begin{aligned} |\widetilde{R}(x,y)| &\leq c \left(1 + \int_{C_{r_1}} |z|^{\alpha} |x-z|^{1-n} |y-z|^{1-n} dz \right) \\ &= c \left(1 + I_1 + I_2 \right), \end{aligned}$$

where c depends on L, α , k and n and

$$\begin{split} I_1 &= \int_{\{|z| < 4h\} \cap C_{r_1}} |z|^{\alpha} |x - z|^{1 - n} |y - z|^{1 - n} dz, \\ I_2 &= \int_{\{|z| > 4h\} \cap C_{r_1}} |z|^{\alpha} |x - z|^{1 - n} |y - z|^{1 - n} dz. \end{split}$$

Now

$$I_{1} \leq \int_{|w|<4} h^{\alpha} |w|^{\alpha} h^{1-n} \left| \frac{x}{h} - w \right|^{1-n} h^{1-n} \left| \frac{y}{h} - w \right|^{1-n} h^{n} dw$$

$$= h^{\alpha+2-n} \int_{|w|<4} |w|^{\alpha} \left| \frac{x}{h} - w \right|^{1-n} \left| \frac{y}{h} - w \right|^{1-n} dw$$

$$\leq h^{\alpha+2-n} F(\xi, \eta),$$

where h = |x - y| and

$$F(\xi,\eta) = 4^{\alpha} \int_{|w| < 4} |\xi - w|^{1-n} |\eta - w|^{1-n} dw$$

and $\xi = x/h$ and $\eta = y/h$. From standard bounds (see, for instance, [M] Chapter 2, Section 11), it is not difficult to see that

$$F(\xi,\eta) \le \text{const.} < \infty,$$

for all $\xi, \eta \in \mathbb{R}^n$, $|\xi - \eta| = 1$. Thus

$$I_1 \le c|x-y|^{\alpha+2-n}.$$

Let us consider now I_2 . Since $|y| = -y_n \le |x - y| = h$, we can deduce $|z| \le \frac{4}{3}|y - z|$ and $|z| \le 2|x - z|$ and thus obtain that

$$I_2 \le c \int_{|z|>4h} |z|^{\alpha+1-n+1-n} dz \le ch^{\alpha+2-n}.$$

Then we conclude

(4.12)
$$|\widetilde{R}(x,y)| \le c|x-y|^{\alpha+2-n},$$

for every $|x|, |y| < r_1/2$, where c depends on L, α , k and n only. Let us go back to the original coordinates system. We observe that if $x \in \Phi^{-1}(B^+_{r_1/2}(0))$ and $y = e_n y_n$, with $y_n \in (-r_1/2, 0)$ then $|\Phi(x) - x|$ is bounded by $c|x - y|^{1+\alpha}$. Namely, since $\Phi(x) \cdot y \leq 0$ and $\Phi(y) = y$, by (4.8) we have

(4.13)
$$c^{-1}|x| \le |\Phi(x)| \le |\Phi(x) - y| \le c|x - y|$$

On the other hand, by (4.9) and (4.13)

(4.14)
$$|\Phi(x) - x| \le \frac{c}{\overline{r}^{\alpha}} |x|^{1+\alpha} \le \frac{c'}{\overline{r}^{\alpha}} |x - y|^{1+\alpha}.$$

We have

$$\begin{split} R(x,y) &= \Gamma_D(x,y) - \Gamma_+(x,y) \\ &= \quad \Gamma_D(x,y) - \Gamma_+(x,y) + \Gamma_+(\Phi(x),\Phi(y)) - \Gamma_+(\Phi(x),\Phi(y)) \\ &= \quad \widetilde{R}(\Phi(x),\Phi(y)) + \Gamma_+(\Phi(x),y) - \Gamma_+(x,y). \end{split}$$

Using (4.8), (4.9), (4.12) and (4.14) we obtain

$$\begin{aligned} &|\Gamma_D(x,y) - \Gamma_+(x,y)| \\ &\leq \quad \frac{c}{\overline{r}^{\alpha}} |x-y|^{\alpha+2-n} + \frac{c}{\overline{r}^{\alpha}} \|\nabla\Gamma_+(\cdot,y)\|_{L^{\infty}(Q_{r_1})} |x-\Phi(x)| \\ &\leq \quad \frac{c}{\overline{r}^{\alpha}} |x-y|^{\alpha+2-n} + \frac{c'}{\overline{r}^{\alpha}} |x-y|^{1+\alpha} h^{1-n} \\ &\leq \quad \frac{c''}{\overline{r}^{\alpha}} |x-y|^{\alpha+2-n}, \end{aligned}$$

where c'' depends on k, n, α and L only. We estimate now the first derivative of R. To estimate the first derivative of \widetilde{R} let us consider a cube $Q \subset B^+_{r_1/4}(x)$ of side $cr_1/4$, with 0 < c < 1, such that $x \in \partial Q$. The following interpolation inequality holds:

$$\|\nabla \widetilde{R}(\cdot, y)\|_{L^{\infty}(Q)} \le c \|\widetilde{R}(\cdot, y)\|_{L^{\infty}(Q)}^{1-\delta} |\nabla \widetilde{R}(\cdot, y)|_{\alpha, Q}^{\delta},$$

where $\delta = \frac{1}{1+\alpha}$, c depends on L only and

$$|\nabla \widetilde{R}|_{\alpha,Q} = \sup_{x,x' \in Q, x \neq x'} \frac{|\nabla \widetilde{R}(x,y) - \nabla \widetilde{R}(x',y)|}{|x - x'|^{\alpha}}.$$

Since, from the piecewise Hölder continuity of $\nabla \Gamma_D$ see (4.3), and also of $\nabla \Gamma_+$, see (3.11), we have that

$$|\nabla \widetilde{R}(\cdot, y)|_{\alpha, Q} \le |\nabla \widetilde{\Gamma}_D(\cdot, y)|_{\alpha, Q} + |\nabla \Gamma_+(\cdot, y)|_{\alpha, Q} \le ch^{-\alpha + 1 - n},$$

where c depends on L only, thus we conclude

$$|\nabla_x \widetilde{R}(x,y)| \le \frac{c}{\overline{r}^{\eta}} h^{(\alpha+2-n)(1-\delta)} h^{(-\alpha+1-n)\delta} = \frac{c}{\overline{r}^{\eta}} h^{1-n+\eta},$$

where $\eta = \frac{\alpha^2}{1+\alpha}$. Thus

(4.15)
$$|\nabla_x \widetilde{R}(x,y)| \le \frac{c}{\overline{r}^{\eta}} |x-y|^{\eta+1-n},$$

where $\eta = \frac{\alpha^2}{1+\alpha}$ and c depends on L only. Concerning Γ_+ we have

$$\begin{aligned} |\nabla_x \Gamma_+(\Phi(x), y) - \nabla_x \Gamma_+(x, y)| \\ &= |D\Phi(x)^T \nabla \Gamma_+(\cdot, y)|_{\Phi(x)} - \nabla_x \Gamma_+(x, y)| \\ &\leq |(D\Phi(x)^T - I) \nabla \Gamma_+(\cdot, y)|_{\Phi(x)}| \\ &+ |\nabla \Gamma_+(\cdot, y)|_{\Phi(x)} - \nabla_x \Gamma_+(x, y)| \\ &\leq \frac{c}{\overline{r}^{\alpha}} \|\nabla \Gamma_+(\cdot, y)\|_{L^{\infty}(Q_{r_1})} |x - \Phi(x)| + |\nabla \Gamma_+(\cdot, y)|_{\alpha, Q} |\Phi(x) - x|^{\alpha} \\ &\leq \frac{c'}{\overline{r}^{\alpha}} h^{1+\alpha} h^{1-n} + \frac{c}{\overline{r}^{\alpha^2}} h^{-\alpha+1-n} h^{(1+\alpha)\alpha} \\ &\leq \frac{c}{\overline{r}^{\alpha^2}} h^{1-n+\alpha^2}, \end{aligned}$$

where c depends on k, n, α and L only.

Proof of Proposition 3.3. Let us fix $\overline{y} \in S_{2\overline{r}}$ and let us consider $f(\overline{y}, \cdot)$. We have that

(4.16)
$$\Delta_w f(\overline{y}, w) = 0 \qquad \text{in } \mathcal{C}\overline{\Omega}_D$$

For $w \in S_{2\overline{r}}$, by (2.6), (3.10) and (3.12) we have

(4.17)
$$|f(\overline{y},w)| \le C(\overline{r},L,M) ||\Lambda_{D_1} - \Lambda_{D_2}|| = \widetilde{\varepsilon}.$$

Let us now estimate $f(\overline{y}, w)$ when $w \in \mathcal{G}$. We define $\mathcal{G}^h = \{x \in \mathcal{G} : \operatorname{dist}(x, \Omega_D) \ge h\}$. For every $w \in \mathcal{G}^h$, we have that

$$(4.18) \qquad |S_{D_1}(\overline{y}, w)| \leq |k-1| \int_{D_1} |\nabla_x \Gamma_{D_1}(x, \overline{y})| |\nabla_x \Gamma_{D_2}(x, w)| dx$$
$$\leq c \int_{D_1} |x-w|^{1-n} dx \leq ch^{1-n}.$$

Similarly $|S_{D_2}(\overline{y}, w)| \leq ch^{1-n}$. Then we conclude that

$$|f(\overline{y}, w)| \le ch^{1-n}$$
 in \mathcal{G}^h

At this stage we shall make use of the three spheres inequality for supremum norms of harmonic functions v, see for instance [K-M], [K]. For every l_1 , l_2 , $1 < l_1 < l_2$ and for every $x \in \mathcal{G} \cup \mathcal{S}_{2\tau} \cup \Omega_{\tau}$ there exists $\tau \in (0, 1]$, depending only on l_1 , l_2 and n such that

$$\|v\|_{L^{\infty}(B_{l_{1}r}(x))} \le \|v\|_{L^{\infty}(B_{r}(x))}^{\tau}\|v\|_{L^{\infty}(B_{l_{2}r}(x))}^{1-\tau}$$

We apply it for $v(\cdot) = f(\overline{y}, \cdot)$ in the ball $B_{\overline{r}}(\overline{x})$, where $\overline{x} \in S_{2\overline{r}}$ be such that $\operatorname{dist}(\overline{x}, \Gamma) = \overline{r}/2$, where $\Gamma = \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) = \overline{r}\} \subset \partial S_{2\overline{r}}, l_1 = 3r = 3\overline{r}/2$ and $l_2 = 4r = 2\overline{r}$, then we obtain

$$(4.19) \|f(\overline{y},\cdot)\|_{L^{\infty}(B_{3\overline{r}/2}(\overline{x}))} \le \|f(\overline{y},\cdot)\|_{L^{\infty}(B_{\overline{r}/2}(\overline{x}))}^{\tau}\|f(\overline{y},\cdot)\|_{L^{\infty}(B_{2\overline{r}}(\overline{x}))}^{1-\tau}$$

For every $\overline{w} \in \mathcal{G}^h$, we denote with γ a simple arc in $\overline{\mathcal{G}} \cup \overline{\Omega_{\overline{r}}} \cup \overline{\mathcal{S}}_{2\overline{r}}$ joining \overline{x} to \overline{w} . Let us define $\{x_i\}$, $i = 1, \ldots, s$ as follows $x_1 = \overline{x}$, $x_{i+1} = \gamma(t_i)$, where $t_i = \max\{t : |\gamma(t) - x_i| = \overline{r}\}$ if $|x_i - \overline{w}| > \overline{r}$, otherwise let i = s and stop the process. By construction, the balls $B_{\overline{r}/2}(x_i)$ are pairwise disjoint, $|x_{i+1} - x_i| = \overline{r}$ for $i = 1, \ldots, s - 1$, $|x_s - \overline{w}| \leq \overline{r}$. For (2.1), there exists β such that $s \leq \beta$. An iterated application of the three spheres inequality (4.19) for $f(\overline{y}, \cdot)$ (see for instance [A-B-R-V] pg.780, [A-DB] Appendix E) gives that for any $r, 0 < r < \overline{r}$

$$(4.20) \|f(\overline{y},\cdot)\|_{L^{\infty}(B_{r/2}(\overline{w}))} \le \|f(\overline{y},\cdot)\|_{L^{\infty}(B_{r/2}(\overline{x}))}^{\tau}\|f(\overline{y},\cdot)\|_{L^{\infty}(\mathcal{G})}^{1-\tau}$$

We can now estimate the right hand side of (4.20) by (4.17) and (4.18) and obtain, for any $r, 0 < r < \overline{r}$

(4.21)
$$\|f(\overline{y},\cdot)\|_{L^{\infty}(B_{r/2}(\overline{w}))} \le c(h^{1-n})^{1-\tau^s} \varepsilon^{\tau^s} \le c(h^{1-n})^A \varepsilon^{\widetilde{\beta}},$$

where $\widetilde{\beta} = \tau^{\beta}$ and $A = 1 - \widetilde{\beta}$. Let $O \in \partial D_1$, as defined in (3.3), that is

$$d(O, D_2) = d_\mu(D_1, D_2).$$

There exists a $C^{1,\alpha}$ neighborhood U of O in $\partial \Omega_D$ with constants \overline{r} and L. Thus there exists a non-tangential vector field $\tilde{\nu}$, defined on U such that the truncated cone

(4.22)
$$C(O,\widetilde{\nu}(O),\theta,\overline{r}) = \left\{ x \in \mathbb{R}^n : \frac{(x-O) \cdot \widetilde{\nu}(O)}{|x-O|} > \cos\theta, \, |x-O| < \overline{r} \right\}$$

satisfies

$$C(O, \widetilde{\nu}(O), \theta, \overline{r}) \subset \mathcal{G},$$

where $\theta = \arctan(1/\overline{L})$. Let us define

$$\lambda_{1} = \min\left\{\frac{\overline{r}}{1+\sin\theta}, \frac{\overline{r}}{3\sin\theta}\right\}$$
$$\theta_{1} = \arcsin\left(\frac{\sin\theta}{4}\right),$$
$$w_{1} = O + \lambda_{1}\nu,$$
$$\rho_{1} = \lambda_{1}\sin\theta_{1}.$$

We have that $B_{\rho_1}(w_1) \subset C(O, \tilde{\nu}(O), \theta_1, \overline{r}), B_{4\rho_1}(w_1) \subset C(O, \tilde{\nu}(O), \theta, \overline{r})$. Let $\overline{w} = w_1$, since $\rho_1 \leq \overline{r}/2$, we can use (4.21) in the ball $B_{\rho_1}(\overline{w})$ and we can approach $O \in \partial D_1$ by constructing a sequence of balls contained in the cone $C(O, \tilde{\nu}(O), \theta_1, \overline{r})$. We define, for $k \geq 2$

$$w_k = O + \lambda_k \nu, \qquad \lambda_k = \chi \lambda_{k-1}, \qquad \rho_k = \chi \rho_{k-1}, \quad \text{with } \chi = \frac{1 - \sin \theta_1}{1 + \sin \theta_1}.$$

Hence $\rho_k = \chi^{k-1} \rho_1$, $\lambda_k = \chi^{k-1} \lambda_1$ and

$$B_{\rho_{k+1}}(w_{k+1}) \subset B_{\rho_{3k}}(w_k) \subset B_{\rho_{4k}}(w_k) \subset C(O,\nu,\theta,\overline{r}).$$

Denoting $d(k) = |w_k - O| - \rho_k = \lambda_k - \rho_k$, we have $d(k) = \chi^{k-1}d(1)$, with $d(1) = \lambda_1(1 - \sin\theta)$. For any $r, 0 < r \le d(1)$, let k(r) be the smallest integer such that $d(k) \le r$, that is

$$\frac{\left|\log \frac{r}{d(1)}\right|}{\left|\log \chi\right|} \le k(r) - 1 \le \frac{\left|\log \frac{r}{d(1)}\right|}{\left|\log \chi\right|} + 1.$$

By an iterated application of the three spheres inequality over the chain of balls $B_{\rho_1}(w_1), \ldots, B_{\rho_{k(r)}}(w_{k(r)})$, we have

(4.23)
$$\|f(\overline{y}, \cdot)\|_{L^{\infty}(B_{\rho_{k(r)}}(w_{k(r)}))} \leq c(h^{1-n})^{A(1-\tau^{k(r)-1})} \varepsilon^{\beta\tau^{k(r)-1}} \\ \leq c(h^{1-n})^{A} \varepsilon^{\beta\tau^{k(r)-1}},$$

for $0 < r < c\overline{r}$, where 0 < c < 1 depends on L only.

Let us consider now f(y, w) as a function of y. First observe that

$$\Delta_y f(y, w) = 0 \quad \text{in } \mathcal{C}\Omega_D, \quad \text{for all } w \in \mathcal{C}\Omega_D.$$

For $y, w \in \mathcal{G}^h$, $y \neq w$, using (3.12), we have

$$|S_{D_1}(y,w)| \le c \int_{D_1} |x-y|^{1-n} |x-w|^{1-n} dx \le ch^{2-n}.$$

Similarly for S_{D_2} . Therefore

$$|f(y,w)| \le ch^{2-2n}$$
 with $y, w \in \mathcal{G}^h$.

Finally, for $y \in S_{2\overline{r}}$ and $w \in \mathcal{G}^h$, using (4.23), we have

$$|f(y,w)| \le c(h^{1-n})^A \varepsilon^{\beta \tau^{k(h)-1}}$$

Proceeding as before, let us fix $w \in \mathcal{G}$ such that $\operatorname{dist}(w, \partial \Omega_D) = h$ and $\tilde{y} \in S_{2\overline{r}}$ such that $\operatorname{dist}(\tilde{y}, \Gamma) = \overline{r}/2$. Taking $r = \overline{r}/2$, $l_1 = 3r$, $l_2 = 4r$, $y_1 = O + \lambda_1 \nu$ and using iteratively the three spheres inequality, we have

$$\|f(y,w)\|_{L^{\infty}(B_{\overline{\tau}/2}(y_1))} \le \|f(y,w)\|_{L^{\infty}(B_{\overline{\tau}/2}(\widetilde{y}))}^{\tau^{\circ}} \|f(y,w)\|_{L^{\infty}(\mathcal{G})}^{1-\tau^{\circ}},$$

where τ and s are the same number established previously. Therefore

$$\begin{aligned} \|f(y,w)\|_{L^{\infty}(B_{\tau/2}(y_1)))}^{\tau^s} &\leq c(h^{2-2n})^{1-\tau^s}(h^{1-n})^{A\tau^s}(\varepsilon^{\beta\tau^{k(h)-1}})^{\tau^s} \\ &\leq c(h^{2-2n})^{1-\gamma}(h^{1-n})^{A\tau^s}(\varepsilon^{\beta\tau^{k(h)-1}})^{\gamma} \\ &\leq c(h^{2-2n})^{A'}(\varepsilon^{\beta\tau^{k(h)-1}})^{\gamma}, \end{aligned}$$

where $\gamma = \tau^{\beta}$, with β as before, so $0 < \gamma < 1$, and $A' = A\tau^s + 1 - \gamma$. Once more, let us apply iteratively the three spheres inequality over a chain of balls contained in a cone with vertex in O and we obtain

(4.24)
$$||f(y,w)||_{L^{\infty}(B_{\rho_k}(y_{k(h)}))} \leq c(h^{2-2n})^{A'(1-\tau^{k(h)-1})} (\varepsilon^{\beta\tau^{k(h)-1}})^{\gamma\tau^{k(h)-1}}$$

Now, from (4.24), choosing $y = w = h\nu(O)$, where $\nu(O)$ is the exterior unit normal to $\partial\Omega_D$ in O, we obtain

(4.25)
$$|f(y,y)| \le ch^{A^{\prime\prime}} (\varepsilon^{\beta \tau^{k(h)-1}})^{\gamma \tau^{k(h)-1}},$$

where $A'' = -(2-2n)\beta A' > 0$. We observe that, for $0 < h < c\overline{r}$, where 0 < c < 1 depends on L, $k(h) \le c |\log h| = -c \log h$, so we can write

$$\tau^{k(h)} = e^{-c \log h \log(\tau)} = h^{-c \log \tau} = h^{c |\log \tau|} = h^F,$$

with $F = c |\log \tau|$. Therefore

$$\begin{aligned} |f(y,y)| &\leq h^{-A''} \varepsilon^{B\tau^{k(h)}} \\ &= e^{-A'' \log h} e^{B\tau^{k(h)} \log \varepsilon} \\ &= e^{-A'' \log h + B' h^F \log \varepsilon}. \end{aligned}$$

Then in (4.25) we obtain that

$$|f(y,y)| \leq \mathrm{e}^{-A'\log h + B'h^F \log \varepsilon} = \frac{\varepsilon^{B'h^F}}{h^{A'}}.$$

Proof of Proposition 3.4. Let us consider $y = h\nu(O)$, where $\nu(O)$ is the exterior outer normal to $\partial\Omega_D$ in O with O defined as in (3.3), $0 < h < \overline{r}_0$, where \overline{r}_0 is the number introduced in Proposition 3.2 and $x \in D_1$ such that |x - y| < r, with $0 < r < \overline{r}_0$. Let us first observe that since $O \in \partial D_1$ and $x \in D_1$, for Γ_{D_1} we have the asymptotic formula (3.14), which says that

$$\left|\nabla_x \Gamma_{D_1}(x,y) - \nabla_x \Gamma_+(x,y)\right| \le c_1 |x-y|^{1-n+\delta}.$$

Furthermore, since we are in the situation in which $x \in D_1$ and $y \notin D_1$, for (3.11), $\Gamma_+(x,y) = 2/(k+1)\Gamma(x,y)$, where $\Gamma(x,y)$ denotes the standard fundamental solution of the Laplace operator. Let us consider now $\Gamma_{D_2}(x,y)$. With our choice of O, x and y, we know that $y \notin D_2$ but we do not have any information on x, that is we do not know in which side of the interface ∂D_2 it is.

Thus we have to distinguish different situations.

If $x \in B_r(O) \cap D_1 \cap D_2$, then we have the asymptotic formula (3.11) for Γ_{D_2} and from Lemma 3.1 of [A] the following formula holds

(4.26)
$$\nabla_x \Gamma_{D_1}(x,y) \cdot \nabla_x \Gamma_{D_2}(x,y) \ge c|x-y|^{2-2n}.$$

Consider now the case $x \in (D_1 \setminus D_2) \cap B_r(O)$. In this region let us consider a smaller ball $B_{\rho}(O)$ centered in O with radius ρ where $0 < \rho < d_{\mu}$. Since the definition of d_{μ} we have $B_{\rho} \cap D_2 = \emptyset$. If x and y are in $B_{\rho}(O)$, we have

(4.27)
$$\begin{cases} \Delta_x \big(\Gamma_{D_2}(x, y) - \Gamma(x, y) \big) = 0 & \text{in } B_\rho(O) \\ \big[\Gamma_{D_2}(x, y) - \Gamma(x, y) \big]_{|\partial B_\rho(O)} \le c \rho^{2-n}. \end{cases}$$

Thus by the maximum principle

(4.28)
$$\left|\Gamma_{D_2}(x,y) - \Gamma(x,y)\right| \le c_1 \rho^{2-n} \qquad \forall x, y \in B_{\rho}(O),$$

and by interior gradient bound

(4.29)
$$\left| \nabla_x \Gamma_{D_2}(x, y) - \nabla_x \Gamma(x, y) \right| \le c_2 \rho^{1-n} \quad \forall x \in B_{\rho/2}(O), \forall y \in B_{\rho}(O).$$

Thus, using Lemma 3.1 of [A], in $B_{\rho/2}(O)$ we obtain the formula

(4.30)
$$\nabla_x \Gamma_{D_1}(x, y) \cdot \nabla_x \Gamma_{D_2}(x, y) \ge c_3 |x - y|^{2-2n} - c_4 \rho^{2-2n}.$$

Let us consider $h \leq \overline{r}_0/2$ and $B_r(O) = \{x \in \mathbb{R}^n : |x - O| < r\}$, with $0 < r < \overline{r}_0$. Then we have

$$|S_{D_{1}}(y,y)|$$

$$= |k-1| \left| \int_{D_{1} \cap B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx + \int_{D_{1} \smallsetminus B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right|$$

$$\geq |k-1| \left| \int_{D_{1} \cap B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right| - |k-1| \left| \int_{D_{1} \smallsetminus B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right|$$

The first term can be estimated as follows

$$\begin{aligned} \left| \int_{D_{1} \cap B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right| \\ &= \left| \int_{(D_{1} \cap D_{2}) \cap B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx + \int_{(D_{1} \smallsetminus D_{2}) \cap B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right| \\ &\geq \left| \int_{(D_{1} \cap D_{2}) \cap B_{r}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx + \int_{(D_{1} \smallsetminus D_{2}) \cap B_{\rho}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right| \\ &- \left| \int_{[(D_{1} \smallsetminus D_{2}) \cap B_{r}(O)] \smallsetminus B_{\rho}(O)} \nabla \Gamma_{D_{1}} \cdot \nabla \Gamma_{D_{2}} dx \right| \end{aligned}$$

In conclusion, choosing $\rho = d_{\mu}/2$ and using (4.26) and (3.12) we obtain

$$\begin{aligned} |S_D(y,y)| &\geq c_1 \int_{[(D_1 \cap D_2) \cap B_r(O)] \cup [(D_1 \setminus D_2) \cap B_{d_{\mu/2}}(O)]} |x-y|^{2-2n} dx \\ &- c_2 \int_{[(D_1 \setminus D_2) \cap B_r(O)] \setminus B_{d_{\mu/2}}(O)} |x-y|^{1-n} |x-y|^{1-n} dx \\ &- c_3 \int_{D_1 \setminus B_r(O)} |x-y|^{1-n} |x-y|^{1-n} dx \\ &\geq c_4 h^{2-n} - c_5 d_{\mu}^{2-2n} - c_7. \end{aligned}$$

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