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TIGHTER UPPER BOUNDS ON THE EXACT COMPLEXITY OF STRING MATCHING*

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Abstract. This paper considers how many character comparisons are needed to find all occurrences of a pattern of length m in a text of length n. The main contribution is to show an upper bound of the form of $n + O(n/m)$ character comparisons, following preprocessing. Specifically, we show an upper bound of $n + \frac{8}{3(m+1)}(n-m)$ character comparisons. This bound is achieved by an online algorithm which performs $O(n)$ work in total and requires $O(m)$ space and $O(m^2)$ time for preprocessing. The current best lower bound for online algorithms is $n + \frac{16}{7m+27}(n-m)$ character comparisons for $m = 16k + 19$, for any integer $k \geq 1$, and for general algorithms is $n + \frac{2}{m+3}(n-m)$ character comparisons, for $m = 2k + 1$, for any integer $k \geq 1$.

Key words. string matching, exact complexity, comparisons, periodicity

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1. Introduction.

String matching is the problem of finding all occurrences of a pattern $p[1 \ldots m]$ in a text $t[1 \ldots n]$. We assume that the characters in the text are drawn from a general (possibly infinite) alphabet unknown to the algorithm. We investigate the time complexity of string matching measuring both the exact number of comparisons and the time complexity counting all operations. As is standard, the time complexity refers to operations performed following preprocessing of the pattern; prepossessing of the text is not allowed. Our goal is to minimize the number of comparisons while still maintaining a total linear-time complexity and a polynomial-in-m preprocessing cost.

Note that if the algorithm is permitted to know the alphabet, then there is a finite automaton which performs string matching by reading each text character exactly once (which can be obtained from the failure function of [KMP77]). However, in this case the running time depends on the alphabet size.

Perhaps the most widely known linear-time algorithms for string matching are the Knuth–Morris–Pratt [KMP77] and Boyer–Moore [BM77] algorithms. We refer to these as the KMP and BM algorithms, respectively. The KMP algorithm makes at most $2n - m + 1$ comparisons and this bound is tight. The exact complexity of the BM algorithm was an open question until recently. It was shown in [KMP77] that the BM algorithm makes at most $6n$ comparisons if the pattern does not occur in the text. Guibas and Odlyzko [GO80] reduced this to $4n$ under the same assumption. Cole [Co91] finally proved an essentially tight bound of $3n - \Omega(n/m)$ comparisons for the BM algorithm, whether or not the pattern occurs in the text. Colussi [Col91] gave a simple variant of the KMP algorithm which makes at most $\frac{3}{2}n$ comparisons. Apostolico and Giancarlo [AG86] gave a variant of the BM algorithm which makes at

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most $2n - m + 1$ comparisons. Crochemore et al. [CCG92] showed that remembering just the most recently matched portion reduces the upper bound of BM from $3n$ to $2n$ comparisons.

Recently, Galil and Giancarlo [GG92] gave a string-matching algorithm which makes at most $\frac{4}{3}n$ comparisons. This was the strongest upper bound for string matching known prior to our work. In fact, [GG92] gave this bound in a sharper form as a function of the period z of the pattern; the bound becomes $n + \min\{\frac{1}{3}, \frac{\min\{z,m-z\}+2}{2m}\}$ $(n - m)$.

Galil and Giancarlo [GG91] gave a lower bound of $n(1 + \frac{1}{2m})$ comparisons. For online algorithms, [GG91] showed an additional lower bound of $n(1+\frac{2}{m+3})$. An online algorithm is an algorithm which examines text characters only in a window of size m sliding monotonically to the right; further, the window can slide to the right only when all matching pattern instances to the left of the window or aligned with the window have been discovered. Recently, Zwick and Paterson gave additional lower bounds, including a bound of $\frac{4n}{3}$ for patterns of length 3 in the general case [ZP92].

Our contribution is a linear-time online algorithm for string matching which makes at most $n(1 + \frac{8}{3(m+1)})$ character comparisons. Our algorithm requires $O(m)$ space and $O(m^2)$ preprocessing time and runs in $O(m + n)$ time overall (exclusive of preprocessing). Independently, Breslauer and Galil discovered a similar algorithm which performs at most $n + O(\frac{n \log m}{m})$ comparisons [BG92]; this algorithm requires $O(m)$ preprocessing space and time and runs in linear time. Recently, Hancart [Ha93] and Breslauer et al. [BCT93] have independently shown an upper and lower bound of $(2 - \frac{1}{m})n$ on the number of comparisons required for string matching when comparisons must involve only text characters in a window of size one sliding monotonically to the right.

Nearly matching lower bounds are given in a companion paper [CHPZ92]. They show the following bounds: for online algorithms, a bound of $n + \frac{16}{7m+27}(n - m)$ character comparisons for $m = 16k + 19$, for any integer $k \geq 1$; and for general algorithms, a bound of $n + \frac{2}{m+3}(n - m)$ character comparisons, for $m = 2k + 1$, for any integer $k \geq 1$.

Even if exponential (in m) preprocessing and exponential space are available, it is not clear that the above upper bound can be achieved (assuming that a result independent of the alphabet size is sought). The difficulty is that text characters which are mismatched may need to be compared repeatedly. In order to minimize the total number of comparisons, this has to be offset by other text characters which do not need to be compared. The hardest patterns to handle are those which have proper suffixes which are also prefixes of the pattern. We refer to such substrings as *presufs*.[1] Our algorithm has two parts: a basic algorithm and a presuf handler. The basic algorithm handles primary patterns, i.e., patterns with no presufs; this is also the core of the algorithm for the general case. The presuf handler copes with presufs; its design constituted the main challenge in this work. Understanding the structure of the presufs was a key ingredient in its design. Understanding this structure also led to the new lower bound constructions given in [CHPZ92].

The flavor of the algorithm is as follows. Initially, the pattern is aligned with the left end of the text. Repeatedly, an attempt to match the pattern against the text is made. When a mismatch is found or the pattern is fully matched the pattern is shifted to the right. The goal is to maximize this shift without missing any possible

[1] Presufs are also called *borders* in the literature. Strings without presufs are called *primary* strings.

matches. The basic algorithm has the property that the length of each shift is at least equal to the number of comparisons since the previous shift (or the start of the algorithm). This results in an algorithm that performs at most n comparisons if the pattern has no presuf (the algorithms of [GG92] and [CP89] also have this property).

The presuf handler cannot quite match the performance of the basic algorithm (which is not surprising given that the lower bounds for this problem are larger than n comparisons). Here the approach is to follow the basic algorithm until a suffix which is also a prefix is matched. The only possible matches in which a new instance of the pattern overlaps the current partially (or fully) matched instance arise with an overlap by a presuf. Ignoring, for the moment, problems introduced by periodic patterns, it is the case that at most one of these overlapping pattern instances can result in a match. An elimination is performed to determine which one, if any, of the overlapping pattern instances might result in a match. Following this elimination, a further nontrivial sequence of comparisons is made; this can lead to one of two situations: another match of a suffix which is also a prefix, or a mismatch which causes a return to the basic algorithm. The presuf handler is invoked at most once for every $\frac{m}{2}$ text characters and performs a number of comparisons at most two greater than the number of characters shifted over. (Actually, there are two possible scenarios: an invocation after $\frac{3}{4}m$ text characters and at most two excess comparisons, or an invocation after $\frac{m}{2}$ text characters and at most one excess comparison.) Periodic patterns have the added difficulty that the presuf handler could be invoked more frequently. In this case, we show the additional fact that if the presuf handler is invoked after fewer than $\frac{m}{2}$ text characters, then the number of comparisons is at most the number of characters shifted over.

This structure of the algorithm of Breslauer and Galil is similar; their analogue of the presuf handler works in a completely different way, however.

Section 2 provides several definitions. The basic algorithm is described in section 3. In section 4, the presuf handler for nonperiodic strings is presented. Section 5 gives a technical construction deferred from section 4. Finally, in section 6, the result is extended to periodic patterns.

We remark here that the properties of strings which we develop in section 4 and later are mostly new and appropriate references are given otherwise.

2. Definitions and preliminaries. A string v is a *presuf* of p if it is both a proper suffix and a proper prefix of p. Let x be the length of the largest presuf of p. The *period* of a pattern p with length m is defined to be $m - x$. x is called the *s-period* (or shift period) of p. A string p is *cyclic* in string v if it is of the form v^k, $k > 1$. A *primitive* string is a string which is not cyclic in any string.

A string p is *periodic* if $p = wv^k$, where w is a (possibly null) proper suffix of v and $k > 1$. The smallest such v is called the *core* of p and the corresponding w is called the *head* of p. Note that the core is primitive. A *cyclic shift* of p is any string vu where $p = uv$. $|v$ and $v|$ refer, respectively, to the leftmost and rightmost characters in string v; on occasion, we will call these characters, respectively, the left end and right end of v. Two characters are said to be *distance* d apart if they are separated by $d - 1$ other characters.

For the rest of the paper, let p be a pattern with length m. Let the text t have length n. $p[i]$ denotes the ith character of p, reading from the left end; i is called the *index* of $p[i]$ in p. The same notation and terminology is used for string t.

The algorithm will be comparing the pattern with substrings of the text with which the pattern is aligned; as the algorithm proceeds, the pattern is shifted to the

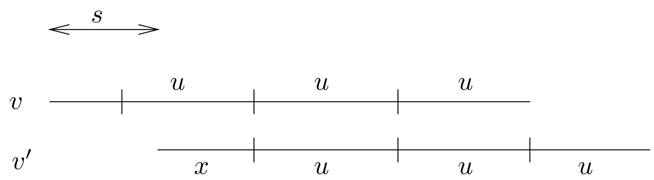

FIG. 1. *Periodicity.*

right across the text. Each possible alignment of the pattern with the text is called an *instance* of the pattern. Note that an instance is not necessarily an occurrence.

For each pair of overlapping instances of the pattern a location at which the two differ, if any, will be precomputed. This location is called the *difference point* of the two instances. Note, however, that for a given pair, a difference point may not exist, but this can happen only if the pattern has a nonempty presuf. Let p_1 and p_2 denote two pattern instances, where $p_1[i]$ is aligned with $p_2[1]$; then dif_i is the difference point if any; i.e., $p_1[dif_i] \neq p_2[dif_i - i + 1]$.

Let q be a pattern instance. Those pattern instances to the right of q, overlapping q, but which do not have a difference point with q are called the *presuf overlaps* of q.

We quote a few standard results concerning strings.

LEMMA 2.1. *Let w be a presuf of string v. If $|w| > \frac{|v|}{2}$, then v is periodic.*

Proof. See Fig. 1. Let $s = |v| - |w|$. Let v' denote string v shifted distance s to the right. Then the portion of v' overlapping v is presuf w which matches the corresponding portion of v. Let u denote the suffix of v' of length s. An easy induction shows that $v = xu^k$ for some $k \geq 2$, where x is a proper suffix of u. ∎

The following appear in different forms in [Lo82] (see Propositions 1.3.2, 1.3.4, and 1.3.5 there).

LEMMA 2.2 (see [LS62, FW65]). *If x and y are two distinct periods of a string v such that $x + y \leq m + \gcd\{x, y\}$, then $\gcd\{x, y\}$ is also a period of v.*

LEMMA 2.3. *Suppose that $v = xy$, where both x and y are presufs of v. Then v is cyclic in some string w of length $\gcd\{|x|, |y|\}$.*

LEMMA 2.4. *If v is periodic and can be expressed both as $x_1 u_1^{k_1}$ and $x_2 u_2^{k_2}$, where x_i is a suffix of u_i, $u_1 > u_2$, and $k_1, k_2 \geq 2$, then either u_1 is cyclic in u_2 or both u_1 and u_2 are cyclic in some smaller string.*

3. The basic algorithm. The algorithm in this section also appears in [Col91] and is also exposed in [GG92]. We describe it again for the sake of completeness.

If all the characters in p are identical, then it is easily seen that the KMP algorithm makes at most n character comparisons. Further, if $m = 2$ and p consists of two distinct characters, then the BM algorithm makes at most n character comparisons. Henceforth, we assume that $m > 2$ and that p has at least two distinct characters.

The algorithm proceeds by eliminating pattern instances as possible matches. It repeatedly performs the following two steps: first, it attempts to match the leftmost surviving pattern instance with the aligned text substring; then, it shifts to the next leftmost surviving pattern instance.

After a shift occurs, the strategy followed depends on the nature of the shift. The order in which pattern characters are compared ensures that all the shifts satisfy one of the following two properties.

1. A shift has size greater than or equal to the number of comparisons made since the previous shift. This is called a *basic shift.*
2. When property 1 is not true, a proper prefix x of p is completely matched with the text after the shift. Moreover, x is also a suffix of p. This is called a *presuf shift.*

Following a basic shift, the basic algorithm is continued; a presuf shift results in a transfer to the presuf handler.

The following observation is the key to the basic algorithm. Consider two overlapping instances of the pattern p. Then comparing either of the two pattern characters at their difference point with the aligned text character is sure to eliminate one of the two pattern instances from being a potential match. As long as the overlap is not a presuf of p, there will be a difference point. This is exactly the notion of *duelling* introduced by Vishkin [Vi85].

More formally, let p_a and p_b be the two leftmost surviving pattern instances, where p_b is not a presuf overlap of p_a. Let d be the difference point of p_a and p_b. $p_a[d]$ is compared with the aligned text character. A match eliminates p_b; a mismatch eliminates p_a.

Next, we give the exact sequence of comparisons made by the above strategy. We precompute the following sequence S. S is the sequence of indices $dif_2, dif_3, \ldots, dif_m$ omitting repetitions and undefined indices. Henceforth, where no ambiguity will result, we will use the sequence S to refer both to the indices it contains and to the corresponding characters in p_a.

The characters in p_a are compared with their corresponding text characters in two passes, stopping if a mismatch is found. In pass 1, those characters in p_a contained in S are compared in sequence. If all of these match, then the remaining pattern characters are compared from right to left in pass 2.

LEMMA 3.1. *If a mismatch occurs at the character given by the kth index in S, then the resulting shift has size at least k.*

Proof. Let the kth index in S be dif_l. Note that $k < l$. Recall that $l \leq dif_l \leq m$ and $p[dif_l] \neq p[dif_l - l + 1]$). Suppose for a contradiction that the shift was of length $j < k$. Let p_a and p_b be the pattern instances as specified in the algorithm above, before this shift. Note that p_b becomes p_a after the shift; i.e., p_b is p_a shifted j units. But then p_a and p_b have a difference point and hence dif_{j+1} is defined. dif_{j+1} is the ith index in S, for some $i \leq j$; hence since $j < k$, dif_{j+1} occurs prior to dif_l in S. Therefore, $p_a[dif_{j+1}]$ would have been matched against the text and one of p_a or p_b eliminated before $p_a[dif_l]$ was compared. The contradiction proves the lemma. ☐

Consequently, all shifts resulting from mismatches in pass 1 are basic shifts. When a basic shift is made, the basic algorithm is restarted. It is easy to see that if all shifts are basic shifts, then the total number of comparisons made is upper bounded by n.

Next, suppose that all comparisons in pass 1 result in matches.

LEMMA 3.2. *Suppose pass* 2 *results in a mismatch at $p_a[l]$. The resulting shift has length at least l.*

Proof. Suppose for a contradiction that the resulting shift has length i, $i < l$. Let p_b be p_a shifted distance i. Then l is a difference point for p_a and p_b; hence one of p_a and p_b would have been eliminated in pass 1, a contradiction. ☐

Consequently, for each shift resulting from pass 2 with length less than the number of comparisons made since the previous shift, a proper prefix of p (which is also a suffix of p) is matched with the text; i.e., it is a presuf shift. The main challenge in minimizing the exact number of comparisons is to handle presuf shifts.

Preprocessing. The sequence S, as defined above, is not unique. We show that a particular instance of S can be precomputed in a manner akin to the computation of the KMP shift function or the BM shift function. The KMP shift function comprises, for each j, $1 < j \leq m$, a number s_j. s_j is the largest i, $i < j$, such that $p[1 \ldots i-1] = p[j-i+1 \ldots j-1]$ and $p[i] \neq p[j]$; note that $i = dif_{j-i+1}$. If no such i exists, then s_j is defined to be zero. Consider the set of all those values of j for which $s_j > 0$. Furthermore, let this set be ordered by the increasing value of $j - s_j + 1$. This provides the sequence S. For every k, $2 \leq k \leq m$, if dif_k is defined, then for some $l \in S$, $k \leq l \leq m$, $p[1 \ldots l-k] = p[k \ldots l-1]$ and $p[l-k+1] \neq p[l]$; hence the value dif_k occurs in S (though not necessarily indexed by k). Finally, it is straightforward to compute S in $O(m)$ time.

4. The presuf handler. In this section, the presuf handler for nonperiodic patterns p is described. This presuf handler also deals with some presuf shifts for periodic p, as specified in the next few paragraphs.

With each presuf shift, we associate a presuf x_1' of p, defined as follows. If p is not periodic, then x_1' is the longest presuf of p. Otherwise, suppose $p = u_p v_p^{i_p}$ is periodic with core v_p and head u_p. Then if the presuf of p matching the text is at least $|v_p|$ long, $x_1' = u_p v_p^{i_p - 1}$. Otherwise, if the above presuf is shorter than $|v_p|$, then x_1' is defined to be the longest presuf of p of length less than $|v_p|$.

In this section, we give an algorithm for handling presuf shifts for the case $|x_1'| < \frac{m}{2}$. The case $|x_1'| \geq \frac{m}{2}$ is considered in section 6. Note that $|x_1'| < \frac{m}{2}$ always holds for nonperiodic p and may hold for periodic p.

Consider the situation immediately following a presuf shift. Some prefix of p, which is also a presuf, matches the text substring that it is aligned with. It is convenient for the presuf handler to assume that the pattern was shifted by $m - |x_1'|$ characters and that x_1' matches the text. Note that this will not be the case if pass 2 in the basic algorithm mismatches before x_1' is completely matched. A simple check will prevent the declaration of any incorrect complete match that might result from the above assumption. To facilitate this check, a variable t_{last} is used. Suppose pass 2 in the basic algorithm ends in a mismatch. Then t_{last} is set to the index of the text character where the mismatch occurred. Otherwise, if no mismatch occurs, $t_{\text{last}} \leftarrow \phi$.

Since $|x_1'| < \frac{m}{2}$, $p = x_1' u x_1'$, for some string u. Let t_A be the substring of the text aligned with the prefix x_1' of p immediately following the presuf shift; note that x_1' matches t_A. Order all the presufs of x_1' by decreasing length and let this order be $x_1, x_2, x_3, \ldots, x_k, x_{k+1}$, where x_k is the smallest nonnull presuf of x_1' and x_{k+1} is the null string and hence a trivial presuf of x_1'. Note that $x_1 = x_1'$. Let the future instances of p (i.e., potential match instances) before its left end slides beyond $t_A|$, in left to right order, be $p_1, p_2, \ldots, p_k$. Let p_{k+1} be the pattern instance whose left end is to the immediate right of $t_A|$. Then p_i, $1 \leq i \leq k+1$, is the pattern instance with the prefix x_i of p aligned with the suffix x_i of t_A. x_i is said to be the presuf associated with p_i. $p_1, p_2, \ldots, p_k, p_{k+1}$ are called the *presuf pattern instances*.

LEMMA 4.1. *If $|x_1'| < \frac{m}{2}$, then at most one of $p_1, \ldots, p_k, p_{k+1}$ can lead to a complete match.*

Proof. This is a proof by contradiction. Suppose some two of them, say p_i and p_j, $i < j$, each result in a complete match. It follows that there is a prefix of p of size $m - |x_i| + |x_j|$ that matches a suffix of p. Since $|x_i| - |x_j| < \frac{m}{2}$, $m - |x_i| + |x_j| > \frac{m}{2}$; also, $|x_i| - |x_j| \leq |x_1'|$. This implies that p is periodic with core of length at most $|x_1'|$, contrary to our assumption. □

The presuf handler begins by eliminating all but at most one of $p_1, p_2, \ldots, p_k, p_{k+1}$.

This is carried out by a procedure that performs $j \le k$ comparisons; at most two of these comparisons are unsuccessful. We seek to minimize the number of unsuccessful comparisons because while successful comparisons can be remembered, unsuccessful comparisons may lead to repeated comparison of some text characters.

The elimination procedure is described in section 4.1. The remainder of the presuf handler procedure for all but two special cases is given in section 4.2, and its analysis is presented in section 4.3. The special cases are handled in section 4.4. Finally, data structure details are described in section 4.5.

4.1. Elimination strategy. Before describing the exact sequence of comparisons made by the elimination strategy, we need to understand some structural properties of these overlapping instances of p.

LEMMA 4.2. *Suppose $x_i = uv^l$ is the ith presuf, where u is a proper suffix of v, v is primitive, and $l \ge 2$. Then $x_{i+1} = uv^{l-1}$.*

Proof. Certainly, uv^{l-1} is a presuf, so the only question is whether there is a presuf x between uv^l and uv^{l-1}. Suppose there is such an x. Since $|uv^{l-1}| < |x| < |uv^l|$ and since x is a prefix of uv^l, the suffix of x of length $|v|$ is a cyclic shift of v. But x is a suffix of uv^l, which implies that a proper cyclic shift of v matches v. By Lemma 2.3, v is cyclic, contrary to our assumption. ∎

LEMMA 4.3. *The presuf pattern instances can be partitioned into $g = O(\log m)$ groups*[2] *$A_1, A_2, \ldots, A_g$. The groups preserve the left-to-right ordering of the pattern instances; i.e., the pattern instances in group A_i are all to the left of those in group A_{i+1}, for $i = 1, \ldots, g-1$. Let B_i be the set of presufs associated with the pattern instances in A_i. Then either $B_i = \{u_iv_i^{k_i}, \ldots, u_iv_i^3, u_iv_i^2\}$ or $B_i = \{u_iv_i^{k_i}, \ldots, u_iv_i\}$ or $B_i = \{u_iv_i^{k_i}, \ldots, u_iv_i, u_i\}$, where $k_i \ge 1$ is maximal, u_i is a proper suffix of v_i, and v_i is primitive.*

Proof. The proof is by construction. The groups are constructed in left-to-right order. Inductively suppose A_i is being built presently and all presuf pattern instances with associated presufs longer than $u_iv_i^{k_i}$ have been placed in groups to the left of A_i.

$\{u_iv_i^{k_i}, \ldots, u_iv_i^2\}$ are all added to B_i. u_iv_i is also added if and only if it is not periodic; otherwise, u_iv_i starts set B_{i+1}. By Lemma 4.2, all presuf pattern instances with associated presufs longer than u_iv_i are in group A_i or by induction in a group to its left. In addition, if u_i is empty and v_i has no presufs, then u_i is also added.

The maximality of k_i can be seen as follows. Suppose k_i is not maximal; i.e., there exists a presuf w of the form $u_iv_i^{k_i+1}$, $k_i+1 \ge 2$. By the inductive hypothesis describing the construction, this presuf would already be in one of the groups $B_1, \ldots, B_{i-1}$. By Lemma 4.2, it follows that w is the smallest presuf in B_{i-1}. w is clearly periodic. By construction, $w = u_{i-1}v_{i-1}^2 = u_iv_i^{k_i+1}$, $k_i + 1 \ge 2$. Then by Lemma 2.4, v_{i-1} must be cyclic, which contradicts the assumption that v_{i-1} is primitive. Thus k_i must be maximal.

This shows that the presuf pattern instances are partitioned into groups. It remains to show that there are only $O(\log m)$ groups. Let x_{j_i} be the leftmost presuf in B_i. If $x_{j_{i+1}} = u_iv_i$, then $|x_{j_{i+1}}| \le \frac{2}{3}|x_{j_i}|$, and otherwise $|x_{j_{i+1}}| \le \frac{1}{2}|x_{j_i}|$. (The latter claim follows because $x_{j_{i+1}}$ is both a prefix and a suffix of x_{j_i} and this prefix and suffix are nonoverlapping.) The $O(\log m)$ bound follows immediately. ∎

LEMMA 4.4. *The groups satisfy the following properties.*

Property 1. *Consider the presufs x_i corresponding to the pattern instances p_i in some group A_j. For $j \ne g$, all of these presufs x_i, except possibly the rightmost one,*

[2] Actually, a sharper bound of $\log_\phi m$ groups is known [KMP77, B94], where ϕ is the golden ratio.

are periodic with the same core and head. For $j = g$, all but the rightmost two presufs are periodic with the same core and head.

Property 2. *Let p_i be the rightmost instance in its group. If x_i is periodic then so is x_{i+1}.*

Property 3. *Suppose p_i is the rightmost instance in its group A_j and x_i is periodic with head u and core v; then $|x_{i+2}| < |v|$. Further, suppose $x_{i+1} = u'(v')^l$, where v' is primitive and u' is a proper suffix of v'. Then $|v'| > |u|$.*

Property 4. *Suppose p_i is the rightmost instance in its group A_j, where $|A_j| > 1$; further, suppose that x_{i-1} is periodic with core v and x_i is not periodic. Then $|x_{i+1}| < |v|$.*

Property 5. *Both p_k and p_{k+1} are in the group A_g.*

Proof. Let p_i be in group A_j.

Property 1 is true by definition. To see Property 2, note that since x_i is periodic, $x_i = uv^l$, where u is a proper suffix of primitive v and $l \geq 2$. But if $l > 2$, then the pattern instance corresponding to either presuf uv^2 or presuf uv would be the rightmost item in A_j. Thus $l = 2$; however, by definition, the pattern instance corresponding to uv is not in A_j only if uv is periodic. Finally, by Lemma 4.2, $x_{i+1} = uv$.

Property 3 can be seen as follows. As in the previous paragraph, $x_i = uv^2$ and $x_{i+1} = uv$. Again uv is periodic; that is, $uv = u'(v')^l$ for some $l \geq 2$, where u' is a proper suffix of v' and v' is primitive. By Lemma 4.2, $x_{i+2} = u'(v')^{l-1}$. Suppose $|v'| \leq |u|$. Since v is primitive, there must be a substring v' of $u'(v')^l = uv$ which straddles the boundary between u and v. Thus the substring of $u'(v')^l$ aligned with the rightmost $|v'|$-sized substring of u is a proper cyclic shift of v'. But since u is a suffix of v this substring is also identical to v'. By Lemma 2.3, v' is cyclic, a contradiction. Thus $|v'| > |u|$ and hence $|x_{i+2}| < |v|$.

Property 4 can be seen as follows. As with the previous properties, it follows that $x_i = uv$, where u is a proper suffix of v and v is primitive. If $|x_{i+1}| \geq |v|$, then $|x_{i+1}| > |x_i|/2$. But x_{i+1} is a presuf of x_i; by Lemma 2.1, x_i would be periodic, a contradiction.

Property 5 can be seen as follows. Since x_k is the smallest nonnull presuf of p, no nonnull prefix of x_k matches a suffix of x_k. Therefore, all strings in B_g have the form $u_g v_g^l$, $0 \leq l \leq k_g$, where u_g is the null string and $v_g = x_k$. Since both x_k and x_{k+1} have this form, Property 5 is true. □

Remark. The elimination strategy described below and the algorithm in section 4.2, which uses this elimination strategy to handle presuf shifts, work for most patterns p. However, there are some patterns for which presuf shifts must be handled differently. The reason for this is made clear in section 5, which gives a technical portion of the analysis of the algorithm in section 4.2. These exception patterns are precisely those in which x_k, the smallest nonnull presuf, is a single character and g, the number of groups, is one. Presuf shifts for these exception patterns are handled separately in section 4.4.

DEFINITION. *A clone set is a set $Q = \{s_1, s_2, \ldots\}$ of strings, with $s_i = uv^{k_i}$, where u is a proper suffix of primitive v and $k_i \geq 0$. A set U of pattern instances is* half-done *if $|U| \leq 2$ or the set of associated presufs forms a clone set.*

The following lemma is the key to our elimination strategy.

LEMMA 4.5. *Consider three presuf pattern instances p_a, p_b, p_c, $a < b < c$. (The order of the indices corresponds to the left-to-right order of the pattern instances.) Suppose the set $\{x_a, x_b, x_c\}$ is not a clone set. Then there exists an index d in p_1*

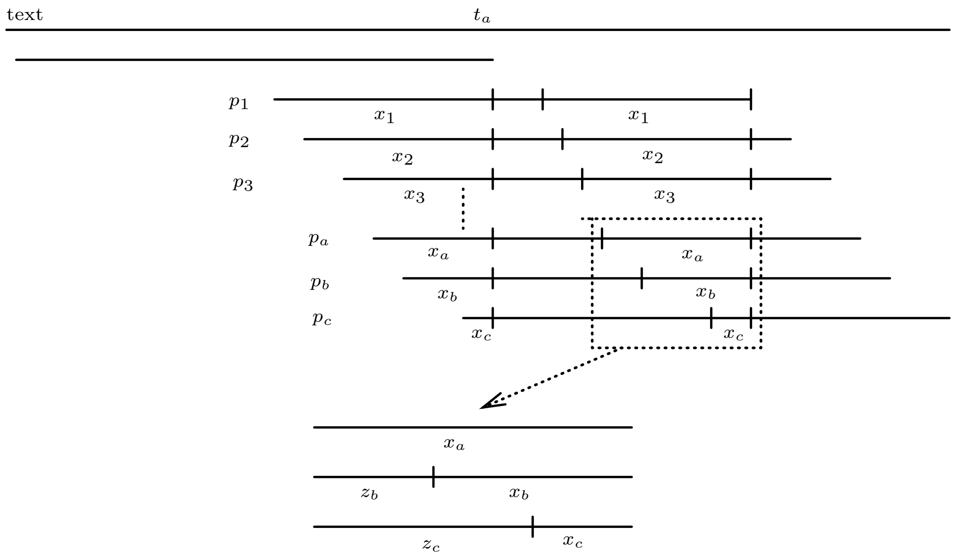

FIG. 2. *Overlapping pattern instances.*

with the following properties. The characters in $p_1, p_2, \ldots, p_a$ *aligned with* $p_1[d]$ *are all equal*; *however, the character aligned with* $p_1[d]$ *in at least one of* p_b *and* p_c *differs from* $p_1[d]$. *Moreover,* $m - |x_a| + 1 \leq d \leq m$; *i.e.,* $p_1[d]$ *lies in the suffix* x_a *of* p_1.

Proof. The substrings of $p_1, \ldots, p_a$ aligned with the suffix x_a of p_1 are all identical to the string x_a. Let the substring of p_b (respectively, p_c) aligned with the suffix x_a of p_1 be y_b (respectively, y_c). See Fig. 2. It suffices to show that at least one of y_b or y_c is not identical to x_a. Suppose for a contradiction that $y_b = y_c = x_a$.

Let $y_b = z_b x_b$ and $y_c = z_c x_c$. Note that z_b is a suffix of z_c. Since $y_b = y_c$, a simple induction shows that $y_b = uv^l$, where u is a proper prefix of v and $l \geq 1$, $|v|$ is either $|x_b| - |x_c|$ or some proper divisor of $|x_b| - |x_c|$, and v is primitive. Since $x_a = y_b$, if $l \geq 2$, x_a is periodic with core v.

First, suppose x_a is periodic with head u and core v. By Lemma 4.2, if $|x_b| > |uv|$, $x_b = uv^h$ for some h, $1 \leq h < l$. If $|x_b| < |uv|$, since $|x_b| - |x_c|$ is a multiple of $|v|$, $|x_b| = |v| + |x_c|$, so $x_b = wv$ for some string w, $|w| < |u|$. But then wv is a prefix of uv, which implies that v is cyclic; this is a contradiction. Thus $x_b = uv^h$. Since $|x_b| - |x_c|$ is a multiple of $|v|$, $x_c = uv^j$ for some j, $0 \leq j < h$, contradicting the fact that $\{x_a, x_b, x_c\}$ is not a clone set.

Consequently, $x_a = uv$. If x_a is not periodic, then $|x_b| < \frac{|x_a|}{2}$ and $|v| \leq |x_b| - |x_c| < \frac{|x_a|}{2}$. But then $|x_a| < 2|v| < |x_a|$, a contradiction. If x_a is periodic, $x_a = u'(v')^k$ for some $k \geq 2$. Also, $|v'| > |u|$ by Property 3 of Lemma 4.4. Hence $|x_b| < |v|$. But $|x_c| \leq |x_b| - |v| < 0$, a contradiction. $\square$.

Lemma 4.5 implies that a comparison of $p_1[d]$ with the aligned text character has the following effect: if it is a mismatch, all of $p_1, \ldots, p_a$ are eliminated, while if it is a match, at least one of p_b and p_c is eliminated.

Lemma 4.5 enables the elimination of essentially all but one group of pattern instances with at most one mismatch. At each step, for the rightmost d yielded by Lemma 4.5, $p_1[d]$ is compared with the aligned text character. If there is a mismatch, the surviving set of pattern instances is half-done, as we show in the following lemma. If there is no such d, the surviving set of pattern instances is half-done by Lemma

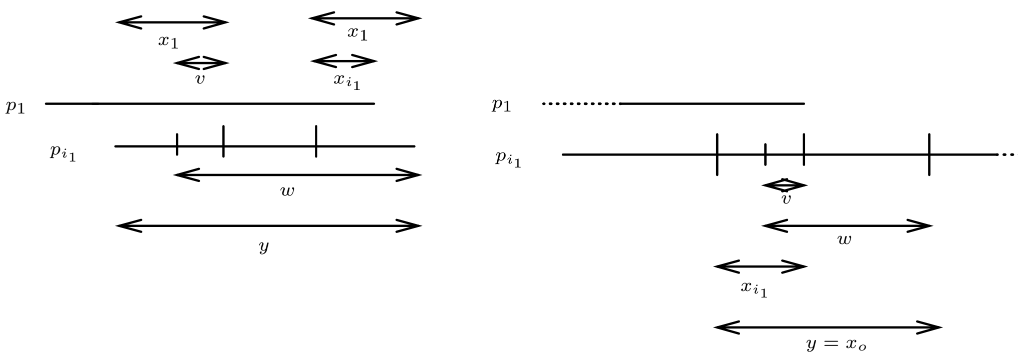

FIG. 3. (a) $p_{i_1} \in A_1$. (b) $p_{i_1} \notin A_1$.

4.5. This procedure comprises Phase 1 of the elimination procedure.

LEMMA 4.6. *If there is a mismatch in Phase* 1 *of the elimination procedure, the set X of surviving pattern instances is half-done.*

Proof. Suppose it was not; i.e., for some subset $\{p_a, p_b, p_c\}$ of X, $\{x_a, x_b, x_c\}$ is not a clone set. The characters in the x_i suffix of p_1, for $i = a, b, c$, match the aligned substring of p_i. Hence the mismatch at $p_1[d]$, which created set X, lies to the left of the suffix x_i of p_1, for $i = a, b, c$. However, by Lemma 4.5, at least one of p_a, p_b, and p_c could have been eliminated by a comparison made within the suffix of p_1 of size $\max\{|x_a|, |x_b|, |x_c|\}$. This contradicts the choice of d as the rightmost index at which a comparison eliminates some pattern instance. ◻

The elimination among the remaining half-done set of pattern instances also requires at most one mismatch.

LEMMA 4.7. *Let $O = \{p_{i_1}, p_{i_2}, \ldots, p_{i_l}\}$, $l \geq 2$, be an uneliminated half-done set and let $p_{i_1} \in A_r$. Then $x_{i_j} = uv^{h-j}$, where u is a proper suffix of primitive v and $h \geq l$. Further, there exists an index d such that the characters in $\{p_{i_1}, p_{i_2}, \ldots, p_{i_{l-1}}\}$ aligned with $p_{i_l}[d]$ are all equal, but differ from the character $p_{i_l}[d]$. $p_{i_l}[d]$ is aligned with or to the left of $p_{i_1}[m]$. In addition, if $p_{i_1} \notin A_1$ then $p_{i_l}[d]$ is to the right of $p_1[m]$ and within distance $|x_o| - |x_{i_1}|$ of $p_1[m]$, where p_o is the rightmost pattern instance in A_{r-1}. If $p_{i_1} \in A_1$, then $p_{i_l}[d]$ is to the right of t_A and aligned with or to the left of $p_1[m]$.*

Proof. Each x_{i_j}, $1 \leq j \leq l$, is of the form uv^{h_j}, for some $h_j \geq 0$, where u is a proper suffix of primitive string v.

See Fig. 3. Let y denote the string p_{i_1} if $p_{i_1} \in A_1$ and the string x_o otherwise. Clearly, y cannot be periodic with core v. If $p_{i_1} \in A_1$, then let w denote the suffix of p_{i_1} of length $m - |x_1| + |v|$. If $p_{i_1} \notin A_1$, then let w denote the substring of p_{i_1} which has length $|x_o| - |x_{i_1}| + |v|$ and which overlaps p_1 in exactly $|v|$ characters. Note that $w \neq uv^{h'}$, where $h' > 0$; otherwise, by Lemma 2.3 and the fact that v is primitive, the suffix of y of length $|w| - |v|$ is cyclic in v and therefore y is periodic with core v, contrary to our assumption.

Let w' be the smallest suffix of w which is not of the form $u'v^{h'}$, with u' a suffix of v and $h' > 0$. Define d to be the index in p_{i_1} corresponding to $|w'$. Clearly, $|w'| > |x_{i_1}| \geq (l-1)|v|$. Therefore, $p_{i_1}[d + (l-1)|v|]$ is a character in w. In addition, if $p_{i_1} \in A_1$, then $|w'| > |x_1|$ and therefore $p_{i_1}[d + (l-1)|v|]$ is aligned with or to the left of $p_1[m]$. The lemma follows if $p_{i_l}[d]$ is aligned with $p_{i_1}[d + (l-1)|v|]$ and the characters in $p_{i_1}, \ldots, p_{i_{l-1}}$ which are aligned with $p_{i_l}[d]$ are all identical and different

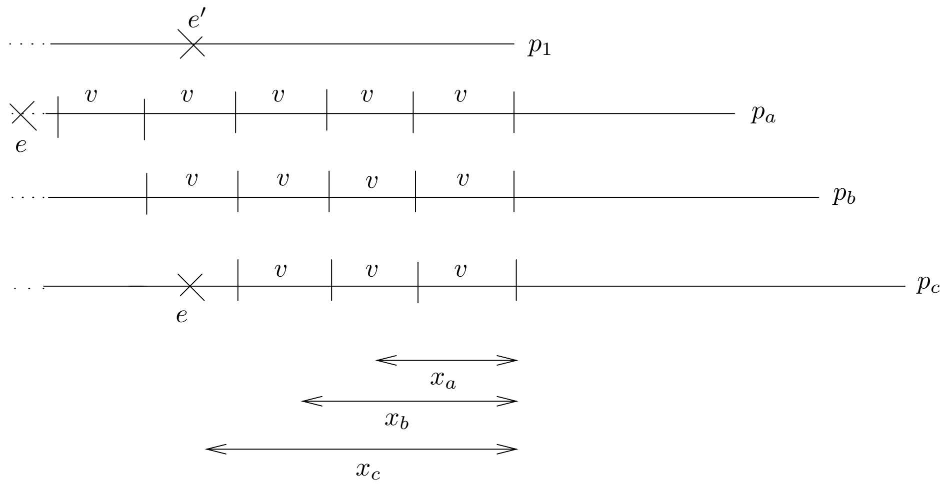

FIG. 4. *The half-done set is complete.*

from $p_{i_l}[d]$. We show that these two claims are indeed true.

First, we show that $|x_{i_j}| - |x_{i_{j+1}}| = |v|$, for $1 \leq j < l$. Suppose for a contradiction that there is a pattern instance $p_b \notin O$ with $x_b = uv^{h''}$, $h'' > 0$, and there are pattern instances $p_a, p_c \in O$, p_a to the left of p_b and p_c to the right of p_b. See Fig. 4. Let $p_a[e]$ be the rightmost character in p_a such that the substring of p_a which starts at $p_a[e]$ and overlaps p_1 is longer than $|x_a|$ and not periodic with core v. Consider the character $p_1[e']$ aligned with $p_c[e]$. The portions of p_a, p_b, and p_c which overlap the suffix of p_1 to the right of e' are all identical. If Phase 1 had stayed to the right of e', then p_a, p_b, and p_c would all have been eliminated by the mismatch at the end of Phase 1. Thus $p_1[e']$ must have been compared in Phase 1. A mismatch at e' eliminates p_a and p_b while a match eliminates p_c. Either way, a contradiction results.

Finally, note that the character in p_{i_j}, $1 \leq j \leq l-1$, which is aligned with $p_{i_l}[d]$ is precisely the character $p_{i_l}[d + (l-j)|v|]$. But $p_{i_l}[d] \neq p_{i_l}[d + |v|] = p_{i_l}[d + 2|v|] = \cdots = p_{i_l}[d + (l-1)|v|]$. ◻

COROLLARY 4.8. *To eliminate all but one of the pattern instances in any half-done set* (*in particular, the Phase* 1 *survivors set*) $\{p_{i_1}, p_{i_2}, \ldots, p_{i_j}\}$, *it suffices to compare a sequence of characters with the property that any two consecutive characters in the sequence are distance* $|v|$ *apart, where* v *is the core of* x_{i_1}. *Further, the pattern instances in this set are eliminated in right-to-left order by this comparison sequence* (*i.e., in decreasing value of* j).

Let p_{i_l} be as in Lemma 4.7; the character in p_{i_1} aligned with $p_{i_l}[d]$ is compared with the aligned text character. A match eliminates p_{i_l}; a mismatch leaves only p_{i_l} surviving. Iteration of this step ends with one pattern instance surviving after at most one mismatch. This comprises Phase 2 of the elimination procedure.

The sequence of comparisons made in Phase 2 is clearly a right-to-left sequence. If p_1 is eliminated in Phase 1, then all comparisons in Phase 2 are made to the right of the characters compared in Phase 1. Otherwise, if p_1 is not eliminated in Phase 1, all comparisons in Phase 2 are made to the left of the characters compared in Phase 1.

We recapitulate the elimination strategy now. In Phase 1, characters in p_1 at indices given by a precomputed sequence S_1 are compared in sequence until a mismatch occurs or until the sequence is exhausted. Associated with a mismatch at the ith comparison given by S_1 is an auxiliary sequence S_{2_i} of indices. If a mismatch occurs at the ith comparison in S_1, Phase 2 begins and comparisons are now made according to the auxiliary sequence S_{2_i}. A mismatch at any index in the relevant auxiliary sequence completes the elimination process as does the exhaustion of that auxiliary sequence. In either case, only one pattern instance from the set $\{p_1, \ldots, p_k, p_{k+1}\}$ survives.

Let $|S_1| = j$. The sequences S_1 and $S_{2_1}, S_{2_2}, \ldots, S_{2_j}$ collectively form a tree ET (the *elimination tree*). ET is a binary tree. Each internal node x of ET stores an index indicating the comparison to be made. Each internal node has two children. The computation continues at the left child if the comparison at x is successful and at the right child otherwise. The computation starts at the root of ET. Each external node stores the one pattern instance to survive the two phases of comparisons leading to that external node. The external nodes are also called terminal nodes. Note that no pattern instance p_i can be the survivor at two distinct terminal nodes of ET. This is because one of the two outcomes of the comparison at the least common ancestor of these two nodes in ET is bound to eliminate p_i. It follows that the size of ET is $O(k)$.

The total number of mismatches occurring in the elimination process is at most two because each phase terminates when a mismatch occurs.

LEMMA 4.9. *All but at most one of $p_1, \ldots, p_k, p_{k+1}$ can be eliminated by making up to k comparisons using the $O(k)$-sized binary comparison tree ET. At most two of these comparisons result in mismatches. The sequence of comparisons made by the elimination strategy consists of two left-to-right sequences. The second sequence is either entirely to the right or entirely to the left of the first one.*

4.2. Strategy for handling presuf shifts. Subsequent to the elimination due to tree ET, the presuf handler proceeds in a manner reminiscent of the basic algorithm. That is, there is a current pattern instance, p_a, which is being matched and which is the leftmost surviving pattern instance. The next leftmost surviving pattern instance, p_b, which has a difference point with p_a is a candidate for elimination. Indeed, a comparison of p_a with the text is made at the difference point.

The analysis of the presuf handler has the following flavor. With a few exceptions, comparisons are charged to distinct text characters. To be precise, for each suffix shift, at most two comparisons are charged to the shift rather than to text characters. Even more precisely, if two comparisons are charged to the shift, the next presuf shift is at distance at least $\frac{3(m+1)}{4}$ to the right, and otherwise it is at distance at least $\frac{m+1}{2}$ to the right. The complexity bound of the algorithm now follows readily.

THEOREM 4.10. *The algorithm performs at most $n + \frac{8}{3(m+1)}(n-m)$ character comparisons.*

There are three ways in which text characters are charged:

(i) The character compared is charged.

(ii) The text character aligned with the left end of the pattern instance eliminated by the comparison is charged.

(iii) The text character to the immediate right of $t_A|$ (i.e., aligned with $|p_{k+1}$) is charged.

The three charging methods do not interact readily. To ensure that no text character is charged twice, the switching from one charging method to the other will occur only at carefully selected points in the algorithm. In addition, mismatches are not charged according to rule (i) since the text characters in question may be compared again.

Basically, charging method (i) is used if a pattern instance is successfully matched (at least up to a suffix which is a presuf). Charging method (iii) is used only for the comparison that eliminates the presuf pattern instance which survives the elimination procedure ET. Charging method (ii) is used otherwise. A partial exception arises for the characters compared by procedure ET; this is discussed further below.

Following the use of procedure ET, the aim is to perform comparisons essentially as in the basic algorithm, that is, to compare the character at the difference point of the two leftmost surviving pattern instances which are not prefix overlaps of each other. The analysis ceases to be as straightforward because of the additional $j \leq k$ comparisons performed by procedure ET; indeed, to cope with this, a modified form of the basic algorithm is needed.

There are two objectives:

1. to avoid repeating comparisons at the text characters successfully compared by procedure ET;

2. to perform essentially j fewer comparisons than in the basic algorithm.

Objective 1 is achieved by keeping a record of the successful comparisons in a bit vector of length roughly m.

The major difficulty, however, is caused by the method used for charging the j comparisons made by procedure ET. It is natural to charge these comparisons to the text characters compared. Unfortunately, this may conflict with charging using method (ii). To avoid this difficulty, a single additional comparison, with text character t_b, is performed before using procedure ET. The following lemma can then be shown.

LEMMA 4.11. *For each text character t_c compared by procedure ET, with at most $\alpha \leq 2$ exceptions, there is a distinct previously uncharged text character $t_{c'}$, with $t_{c'}$ aligned with or to the left of t_c and to the right of $|p_{k+1}$, such that the pattern instance q_c whose left end is aligned with $t_{c'}$ mismatches either the text character t_b or some text character matched in procedure ET.*

Let β be the number of mismatches performed by procedure ET. Then, in addition, $\alpha + \beta \leq 2$.

The lemma is proven by specifying a transfer function f, which associates c with c'. The form of f depends on the sequence of comparisons performed by procedure ET. The proof of the lemma is quite nontrivial; it is deferred until section 5.

Lemma 4.11 is used as follows. Let q_e be the next pattern instance to match the text (or at least to have a suffix, which is also a presuf, matching the text). All pattern instances to the left of q_e, eliminated by comparisons made after the use of procedure ET, are charged used charging method (ii). Using the transfer function, those comparisons to the left of $|q_e$ made by procedure ET are charged to text characters which are not otherwise charged. By contrast, text characters aligned with q_e are charged using charging method (i). There will be no more than two comparisons performed by the presuf handler that are not thereby charged to a text character; these comparisons are charged to the presuf handler itself.

The algorithm requires a total of five subphases, whose details depend on exactly how q_e arises.

It is helpful to distinguish three scenarios that may ensue. To this end, let p_e denote the presuf pattern instance to survive the elimination using tree ET. In addition, let t_a denote the text character $t_A|$.

The three scenarios follow:

1. All pattern instances overlapping p_e are eliminated apart from its presuf overlaps, and p_e or at least a suffix of p_e is matched.

2. p_e is eliminated. In addition, there is some pattern instance q_c overlapping p_e such that all pattern instances overlapping q_c are eliminated apart from its presuf overlaps; further, q_c or at least a suffix of q_c is matched.

3. p_e is eliminated, as are all pattern instances overlapping p_e. Let q_d denote the leftmost surviving pattern instance in this case.

The first scenario causes no problems from the perspective of the analysis. It suffices to ensure that none of the successful comparisons made by ET are repeated. The third scenario is handled by using charging scheme (iii) for the comparison which eliminates p_e and charging scheme (ii) for the remaining comparisons in the post-ET phase. Lemma 4.11 ensures that for each comparison made by ET (with at most two exceptions) with a text character strictly between t_a and $|q_d$, there is a distinct pattern instance whose left end lies strictly between $|p_{k+1}$ and $|q_d$ and which is eliminated by the comparisons made by ET plus the one other comparison at text character t_b. Again, this leads to the desired complexity bound without difficulty.

The second scenario provides the greatest difficulty. In order to avoid unnecessary comparisons, the locations of successful comparisons are recorded. Then if a difference point occurs at one of these matched text characters, the present pattern instance p_b (see the first paragraph of the subsection) can be removed without further comparisons. However, following a mismatch, it is not clear how to maintain this property: with only linear storage, it is not clear how to ensure that the current pattern instance following the mismatch agrees with the text on a previously matched character, at least if the total work bound is to be linear. (There is no problem if exponential-in-m space is available for precomputed structures.) To avoid this difficulty, only the successful comparisons since the last mismatch are recorded.

In fact, this is not quite good enough. It appears necessary to keep track of the characters compared by procedure ET regardless of how many characters are compared. This avoids subsequent comparison of these characters. Indeed, any pattern instances mismatching on one or more of these characters are eliminated immediately after the computation with procedure ET. This is done with the help of precomputed information.

With this motivation, we proceed with a precise description of the presuf handler procedure. It proceeds in five steps.

Step 1 (before the use of tree ET). The characters in $p_1, \ldots, p_k$ aligned with $p_1[m]$, the rightmost character in p_1, are identical. If the character in p_{k+1} aligned with $p_1[m]$ is also identical to it, then $p_1[m]$ is compared with the aligned text character. A mismatch eliminates all of $p_1, \ldots, p_k, p_{k+1}$ and the basic algorithm is restarted with $|p_a$ placed immediately to the right of $|p_{k+1}$. A match is not immediately beneficial since it does not eliminate any of $p_1, \ldots, p_k, p_{k+1}$. However, it ensures the elimination of sufficiently many appropriate pattern instances for scenarios 2 and 3 described above.

Step 2. The elimination strategy using tree ET is applied to the pattern instances $p_1, \ldots, p_k, p_{k+1}$.

Following Step 2, at most one presuf pattern instance survives. Call it p_e. Let Q denote the set of pattern instances which overlap p_e and have their left end to the right of $|p_{k+1}$. In the elimination process, some elements of Q may have also been eliminated from being potential matches. They need not be reconsidered. Indeed, since the characters successfully matched in Step 2 must not be compared anew, it appears that these pattern instances must not be considered anew. To this end, a subset Q_x of Q is associated with each terminal node x in ET.

Let T_x denote the indices of the text characters successfully compared in Steps 1 and 2. Q_x contains those pattern instances in Q which match at all the text indices in T_x, except possibly the last. This seemingly odd exception is necessary in order to store Q_x efficiently. Actually, Q_x satisfies further constraints, but they are not needed for this section. The complete definition of Q_x and the method for computing it are described in section 4.5. Here it suffices to work with the following property: all but at most two of the comparisons in Steps 1 and 2 are successful and are remembered by pattern instances in Q_x.

Suppose that the elimination process terminates at terminal node x. Let $Q' = \{p_e\} \cup Q_x$. The elimination procedure of Step 3 is applied to the pattern instances in Q'.

Step 3. This step eliminates among the elements of Q'. q_c will denote the leftmost pattern instance to survive Step 3. If $q_c = p_e$, then every surviving pattern instance overlapping q_c will be a presuf overlap of q_c.

The strategy used here is similar to the one for the basic algorithm. One of two overlapping pattern instances is eliminated by comparing at the difference point of the two instances.

To prevent repeated comparisons of text characters to the right of t_a, two additional data structures are used. The first is a bit vector $BV[1 \dots 2m]$. $BV[i] = 1$ if the ith text character to the right of t_a has been successfully compared in Steps 1 and 2 or in Step 3 since the last mismatch. The second is a list LBV; it stores the indices of the bits in BV set to one in Step 3 since the last mismatch. Initially, LBV is empty.

The elimination procedure for Step 3 follows. Let q_a and q_b denote the two leftmost uneliminated pattern instances in Q'. Suppose that q_b lies i units to the right of q_a. The reader is advised to refer to section 2 to review the definition of dif_{i+1}. If dif_{i+1} is undefined, then q_b is removed from Q'.

If dif_{i+1} is defined, then the bit in BV corresponding to the text character aligned with $q_a[dif_{i+1}]$ is read. If this bit is 1, then q_b is removed from Q'. (q_b can be eliminated since it does not match an already compared text character.) Otherwise, $q_a[dif_{i+1}]$ is compared with the aligned text character. If the two characters are equal, the corresponding bit in BV is set, the bit's index is added to LBV, and q_b is eliminated. If they are not equal, then the bits in BV at all indices currently in LBV are reset to 0, LBV is reset to empty, and q_a is eliminated.

The elimination procedure is iterated until only one pattern instance remains in Q'. Let q_c denote this remaining pattern instance.

Step 4. In this step, either all pattern instances overlapping q_c, apart from presuf overlaps, are eliminated or q_c is eliminated.

Let Q'' be the set of pattern instances whose left end lies to the right of $p_e|$ but not to the right of $q_c|$. The following step is repeated until either q_c is eliminated or $Q'' = \phi$. Let q_d be the leftmost pattern instance in Q''. Suppose q_d lies i units to the right of q_c. If dif_{i+1} does not exist, then q_d is removed from Q''. Otherwise, the following bit in BV is read: the bit corresponding to the text character aligned with $q_c[dif_{i+1}]$. If this bit is 1, q_d is eliminated. If it is 0, $q_c[i]$ and the aligned text character are compared. If they match, then the corresponding bit in BV is set, its index is added to LBV, and q_d is eliminated. Otherwise, q_c is eliminated and Step 4 comes to an end.

If q_c is eliminated, then LBV is reset to be empty, BV is reset to 0, and the basic algorithm is restarted with $p_a = q_d$. Otherwise, Step 5 is performed.

Step 5. This step seeks to complete the match of q_c. If at least a presuf of q_c is matched, the complete match or the partial match results in a new presuf shift. Otherwise, the basic algorithm is resumed with $|p_a$ immediately to the right of $q_c|$.

Step 5 compares the characters in q_c to the right of t_a, apart from those matched in Steps 1 and 2, and those matched in Steps 3 and 4 following the most recent mismatch. (Incidentally, there was no mismatch in Step 4 since q_c survived Step 4 if Step 5 is performed.) These characters are identified with the help of bit vector BV. They are matched in right-to-left order until either a mismatch occurs or they are all matched.

If they all match q_c is declared a complete match if either $t_{\text{last}} = \phi$ or t_{last} lies to the left of $|q_c$. Recall that t_{last} is the index of the text character mismatched, if any, immediately prior to the most recent presuf shift (if there was no mismatch, $t_{\text{last}} = \phi$).

Next, BV is reset to zero, LBV is reset to be empty, and t_{last} is updated as follows. If the above right-to-left pass results in a mismatch, then t_{last} is set to the index of the text character at which the mismatch occurs. Otherwise, t_{last} retains its value unless $|q_c$ is to its right. In the latter case, $t_{\text{last}} := \phi$.

The present situation is identical to that preceding a presuf shift in the basic algorithm. This resulting shift is treated in the same way; it too is called a presuf shift.

4.3. The analysis. The comparison complexity of the algorithm of section 4.2 is given by the following lemma.

LEMMA 4.12. *If p is not a special case pattern and $|x'_1| < \frac{m}{2}$ for each presuf shift, then the comparison complexity of the algorithm is bounded by $n(1 + \frac{8}{3(m+1)})$.*

Proof. We give a charging scheme to account for the comparisons made by the algorithm. This scheme charges almost every comparison to a distinct text character. The only exceptions are a few of the comparisons made by the presuf shift handler. For each presuf shift, depending on the distance between this presuf shift and the next one, the charging scheme fails to charge for up to two of the comparisons made by the presuf shift handler. We refer to the number of comparisons which the charging scheme fails to charge to distinct text characters as the *overhead* of the presuf shift. If a presuf shift has an overhead of two, we show that the next presuf shift must occur at least distance $\frac{3(m+1)}{4}$ to the right of the current presuf shift. The comparison complexity of our algorithm now follows from the fact that any two consecutive presuf shifts must occur at least distance $\frac{m+1}{2}$ apart.

Charging scheme. The charging scheme charges in phases. The phases begin and end at shifts and at reversions to the basic algorithm. There are four types of phases; for each phase type, a different charging scheme is used:

1. a phase beginning and ending with a basic shift;
2. a phase beginning in the basic algorithm and ending with a presuf shift;
3. a phase beginning with a presuf shift and ending with a reversion to the basic algorithm;
4. a phase beginning and ending with a presuf shift.

Consider any phase and let q_1 and q_2 refer to the leftmost surviving pattern instances at the beginning and end of the phase, respectively. Note that for Type 3 and Type 4 phases, q_1 is a presuf overlap of the pattern instance q', the leftmost uneliminated pattern instance prior to the presuf shift which initiated this phase.

Specifically, the prefix x_1' of q_1 is aligned with the suffix x_1' of q'. (Recall from the start of section 4 that on a presuf shift, we assume that the suffix x_1' of q_1 matches the text.)

The charging scheme obeys the following properties.

1. At the start of a Type 1 or Type 2 phase, only text characters to the left of q_1 have been charged.

2. At the start of a Type 3 or Type 4 phase, only text characters aligned with or to the left of the prefix x_1' of q_1 have been charged.

Type 1 *phase.* Suppose i comparisons were made in this phase. These i comparisons are charged to text characters which are aligned with q_1 but to the left of $|q_2$. By Lemma 3.1, $|q_2$ lies at least i characters to the right of $|q_1$. Thus each text character aligned with q_1 and to the left of $|q_2$ is charged at most once in this process. Clearly, property 1 holds at the start of the next phase.

Type 2 *phase.* In each comparison, a distinct character in q_1 is compared with the aligned text character. Each of these comparisons is charged to the text character compared. Thus each text character aligned with q_1 is charged at most once in this process. Clearly, property 2 holds at the start of the next phase.

The charging scheme for Type 3 and Type 4 phases is more involved. Before describing the scheme, we mention the ranges of the text characters charged in each case.

Type 3 *phase.* The text characters charged lie to the right of the right end of the prefix x_1' of q_1 and to the left of $|q_2$. Each text character in this range is charged at most once. Clearly, property 1 holds at the start of the next phase.

Type 4 *phase.* The text characters charged lie to the right of the right end of the prefix x_1' of q_1 and are aligned with or to the left of the rightmost character in the prefix x_1' of q_2. Each text character in this range is charged at most once. Clearly, property 2 holds at the start of the next phase.

Clearly, the ranges of the text characters charged for different phases are disjoint. Next, we specify the charging scheme for Type 3 and Type 4 phases and justify the claims regarding the overhead.

Consider a presuf shift which initiates a new Type 3 or Type 4 phase. Let q' be the leftmost uneliminated pattern instance immediately before the presuf shift. Recall that t_a is the text character aligned with $q'|$. Consider the comparisons made by the current use of the presuf shift handler. If a mismatch occurs in Step 1, the current phase ends immediately and the basic algorithm is resumed. The presuf shift in this case has overhead 1 and the next presuf shift occurs at least distance $m+1$ to the right. Next, suppose that the comparison in Step 1 is successful. Let p_e be the presuf pattern instance that survives the elimination using tree ET in Step 2. After the presuf shift handler finishes, one of the three scenarios mentioned in section 4.2 ensues. We consider each in turn.

1. All pattern instances overlapping p_e are eliminated, apart from its presuf overlaps, and p_e or at least a suffix of p_e is matched. This is a Type 4 phase.

All comparisons made by the presuf shift handler, except the unsuccessful comparisons in Step 2, are charged to the text characters compared. The bit vector BV ensures that each of these comparisons involves a different text character. Thus each text character which lies to the right of t_a and is aligned with or to the left of $p_e|$ is charged at most once. At most two comparisons in Step 2 are unsuccessful, so this shift has overhead at most 2.

Consider the situation when there are exactly two mismatches in Step 2. p_1 is clearly eliminated in this case. In addition, we show in the next paragraph that if x_1 is periodic, with core v and head u, say, then all pattern instances whose associated presufs have the form uv^o, $o \geq 1$, are also eliminated. Let x_e be the presuf associated with p_e. It follows that $x_1 = x_e w x_e$ for some nonempty string w. Since $p = x_1 z x_1$, for some nonempty string z, $|x_e| \leq \frac{m-3}{4}$. This guarantees that the next presuf shift occurs at least distance $\frac{3(m+1)}{4}$ to the right. If there is just one mismatch in Step 2, then since $|x_1'| < \frac{m}{2}$, the next presuf shift occurs at least distance $\frac{m+1}{2}$ to the right.

To see that two mismatches in Step 2 eliminate all presuf pattern instances with associated presufs of the form uv^o, $o \geq 1$, it suffices to show that at most one such pattern instance survives the first mismatch; the second mismatch will surely eliminate this pattern instance. Suppose two pattern instances p_{i_1} and p_{i_2}, $i_1 < i_2$, $i_1, i_2 \neq 1$, $x_{i_1} = uv^{o_1}$, $x_{i_2} = uv^{o_2}$, $o_1, o_2 \geq 1$, survive the first mismatch, which occurs at text character t_x, say. The portions of p_1 and p_{i_1} to the right of t_x match each other while the characters in p_1 and p_{i_1} aligned with t_x are different. This implies that p_1 and p_{i_2} have a difference point strictly between t_x and t_b; more precisely, the character in p_1 which is distance $(o_2 - o_1)|v|$ to the right of t_x is a difference point. Therefore, either p_1 or p_{i_2} would have been eliminated before the first mismatch, which is a contradiction.

2. p_e is eliminated. In addition, there is some pattern instance q_c overlapping p_e such that all pattern instances overlapping q_c are eliminated apart from its presuf overlaps; further, q_c or at least a suffix of q_c is matched. This is also a Type 4 phase.

Each comparison in Steps 1 and 2 with a text character to the left of $|q_c$ for which function f is defined is charged to the text character specified by the function f, called its f value; f values are distinct by definition. Comparisons in Step 3 fall into one of three categories (see Lemma 4.11 and the following paragraph):

1. comparisons which eliminate pattern instances whose left ends lie to the right of $|p_{k+1}$ and to the left of $|q_c$;
2. comparisons which eliminate pattern instances whose left ends are aligned with or to the right of $|q_c$;
3. the comparison which eliminates p_e.

Each comparison in the first category is charged to the text character aligned with the left end of the pattern instance eliminated. By the definition of the function f, these text characters do not occur in the range of f values. Comparisons in the second category, along with the comparisons made in Steps 4 and 5 and those successful comparisons in Steps 1 and 2 that involve text characters overlapping q_c, are charged to the text characters compared. BV ensures that each of these comparisons involves a distinct text character. Thus each text character which lies to the right of $|p_{k+1}$ and is aligned with or to the left of $q_c|$ is charged at most once. The comparison that eliminates p_e is charged to the text character aligned with $|p_{k+1}$. Since all f values lie to the right of $|p_{k+1}$ and all pattern instances eliminated by comparisons in the first category are with left ends to the right of $|p_{k+1}$, this text character is charged exactly once. The two comparisons in Step 2 lacking f values constitute the overhead of this presuf shift. Since p_e is eliminated, the next presuf shift occurs at least distance $m+1$ to the right of the current presuf shift.

3. p_e is eliminated, as are all pattern instances overlapping p_e. This is a Type 3 phase.

Let q_d denote the leftmost surviving pattern instance. All comparisons in Steps 1 and 2 for which function f is defined are charged to their f values. f values are distinct by definition. Excluding the comparison which eliminates p_e, each comparison in Steps 3 and 4 eliminates some pattern instance whose left end lies to the right of $|p_{k+1}$ and to the left of $|q_d$. Each such comparison is charged to the text character aligned with the left end of the pattern instance eliminated. These text characters cannot occur in the range of the function f and hence are charged only once. Thus each text character which lies to the right of $|p_{k+1}$ and to the left of $|q_d$ is charged at most once. The comparison that eliminates p_e is charged to the text character aligned with $|p_{k+1}$. The two comparisons in Step 2 lacking f values constitute the overhead of this presuf shift. Since p_e is eliminated, the next presuf shift occurs at least distance $m+1$ to the right of the current presuf shift. ∎

The following lemma is shown in section 4.5.

LEMMA 4.13. *The total space used by the algorithm for the case when $|x'_1| < \frac{m}{2}$ for all presuf shifts is $O(m)$. Further, for any terminal node x of ET, Q_x can be obtained in $O(m)$ time. The preprocessing required by the algorithm can be accomplished in $O(m^2)$ time.*

LEMMA 4.14. *Suppose that p is not a special-case pattern and $|x'_1| < \frac{m}{2}$ for all presuf shifts. Then the total time taken by the algorithm is $O(n+m)$, following preprocessing of the pattern, which takes $O(m^2)$ time.*

Proof. By Lemma 4.12, the number of character comparisons made is $O(n)$. It remains to count the time spent in all other operations. The basic algorithm makes only character comparisons. Next, consider the presuf handler of section 4.2. Steps 1 and 2 make only character comparisons. Following Step 2, computing Q_x takes $O(m)$ time by Lemma 4.13. Steps 3 and 4 take $O(m)$ time because $|Q'|, |Q''| = O(m)$ and each of the operations in these steps, except the operations used for resetting BV, leads to the removal of a pattern instance from one of Q'' or Q'. Further, the total time spent by Steps 3 and 4 in resetting BV is bounded by the time taken by these steps to set bits in BV, which is $O(m)$. Clearly, Step 5 takes $O(m)$ time. Thus the total time taken by the presuf handler of section 4.2 is $O(m)$. Since any two presuf shifts occur at least $m-|x'_1| > \frac{m}{2}$ distance apart, the total time taken by the algorithm is $O(n+m)$. ∎

4.4. Handling presuf shifts for special-case patterns. As mentioned in section 4.1, a different algorithm is needed to handle presuf shifts for patterns for which $|x_k| = 1$ and $g = 1$. We give an algorithm which handles presuf shifts for such patterns when $|x'_1| < \frac{m}{2}$. (Recall that x'_1 for a presuf shift was defined towards the start of section 4.) The case where $|x'_1| \geq \frac{m}{2}$ is handled in section 6.

The goal of this algorithm is to reach one of the following two situations:

1. the identification of a pattern instance q_c satisfying the following property: no pattern instance q_d which precedes q_c survives and a pattern instance overlapping q_c survives only if it is a presuf overlap of q_c;

2. a return to the basic algorithm.

Further, this is achieved with at most two mismatches.

Let $x_k = b$. Any character other than b is called a non-b character. Since we assume that the pattern contains at least two different characters, it contains a non-b character. Let $p[j]$ be the leftmost non-b character in p and let t_c denote the text character aligned with $|p_{k+1}$. Let t_d be the leftmost non-b text character, if any, to the right of, and including, t_c.

By the definition of special-case patterns, all presufs consist solely of b's. Therefore, $p_1[j]$ lies to the right of t_a. Note that no complete match can occur with one of $p[1 \dots j-1]$ aligned with t_d. Thus if t_d lies to the left of $p_1[j]$, then the next potential match instance of p would have its left end to the right of t_d. Otherwise, the next potential match instance of p has $p[j]$ aligned with t_d. Also notice that if t_d does not exist, then there are no more complete matches. These observations lead to the following three-step procedure.

Step 1. This step locates t_d and then eliminates all but at most one pattern instance q_c overlapping t_d. Starting at t_c, a left-to-right scan of the text is performed to locate t_d (i.e., each text character is compared to b; t_d is the character at which the first mismatch occurs). If t_d does not exist, the algorithm halts. If t_d exists and lies to the left of $p_1[j]$, then the basic algorithm is restarted with $|p$ placed to the immediate right of t_d. Otherwise, q_c is chosen to be the pattern instance such that $q_c[j]$ is aligned with t_d. q_c is the next potential match instance to be considered.

Step 2. In this step, either q_c is eliminated or all pattern instances overlapping q_c, except for presuf overlaps of q_c, are eliminated. This is done using the basic algorithm, slightly modified to account for the matched prefix. Suppose the leftmost difference points are used in the sequence S in the basic algorithm, as against any arbitrary difference points. Then $dif_2, \ldots, dif_j$ are all equal to j and $dif_{j+1}, \ldots, dif_m$ are all greater than j, whenever defined. In Step 2, the characters in q_c to the right of $q_c[j]$ which are at the indices given by S are compared with the aligned text characters in the order in which they appear in S. This continues until either a mismatch occurs or the sequence is exhausted. A mismatch leads to the basic algorithm with $|p$ shifted to the right of q_c by distance at least $j-1$ plus the number of comparisons made in this step. If no mismatch occurs, then Step 3 follows.

Step 3. Characters in q_c which are not yet matched are compared from right to left with their aligned text characters until a mismatch occurs or q_c is fully matched. The present situation is now identical to the situation at the beginning of a presuf shift and is handled in the same way.

The comparison complexity of the above algorithm is determined by the following lemma.

LEMMA 4.15. *If p is a special-case pattern and $|x'_1| < \frac{m}{2}$ for each presuf shift, then the comparison complexity of the algorithm is $n(1 + \frac{2}{m+1})$.*

Proof. We give a charging scheme to account for the comparisons made by the algorithm for handling special-case patterns. The definition of a phase, the charging scheme for Type 1 and Type 2 phases, and the ranges of text characters charged in each phase type remain the same as in Lemma 4.12. Only the charging scheme for Type 3 and Type 4 phases needs to be modified in accordance with the presuf shift handler for special-case patterns.

Consider a presuf shift which initiates a new Type 3 or Type 4 phase. We show that it has an overhead of at most one. The comparison complexity of the algorithm now follows from the fact that $|x'_1| < \frac{m}{2}$ and therefore any two consecutive presuf shifts must occur at least $\frac{m+1}{2}$ characters apart.

Charging scheme for the presuf shift handler. Let q' and q_1 be the leftmost uneliminated pattern instances immediately before and after the presuf shift, respectively. Recall that t_a is the text character aligned with $q'|$.

We show that presuf shifts have overhead at most one for these patterns. Let q_c be the leftmost pattern instance which survives Step 1. All successful comparisons in Step 1 are charged to the text characters compared. These text characters lie to the left of $q_c[j]$, where j is the least index such that $p[j]$ differs from $p[m]$. The lone unsuccessful comparison in Step 1 constitutes the overhead of this shift. Now consider two cases.

1. Suppose q_c survives Step 2. All comparisons made in Steps 2 and 3 are charged to the text characters compared. Thus each text character which lies to the right of t_a and is aligned with or to the left of $q_c|$ is charged at most once. All future comparisons will be charged to text characters to the right of $q_c|$.

2. Suppose q_c does not survive Step 2. Each successful comparison in Step 2 eliminates some pattern instance lying entirely to the right of $q_c[j]$ and is charged to the text character aligned with the left end of that pattern instance. The unsuccessful comparison which eliminates q_c in Step 2 is charged to the text character aligned with $q_c[j]$. Thus each text character lying strictly between t_a and $|q_d$ is charged at most once, where q_d is the leftmost surviving pattern instance at the end of Step 2. All future comparisons will be charged to text characters aligned with or to the right of $|q_d$. ∎

LEMMA 4.16. *Suppose that p is a special-case pattern and $|x'_1| < \frac{m}{2}$ for all presuf shifts. Then the total time taken by the algorithm is $O(n+m)$, following preprocessing of the pattern, which takes $O(m^2)$ time. The total space used by the algorithm is $O(m)$.*

Proof. The lemma, except for the preprocessing time, is obvious from the above description. Since no extra preprocessing is required for special-case patterns, the lemma follows from Lemma 4.14. ∎

THEOREM 4.17. *Suppose for all presuf shifts that $|x'_1| < \frac{m}{2}$. Then the total space used by the algorithm is $O(m)$ and the total time taken by the algorithm, after preprocessing, is $O(n+m)$. The preprocessing required by the algorithm takes $O(m^2)$ time.*

Proof. The proof follows from Lemmas 4.14 and 4.16. ∎

4.5. Data structure details. We prove Lemma 4.13 in this section. The following data structures are used by the algorithm:

1. the array S used in the basic algorithm;
2. an array, indexed by i, storing dif_i, $2 \le i \le m$, used by the presuf shift handler;
3. BV and LBV, the bit vector and its associated list;
4. ET, the elimination tree;
5. Q_x, for each terminal node x of ET, as defined after Step 2 in section 4.2.
   Of these, the first three have size $O(m)$ by definition. By Theorem 4.9, ET also has size $O(m)$.

It remains to show how to represent Q_x, for each terminal node x of ET, using $O(m)$ space overall. The following definitions are helpful. Let t_b be the text character aligned with $p_1|$. Let Q refer to the set of pattern instances which overlap p_{k+1}, have left ends to the right of $|p_{k+1}$ and either match or do not overlap t_b.

Before showing how to maintain Q_x, it is helpful to recapitulate some structural properties of ET. ET is a binary tree with each internal node having two children. At each internal node y, a character c_y in p is potentially compared with the text character tc_y. A successful comparison leads to the left child of y while a mismatch leads to the right child. Comparisons are made starting at the root of ET and continuing until a terminal node (a leaf) is reached. A node in ET lies in the right subtree of at most two of its ancestors.

Node x is said to be a *failing descendant* of node y if x is a proper descendant of y and lies in the right subtree of y. A terminal node x can be a failing descendant of at most two nodes in ET. Let $p(x)$ denote the parent of x. For each terminal node x, let $Anc(x)$ be defined as follows. If both children of $p(x)$ are terminal nodes and

$p(x)$ is the right child of $p(p(x))$, then $Anc(x)$ is the set of proper ancestors of $p(x)$. Otherwise, $Anc(x)$ is the set of proper ancestors of x.

$q \in Q$ is said to *occur* at terminal node x of ET if $q \in Q_x$. In section 4.2, we tentatively defined Q_x to be the set of pattern instances in Q which match at all text characters compared successfully at nodes in $Anc(x)$ (actually, the definition was not this precise). Now we refine this definition by letting Q_x satisfy some additional constraints. Informally, q should occur at x if it is consistent with all comparisons made at nodes in $Anc(x)$. This motivates the following characterization of Q_x. Let $Y \subset Anc(x)$ consist of those nodes with respect to which x is a failing descendant. Then Q_x is the maximal subset of Q such that each $q \in Q_x$ satisfies the following properties:

1. $\forall y \in Anc(x) - Y$, the character in q aligned with tc_y, if any, matches the character c_y;
2. $\forall y \in Y$, the character in q aligned with tc_y, if any, is different from c_y.

ET may have $\theta(m)$ terminal nodes. Even though $|Q_x| < m$ for each terminal node x, storing Q_x explicitly for each terminal node x could require $\Omega(m^2)$ space overall. We show how to store the sets Q_x so that $O(m)$ space is used in total and any particular Q_x can be retrieved in $O(m)$ time.

Let $l_1, l_2, \ldots, l_h$, in that order, be the nodes along the leftmost path in ET starting at the root and ending at the terminal node l_h. Define the right subtree of l_i to be the subtree rooted at the right child of l_i. Note that $tc_{l_1}, \ldots, tc_{l_{h-1}}$ form a right-to-left sequence. We show how to maintain Q_x, for all terminal nodes x in the right subtrees of $l_1, \ldots, l_{h-1}$, in $O(m)$ space altogether. Only the terminal node l_h remains and Q_{l_h} can be stored explicitly in $O(m)$ space.

We mark some of the nodes $l_1, \ldots, l_{h-1}$. Node l_i is marked if its right child is neither a terminal node nor the parent of two terminal nodes. Thus node l_i is marked if Phase 2 could make at least two comparisons following a mismatch at tc_{l_i}. Let $l'_1, \ldots, l'_s$, in that order, be the nodes marked.

The following lemmas are helpful.

LEMMA 4.18. *Consider terminal nodes x_1 and x_2 of ET and let their least common ancestor be y. Suppose at most one of the following is true: first, y is the parent of both x_1 and x_2, and second, y is the right child of $p(y)$. If q occurs at x_1 and at x_2, then q does not overlap tc_y.*

Proof. Clearly, $y \in Anc(x_1)$ and $y \in Anc(x_2)$. Suppose q overlaps tc_y. Let c be the character in q aligned with tc_y. Without loss of generality, assume that x_1 is a failing descendant of y. Then x_2 is not a failing descendant of y. By the definition of Q_{x_1}, $c \neq c_y$. By the definition of Q_{x_2}, $c = c_y$, a contradiction. ∎

COROLLARY 4.19. *Let $i \geq 1$ be the smallest number such that $q \in Q$ does not overlap tc_{l_i}. q can occur at terminal nodes in the right subtrees of at most one of $l_1, \ldots, l_{i-1}$. Further, if q occurs at some terminal node in the subtree rooted at l_i, it cannot occur at terminal nodes in the right subtrees of any of $l_1, \ldots, l_{i-1}$.*

LEMMA 4.20. *Let $i \geq 1$ be the smallest number such that $q \in Q$ does not overlap tc_{l_i}. Suppose q occurs at a terminal node in the subtree rooted at l_i. Then q occurs at all terminal nodes in the right subtrees of each of those nodes among $l_i, \ldots, l_{h-1}$ which are unmarked. Further, q occurs at l_h.*

Proof. Clearly, the characters in q which overlap $tc_{l_1}, \ldots, tc_{l_{i-1}}$ match the characters $c_{l_1}, \ldots, c_{l_{i-1}}$, respectively. Further, q does not overlap $tc_{l_i}, \ldots, tc_{l_{h-1}}$. Therefore, q occurs at l_h. In addition, if a terminal node x is in the right subtree of an unmarked node l_j, $j \geq i$, then either $l_j = p(x)$ or $l_j = p(p(x))$ and $p(x) \notin Anc(x)$. From the definition of Q_x, q must occur at x. ∎

LEMMA 4.21. *Consider marked node l'_i, $1 \le i \le s$, and let j be the smallest number such that $tc_{l'_i}$ is to the left of the suffix x_j of p_1. p_{j-1} must be the rightmost pattern instance in its group. In addition, p_j is the leftmost presuf pattern instance to survive a mismatch at $tc_{l'_i}$.*

Proof. Since l'_i is marked, at least three presuf pattern instances must survive a mismatch at $tc_{l'_i}$. Let the leftmost three such pattern instances be p_a, p_b, and p_c (listed in left-to-right order). Let A_w be the group containing p_j. Write x_j as uv^e, where $e \ge 1$, u is a proper suffix of primitive v, and all presufs associated with A_w have the form $uv^{e'}$, $e' \ge 1$.

By Lemma 4.5, successful comparisons within the suffix x_j of p_1 suffice to eliminate all but at most two of the pattern instances in the groups $A_{w+1}, \ldots, A_g$. (At most two pattern instances in $A_{w+1}, \ldots, A_g$ can form a half-done set with p_j.) Therefore, $p_a \in A_w$ and $x_a = uv^{e_a}$, $e_a \ge 1$. Since p_a, p_b, and p_c all survive the mismatch at $tc_{l'_i}$, $\{p_a, p_b, p_c\}$ is a half-done set and therefore $e_a \ge 2$. It follows that $x_b = uv^{e_b}$, $e_b \ge 0$, and $x_c = uv^{e_c}$, $e_c \ge 0$.

Next, suppose p_{j-1} is not the rightmost pattern instance in its group. Then $p_{j-1} \in A_w$ and x_{j-1} has the form uv^{e+1}, $e+1 \ge 2$. We show that p_b would have been eliminated by a comparison to the right of $tc_{l'_i}$, which is a contradiction. Note that p_{j-1} and p_a have a difference point, which is aligned with or to the right of $tc_{l'_i}$ and aligned with p_1. Let $p_{j-1}[d]$ be the rightmost such difference point. Clearly, $p_{j-1}[d]$ is to the left of the suffix x_a of p_1. $p_{j-1}[d+(e_a-e_b)|v|]$ is a difference point of p_{j-1} and p_b which is aligned with p_1. A match at this difference point would have eliminated p_b.

Finally, suppose $p_a \neq p_j$. Then p_j and p_a have a difference point, which is aligned with or to the right of $tc_{l'_i}$ and aligned with p_1. Let $p_j[d]$ be the rightmost such difference point. An argument similar to the one in the previous paragraph shows that p_j and p_b have a difference point to the right of $p_j[d]$ and aligned with p_1; a match at this difference point would have eliminated p_b, which is a contradiction. ∎

COROLLARY 4.22. *Consider marked nodes l'_{i_1} and l'_{i_2}, $1 \le i_1 < i_2 \le s$. Let x_{i-1} and x_{j-1} be the smallest suffixes (which are also presufs) of p_1 which overlap $tc_{l'_{i_1}}$ and $tc_{l'_{i_2}}$, respectively. Then $i \neq j$.*

Proof. If $i = j$, then by Lemma 4.21, p_j is the leftmost presuf pattern instance to survive the mismatches at both $tc_{l'_{i_1}}$ and $tc_{l'_{i_2}}$. But since a p_j survives a mismatch at $tc_{l'_{i_1}}$, it cannot survive a match at $tc_{l'_{i_1}}$ and therefore it cannot survive a mismatch at $tc_{l'_{i_2}}$. ∎

LEMMA 4.23. *The size of the presuf corresponding to the rightmost pattern instance in A_j, $1 \le j \le g$, is at most $\frac{m}{(3/2)^j}$.*

Proof. For $j = 1$, the claim is clearly true. Assume that the claim is true for A_{j-1}; i.e.; the size of the presuf corresponding to rightmost pattern instance p_e in A_{j-1} is less than $\frac{m}{(3/2)^{j-1}}$. x_e has either the form uvv or the form uv, where u is a proper suffix of v. In the former case, $x_{e+1} = uv$, and in the latter case, $x_e = x_{e+1}zx_{e+1}$ for some nonempty string z (since uv is not periodic). Thus $|x_{e+1}| < \frac{2|x_e|}{3}$. The claim follows. ∎

LEMMA 4.24. *The number of pattern instances in Q which overlap t_b and are entirely to the right of $tc_{l'_i}$ is less than $\frac{m}{(3/2)^{s-i+1}}$, for all i, $1 \le i \le s$.*

Proof. From Lemma 4.21 and Corollary 4.22, the rightmost presuf pattern instance p_j such that the suffix x_j of p_1 overlaps $tc_{l'_i}$ must be the rightmost pattern instance in some group $A_{j'}$, $j' \geq s-i+1$. The lemma follows from Lemma 4.23. □

Consider the right subtrees of $l'_1, \ldots, l'_s$. Note that the comparisons made in each of these subtrees are aimed at eliminating half-done sets whose leftmost pattern instances are in distinct groups. Each of these comparisons is made to the right of $p_1[m]$, as described in Lemma 4.7 and Corollary 4.8.

LEMMA 4.25. *The number of pattern instances in Q which are entirely to the right of t_b and overlap some text character compared in the right subtree of l'_i is at most $\frac{m}{(3/2)^{s-i+1}}$, for all i, $1 \leq i \leq s$.*

Proof. Recall from Lemma 4.7 that a half-done set whose leftmost pattern instance is in group A_j, $j > 1$, is eliminated in Phase 2 of the elimination strategy by making comparisons at text characters which are at most distance $|x_{j'-1}| - |x_{j'}|$ to the right of t_b, where $p_{j'}$ is the leftmost presuf pattern instance in A_j. From Lemma 4.23, $|x_{j'-1}| < \frac{m}{(3/2)^{j-1}}$, and the lemma follows. □

LEMMA 4.26. *Consider a marked node l'_i and the set of terminal nodes in its right subtree. If a pattern instance q occurs at two of these terminal nodes, say w and y, then q occurs at all terminal nodes in the subtree rooted at the least common ancestor z of w and y.*

Proof. By Lemma 4.18, q does not overlap the character tc_z. Since comparisons made in the right subtree of l'_i constitute a right-to-left sequence, q does not overlap $tc_{z'}$, where z' is any descendant of z. The lemma now follows immediately from the definition of the sets Q_x. □

We are now ready to describe the data structure for storing the Q_x's. The following subsets of Q are required: $Z_1, \ldots, Z_{h-1}$, $Y_1, \ldots, Y_{h-1}$ and $W_1, \ldots, W_s$. The Z and the Y sets are used for terminal nodes which lie in the right subtrees of the unmarked nodes among $l_1, \ldots, l_{h-1}$. The W sets are used for terminal nodes which lie in the right subtrees of marked nodes.

The Z sets are defined first. For each i, $1 \leq i \leq h-1$, where l_i is unmarked, define Z_i to be the set of pattern instances in Q which overlap $tc(l_i)$ and occur only at terminal nodes in the right subtree of l_i. Clearly, $\sum_{i=1}^{h-1} Z_i = O(m)$.

The Y sets are defined next. For each i, $1 \leq i \leq h-1$, define Y_i to be the set of pattern instances $q \in Q$ with the following properties.

1. q does not overlap tc_{l_i}.
2. If $i > 1$, q overlaps $tc_{l_{i-1}}$.
3. q occurs at a terminal node in the subtree rooted at l_i.

Clearly, $\sum_{i=1}^{h-1} Y_i = O(m)$. Further, each pair of Y sets is disjoint and Z_i is disjoint from $Y_1, \ldots, Y_i$. The following lemma explains the significance of the Y and Z sets.

LEMMA 4.27. *If terminal node x is in the right subtree of unmarked node l_i, $Q_x = Y_1 \cup Y_2 \cup \cdots \cup Y_i \cup Z_i$.*

Proof. Suppose $q \in Q_x$. If q overlaps tc_{l_i} then by Corollary 4.19, $q \in Z_i$. If q does not overlap tc_{l_i}, then clearly q must be in some Y_j, $j \leq i$.

Next, suppose $q \in Y_j$, $j \leq i$. By Lemma 4.20, $q \in Q_x$. Finally, if $q \in Z_i$, then $q \in Q_x$ since the only internal node (if any) in the right subtree of l_i is not in $Anc(x)$. □

Finally, the W sets are defined. For each i, $1 \leq i \leq s$, W_i consists of those pattern instances which occur at some terminal node in the right subtree of marked node l'_i. Let W'_i denote the set obtained from W_i by removing those pattern instances which do not overlap any of the text characters compared in the right subtree of l'_i.

LEMMA 4.28. $\sum_{i=1}^{s}|W'_i| = O(m)$.

Proof. Split W'_i into two disjoint subsets, W_i^1 and W_i^2. W_i^1 consists of those pattern instances which overlap $tc_{l'_i}$ and W_i^2 consists of pattern instances which do not overlap $tc_{l'_i}$.

By Lemma 4.18, pattern instances in W_i^1 occur only at terminal nodes in the right subtree of l'_i. Therefore, it suffices to show that $\sum_{i=1}^{s}|W_i^2| = O(m)$. From Lemmas 4.24 and 4.25, it follows that $\sum_{i=1}^{s}|W_i^2| = \sum_{i=1}^{s}(2\frac{m}{(3/2)^{s-i+1}}) = O(m)$. ∎

Consider some i, $1 \leq i \leq s$. The manner in which W_i is maintained so as to facilitate the recovery of Q_x for each terminal node x in the right subtree of marked node l'_i remains to be shown. Clearly, pattern instances in $W_i - W'_i$ occur at all such nodes x and can be stored implicitly in constant space by just storing the rightmost text position compared in the right subtree of l'_i. For the terminal nodes x in l_i's right subtree, we show how to store the pattern instances in $Q_x \cap W'_i$ using a total of $O(|W'_i|)$ space (summing over all x). The linear-space bound then follows from Lemma 4.28.

At each internal node y in the right subtree T of l'_i, a set Com_y is stored. At each terminal node x in T, a set $Spec_x$ is stored. For each $q \in W'_i$, if q occurs only at terminal node x, then it is added to $Spec_x$. Otherwise, if q occurs at more than one terminal node in T, then q is added to the set Com_y, where y is the least common ancestor of those terminal nodes at which q occurs. Clearly, all Com and $Spec$ sets are disjoint and therefore the total space taken by them is $O(|W'_i|)$. The following lemma shows how Q_x can be retrieved from the Com and $Spec$ sets, for each terminal node x in T.

LEMMA 4.29. *For each terminal node $x \in T$, $Q_x = (W_i - W'_i) \cup Com_{y_1} \cup Com_{y_2} \cup \cdots \cup Com_{y_j} \cup Spec_x$, where $y_1, \ldots, y_j$ are the proper ancestors of x in T.*

Proof. The proof follows immediately from Lemma 4.26. ∎

To compute Q_x as an ordered list, it suffices to maintain each of the Y, Z, Com, and $Spec$ sets as ordered lists which are then appended together in $O(m)$ time according to either Lemma 4.27 or Lemma 4.29, as the case may be.

This concludes the data structure description. We remark that all of the data structures mentioned at the beginning of this section can be computed using naïve algorithms in $O(m^2)$ time.

5. The transfer function f.

Before giving the definition of the function f, we prove a number of preliminary lemmas.

5.1. Preliminary lemmas.

These lemmas describe some properties of periodic strings and the distribution of text characters compared in Step 2 (the elimination-tree phase) of the presuf handler described in section 4.2.

Let $V = \{p_1, p_2, \ldots, p_k, p_{k+1}\}$. Consider the set of pattern instances in V which are rightmost in their respective groups. Let p_i be a pattern instance in this set. We introduce a function $h(x_i)$ which is central to the analysis.

DEFINITION. *If $i < k$, then $h(x_i)$ is defined by one of the following three cases:*

1. *x_i is periodic. Then x_{i+1} is also periodic. Let u and v be the head and core, respectively, of x_i. Let w be the core of x_{i+1}. $h(x_i)$ is defined to be the suffix of p_1 of length $|v| + |w|$.*

2. *$x_i = uvu$ is not periodic, where $|u|$ is its s-period. Further, x_{i+1} is periodic with core w. $h(x_i)$ is defined to be the suffix of p_1 of length $|v| + |u| + |w|$.*

3. *$x_i = uvu$ is not periodic, where $|u|$ is its s-period. Further, x_{i+1} is not periodic. $h(x_i)$ is defined to be the suffix of p_1 of length $|u|$.*

If $i = k+1$, then $h(x_i)$ is defined to be the empty string. Note that $i \neq k$ as p_k and p_{k+1} are both in the same group.

The first two lemmas consider the case when $i < k$ and x_{i+1} is periodic with core w. They show that $h(x_i)$ cannot be periodic with core w.

LEMMA 5.1. *Suppose $x_i = uv^2$, where v is the core of x_i. Further, suppose $x_{i+1} = uv = w'w^{k1}$ is periodic with core w, $|w| < |v|$. Then $h(x_i)$ is not periodic with core w.*

Proof. w is a suffix of v. Since v is primitive, $|v|$ is not a multiple of $|w|$. If $h(x_i)$ were periodic with core w, then the prefix of $h(x_i)$ of size $|w|$ would have the form xy, with x a proper suffix of w and y a proper prefix of w. But this prefix of $h(x_i)$ is a suffix of v and hence is the string w. This implies that w is cyclic and cannot be the core of x_{i+1}, a contradiction. □

LEMMA 5.2. *Suppose $x_i = uvu$ is not periodic, where $|u|$ is the s-period of x_i. Suppose the string $x_{i+1} = u = w'w^{k1}$ is periodic with core w, $|w| < |u|$. Then $h(x_i)$ is not periodic with core w.*

Proof. w is a suffix of u. vu is primitive; otherwise, x_i would be periodic. Suppose $h(x_i)$ is periodic with core w. Then $|vu|$ is not a multiple of $|w|$. Therefore, the prefix of $h(x_i)$ of size $|w|$ is of the form xy, with x a proper suffix and y a proper prefix of w. But this prefix of $h(x_i)$ is a suffix of u and hence is the string w. This implies that w is cyclic and cannot be the core of x_{i+1}, a contradiction. □

The next lemma describes the order in which pattern instances in a half-done set are eliminated in Step 2 of the presuf shift handler.

LEMMA 5.3. *Let $p_{i_1}, \ldots, p_{i_r}$, $r \geq 3$, be pattern instances in V comprising a half-done set. For any l, $3 \leq l \leq r$, if p_{i_1} and p_{i_l} both survive at any instant in Step 2, then $p_{i_1}, \ldots, p_{i_{l-1}}$ also survive at that instant.*

Proof. We show that the lemma is true for any instant in Phase 1 and at the end of Phase 1. For Phase 2, the lemma follows from Corollary 4.8.

Consider the rightmost position e such that $p_{i_1}[e]$ is to the left of t_b (recall that t_b is the text character aligned with $p_1[m]$) and different from the character in p_{i_l} aligned with it. The portions of $p_{i_1}, \ldots, p_{i_l}$ whose left and right ends are aligned with $p_{i_1}[e+1]$ and t_b, respectively, are identical and periodic with core v, where v is the core of x_{i_1}. The portions of $p_{i_1}, \ldots, p_{i_{l-1}}$ whose left and right ends are aligned with $p_{i_1}[e]$ and t_b, respectively, are identical. Therefore, a comparison to the right of $p_{i_1}[e]$ eliminates none or all of $p_{i_1}, \ldots, p_{i_l}$ depending upon whether it succeeds or fails. A comparison at $p_{i_1}[e]$ eliminates either p_{i_l} or all of $p_{i_1}, \ldots, p_{i_{l-1}}$. Thus if p_{i_1} and p_{i_l} survive at any instant in Phase 1 or at the end of Phase 1, then all comparisons made until that instant are to the right of $p_{i_1}[e]$. Each of these comparisons eliminates none or all of $p_{i_1}, \ldots, p_{i_l}$. □

The next lemma establishes that if all comparisons in the suffix x_i of p_1 are successful, then at most two pattern instances to the right of p_i survive.

LEMMA 5.4. *Suppose all comparisons made by S_1 within the suffix x_i of p_1 result in matches. Then at most two instances in V among those lying to the right of p_i survive. Further, if two instances p_y and p_z survive, then $\{x_i, x_y, x_z\}$ is a clone set.*

Proof. First, suppose x_i is periodic. Then by the manner in which groups were defined, x_i has the form uv^2. Let $p_a, p_b \in V$, $a, b > i$. If $\{x_i, x_a, x_b\}$ is not a clone set then by Lemma 4.5, successful comparisons in the suffix x_i of p_1 suffice to eliminate one of p_a, p_b. Thus two or more pattern instances in V to the right of p_i can survive only if their presufs form a clone set with x_i. But the only candidates are the pattern instances p_y and p_z whose presufs are uv and u, respectively.

Second, suppose x_i is not periodic. Then it is of the form uvu, where u is its s-period. For no two pattern instances p_y and p_z, $y, z > i$, can $\{x_i, x_y, x_z\}$ be a clone set. By Lemma 4.5, one of p_y, p_z, for every such y and z, can be eliminated by a successful comparison made within the suffix x_i of p_1. Thus in this case, at most one pattern instance in V to the right of p_i survives. □

The next two lemmas establish that if all comparisons within $h(x_i)$ are successful, then at most two pattern instances in V to the right of p_i survive.

LEMMA 5.5. *Suppose x_{i+1} is periodic with core w and all comparisons made by S_1 within $h(x_i)$ result in matches. Then at most two instances in V among those to the right of x_i survive.*

Proof. Since the case $i = k + 1$ is vacuous, we assume that $i < k$.

The proof is based on Lemmas 5.1, 5.2, 5.3, and 5.4. Let A_s be the group containing p_i. Let $p_{i+1}, \ldots, p_y$ be the pattern instances in group A_{s+1}. Consider the set V' of pattern instances in V which are to the right of p_i and which survive successful comparisons in the suffix x_y of p_1. By Lemma 5.4, with at most one exception (call it p_o), the pattern instances in V' form a half-done set. By Lemma 5.3, the presufs corresponding to the pattern instances in this half-done set comprise the set $\{w'w^{k2}, \ldots, w'w^{k3+1}, w'w^{k3}\}$, where $k3$ equals 0, 1, or 2. Let $V' = \{p_{i_1}, p_{i_2}, \ldots, p_{i_j}, p_o\}$. We show that successful comparisons in $h(x_i)$ eliminate all but at most one of $\{p_{i_1}, p_{i_2}, \ldots, p_{i_j}\}$.

Note that $\{p_{i_1}, p_{i_2}, \ldots, p_{i_j}\}$ is a half-done set. By Lemmas 5.1 and 5.2, the suffix $h(x_i)$ of p_1 (and of x_i) is not periodic with core w. Let the rightmost suffix of p_1 which is longer than $|x_{i+1}|$ and not periodic with core w begin at $p_1[e]$; $p_1[e]$ lies in $h(x_i)$. Consider the largest h, $1 \leq h \leq j$, such that p_{i_h} survives all comparisons made to the right of $p_1[e]$. Then by Lemma 5.3, $p_{i_1}, \ldots, p_{i_{h-1}}$ also survive these comparisons while $p_{i_{h+1}}, \ldots, p_{i_j}$ are eliminated. If $h \leq 1$, then we are done. Otherwise, as shown in the next paragraph, the characters in $p_{i_1}, \ldots, p_{i_{h-1}}$ aligned with $p_1[e]$ are identical to each other yet different from $p_1[e]$. Hence there will be a comparison involving $p_1[e]$, which by assumption is a match; this leaves only p_{i_h} and p_o uneliminated.

Since the rightmost eligible character is always chosen by the elimination strategy for comparison, the portions of $p_{i_1}, \ldots, p_{i_h}$ aligned with the suffix of p_1 which lies to the right of $p_1[e]$ match that suffix. Suppose for some r, $1 \leq r < h$, $a = p_{i_r}[c] \neq p_{i_r}[c + |w|] = b$, where $p_{i_r}[c]$ is aligned with $p_1[e]$. Since $p_{i_{r+1}}$ is $|w|$ units to the right of p_{i_r}, the character in $p_{i_{r+1}}$ aligned with $p_{i_r}[c + |w|] = b$ is an a, a contradiction. Therefore, the characters in $p_{i_1}, \ldots, p_{i_{h-1}}$ aligned with $p_1[e]$ are all equal to the character $p_1[e + |w|]$. However, from the definition of e, $p_1[e + |w|] \neq p_1[e]$. This proves the lemma. □

LEMMA 5.6. *Suppose x_{i+1} is not periodic and all comparisons made by S_1 within the suffix $h(x_i)$ of p_1 result in matches. Then at most two instances in V among those to the right of p_i survive.*

Proof. Since the case $i = k + 1$ is vacuous, we assume that $i < k$.

By the manner in which groups were defined, x_i is not periodic. Since x_{i+1} is not periodic, p_{i+1} is the rightmost instance in its group. Thus x_{i+1} cannot form a clone set with any two of its presufs. By Lemma 5.4, at most one pattern instance to the right of p_{i+1} survives successful comparisons in the suffix $x_{i+1} = h(x_i)$ of p_1. □

The following lemma relates the length of the presufs x_i and x_{i+2} with the suffix $h(x_i)$ of p_1 for $i \leq k - 1$.

LEMMA 5.7. $|x_{i+2}| + |h(x_i)| \leq |x_i|$.

Proof. First, suppose $x_i = uvu$ is not periodic, where u is its s-period. Then $|x_{i+2}| < |u|$. If x_{i+1} is not periodic, then $h(x_i) = u$ and $|x_{i+2}| + |h(x_i)| < 2|u| < |x_i|$. If x_{i+1} is periodic with core w, then $|h(x_i)| = |u| + |v| + |w|$ and $x_{i+2} = |u| - |w|$. This implies that $|x_{i+2}| + |h(x_i)| = |x_i|$.

Next, suppose x_i is periodic with core v and head u. Then x_{i+1} is also periodic, say with core w. Thus $|h(x_i)| = |v|+|w|$ and $|x_{i+2}| = |v|+|u|-|w|$. Then $|x_{i+2}|+|h(x_i)| = 2|v| + |u| = |x_i|$. ∎

DEFINITIONS. *Let the term* misfit *refer to any character that differs from the rightmost character of* p. *If* $|x_k| > 1$, *let* r_i *be the number of pattern instances in* V *which lie to the right of* p_i. *Otherwise, if* $|x_k| = 1$, *let* r_i *be one more than the number of pattern instances in* V *which lie to the right of* p_i *and do not belong to the rightmost group. For convenience, we define* r_i *to be* 0 *if* $|x_k| = 1$ *and* p_i *belongs to the rightmost group.*

We provide some lower bounds on the number of occurrences of misfit characters in the presufs of p and in the cores of periodic presufs.

LEMMA 5.8. *Let* $|x_k| > 1$. *Let* p_j, $j \leq k$, *be any pattern instance in* V. *Then* x_j *contains at least* r_j *instances of the string* x_k *and hence* r_j *misfit characters.*

Proof. Since x_k is the smallest nonnull suffix of p that matches a prefix of p, no nonnull suffix of x_k matches a prefix of x_k. Hence all instances of x_k in any string are disjoint. Since x_k itself contains x_k and $r_k = 1$, the lemma is true for $j = k$. Next, suppose $j < k$ and assume inductively that x_{j+1} contains at least r_{j+1} instances of x_k. Then since x_{j+1} is a proper prefix and a proper suffix of x_j, x_j must contain at least $r_{j+1} + 1 = r_j$ instances of x_k. Since the first character of x_k differs from its last character, x_j has at least r_j misfit characters. ∎

LEMMA 5.9. *Suppose* $|x_k| = 1$. *Let* p_j *be any pattern instance in* V. *Then* x_j *has at least* r_j *misfit characters.*

Proof. If p_j belongs to the rightmost group, then $r_j = 0$ and the lemma holds trivially. Therefore, suppose p_j is not in the rightmost group. Let p_y be the rightmost pattern instance not in the rightmost group. x_y contains at least one misfit character; otherwise, it would be in the rightmost group. Since $r_y = 1$, the lemma is true for $j = y$. Next, assume that $j < y$ and assume inductively that x_{j+1} contains at least r_{j+1} misfit characters. Then since x_{j+1} is a proper prefix and a proper suffix of x_j, x_j must have at least $r_{j+1} + 1 = r_j$ misfit characters. ∎

LEMMA 5.10. *Let* p_j *be any instance in* V *and suppose* x_j *is periodic with head* u *and core* v, $|v| > 1$. *Then* v *contains a misfit character.*

Proof. If v does not contain a misfit character, then x_j does not contain a misfit character either. This implies that all the characters in x_j are identical. This contradicts the assumption that $|v| > 1$. ∎

We conclude this section of lemmas with two key lemmas, the *h-suffix mapping lemma* and the *half-done set mapping lemma*. In the h-suffix mapping lemma, a set $R_1(i)$ of text characters is defined for each i, $1 \leq i \leq k - 1$, such that p_i is the rightmost instance in its group. In the half-done set mapping lemma, a set $R_2(O)$ of text characters is defined for a half-done set O consisting of pattern instances from V. These two sets are used as ranges for the f function.

Recall that $V = \{p_1, \ldots, p_k\}$ and t_b is the text character aligned with the rightmost character of p_1.

Let $i \leq k-1$ and p_i be the rightmost instance in its group. Let h'_i be the suffix of length $|x_{i+2}|$ of the prefix x_i of p. Let $R_1(i)$ be the set of text characters with which $|p$ is aligned when some misfit character in h'_i is aligned with t_b.

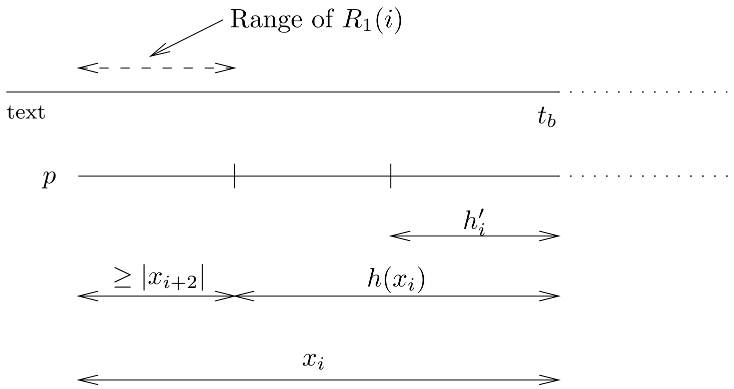

FIG. 5. *The h-suffix mapping lemma.*

LEMMA 5.11 (the h-suffix mapping lemma). $|R_1(i)| \geq r_i - 2$. *All text characters in* $R_1(i)$ *lie strictly to the left of* $|h(x_i)$ *but within the suffix* x_i *of* p_1.

Proof. (See Fig. 5.) By Lemmas 5.8 and 5.9, x_{i+2} and hence h'_i contain at least $r_{i-2} = \max\{0, r_i - 2\}$ misfit characters. Therefore, $|R_1(i)| \geq r_i - 2$. By Lemma 5.7, the left end of any pattern instance in which h'_i overlaps t_b is strictly to the left of $|h(x_i)$ and within the suffix x_i of p_1. ∎

DEFINITION. *Let* $O \subset V$ *be a half-done set consisting of the pattern instances* $\{p_{h_1}, \ldots, p_{h_j}\}$, $j \geq 3$, $p_{h_1} = p_1$. *Let the head and core of* x_{h_1} *be denoted* u *and* v, *respectively. Let* $v = u'u$. *Suppose* $|v| > 1$. *Further, suppose* p_{h_i} *is* $p_{h_{i-1}}$ *shifted distance* $|v|$ *to the right, for* $1 < i \leq j$. *Let* i_c, $2 \leq c \leq j$, *be the largest index such that* $p_{h_1}[i_c]$ *is different from the character in* p_{h_c} *aligned with it* (*such an index exists by Lemma* 4.7). *Note that* $i_c - i_{c-1} = |v|$, *for all* c, $3 \leq c \leq j$, *and that* $p_{h_1}[i_2] = p_{h_1}[i_3] = \cdots = p_{h_1}[i_j]$. *The text character* t_{i_c} *aligned with* $p_{h_1}[i_c]$ *is called the* characteristic character *of* p_{h_c}.

Let d be the leftmost character in the prefix uu' of p which differs from $p_{h_1}[i_2]$. Define $R_2(t_{i_c})$, $3 \leq c \leq j$, to be the text character with which $|p$ is aligned when d is aligned with t_{i_c}. In addition, define $R_2(O_e)$ to be the set of text characters $R_2(t_{i_c})$, $3 \leq c \leq e \leq j$. For convenience, let $R_2(O)$ denote $R_2(O_j)$.

LEMMA 5.12 (the half-done set mapping lemma). *All text characters in* $R_2(O_j)$ *are distinct.* $R_2(t_{i_c})$ *is aligned with or to the left of* t_{i_c} *and strictly to the right of* $t_{i_{c-1}}$, *for* $3 \leq c \leq j$. *All characters in* $R_2(O_c)$ *are aligned with or to the left of the characteristic character of* p_{h_c}; *in addition, they are strictly to the right of* $|p_{k+1}$, *for* $3 \leq c \leq j$.

Proof. By construction, $R_2(t_{i_c})$ is aligned with or to the left of t_{i_c}, $3 \leq c \leq j$. In addition, $R_2(t_{i_c})$ is at most distance $|v| - 1$ to the left of t_{i_c}. Since $i_c - i_{c-1} = |v|$, $R_2(t_{i_c})$ is strictly to the right of $t_{i_{c-1}}$.

The only part of the lemma still unproven is the claim that all characters in $R_2(O_j)$ are strictly to the right of $|p_{k+1}$. Note that t_{i_3} is distance $|v|$ to the right of t_{i_2}, $R_2(t_{i_3})$ is at most distance $|v| - 1$ to the left of t_{i_3}, and all characters in $R_2(O_j)$ are aligned with or to the right of $R_2(t_{i_3})$. Since t_{i_2} is aligned with or to the right of $|p_{k+1}$, the lemma follows. ∎

Note that p_{h_c} is eliminated by the time a comparison is made strictly to the left of its characteristic character, for $3 \leq c \leq j$. Further, if a successful comparison eliminates p_{h_c}, then this comparison must involve its characteristic character.

5.2. The transfer function f. Let C be the set of text characters involved in comparisons in Steps 1 and 2 of the presuf shift handler of section 4.2. For each character $t_c \in C$, with at most two exceptions, we define $f(t_c)$ to be a text character t_d satisfying the following properties.

1. t_d is to the right of $|p_{k+1}$.
2. t_d either coincides with t_c or lies to the left of t_c.
3. The pattern instance whose left end is aligned with t_d is eliminated as a result of comparisons in Steps 1 and 2 of the presuf shift handler.
4. For every distinct $t_{c_1}, t_{c_2} \in C$, $f(t_{c_1}) \neq f(t_{c_2})$.

Furthermore, the mismatches, if any, are always included among the exceptions. We refer to the above properties as Properties 1, 2, 3 and 4, respectively.

Since patterns with $g = 1$ and $|x_k| = 1$ are special-case patterns, we assume that $g > 1$ if $|x_k| = 1$. Further, if $p_1[m]$ does not match the text, then Steps 1 and 2 of the presuf shift handler together make at most one comparison. Therefore, we also assume that $p_1[m]$ matches the text. Let p_l be the rightmost pattern instance in A_1. Let p_r be the rightmost pattern instance in V outside A_g, if any.

We split the sequence C' of comparisons made in Steps 1 and 2 of the presuf shift handler into three disjoint classes as follows.

1. Class 1 consists of the comparison in Step 1. In addition, if $|x_k| = 1$, then Class 1 contains the comparisons which comprise the smallest prefix of C' having the following property: either the last comparison in this prefix is unsuccessful or following that comparison, exactly one pattern instance in A_g survives.

2. Class 2 consists of the comparisons in C' which follow all Class 1 comparisons and are made in the suffix $h(x_l)$ of p_1.

3. Class 3 consists of comparisons in C' which follow all Class 2 comparisons.

Note that if Class 1 contains an unsuccessful comparison, then Class 2 is empty because no further comparisons are made in the suffix $h(x_l)$ of p_1. Thus Classes 1 and 2 together have at most one unsuccessful comparison. The only other possibly unsuccessful comparison is the last comparison in Class 3. We do not define an f value for the last comparison in Class 3. In addition, one other comparison may not receive an f value. If Classes 1 and 2 contain an unsuccessful comparison, then this comparison does not receive an f value. If all comparisons in Classes 1 and 2 are successful, then one successful comparison in one of the three classes may not receive an f value. All other comparisons receive f values. Thus f values are never defined for mismatches and at most two comparisons in C' do not receive f values.

We define f values for each class in turn. f values for Class 2 comparisons are always defined using the set $R_1(l)$. These f values are aligned with the suffix x_l of p_1 and to the left of $h(x_l)$. f values for Class 3 comparisons are defined in one of three ways. If all comparisons in Classes 1 and 2 are successful, then these f values are to the left of the suffix x_l of p_1. If Class 2 contains a mismatch, then these f values are defined using the set $R_1(l)$. If Class 2 is empty and Class 1 contains a mismatch, then these f values are aligned with or to the right of the suffix x_{r+1} of p_1. f values for Classes 2 and 3 are easily seen to be distinct. f values for Class 1 comparisons are aligned either with the suffix x_r of p_1 or with the suffix x_{r-1} of p_1; in Lemma 5.17, we show that these f values do not clash with the f values for Classes 2 and 3.

Classes 2 *and* 3. We consider three cases.

Case 1. Class 2 contains an unsuccessful comparison.

Classes 2 and 3 together contain at most $r_l - 1$ comparisons in addition to this unsuccessful comparison. To see this, note that $r_l - 1$ comparisons in addition to the comparisons in Class 1 suffice to eliminate all but one of the pattern instances in V to the right of p_l. Further, excluding the unsuccessful Class 2 comparison and the last comparison in Class 3, all other comparisons in Classes 2 and 3 are successful. f is defined to map the text characters involved in these $r_l - 2$ successful comparisons to the text characters in $R_1(l)$ in some arbitrary order. By the h-suffix mapping lemma and the fact that all Class 3 comparisons are to the right of $p_1[m]$ in this case, all the text characters in $R_1(l)$ lie to the left of all the text characters involved in Class 2 and Class 3 comparisons. Clearly, Properties 2, 3, and 4 are true for these f values. Property 1 follows from the fact that $|x_l| \le |x_1| < \frac{m}{2}$, and hence $|p_{k+1}$ is to the left of the suffix x_l of p_1.

Case 2. All comparisons in Classes 1 and 2 are successful.

There are at most r_l comparisons in Class 2, all of which are successful. f is defined to map the text characters involved in $r_l - 2$ of these r_l comparisons to the text characters in $R_1(l)$ in some arbitrary order. As in Case 1, Properties 1, 2, 3, and 4 are satisfied by these f values. This leaves at most s Class 2 comparisons for some $s \le 2$.

Next, we define f values for Class 3 comparisons and s Class 2 comparisons. These f values will be defined for all comparisons in Class 3 plus the s comparisons in Class 2, with at most two exceptions. These f values will be to the left of the suffix x_l of p_1 and thus clearly distinct from f values for Class 2 comparisons.

Following Class 2 comparisons, at most $\min\{r_l, 2\} - s$ of the pattern instances to the right of p_l survive along with pattern instances in A_1. Let O' denote the following set of $\min\{r_l, 2\}$ pattern instances: those pattern instances to the right of p_l which survive Class 1 and Class 2 comparisons and those pattern instances which are eliminated by one of the s Class 2 comparisons under consideration. Let O refer to the largest half-done set consisting of pattern instances in A_1 and O'. Redefine O' by removing pattern instances in it which are also in O. Considering comparisons which eliminate pattern instances in O and O' is equivalent to considering Class 3 comparisons plus the s Class 2 comparisons.

Let $O = \{p_{h_1}, \dots, p_{h_e}\}$. If $l = 1$, then the number of comparisons in Class 3 plus s is at most $2 - s + s = 2$. In this case, we do not define f values for the comparisons in Class 3 and the s comparisons in Class 2. Therefore, suppose that $l > 1$. Each successful comparison which eliminates a pattern instance in O involves the characteristic character of the pattern instance eliminated. Let v and u be the core and head, respectively, of x_{h_1} and let $v = u'u$. $|v| > 1$ because either $|x_k| > 1$ or $|x_k| = 1$ and $g > 1$. By Lemma 5.10, v contains a misfit character.

First, consider successful comparisons which eliminate pattern instances in O. If $|O| \le 2$, there is at most one such comparison in Class 3 and we do not define an f value for it. Therefore, suppose $|O| > 2$. There are two subcases depending on the location of the characteristic character t_{i_e} of p_{h_e}.

Subcase 2a. Either O' is not empty and t_{i_e} is strictly to the left of the left end of the suffix $x_{h_{e-1}}$ of p_1 or O' is empty and t_{i_e} is strictly to the left of the left end of the suffix $x_{h_{e-2}}$ of p_1.

For $3 \le c \le e$, if a successful comparison is made at t_{i_c}, $f(t_{i_c})$ is defined to be $R_2(t_{i_c})$. By the half-done set mapping lemma, these f values are strictly to the left of the suffix $x_{h_{e-1}}$ of p_1 if O' is not empty and strictly to the left of the suffix $x_{h_{e-2}}$ of p_1 if O' is empty. A simple case analysis (O' equals 0, 1, 2) shows that these f values are strictly to the left of the left end of the suffix x_l of p_1, as claimed. Properties 1, 2, and 4 for these f values follow easily from the half-done set mapping lemma while Property 3 follows from the definition of the set $R_2(O)$. At most one successful comparison eliminating pattern instances in O does not have an f value: the one eliminating p_{h_2}.

Subcase 2b. Either O' is not empty and t_{i_e} is aligned with some character in the suffix $x_{h_{e-1}}$ of p_1 or O' is empty and the characteristic character of p_{h_e} is aligned with some character in the suffix $x_{h_{e-2}}$ of p_1.

In the first case, t_{i_c}, the characteristic character of p_{h_c}, is aligned with some character in the suffix $x_{h_{c-1}}$ of p_1, for $2 \le c \le e$. Consider the set R'_2 of $e-2$ text characters with which $|p$ is aligned when the rightmost misfit character in the prefix x_{h_c} of p, $1 \le c \le e-2$, is aligned with t_b. Since v contains a misfit character, the cth leftmost text character in R'_2 is aligned with some character in the suffix x_{h_c} of p_1 and is strictly to the left of the left end of the suffix $x_{h_{c+1}}$ of p_1. A successful comparison at t_{i_c}, $3 \le c \le e$, is mapped by f to the $(c-2)$nd leftmost character in R'_2. Clearly, all characters in R'_2 are distinct and $f(t_{i_c})$ is strictly to the left of t_{i_c}. All characters in R'_2 are aligned with some character in the suffix x_1 of p_1. Thus Properties 1, 2, 3, and 4 are satisfied by these f values. These f values are strictly to the left of the left end of the suffix $x_{h_{e-1}}$ of p_1. Since O' is not empty, $h_{e-1} \le l$. Therefore, these f values are strictly to the left of the left end of the suffix x_l of p_1. Again, the only successful comparison without an f value, if any, is the one eliminating p_{h_2}.

In the second case, t_{i_c}, $4 \le c \le e$, is aligned with some character in the suffix $x_{h_{c-2}}$ of p_1. f values are not defined for the two leftmost comparisons under consideration. The remaining comparisons involve text characters aligned with some character in the suffix x_{h_2} of p_1. A successful comparison at t_{i_c}, $4 \le c \le e$, is mapped by f to the $(c-3)$rd leftmost character in R'_2. Clearly, $f(t_{i_c})$ is strictly to the left of t_{i_c}. As in the first case, Properties 1, 2, 3, and 4 are satisfied by these f values. All of these f values are to the left of the left end of the suffix $x_{h_{e-2}}$ of p_1. Since $h_{e-2} \le l$ for this case, these f values are strictly to the left of the left end of the suffix x_l of p_1. The only successful comparisons without f values, if any, are those eliminating p_{h_3} and p_{h_2}. This ends Subcase 2b.

Before we define f values for comparisons which eliminate pattern instances in O', we need a lemma which will be used later when Class 3 comparisons are defined. This lemma can be verified easily from the above description.

LEMMA 5.13. *If $O \subset A_1$ and $|O'| \le 1$, then the f values defined in Subcases* 2a *and* 2b *are to the left of the suffix x_{l-1} of p_1.*

Next, consider successful comparisons which eliminate pattern instances in O'. If $|O'| < 2$ or no successful comparison eliminates a pattern instance in O', then no further f values are defined. Therefore, suppose $|O'| = 2$ and a successful comparison is made to eliminate one of the pattern instances in O'. By Lemma 4.5, this comparison involves a text character t_c which is aligned with some character in the suffix x_{h_e} of p_1. $f(t_c)$ is defined to be the text character with which $|p$ is aligned when the rightmost misfit character in the prefix $x_{h_{e-1}}$ of p_1 is aligned with t_b. Since v contains a misfit character, $f(t_c)$ is aligned with some character in the suffix $x_{h_{e-1}}$ of p_1 and is strictly to the left of the suffix x_{h_e} of p_1. $f(t_c)$ is thus to the right of and distinct from all f values defined previously for comparisons which eliminate pattern instances in O. Further, $f(t_c)$ is to the left of t_c because t_c is aligned with the suffix x_{h_e} of p_1. Since $|O'| = 2$, $l = h_e$, and therefore $f(t_c)$ is strictly to the left of the suffix x_l of p_1, as claimed. Now Properties 1, 2, 3, and 4 are easily seen to be true for all Class 2 and 3 comparisons.

LEMMA 5.14. *For Case* 2, f *values have been defined for all but two of the comparisons in Classes* 2 *and* 3. *Further, the omitted comparisons include mismatches, if any.*

Proof. We just need to show that at most two of the comparisons among those which eliminate pattern instances in O and O' do not receive f values, for the mismatches never receive f values.

If $|O| < 2$, then there are at most two comparisons which eliminate pattern instances in O and O'. If O' is empty, then f values are defined for all but the last two comparisons which eliminate pattern instances in O. Therefore, suppose O' is not empty and $|O| \geq 2$. The only possible successful comparisons for which an f value might not be defined are those which eliminate p_{h_2} or one of the pattern instances in O'. There are two cases.

First, suppose $|O'| = 1$. Let $O' = \{p_z\}$. The only possible successful comparisons for which an f value is not defined are those which eliminate p_{h_2} or p_z. We show that if one of these successful comparisons actually occurs, then there can be at most one mismatch, and if both these successful comparisons occur, then there are no mismatches. (Recall that all comparisons in Classes 1 and 2 are successful.) Suppose p_{h_2} is eliminated by a successful comparison. p_{h_1} must be alive immediately before this comparison and $p_{h_3} \dots p_{h_e}$ must have been eliminated prior to this comparison. This implies that no mismatch could have occurred before this comparison and only the pattern instances p_{h_1} and p_z survive this comparison. Therefore, if p_{h_2} is eliminated by a successful comparison, then there is at most one unsuccessful comparison, and if both p_{h_2} and p_z are eliminated by successful comparisons, then there are no unsuccessful comparisons. Next, suppose p_z is eliminated by a successful comparison but no successful comparison eliminates p_{h_2}. Each of the other comparisons in Class 3 eliminates some pattern instance in O and the first such unsuccessful comparison eliminates all but one of the instances in O. Therefore, there is at most one unsuccessful comparison in this case.

Second, suppose $|O| \geq 2$ and $|O'| = 2$. If one of the pattern instances in O' is eliminated by a successful Class 2 or 3 comparison, then an f value is defined for this comparison. From this point onwards, $|O| \geq 2$ and $|O'| = 1$. Therefore, the argument in the previous paragraph applies. On the other hand, if no successful Class 2 or 3 comparison eliminates a pattern instance in O', then the first comparison in Class 3 must be unsuccessful. This comparison leaves at most two pattern instances uneliminated and thus there are at most two comparisons which eliminate pattern instances in O and O'. ◻

Case 3. Class 1 contains an unsuccessful comparison.

In this case, $|x_k| = 1$ and Class 2 is empty as mentioned before. We define f values for all but the last of the comparisons in Class 3. These f values are aligned with or to the right of the suffix x_{r+1} of p_1. All Class 3 comparisons are made to the right of $p_1[m]$ in this case. Each such successful comparison matches an instance of the character x_k in p_{r+1} against a text character t_c; t_c is aligned with a non-x_k character in some pattern instance p_s, $s > r$. f is defined to map a text character t_c matched successfully by a Class 3 comparison to the text character with which $|p$ is aligned when the leftmost non-x_k character in p is aligned with t_c. Clearly, these f values are aligned with or to the right of the suffix x_{r+1} of p_1 and Properties 1, 2, 3, and 4 are satisfied by these f values.

This finishes the description of the f function for Classes 2 and 3. The following lemma is obvious from the above description.

LEMMA 5.15. *f values for Class 2 and 3 comparisons belong to one of the following sets of text characters:*
- (i) *the set $R_1(l)$;*
- (ii) *the set of text characters to the left of the suffix x_l of p_1 and to the right of $|p_{k+1}$;*
- (iii) *the set of text characters aligned with or to the right of the suffix x_{r+1} of p_1.*

Further, an f value can be in set (i) *only if $r_l - 2 > 0$ and in set* (iii) *only if Class* 1 *contains an unsuccessful comparison.*

Class 1. We consider two cases, $|x_k| > 1$ and $|x_k| = 1$.

Case 1. $|x_k| > 1$.

The only comparison in Class 1 matches t_b with $p_1[m]$. $f(t_b)$ is defined to be t_b. This mapping satisfies Property 3 because the leftmost character in p is a misfit character in this case. If $g = 1$, then all other comparisons in C' are made to the left of t_b, and therefore all other f values are to the left of t_b. If $g > 1$, then all other f values are either to the left of the suffix x_l of p_1 or to the left of the suffix $h(x_l)$ of p_1. Since $|h(x_l)| \geq 1$ if $g > 1$, these f values are to the left of t_b. Therefore, Property 4 is satisfied by all f values. Properties 1 and 2 are obvious for $f(t_b)$.

Case 2. $|x_k| = 1$.

$g > 1$ by assumption. f values are defined for all comparisons in Class 1 unless either Class 1 contains an unsuccessful comparison or $r = l = 1$. If Class 1 contains an unsuccessful comparison or if $r = l = 1$, then one comparison in Class 1 does not receive an f value. However, in these cases, there is at most one comparison in Classes 2 and 3 for which an f value was not defined earlier. To see this, note that if the last Class 1 comparison is successful and $r = l = 1$, then Classes 2 and 3 together can have at most one comparison, and if the last Class 1 comparison is unsuccessful, then Class 2 is empty and Case 3 must hold for Class 3 comparisons.

All f values defined for Class 1 comparisons will be to the left of the suffix x_{r+1} of p_1 and aligned with either the suffix x_{r-1} of p_1 or the suffix x_r of p_1. Clearly, if $p_l \neq p_r, p_{r-1}$, then these f values are distinct from all f values defined earlier for Class 2 and 3 comparisons. If $p_l = p_{r-1}$, then $r_l - 2 = 0$ and all f values for Class 2 and 3 comparisons are either to the left of the suffix x_{r-1} of p_1 or aligned with or to the right of the suffix x_{r+1} of p_1. Therefore, f values for comparisons in Class 1 are distinct from f values for comparisons in Classes 2 and 3 in this case. If $p_l = p_r$, then $r_l - 2 < 0$ and, by Lemma 5.15, all f values for Class 2 and 3 comparisons are either to the left of the suffix x_r of p_1 or aligned with or to the right of the suffix x_{r+1} of p_1. In this case, we show that if an f value for some Class 1 comparison is aligned with the suffix x_{r-1} of p_1 and to the left of the suffix x_r of p_1, then all f values for Class 2 and Class 3 comparisons are to the left of the suffix x_{r-1} of p_1. Thus all f values are distinct.

The following lemma describes the distribution of Class 1 comparisons.

LEMMA 5.16. *Let d be the rightmost misfit character in p_1. Each Class* 1 *comparison involves a text character which is aligned with or to the right of d.*

Proof. Suppose two pattern instances $p_{j_1}, p_{j_2} \in A_g$, $j_1 < j_2$, are left uneliminated by comparisons made at or to the right of d. Let c_{j_1} and c_{j_2} be the portions of p_{j_1} and p_{j_2}, respectively, which overlap the suffix z of p_1 starting at d. Then $c_{j_1} = c_{j_2} = z$. Consider the characters in p_{j_1} and p_{j_2} aligned with the $(j_2 - j_1)$th character to the right of d in p_1. Clearly, the first of these matches x_k while the second is a misfit

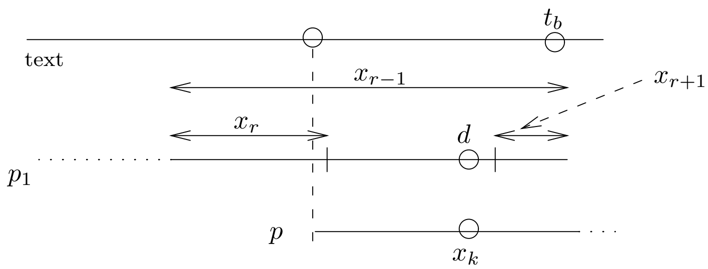

FIG. 6. *The set* R_3, $r > 1$.

character. This is a contradiction. □

Note that x_r contains a misfit character. Further, its suffix and prefix x_{r+1} are disjoint. Each contains at least $k - r$ instances of x_k. We define a set R_3 of text characters which serves as the range of f values for Class 1 comparisons. The definition has the following property. All characters in R_3 are to the left of the suffix x_{r+1} of p_1. If all successful Class 1 comparisons are made to the right of d or if $r = 1$, then all characters in R_3 are aligned with the suffix x_r of p_1. If a successful Class 1 comparison is made at d and $r > 1$, then characters in R_3 are aligned with the suffix x_{r-1} of p_1.

First, suppose all successful Class 1 comparisons are made to the right of d. Each successful comparison matches an occurrence of x_k to the right of d against the text. R_3 is defined to be the set of text characters with which $|p$ is aligned when d', the leftmost misfit character in p, is aligned with one of the text characters matched by a Class 1 comparison. Clearly, all characters in R_3 are aligned with the suffix x_r of p_1. f is defined to map the text characters compared by Class 1 comparisons to the text characters in R_3 in some arbitrary order. Properties 2, 3, and 4 readily follow for these f values. Property 1 follows from the fact that $|x_r| \leq |x_1| < \frac{m}{2}$.

Next, suppose a successful Class 1 comparison is made at d. In this case, all comparisons in Class 1 are successful and there are at most $k + 1 - r$ comparisons in Class 1. R_3 is defined differently depending upon whether $r = 1$ or $r > 1$.

First, suppose $r = 1$. R_3 is defined to contain the $k - r$ text characters with which $|p$ is aligned when one of $k - r$ instances of x_k to the left of d' (recall that d' is the leftmost misfit character in p) is aligned with d. The text characters in R_3 are clearly aligned with the suffix x_r of p_1. f is defined to map up to $k - r$ of the text characters compared by Class 1 comparisons to the $k - r$ text characters in R_3 in some arbitrary order. All of these f values are distinct and are aligned with or to the left of d. Properties 2, 3, and 4 immediately follow for these f values. Property 1 follows from the fact that $|x_r| \leq |x_1| < \frac{m}{2}$.

Next, suppose $r > 1$. See Fig. 6. When p is placed with $|p$ aligned with the left end of the suffix x_{r-1} of p_1, there exist at least $2(k - r) \geq k - r + 1$ instances of x_k to the left of d in p. R_3 is defined to be the set of $2(k - r)$ text characters with which $|p$ is aligned when one of these $2(k - r)$ instances of x_k is aligned with d. Clearly, characters in R_3 are aligned with the suffix x_{r-1} of p_1. f is defined to map the text characters compared by Class 1 comparisons to some $k - r + 1$ of the $2(k - r)$ text characters in R_3 in some arbitrary order. Properties 2 and 3 readily follow for these f values. Property 1 follows from the fact that $|x_{r-1}| \leq |x_1| < \frac{m}{2}$. The distinctness of these f values from the f values for Classes 2 and 3 follows from the following lemma.

LEMMA 5.17. *If $r > 1$, $p_l = p_r$, and a successful Class* 1 *comparison is made at* d, *then* f *values for Class* 2 *and* 3 *comparisons are to the left of the suffix* x_{r-1} *of* p_1.

Proof. At most one pattern instance $p_{r'} \in A_g$ survives a successful comparison at d. Further, Class 1 does not contain an unsuccessful comparison. Since $r_l = 1$, Class 2 contains at most one comparison. If this comparison is unsuccessful, then only $p_{r'}$ survives; no f values are defined for Class 2 or 3 comparisons in this case. If the sole Class 2 comparison is a successful one, then Case 2 must hold for all Class 2 and 3 comparisons. Since $r_l = 1$, $r_l - 2 < 0$ and, by Lemma 5.15, no f values are defined using the set $R_1(l)$. Therefore, all f values for Class 2 and 3 comparisons in this case are defined as in Subcases 2a and 2b. Refer to these subcases. Note that, in this case, O consists of the pattern instances in A_1 and $O' = \{p_{r'}\}$. From Lemma 5.13, all f values for Class 2 and 3 comparisons are to the left of the suffix x_{r-1} of p_1 in this case. $\square$

This concludes the definition of the f function.

6. Presuf shifts with $|x'_1| \geq \frac{m}{2}$. This case can occur only for periodic patterns. Therefore, assume that p is periodic and has the form $u_p v_p^{i_p}$, where v_p and u_p are the core and head of p, respectively, and $i_p \geq 2$.

Recall that the lower bound of $\frac{m+1}{2}$ on the distance between consecutive presuf shifts was crucial in deriving the comparison complexity for the case where $|x'_1| < \frac{m}{2}$. This lower bound does not hold if $|x'_1| \geq \frac{m}{2}$. Consecutive presuf shifts can occur distance $|v_p| \ll \frac{m+1}{2}$ apart. Even a single mismatch per presuf shift leads to a large comparison complexity. Since the problem in this case lies only in the frequency of occurrence of presuf shifts, we use the same basic algorithm, changing only the presuf shift handler. The new presuf shift handler ensures that either two consecutive presuf shifts are at least distance $\frac{m+1}{2}$ apart or no mismatch occurs between consecutive presuf shifts. In fact, we show the following stronger claim about the performance of the presuf shift handler. A presuf shift has overhead 0 if the next presuf shift occurs a distance less than $\frac{m+1}{2}$ ahead, overhead 1 if the next presuf shift occurs a distance less than $\frac{3(m+1)}{4}$ ahead, and overhead at most 2 otherwise. A comparison complexity of $n(1 + \frac{8}{3(m+1)})$ comparisons follows.

6.1. The presuf shift handler for $|x'_1| \geq \frac{m}{2}$. Before describing the presuf shift handler, we recall some definitions and assumptions made in section 4. Let t_A refer to the portion of the text with which the prefix x'_1 of p is aligned following the shift. We assume that prefix x'_1 of p matches t_A following a presuf shift and that the variable t_{last} has been appropriately set to prevent this assumption from leading to an incorrect inference. Let t_a refer to the rightmost character in t_A.

As for the case where $|x'_1| < \frac{m}{2}$, the presuf shift handler considers all presuf pattern instances, i.e., those pattern instances in which a prefix (possibly null) of p matches some suffix of t_A. In a presuf pattern instance, the prefix matching a suffix of t_A is a presuf of p and is called the presuf corresponding to this presuf pattern instance. Presuf pattern instances are of two types. The first type consists of those presuf pattern instances whose corresponding presufs have the form $u_p v_p^l$, $1 \leq l \leq i_p - 1$. The second type consists of those presuf pattern instances whose corresponding presufs are less than $|v_p|$ in length. We identify a presuf pattern instance p'_α of the second type as follows. If $|u_p| > 0$, then p'_α is the presuf pattern instance corresponding to the presuf u_p. If $|u_p| = 0$, then p'_α is the presuf pattern instance corresponding to the null presuf; i.e., $|p'_\alpha$ is to the immediate right of t_a. The following observation enables us to work with only presuf pattern instances of the second type while making comparisons within a text window γ of length $|v_p|$ to the right of t_a.

LEMMA 6.1. *A presuf pattern instance of the first type matches all text characters in the window γ if and only if p'_α matches all characters in that window.*

Proof. The portion of p'_α which overlaps γ is identical to v_p, as is the corresponding portion of any presuf pattern instance of the first type. ∎

If p'_α is eliminated by comparisons in γ, then so are all presuf pattern instances of the first type. This forces the next presuf shift to occur at least distance $m - |v_p| \geq \frac{m}{2} + 1$ to the right. If p'_α is not eliminated by comparisons in γ, then presuf pattern instances of the first type also survive, and therefore the next presuf shift can occur as little as distance $|v_p|$ to the right. In this case, it is important to ensure that no mismatches are made in γ.

The presuf shift handler has five steps and works broadly as follows. As in the presuf shift handler of section 4.2, the first two steps identify a presuf pattern instance p'_e with the following property: all presuf pattern instances that survive the first two steps are presuf overlaps of p'_e. This is accomplished by making comparisons in a manner similar to the presuf shift handler of section 4.2, but with a single difference. This difference is aimed at ensuring that the first mismatch eliminates p'_α. Steps 3, 4, and 5 are, however, identical to the corresponding steps of the earlier presuf shift handler.

Steps 1 and 2 proceed as follows to determine p'_e. They consider only presuf pattern instances of the second type and eliminate all but one of these. The survivor determines p'_e; i.e., if the survivor is p'_α, then p'_e is the leftmost presuf pattern instance, and otherwise p'_e is the survivor itself. We show that in order to eliminate among presuf pattern instances of the second kind, it suffices to consider suitable prefixes of these presuf pattern instances. We need the following definitions in order to describe Steps 1 and 2 in detail. Consider the leftmost presuf pattern instance of the second type and let β be the length of the corresponding presuf. Suppose there are $k''+1$ presuf pattern instances of the second type. We define $p''_1, \dots, p''_{k''}, p''_{k''+1}$ such that p''_j, $1 \leq j \leq k''+1$, is the prefix of length $m'' = \beta + |v_p|$ of the jth leftmost presuf pattern instance of the second type. Let x''_j, $1 \leq j \leq k''+1$, be the presuf corresponding to the jth leftmost presuf pattern instance of the second type. We call x''_j the presuf corresponding to p''_j. Let p''_α refer to the length m'' prefix of p'_α and p'' refer to the length m'' prefix of p. The following lemma shows that in order to eliminate all but one of the presuf pattern instances of the second kind, it suffices to consider only $p''_1, \dots, p''_{k''}, p''_{k''+1}$.

LEMMA 6.2. *At most one of $p''_1, \dots, p''_{k''}, p''_{k''+1}$ matches γ.*

Proof. Let x and y be the portions overlapping γ in some two of $p''_1, \dots, p''_{k''+1}$. Then x and y are different cyclic shifts of v_p. If $x = y$, then v_p is cyclic, a contradiction. ∎

Let $V'' = \{p''_1, \dots, p''_{k''+1}\}$. Note that Lemmas 4.3–4.7 continue to hold if p_j, x_j, V, p, and m are replaced by p''_j, x''_j, V'', p'', and m'', respectively, for $1 \leq j \leq k''+1$. Henceforth, these substitutions are implicit in all references to these lemmas. The elements in V'' are divided into groups $A''_1, \dots, A''_{g''}$ in accordance with Lemma 4.3.

Remark. The presuf shift handler being described does not work for patterns for which $|x''_{k''}| = 1$ and $g'' = 1$. Presuf shifts for these exception patterns are handled separately in section 6.5.

With this background, we describe the five steps of the presuf shift handler.

Step 1. The characters in $p''_1, \dots, p''_{k''}$ aligned with $p''_1[m'']$, the rightmost character in p''_1, are identical. If the character in $p''_{k''+1}$ aligned with $p''_1[m]$ is also identical to it, then $p''_1[m]$ is compared with the aligned text character. A mismatch eliminates all of $p''_1, \ldots, p''_{k''}, p''_{k''+1}$ and the basic algorithm is restarted with $|p$ placed immediately to the right of $|p_{k''+1}$. A match leads to Step 2.

Step 2. All but one of $p''_1, \ldots, p''_{k''+1}$ are eliminated in this step by making up to k'' comparisons, at most two of which are unsuccessful. Further, p''_α is eliminated by the first unsuccessful comparison. As in Step 2 of section 4.2, there are two phases.

Phase 1 is identical to Phase 1 of section 4.2; i.e., at every step the rightmost character c in p''_1 having the following property is compared with the aligned text character: the character aligned with c in at least one of the surviving elements in V'' is different from c. By Lemma 4.6, the outcome of Phase 1 is a half-done set O.

LEMMA 6.3. *$p''_\alpha \in O$ if and only if all of the comparisons in Phase* 1 *are successful.*

Proof. All comparisons in Phase 1 are made in the suffix x''_1 of p_1 because, by Lemma 4.5, successful comparisons in that suffix leave a half-done set uneliminated. x''_1 is a presuf of p''_1 and therefore a suffix of v_p. By the manner in which p'_α is defined, the portion of p''_α that overlaps γ is identical to the string v_p. Therefore, a mismatch in Phase 1 eliminates both p''_1 and p''_α while a match in Phase 1 eliminates neither. The lemma follows. ∎

If Phase 1 ends with a mismatch or if $p''_\alpha = p''_1$, then Phase 2 is identical to Phase 2 of section 4.2; i.e., all but one of the elements in O are eliminated by making comparisons according to a right-to-left sequence. Note that in both cases, the first mismatch eliminates p''_α. Otherwise, if $p''_\alpha \neq p''_1$ and all comparisons in Phase 1 are successful, then we modify Phase 2 as follows so as to ensure that the first mismatch eliminates p''_α.

Phase 2 proceeds exactly as Phase 2 of Step 2 in section 4.2 until p''_α becomes the rightmost element in O. Any mismatch in this process terminates Phase 2 and eliminates p''_α and all elements in O to the left of p''_α. If no mismatch occurs in this process, then let the surviving elements in O be $\{p''_{h_1}, \ldots, p''_{h_e}\}$, where $p''_{h_1} = p''_1$ and $p''_{h_e} = p''_\alpha$. These elements are eliminated using a left-to-right sequence of comparisons instead of the right-to-left sequence used in Step 2 of section 4.2. This left-to-right sequence ensures that a mismatch eliminates p''_α. Let d_e be the leftmost character in p''_{h_e} such that p''_{h_e} differs from the aligned character in $p''_{h_{e-1}}$. By Lemma 6.2, d_e is aligned with or to the left of $p''_1[m'']$ and to the right of t_a. If $e = 2$, then a comparison at d_e terminates Phase 2 with a mismatch eliminating p''_{h_e} and a match eliminating p''_{h_1}. Suppose $e > 2$. Then x''_{h_1} is periodic with core, say, v. Let d_j, $2 \le j \le e-1$, be the character in p''_{h_e} which is distance $(e-j)|v|$ to the left of d_e. The characters $d_2, \ldots, d_e$ are compared with the aligned text characters in sequence until either a mismatch occurs or the sequence is exhausted. The following lemma shows that at most one element of O survives these comparisons.

LEMMA 6.4. *A mismatch at d_j leaves only $p_{i_{j-1}}$ uneliminated. A match at d_j eliminates $p_{i_{j-1}}$.*

Proof. If $e = 2$, then the lemma is clearly true. Suppose $e > 2$. From the definition of d_e, it follows that the prefix of p''_{h_e} ending at d_e is periodic with core of size $|v|$ while the prefix of $p''_{h_{e-1}}$ is not; therefore, $d_2 = d_3 = \cdots = d_e \neq d_{e+1}$, where d_{e+1} is the character which is distance $|v|$ to the right of d_e. It follows that the characters in $p''_{h_j}, \ldots, p''_{h_e}$ aligned with d_j are identical to each other but different from the character in $p''_{h_{j-1}}$ aligned with d_j. Therefore, a match at d_j eliminates $p''_{h_{j-1}}$. A mismatch at d_j eliminates $p''_{h_j}, \ldots, p''_{h_e}$ and the preceding successful comparisons at

$d_2, \ldots, d_{j-1}$ eliminate $p''_{h_1}, \ldots, p''_{h_{j-2}}$. The lemma follows. □

This completes Step 2. At most k'' comparisons are made in this step, at most two of which result in mismatches. Further, p''_α survives only if there are no mismatches. The sequence of comparisons made in Step 2 can be represented by a tree ET'', akin to the tree ET of section 4.2. The only difference between ET'' and ET is that the sequence corresponding to a portion of Phase 2 may now be a left-to-right sequence if Phase 1 does not end in a mismatch and $p''_\alpha \neq p''_1$. We conclude Step 2 with the following lemma.

LEMMA 6.5. *All but at most one of $p''_1, \ldots, p''_{k''}, p''_{k''+1}$ can be eliminated by making up to k'' comparisons using the $O(k'')$-sized binary comparison tree ET''. At most two of these comparisons result in mismatches. If p''_α survives, then no comparisons result in mismatches. Moreover, the sequence of comparisons made by the elimination strategy consists of two sequences: a right-to-left sequence followed by either another right-to-left sequence or a left-to-right sequence.*

We describe Steps 3, 4, and 5 next. Let p''_e be the only element of V'' to survive Steps 1 and 2. If $p''_e = p''_\alpha$, then define p'_e to be the leftmost presuf pattern instance. If $p''_e \neq p''_\alpha$, then let p'_e be the presuf pattern instance of which p''_e is a prefix; i.e., p'_e and p''_e have their left ends aligned. Clearly, p'_e is the leftmost presuf pattern instance to survive Steps 1 and 2. Let Q denote the set of pattern instances which overlap p'_e and have their left end to the right of $|p''_{k''+1}$. In the elimination process, some elements of Q may also have been eliminated from being potential matches. They need not be reconsidered. To this end, a subset Q_x of Q consisting of pattern instances consistent with comparisons in Steps 1 and 2 is associated with each terminal node x in ET''. The maintenance of Q_x is similar to the description in section 4.5 and is described in section 6.4. Suppose that the elimination process terminates at terminal node x. Let $Q' = \{p'_e\} \cup Q_x$. Steps 3, 4, and 5 are now identical to the corresponding steps in section 4.2.

6.2. Comparison complexity. In order to determine the comparison complexity, we need to define a transfer function f'' akin to the transfer function f defined in section 5. We state the following lemma describing the properties of f''. The proof of this lemma is deferred to section 6.3.

LEMMA 6.6. *Let C be the set of text characters involved in comparisons in Steps* 1 *and* 2 *of the presuf shift handler of section* 6.1. *For each character $t_c \in C$, with at most two exceptions, there exists a text character $f''(t_c) = t_d$ satisfying the following properties:*

1. *t_d is to the right of $|p''_{k''+1}$.*
2. *t_d either coincides with t_c or lies to the left of t_c.*
3. *The pattern instance whose left end is aligned with t_d is eliminated as a result of comparisons in Steps* 1 *and* 2 *of the presuf shift handler.*
4. *For every distinct $t_{c_1}, t_{c_2} \in C$, $f(t_{c_1}) \neq f(t_{c_2})$.*

Furthermore, mismatches, if any, are always included among the exceptions.

The following lemma determines the comparison complexity of the algorithm.

LEMMA 6.7. *If p is not a special-case pattern, then the comparison complexity of the algorithm is bounded by $n(1 + \frac{8}{3(m+1)})$.*

Proof. A presuf shift occurs either with $|x'_1| < \frac{m}{2}$ or with $|x'_1| \geq \frac{m}{2}$. In the former case, it was shown in Lemma 4.12 that a presuf shift can have overhead at most two and that an overhead of two implies that the next presuf shift occurs at least distance $\frac{3(m+1)}{4}$ to the right. Further, $\frac{m+1}{2}$ is a lower bound on the distance between two consecutive presuf shifts in this case. We show similar properties for presuf shifts with $|x'_1| \geq \frac{m}{2}$. Specifically, we show that a presuf shift can have overhead at most two. Further, we show that an overhead of one forces the next presuf shift to occur at least distance $\frac{m+1}{2}$ to the right and an overhead of two forces the next presuf shift to occur at least distance $\frac{3(m+1)}{4}$ to the right. We show the above by giving a charging scheme for the presuf shift handler of section 6.1. The comparison complexity of the algorithm now follows.

Charging scheme. As in Lemma 4.12, the run of the algorithm is divided into phases; a phase can be of one of four types. The ranges of the text characters charged in each phase type remain exactly the same as in Lemma 4.12. The charging scheme for Type 1 and Type 2 phases also remains exactly the same. Only the charging scheme for Type 3 and Type 4 phases is modified in accordance with the presuf shift handler of section 6.1.

We consider a single phase, which could be a Type 3 or a Type 4 phase. We assume that this phase begins with a presuf shift with $|x'_1| \geq \frac{m}{2}$. Let q_1 and q_2 refer to the leftmost surviving pattern instances at the beginning and end of that phase, respectively. Note that q_1 is a presuf overlap of the pattern instance q', the leftmost uneliminated pattern instance prior to the presuf shift which initiated this phase. Specifically, the prefix x'_1 of q_1 is aligned with the suffix x'_1 of q' (recall that on a presuf shift, we assume that the prefix x'_1 of q_1 matches the text). Recall that t_a is the text character aligned with $q'|$.

Consider the comparisons made by the current use of the presuf shift handler of section 6.1. If a mismatch occurs in Step 1, the current phase ends immediately and the basic algorithm is resumed. The presuf shift in this case has overhead one and the next presuf shift occurs at least distance $m+1$ to the right. Next, suppose that the comparison in Step 1 is successful. Let p'_e be the presuf pattern instance to survive the elimination using tree ET'' in Step 2. After the presuf shift handler finishes, one of three scenarios ensues. We consider each in turn.

1. All pattern instances overlapping p'_e are eliminated apart from its presuf overlaps, and p'_e or at least a suffix of p'_e is matched. This is a Type 4 phase. We consider two cases, depending upon whether p'_e is a presuf pattern instance of the first or the second type.

First, suppose p'_e is of the first type; i.e., it is the leftmost presuf pattern instance. Then no mismatches are made in Steps 1, 2, 3, 4, or 5. All comparisons made by the presuf shift handler are charged to the text characters compared. The bit vector BV ensures that each of these comparisons involves a different text character. Thus each text character which lies to the right of t_a and is aligned with or to the left of $p'_e|$ is charged at most once. In this case, the overhead of this presuf shift is zero.

Next, suppose p'_e is of the second type. Then p''_α is eliminated in Step 2. All comparisons in Steps 1, 3, 4, and 5 and all but at most two comparisons in Step 2 are successful. Each successful comparison is charged to the text character compared. The bit vector BV ensures that each of these comparisons involves a different text character. Thus each text character which lies to the right of t_a and is aligned with or to the left of $p'_e|$ is charged at most once. At most two comparisons in Step 2 are unsuccessful, so this shift has overhead at most two. If there are two mismatches in Step 2, then we claim that p''_1 is eliminated; in addition, if x''_1 is periodic, with core v and head u, say, then all elements in V'' whose associated presufs have the form uv^o, $o \geq 1$, are also eliminated. (This can be shown in a manner similar to the corresponding proof in Lemma 4.12.) Let p''_e be the m''-length prefix of p'_e and let x''_e be the presuf associated with p''_e. From the above, it follows that $x''_1 = x''_e z x''_e$, for some nonempty string z. Since $|x_1''| < |v_p|$, $p_e'' = x_1''wx_1''$ for some nonempty string w. Therefore, $|x_e''| \leq \frac{m''-3}{4} \leq \frac{m-3}{4}$. This guarantees that the next presuf shift occurs at least distance $\frac{3(m+1)}{4}$ to the right. If there is just one mismatch in Step 2, then since $|x_1''| < |v_p| \leq \frac{m}{2}$, the next presuf shift occurs at least distance $\frac{m+1}{2}$ to the right.

2. p_e' is eliminated. In addition, there is some pattern instance q_c overlapping p_e', such that all pattern instances overlapping q_c are eliminated apart from its presuf overlaps; further, q_c or at least a suffix of q_c is matched. This is also a Type 4 phase.

Each comparison in Steps 1 and 2 with a text character to the left of $|q_c$ for which function f'' is defined is charged to the text character specified by the function f'', called its f'' value; f'' values are distinct by definition. Comparisons in Step 3 fall into one of three categories:

1. comparisons which eliminate pattern instances whose left ends lie to the right of $|p''_{k''+1}$ and to the left of $|q_c$;
2. comparisons which eliminate pattern instances whose left ends lie to the right of $|q_c$;
3. the comparison which eliminates p_e'.

Each comparison in the first category is charged to the text character aligned with the left end of the pattern instance eliminated. By the definition of the function f'', these text characters do not occur in the range of f'' values. Comparisons in the second category, along with the comparisons made in Steps 4 and 5 and those successful comparisons in Steps 1 and 2 that involve text characters overlapping q_c, are charged to the text characters compared. BV ensures that each of these comparisons involves a distinct text character. Thus each text character which lies to the right of $|p''_{k''+1}$ and is aligned with or to the left of $q_c|$ is charged at most once. The comparison that eliminates p_e' is charged to the text character aligned with $|p''_{k''+1}$. Since all f'' values lie to the right of $|p''_{k''+1}$ and all pattern instances eliminated by comparisons in the first category have left ends to the right of $|p''_{k''+1}$, this text character is charged exactly once. The two comparisons in Step 2 lacking f'' values constitute the overhead of this presuf shift. Since p_e' is eliminated, the next presuf shift occurs at least distance $m + 1$ to the right of the current presuf shift.

3. p_e' is eliminated as are all pattern instances overlapping p_e'. This is a Type 3 phase.

Let q_d denote the leftmost surviving pattern instance. All comparisons in Steps 1 and 2 for which function f'' is defined are charged to their f'' values. f'' values are distinct by definition. Excluding the comparison which eliminates p_e', each comparison in Steps 3 and 4 eliminates some pattern instance whose left end lies to the right of $|p''_{k''+1}$ and to the left of $|q_d$. Each such comparison is charged to the text character aligned with the left end of the pattern instance eliminated. These text characters cannot occur in the range of the function f'' and hence are charged only once. Thus each text character which lies to the right of $|p''_{k''+1}$ and to the left of $|q_d$ is charged at most once. The comparison that eliminates p_e' is charged to the text character aligned with $|p''_{k''+1}$. The two comparisons in Step 2 lacking f'' values constitute the overhead of this presuf shift. Since p_e' is eliminated, the next presuf shift occurs at least distance $m + 1$ to the right of the current presuf shift. $\square$

6.3. The transfer function f''. In this section, we prove Lemma 6.6. The definition of the function f'' is similar to that of the function f in section 5. This is hardly surprising since the elimination procedure ET'' is similar to the elimination process ET, the only difference between the two being that the former switches to a left-to-right comparison sequence in some cases.

First, note that each of the definitions and lemmas in section 5.1 continue to hold if p''_j, x''_j, V'', p'', A'', g'', k'', and m'' replace p_j, x_j, V, p, A, g, k, and m, respectively, for $1 \leq j \leq k'' + 1$.

Since patterns with $g'' = 1$ and $|x''_k| = 1$ are special-case patterns, we assume that $g'' > 1$ if $|x''_k| = 1$. If $p''_1[m'']$ does not match the text, then Steps 1 and 2 of the presuf shift handler make at most one comparison. Therefore, we also assume that $p''_1[m]$ matches the text. Let p''_l be the rightmost element in A''_1. Let p''_r be the rightmost element in V'' outside $A''_{g''}$, if such a pattern instance exists.

As in section 5.2, we split the sequence C' of comparisons made in Steps 1 and 2 of the presuf shift handler into three classes as follows.

1. Class 1 consists of the comparison in Step 1. In addition, if $|x''_k| = 1$, then Class 1 contains the comparisons which comprise the smallest prefix of C' having the following property: either the last comparison in that prefix is unsuccessful or following that comparison, exactly one pattern instance in $A''_{g''}$ survives.

2. Class 2 consists of the comparisons in C' which follow all Class 1 comparisons and are made in the suffix $h(x''_l)$ of p''_1.

3. Class 3 consists of comparisons in C' which follow all Class 2 comparisons.

f'' values are defined by considering 3 cases.

Case 1. Suppose Phase 1 of Step 2 terminates with a mismatch or $p''_\alpha = p''_1$. Then ET'' eliminates among elements in V'' exactly as ET eliminates among the elements of V. Therefore, f'' values for comparisons are defined exactly as in section 5.2 with p''_j, x''_j, V'', p'', A'', g'', k', and m'' replacing p_j, x_j, V, p, A, g, k, and m, respectively, for $1 \leq j \leq k'' + 1$.

Case 2. Suppose $p''_\alpha = p''_2$ or the half-done set left uneliminated by Phase 1 has at most two elements. The only difference between the way ET'' eliminates among the elements in V'' and ET eliminates among elements in V is in the last comparison of Step 2. Note that in section 5.2, the last comparison in Step 2 is not given an f value. Therefore, f'' values for comparisons in this case are again defined exactly as in section 5.2 with p''_j, x''_j, V'', p'', A'', g'', k'', and m'' replacing p_j, x_j, V, p, A, g, k, and m, respectively, for $1 \leq j \leq k'' + 1$.

Case 3. Suppose all comparisons in Phase 1 are successful, $p''_\alpha \neq p''_1, p''_2$, and the half-done set which survives Phase 1 has at least three elements. The only difference between the way ET'' eliminates among the elements in V'' and ET eliminates among the elements in V is in the portion of Phase 2 that makes comparisons according to a left-to-right sequence. As we will show in Lemma 6.11, this left-to-right sequence involves only text characters to the left of the suffix x''_1 of p''_1. Consequently, Class 1 and Class 2 comparisons are not affected by this sequence.

f'' values for comparisons in Class 1 are defined exactly as in section 5.2 with p''_j, x''_j, V'', p'', A'', g'', k'', and m'' replacing p_j, x_j, V, p, A, g, k, and m, respectively, for $1 \leq j \leq k'' + 1$. Consider Class 2 comparisons next. At most one element of V'' will survive a mismatch in Class 2, if any, because Phase 1 has no mismatches. Therefore, if a mismatch occurs in Class 2, then Class 3 is empty and f'' values for Class 2 comparisons are defined exactly as in section 5.2 with the appropriate substitutions mentioned above. Otherwise, if all comparisons in Class 2 are successful, then f'' values for all but some s, $s \leq 2$, of the comparisons in Class 2 are defined in the same manner. It remains to define f'' values for Class 3 comparisons and s Class 2 comparisons when all comparisons in Class 2 are successful. This involves modifying only Case 2 of the definition of f values for Class 2 and Class 3 comparisons in section 5.2. We define f'' values for all but two of these comparisons. The range of these f''

values is the same as the range of the f values defined for this subcase, i.e., to the left of the suffix x''_l of p''_1 and to the right of $|p''_{k+1}$.

Following Class 1 and 2 comparisons, at most $\min\{r_l, 2\} - s$ of the elements of V'' to the right of p''_l survive along with the elements in A''_1. Let O' refer to the set of $\min\{r_l, 2\}$ elements in V'' which includes elements which survive comparisons in $h(x''_l)$ and elements which are eliminated by one of the s Class 2 comparisons under consideration. Let O refer to the largest half-done set consisting of elements in A''_1 and O'. Redefine O' by removing pattern instances in it which are also in O. Considering comparisons which eliminate pattern instances in O and O' is equivalent to considering Class 3 comparisons plus s of the Class 2 comparisons. Let $O = \{p''_{h_1}, \ldots, p''_{h_e}\}$. Let v and u be the core and head, respectively, of x''_{h_1} and let $v = u'u$. $|v| > 1$ because either $|x''_k| > 1$ or $|x''_k| = 1$ and $g' > 1$. By Lemma 5.10, v contains a misfit character. If $l = 1$, then the number of comparisons in Class 3 plus s is at most $2 - s + s = 2$. In this case, we do not define an f'' value for the comparisons in Class 3 and the s comparisons in Class 2. Therefore, suppose that $l > 1$.

The comparisons given by tree ET'' in this case form two sequences; the first sequence which includes Phase 1 and part of Phase 2 is a right-to-left sequence and the second sequence is a left-to-right sequence. The following lemmas show some properties which are necessary for defining f''.

LEMMA 6.8. *The portion of p''_α which overlaps the suffix x''_i, $1 \le i \le \alpha$, of p''_1 matches x''_i.*

Proof. x''_i is a suffix of v_p. The length $|v_p|$ substring of p''_α which is to the immediate right of t_a is identical to v_p. ∎

LEMMA 6.9. *$p''_\alpha \in O$ and $|O| \ge 3$.*

Proof. Since all comparisons in Phase 1 are successful, p''_1 survives Phase 1. By Lemma 6.3, p''_α also survives. If $p''_\alpha \notin O$, then p''_1 and p''_α do not form a half-done set with any other element in V''. Therefore, the cardinality of the half-done set which survives Phase 1 would be at most 2, which is a contradiction. Thus $p''_\alpha \in O$. p''_2 must form a half-done set along with p''_1 and p''_α; otherwise, no other element in V'' forms a half-done set with p''_1 and p''_α and, consequently, at most two elements in V'' would survive the successful Phase 1 comparisons. By Lemma 6.8, p''_1 and p''_α survive successful comparisons in $h(x''_l)$, and then by Lemma 5.3, p''_2 also survives these comparisons. Therefore, $p''_2 \in O$ also. Since $p''_\alpha \ne p''_1, p''_2$, the lemma follows. ∎

COROLLARY 6.10. *The half-done set which survives Phase* 1 *must be a subset of O.*

Proof. Both p''_1 and p''_α survive successful comparisons in Phase 1 and both are elements of O. The only elements in V'' which can form a half-done set with p''_1 and p''_α are those in O. ∎

LEMMA 6.11. *The leftmost character compared by ET'' in the first (right-to-left) sequence is at least distance $|v|$ to the right of the rightmost character compared in the second (left-to-right) sequence. The rightmost character compared in the latter sequence is to the left of the suffix x''_1 of p''_1.*

Proof. Let d'' be the rightmost position in p''_α such that $p''_\alpha[d'']$ is aligned with some character in p''_1 and $p''_\alpha[d''] \ne p''_\alpha[d'' + |v|]$. Such an index exists by Lemma 4.7. All comparisons in the second sequence are aligned with or to the left of $p''_\alpha[d'']$. All characters in the first sequence compared in Phase 2 are aligned with or to the right of $p''_\alpha[d'' + |v|]$. All characters compared in Phase 1 involve characters in the suffix x''_1 of p''_1. By Lemma 6.8, $p''_\alpha[d'']$ is to the left of the suffix x''_1 of p''_1. The lemma follows. ∎

COROLLARY 6.12. *Successful comparisons which eliminate elements of O are made at least distance $|v|$ apart.*

LEMMA 6.13. *The portion of p''_{h_e} that overlaps the suffix $x''_{h_{e-2}}$ of p''_1 matches that suffix.*

Proof. Since $p''_\alpha \in O$ and $p''_\alpha \neq p''_1, p''_2$, it follows from Lemma 6.8 that $(uu')^2$ is a suffix of $p''[1 \dots m'' - |x''_1|]$. Therefore, the portion of p''_{h_e} that overlaps the suffix $x''_{h_{e-2}}$ of p''_1 matches that suffix. ∎

COROLLARY 6.14. *All successful comparisons which eliminate an element of O are made to the left of the suffix $x''_{h_{e-2}}$ of p''_1.*

We now define the f'' function for this case.

First, consider comparisons which eliminate elements of O'. From Corollary 6.10, it follows that all elements of O' must be eliminated by Phase 1 comparisons. These comparisons have to be successful because all comparisons in Phase 1 are successful. If $|O'| = 2$, then, by Lemma 4.5, the first such comparison is made in the suffix x''_{h_e} of p''_1. If $|O'| = 2$ or $|O'| = 1$, then, by Lemma 6.13, the portion of p''_{h_e} which overlaps the suffix $x''_{h_{e-1}}$ of p''_1 matches that suffix and therefore, by Lemma 4.5, the last comparison which eliminates an element of O' is made in the suffix $x''_{h_{e-1}}$ of p''_1. Consider the text characters t_c and t'_c with which $|p''$ is aligned when the rightmost misfit characters in the prefixes $x''_{h_{e-1}}$ and $x''_{h_{e-2}}$, respectively, of p'' are aligned with t_b. Since v is a suffix of $x''_{h_{e-1}}$ and $x''_{h_{e-2}}$ and since v contains a misfit character, t_c is aligned with the suffix $x''_{h_{e-1}}$ of p''_1 and to the left of the suffix x''_{h_e} of p''_1 while t'_c is aligned with the suffix $x''_{h_{e-2}}$ of p''_1 and to the left of the suffix $x''_{h_{e-1}}$ of p''_1. If $|O'| = 2$, then f'' is defined to map the text characters involved in comparisons which eliminate elements of O' to the text characters t_c and t'_c. If $|O'| = 1$, then f'' is defined to map the text character involved in the comparison which eliminates the only element of O' to the text character t'_c. A simple case analysis ($p''_l = p''_{h_e}, p''_{h_{e-1}}, p''_{h_{e-2}}$) shows that these f'' values are to the left of p''_l, as claimed. The two f'' values are clearly distinct and to the left of their respective text characters. Further, they are aligned with the suffix x''_1 of p''_1. Since $|x''_1| < \frac{m''}{2}$, these f'' values are to the right of $|p''_{k''+1}$.

Next, consider comparisons which eliminate elements of O, excluding the leftmost and the last such comparison. The remaining comparisons must be successful. f is defined to map the text character t_c involved in such a comparison to the text character with which $|p''$ is aligned when the leftmost character in p'' which differs from t_c is aligned with t_c. Clearly, $f''(t_c)$ is aligned with or to the left of t_c. Since uu' contains at least two characters, $f''(t_c)$ is at most distance $|v| - 1$ to the left of t_c. It follows from Corollary 6.12 that $f''(t_c)$ is distinct from the f'' values for all other text characters involved in successful comparisons which eliminate elements of O. By Corollary 6.14, these f'' values are to the left of f'' values for successful comparisons which eliminate elements of O' and therefore to the left of p''_l. Only the leftmost text character involved in a comparison which eliminates an element of O is within distance $|v|$ of t_a; the rest are at least distance $|v| + 1$ to the right of t_a. Therefore, these f'' values are to the right of $|p''_{k''+1}$.

This concludes the definition of the transfer function f''.

6.4. Data-structure details. It remains to describe the maintenance of the sets Q_x for each terminal node x of tree ET''. These sets can be maintained exactly as described in section 4.5 but with the following difference: the sequence of comparisons corresponding to Phase 2 in Step 2 of the elimination strategy using ET'' is a left-to-right sequence if all comparisons in Phase 1 are successful.

As in section 4.5, let $l_1, \ldots, l_h$ be the nodes, in order of appearance, on the leftmost path from the root of ET''. Consider the largest i such that $tc_{l_i}, \ldots, tc_{l_{h-1}}$ (recall from section 4.5 that tc_x is the text character compared at node x of ET'') is a left-to-right sequence. For all terminal nodes in ET'' which are not in the subtree rooted at l_i, the data structure is maintained exactly as in section 4.5. Q_{l_h} can be stored explicitly. It remains to describe the data structure for terminal nodes in the right subtrees of $l_i, \ldots, l_{h-1}$.

Note that if a mismatch occurs at tc_{l_j}, $i \le j \le h-1$, at most two elements in V'' survive. Therefore, for each terminal node x in the right subtree of l_j, either $p(x)$ or $p(p(x))$ equals l_j, where $p(x)$ is the parent of x. From the definition of the sets Q_x in section 4.5, it follows that for terminal nodes x and y in the right subtree of l_j, $Q_x = Q_y$. The following lemma is crucial.

LEMMA 6.15. *Let terminal node x_1 be in the right subtree of l_{j_1} and terminal node x_2 be in the right subtree of l_{j_2}, $i \le j_1, j_2 \le h-1$, $j_2 > j_1$. If $q \in Q_{x_1}$ and $q \in Q_{x_2}$, then q occurs at all terminal nodes in the right subtrees of $l_i, \ldots, l_{j_1}$.*

Proof. Clearly, q cannot overlap $tc_{l_{j_1}}$. Since $tc_{l_i}, \ldots, tc_{l_{h-1}}$ form a left-to-right sequence, q cannot overlap $tc_i, \ldots, tc_{j_1}$. Further, since q occurs at some terminal node in the subtree rooted at l_i, characters in q which overlap $tc_{l_1}, \ldots, tc_{l_{i-1}}$ match the characters $c_1, \ldots, c_{h-1}$, respectively. The lemma follows from the definition of the sets Q_x. ∎

COROLLARY 6.16. *Suppose q occurs at some terminal node in the subtree T rooted at l_i. Further, suppose j is the largest number, if any, such that $i \le j \le h-1$ and q does not overlap tc_{l_j}. Then q occurs at all terminal nodes in the right subtrees of $l_i, \ldots, l_j$.*

Corollary 6.16 immediately gives a linear-space scheme for storing the sets Q_x for terminal nodes x in the subtree T rooted at l_i. Two sets Com_j and $Spec_j$ are maintained at each node l_j, $i \le j \le h-1$. A pattern instance q is added to Com_j if it occurs at some terminal node in T and overlaps $tc_{l_{j+1}}$ but not tc_{l_j}. A pattern instance q is added to $Spec_j$ if it overlaps tc_{l_j} and occurs at a terminal node in the right subtree of l_j. Each q can be added to at most one Com set and one $Spec$ set; thus, the total space used is linear. Q_x is readily seen to equal $Com_j \cup Com_{j+1} \cup \cdots \cup Com_{h-1} \cup Spec_j$. Note that each pair of Com sets is disjoint and Com_k is disjoint from $Spec_j$, for each $j \le k \le h-1$. In order to obtain Q_x as a sorted list, it suffices to maintain each of the Com and $Spec$ sets as ordered lists which are then appended together. Thus obtaining any particular Q_x takes $O(m)$ time. Q_{l_h} is stored explicitly and hence can be obtained as a list in constant time.

6.5. Presuf shift handler for special-case patterns. We describe the presuf shift handler for patterns for which $|x''_k| = 1$ and $g'' = 1$. This presuf shift handler leads to an overhead of at most two per presuf shift. We show that if a presuf shift has overhead two, then the next presuf shift must occur distance at least $\frac{3(m+1)}{4}$ to the right, and if a presuf shift has overhead one, then the next presuf shift must occur distance at least $\frac{m+1}{2}$ to the right. A comparison complexity of $n(1 + \frac{8}{3(m+1)})$ follows.

Let $b = x''_k$. p contains at least two different characters. Therefore, v_p and p'' both contain at least two different characters. Let $p''[j]$ and $p''[j']$ be, respectively, the leftmost and rightmost characters in p'' which differ from b. Let t_c be the text character to the immediate right of t_a.

We consider two cases, namely $|x''_1| < \frac{|v_p|}{2}$ and $|x''_1| \ge \frac{|v_p|}{2}$. The former case has the advantage that if all presuf pattern instances of the first type (recall that presuf pattern instances were classified into two types in section 6) are eliminated, then the next presuf shift occurs distance at least $\frac{3(m+1)}{4}$ to the right. The absence of this property in the latter case makes it more complicated.

Case 1. $|x''_1| < \frac{|v_p|}{2}$.

Step 1. Step 1 locates the leftmost non-b text character t_d to the right of t_a. Following Step 1, either the basic algorithm is resumed or p'_e, the leftmost surviving pattern instance, is determined and Step 2 follows. This is done as follows. Text characters to the right of t_a and to the left of $p''_\alpha[j]$ are compared from left to right with the character b. A mismatch in this process terminates Step 1. If no mismatch occurs, then $p''_\alpha[j]$ is compared with the aligned text character. A match terminates Step 1. In case of a mismatch, text characters aligned with or to the right of $p''_\alpha[j]$ are compared from left to right with the character b. Step 1 then terminates when a mismatch occurs or when the right end of the text is reached.

One of the following situations now holds:

1. t_d is to the left of $p''_1[j]$. $p''_1, \ldots, p''_{k+1}$ are eliminated and the basic algorithm is resumed with $|p$ placed to the right of the text character that mismatched.

2. t_d is aligned with $p''_i[j]$, $i \neq \alpha$. p'_e is the pattern instance whose left end is aligned with $|p''_i$.

3. t_d is aligned with $p''_\alpha[j]$ and $t_d = p''_\alpha[j]$. p'_e is defined to be the leftmost presuf pattern instance.

4. t_d is aligned with $p''_\alpha[j]$ but $t_d \neq p''_\alpha[j]$. The basic algorithm is resumed with $|p$ immediately to the right of t_e.

5. t_d exists but does not satisfy any of the above cases. p'_e is the pattern instance such that $p'_e[j]$ is aligned with t_d.

6. t_d does not exist. There are no further occurrences of the pattern in the text and the algorithm terminates.

Steps 2 *and* 3. Let q_c denote p'_e. Then Steps 2 and 3 are identical to the corresponding steps in the presuf shift handler for special-case patterns described in section 4.4.

Note that at most two mismatches are made in Step 1 and the first mismatch eliminates p''_α.

LEMMA 6.17. *If p is a special-case pattern and $|x''_1| < \frac{|v_p|}{2}$, then the comparison complexity of the algorithm is $n(1 + \frac{8}{3(m+1)})$.*

Proof. We give charging strategies to show that a presuf shift can have overhead at most two. Further, we show that an overhead of one forces the next presuf shift to occur at least distance $\frac{m+1}{2}$ to the right and an overhead of two forces the next presuf shift to occur at least distance $\frac{3(m+1)}{4}$ to the right. The lemma follows.

As in Lemma 4.12, the run of the algorithm is divided into phases; a phase can be of one of four types. The range of text characters charged in each type of phase remains exactly the same as in Lemma 4.12. The charging scheme for Type 1 and Type 2 phases also remains exactly the same. Only the charging scheme for Type 3 and Type 4 phases is modified in accordance with the presuf shift handlers described above.

We consider a single phase, which could be a Type 3 or a Type 4 phase. We assume that this phase begins with a presuf shift with $|x'_1| \geq \frac{m}{2}$.

The charging scheme. Let q_c be the leftmost pattern instance which survives Step 1. Note that q_c is the leftmost presuf pattern instance if and only if no mismatches occur in Step 1. All successful comparisons in Step 1 are charged to the text characters compared. These text characters lie to the left of $q_c[j]$ if q_c is not the leftmost presuf

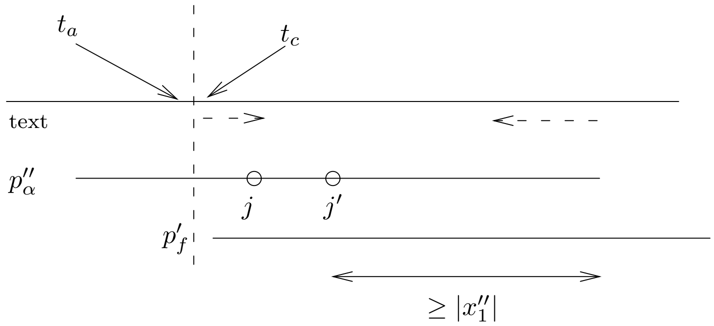

FIG. 7. *Step* 1 *of Case* 2.

pattern instance and are aligned with or to the left of $q_c[j]$ otherwise. If unsuccessful comparisons occur in Step 1, then these comparisons constitute the overhead of this shift. Otherwise, if all comparisons in Step 1 are successful, the only possible comparison which constitutes the overhead of this shift is the comparison in Step 2 which eliminates q_c. Thus the overhead is at most two. Since the first mismatch in Steps 1 and 2 eliminates all presuf pattern instances of the first type and since $|x''_1| < \frac{|v_p|}{2}$, either the overhead is zero or the next presuf shift occurs distance at least $\frac{3(m+1)}{4}$ to the right.

Now consider two cases.

1. Suppose q_c survives Step 2. All comparisons made in Steps 2 and 3 are charged to the text characters compared. Thus each text character which lies to the right of t_a and is aligned with or to the left of $q_c|$ is charged at most once over Steps 1, 2, and 3. All future comparisons will be charged to text characters to the right of $q_c|$.

2. Suppose q_c does not survive Step 2. Each successful comparison in Step 2 eliminates some pattern instance lying entirely to the right of $q_c[j]$ and is charged to the text character aligned with the left end of that pattern instance. The unsuccessful comparison which eliminates q_c in Step 2 is charged to the text character aligned with $q_c[j]$ if q_c is not the leftmost presuf pattern instance. Thus each text character lying between t_a and $|q_d$ is charged at most once, where q_d is the leftmost surviving pattern instance at the end of Step 2. All future comparisons will be charged to text characters aligned with or to the right of $|q_d$. $\square$

Case 2. $|x''_1| \geq \frac{|v_p|}{2}$.

There are five steps in the presuf shift handler for this case. At most five mismatches are made in these steps. We show that three of these mismatches can be charged to unmatched text characters; consequently, the overhead of the current presuf shift is at most two. Further, the first mismatch in Step 1 eliminates p''_α and the second mismatch eliminates all of the presuf pattern instances.

Step 1. Step 1 eliminates all but at most one of $p''_1, \ldots, p''_{k+1}$ as follows. See Fig. 7. The following sequence of text characters is compared with the aligned characters in p''_α: t_b, followed by the text characters strictly between t_b and $p''_\alpha[j']$ considered right to left, followed by the text characters strictly between t_a and $p''_\alpha[j]$ considered left to right. Step 1 terminates when the first mismatch occurs or when this sequence is exhausted.

Let p'_e be the leftmost surviving presuf pattern instance following Step 1. Consider the pattern instance p'_f, $|p'_f$ aligned with the text character to the immediate right of

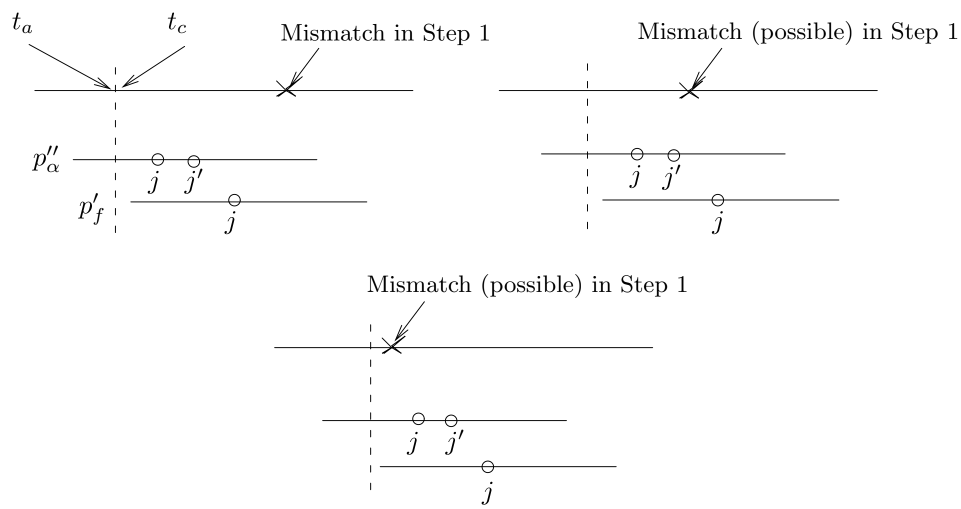

FIG. 8. *Possible outcomes of Step* 1.

t_c. Let t_e be the text character at which the mismatch occurred, if any. Note that since $j \geq |x''_1| + 1$ and $|x''_1| \geq \frac{|v_p|}{2}$, by Lemma 6.8, $p'_f[j]$ must be to the right of $p''_\alpha[j']$. The outcome of Step 1 depends upon which of the following two cases occurs (see Fig. 8).

Case 1.1. $p'_f[j]$ is aligned with or to the left of t_e (first diagram in Fig. 8). Clearly, $p''_\alpha[j']$ is to the left of t_e. We show that a transfer function similar to the function f'' of section 6.1 (see Lemma 6.6) can be used to account for the comparisons made in Step 1. In this case, the rest of the steps are identical to Steps 3, 4, and 5 of the presuf shift handler of section 6.1.

Case 1.2. Either there is no mismatch in Step 1 or $p'_f[j]$ is to the right of t_e (second and third diagrams in Fig. 8). The leftmost surviving pattern instance with left end to the right of t_c has its jth character to the right of t_b; we show this claim in the next paragraph. Step 2 follows in this case.

Recall that $p'_f[j]$ is to the right of $p''_\alpha[j']$. The mismatch, if any, in Step 1 occurs to the left of $p'_f[j]$. Therefore, all text characters aligned with or to the right of $p'_f[j]$ and to the left of (and including) t_b are identical to b. The claim follows.

Step 2. If p'_e does not extend to the right of t_b, then no comparisons are made in this step (this happens if and only if p'_e is the leftmost presuf pattern instance). Otherwise, Step 2 attempts to extend the match of p'_e. Characters in p'_e to the right of t_b (if any) are compared from left to right until a mismatch occurs or a non-b character is matched against the text. To see that p'_e will have a non-b character to the right of t_b if it extends to the right of t_b, note that the distance between t_b and t_c equals $|v_p| - 1$ and that p'_e has at least two non-b characters distance $|v_p|$ apart, neither of which can be to the left of t_c.

The successful comparisons in this step will be charged to the text characters compared. Clearly, all of these text characters are to the right of the text characters compared in Step 1.

Step 3. A pattern instance p'_g with the following properties is determined in this step.

1. p'_g is the leftmost surviving pattern instance.

2. All surviving pattern instances which overlap $p'_g[i]$ are presuf overlaps of p'_g, where i is defined as follows. If $p'_g \neq p'_e$, $i = j$. If $p'_g = p'_e$ and p'_e is the leftmost presuf pattern instance, then $p'_g[i]$ is the character aligned with t_b. Otherwise, if $p'_g = p'_e$ and p'_e is not the leftmost presuf pattern instance, then $p'_g[i]$ is the leftmost non-b character in p'_g which is to the right of t_b.

All text characters compared successfully in this step will be distinct from all text characters compared successfully in Steps 1 and 2. There are two cases depending upon the outcome of Step 2.

Case 2.1. p'_e is eliminated in Step 2. At most two mismatches could have occurred in Steps 1 and 2. Note that p'_e cannot be the leftmost presuf pattern instance in this case. The leftmost surviving pattern instance must have its left end to the right of t_c. As shown in Step 1, its jth character must be to the right of t_b. There are two subcases.

Case 2.1a. Step 2 terminates in a mismatch at a non-b character t_h in p'_e. Then, starting at t_h, a left-to-right pass is made in which each text character is compared with b. This pass ends when a mismatch occurs or when the right end of the text is reached. In the latter case, there are no further occurrences of the pattern and the algorithm terminates. In the former case, p'_g is defined to be the pattern instance in which $p'_g[j]$ is aligned with the text character t_x at which the mismatch occurs. Since all text characters strictly between t_b and t_x are identical to b, p'_g is the leftmost surviving pattern instance and all pattern instances to the right of p'_g which overlap $p'_g[j]$ are eliminated. Note that the number of mismatches made in Steps 1–3 is at most three in this case.

Case 2.1b. Step 2 terminates in a mismatch at a character t_h in p'_e which is a b. p'_g is defined to be the pattern instance in which $p'_g[j]$ is aligned with the text character at which the mismatch occurs. As in the previous case, p'_g is the leftmost surviving pattern instance and all pattern instances to the right of p'_g which overlap $p'_g[j]$ are eliminated. The number of mismatches made in Steps 1–3 is at most two in this case.

Case 2.2. p'_e survives Step 2. At most one mismatch could have occurred so far. There are two subcases.

Case 2.2a. p'_e is not the leftmost presuf pattern instance; i.e., it extends to the right of t_b. Let t_x be the rightmost text character matched in Step 2. t_x must be a non-b character. Consider the pattern instance p'_h, where $p'_h[j]$ is aligned with t_x.

Clearly, all pattern instances to the right of p'_h which overlap t_x are eliminated since each has a b aligned with t_x. We claim that all pattern instances strictly between p'_e and p'_h have also been eliminated. This is shown as follows. All pattern instances to the right of p'_e which overlap t_c have been eliminated in Step 1. Recall from Step 1 that the leftmost surviving pattern instance after Step 1 with left end to the right of t_c has its jth character to the right of t_b. Since all text characters to the right of t_b and up to but not including t_x are identical to b, the claim follows.

If p'_e and p'_h lack a difference point or if $p'_h[j]$ does not match t_x, then $p'_g = p'_e$. Otherwise, if p'_e and p'_h have a difference point, the character in p'_e at that difference point is compared with the aligned text character and one of p'_e and p'_h is eliminated; the difference point itself is to the right of $p'_h[j]$. Let p'_g denote the survivor. Clearly, p'_g is the leftmost surviving pattern instance in both cases. Further, all pattern instances to the right of p'_g which overlap t_x (note that t_x is aligned with $p_g[i]$) have either been eliminated or are presuf overlaps of p'_g.

At most two mismatches are made in Steps 1–3 in Case 2.2a.

Case 2.2b. Second, suppose p'_e is the leftmost presuf pattern instance. Recall that $p'_e[m]$ is aligned with t_b. In this case, no comparisons are made in Step 2. Consider the pattern instance p'_h, where $p'_h[j]$ is to the immediate right of t_b. Recall from Step 1 that p'_h is the leftmost surviving pattern instance with left end to the right of t_c. The only surviving pattern instances to the right of p'_e which overlap t_c are presuf overlaps of p'_e. Therefore, p'_h is the leftmost surviving pattern instance, barring p'_e and its presuf overlaps. In addition, note that any pattern instance which overlaps p'_e but not $p''_\alpha[j']$ and has its jth character to the right of t_b is a presuf overlap of p'_e.

If $|p'_h$ is to the right of $p''_\alpha[j']$, then p'_h is a presuf overlap of p'_e as are all pattern instances to the right of p'_h which overlap p'_e. Step 5 follows with $p'_g = p'_e$ in this case.

Otherwise, if p'_h overlaps $p''_\alpha[j']$ then p'_h is not a presuf overlap of p'_e. The character $p''_\alpha[j']$ is then compared with the text. A match eliminates all pattern instances which overlap $p''_\alpha[j']$ but are not presuf overlaps of p'_e. (This can be seen from the following two facts: (a) all pattern instances with left end to the right of t_c which survive Step 1 have their jth character to the right of $p''_\alpha[j']$, and (b) all surviving pattern instances which overlap t_c are presuf overlaps of p'_e.) Clearly, all surviving pattern instances which overlap t_b are presuf overlaps of p'_e. In this case, Step 5 follows with $p'_g = p'_e$. Otherwise, if a mismatch occurs at $p''_\alpha[j']$, p'_e is eliminated as are all its presuf overlaps which overlap t_c. Text characters to the right of t_b are now compared from left to right with the character b until either a mismatch occurs or the right end of the text is reached. In the former case, Step 4 follows with p'_g denoting the pattern instance such that $p'_g[j]$ is aligned with the text character at which the mismatch occurs. Clearly, p'_g is the leftmost surviving pattern instance and all pattern instances which overlap $p'_g[j]$ have been eliminated. In the latter case (i.e., the right end of the text is reached), the algorithm terminates as there are no further occurrences of the pattern.

At most two mismatches are made in Steps 1–3 in Case 2.2b.

Step 4. In this step, either all surviving pattern instances which overlap p'_g are eliminated (except for presuf overlaps) or p'_g is eliminated. In the latter case, the basic algorithm is resumed with the leftmost surviving pattern instance. In the former case, Step 5 follows. All comparisons in this step are to the right of all text characters matched in the previous steps. In addition, the left end of each pattern instance eliminated in this step is also to the right of any text character matched in one of the previous steps.

In Step 4, difference-point comparisons are used. This step has a number of iterations. In each iteration, a different pattern instance overlapping p'_g but strictly to the right of $p'_g[i]$ is considered. If it is a presuf overlap of p'_g, then nothing is done. Otherwise, if it is not a presuf overlap of p'_g, the character in p'_g at the difference point of the two pattern instances is considered. If the text character aligned with this character has not been successfully compared earlier (this is ascertained using a bit vector), the two characters are compared. Step 4 ends when a mismatch occurs or when all pattern instances overlapping p'_g (excluding presuf overlaps) are eliminated.

If no mismatch occurs in Step 4, then all comparisons in this step will be charged to the text characters compared; otherwise, they will be charged to left ends of the pattern instances eliminated. In both cases, the text characters charged are to the right of all text characters matched in previous steps.

Step 5. This step attempts to complete the match of p'_g. Characters in p'_g which have not yet been matched are compared with the aligned text characters from right to left until a mismatch occurs or all of its characters are matched. In either case, another presuf shift follows. All comparisons in this step will be charged to the text characters compared.

LEMMA 6.18. *If p is a special-case pattern with $|x_1''| \geq \frac{|v_p|}{2}$, then the comparison complexity of the algorithm is $n(1 + \frac{8}{3(m+1)})$.*

Proof. We give charging strategies to show that a presuf shift can have overhead at most two. Further, we show that an overhead of one forces the next presuf shift to occur at least distance $\frac{m+1}{2}$ to the right and an overhead of two forces the next presuf shift to occur at least distance $\frac{3(m+1)}{4}$ to the right. The lemma follows.

As in Lemma 4.12, the run of the algorithm is divided into phases; a phase can be of one of four types. The range of text characters charged in each type of phase remains exactly the same as in Lemma 4.12. The charging scheme for Type 1 and Type 2 phases also remains exactly the same. Only the charging scheme for Type 3 and Type 4 phases is modified in accordance with the presuf shift handler described above.

We consider a single phase, which could be a Type 3 or a Type 4 phase. We assume that this phase begins with a presuf shift with $|x_1'| \geq \frac{m}{2}$.

The charging scheme. We consider two cases.

Case A. Suppose a mismatch occurs in Step 1 at a text character t_e to the right of $p''_\alpha[j']$ and $p'_f[j]$ is aligned with or to the left of t_e (i.e., Case 1.1 in Step 1 holds).

Each successful comparison in Step 1 matches the character b against the text. The charging scheme for this case is identical to the charging scheme in Lemma 6.7 with the function f'' defined as follows. f'' is defined to map each text character compared successfully in Step 1 to the text character which is distance $j-1$ to its left. This definition of f'' is easily verified to satisfy all four required properties of f'' stated in Lemma 6.6. Further, only the last Step 1 comparison can possibly be unsuccessful and might not receive an f'' value. From the above charging scheme, it follows that the overhead of the current presuf shift is at most one.

Case B. Suppose all comparisons in Step 1 are successful or $p'_f[j]$ is to the right of t_e, the character at which the mismatch in Step 1 occurs (i.e., Case 1.2 in Step 1 holds).

Recall from Step 1 that the leftmost surviving pattern instance completely to the right of t_c must have its jth character to the right of t_b. There are three subcases to consider.

Subcase B1. Suppose p'_e (the leftmost of the presuf pattern instances to survive Step 1) survives Steps 2, 3, and 4.

All successful comparisons in Steps 1, 2, 3, and 4 and all comparisons in Step 5 are charged to the text characters compared. All of these comparisons involve distinct text characters, and thus each text character which is to the right of t_a and aligned with p'_e is charged at most once. Further, at most one mismatch is made in Step 1 and no mismatches are made in Steps 2, 3, and 4 (otherwise, p'_e would be eliminated). In addition, a mismatch in Step 1 eliminates p''_α, thus forcing the next presuf shift to occur at least distance $\frac{m+1}{2}$ to the right. Thus the current presuf shift has overhead at most one and an overhead of one forces the next presuf shift to occur distance at least $\frac{m+1}{2}$ to the right.

Subcase B2. Suppose p'_e is eliminated in one of Steps 2, 3, and 4; further, suppose p'_e is the leftmost presuf pattern instance.

In this case, the next presuf shift occurs distance at least $m+1$ to the right. We show an overhead of at most two for this case.

All comparisons in Step 1 are successful and are charged to the text characters compared. No comparisons are made in Step 2. p'_e must be eliminated in Step 3 since

Step 5 follows directly from Step 3 otherwise (see Case 2.2b in Step 3). All successful comparisons in Step 3 are charged to the text characters compared. At most two mismatches are made in Step 3 and these constitute the overhead of this shift. All text characters matched in Steps 1–3 are to the left of $p'_g[j]$. If p'_g survives Step 4, then all comparisons in Step 4 are charged to the text characters compared. If p'_g does not survive Step 4, then the comparison which eliminates p'_g is charged to the text character aligned with $p'_g[j]$ and all other comparisons in Step 4 are charged to the left ends of the respective pattern instances eliminated. (From the definition of p'_g in Step 3, note that the left ends of these pattern instances are to the right of $p'_g[j]$.) Thus all text characters charged in Step 4 are distinct and are aligned with or to the right of $p'_g[j]$. All comparisons in Step 5 are charged to the text characters compared. These text characters are distinct from all text characters matched in the previous steps. Therefore, if p'_g survives Step 4, then each text character to the right of t_a and aligned with or to the left of $p'_g|$ is charged at most once. Otherwise, if p'_g is eliminated in Step 4 and p'_l is the leftmost surviving pattern instance following Step 4, each text character strictly between t_a and $|p'_l$ is charged at most once.

Subcase B3. Suppose p'_e is eliminated in one of Steps 2, 3, and 4 and p'_e is not the leftmost presuf pattern instance.

In this case, the next presuf shift occurs distance at least $m+1$ to the right. We show an overhead of at most two for this case.

All successful comparisons in Steps 1 and 2 and all comparisons in Step 5 are charged to the text characters compared. If p'_e does not survive Step 2 (Case 2.1 of Step 3), then all successful comparisons in Step 3 are charged to the text characters compared. Otherwise, if p'_e survives Step 2 (Case 2.2a of Step 3), there is at most one comparison in Step 3 and it is accounted for later. If p'_g survives Step 4, then all successful comparisons in Step 4 are charged to the text characters compared. In this case, each text character to the right of t_a and aligned with or to the left of $p'_g|$ is charged at most once. Otherwise, if p'_g is eliminated in Step 4, each successful comparison in Step 4 (except the one which eliminates p'_g) is charged to the text character aligned with the left end of the pattern instance eliminated by this comparison; this text character is to the right of $p'_g[j]$. In this case, each text character strictly between t_a and $|p'_l$ is charged at most once, where p'_l is the leftmost surviving pattern instance after p'_g is eliminated.

At most four comparisons have not yet been accounted for. These include the mismatch in Step 1, the mismatch in Step 4 (which eliminates p'_g), and either the mismatches in Steps 2 and 3 or the only comparison in Step 3, depending on whether or not p'_e survives Step 2. Note that if mismatches occur in all of Steps 2, 3, and 4, then the text character aligned with $p'_g[j]$ is not charged for any comparison. In addition, we show that the text character aligned with $p''_\alpha[j]$ is also not charged for any comparison. An overhead of two for the current presuf shift follows immediately.

Clearly, $p''_\alpha[j]$ is not compared in Step 1. All comparisons in Steps 2, 3, and 4 are made to the right of t_b and hence to the right of $p''_\alpha[j]$. Further, $p'_g[i]$ (i as defined in Step 3) is aligned with or to the right of t_b. Therefore, the text character aligned with $p''_\alpha[j]$ is not charged for any of the comparisons made in Steps 1–4. Consider Step 5 next. If p'_e survives Steps 2 and 3 and is eliminated in Step 4, then the presuf shift handler terminates after Step 4 and the basic algorithm is resumed. Therefore, suppose that p'_e is eliminated in Step 2 or Step 3. From the definition of p'_g in Step 3, $p'_g \neq p'_e$ and therefore $i = j$. To show that all comparisons in Step 5 are made to the right of $p''_\alpha[j]$, it suffices to show that $|p'_g$ is to the right of $p''_\alpha[j]$.

We show this by considering two cases. First, suppose $p''_\alpha \neq p''_1$. Then, by Lemma 6.8, it follows that the character in p'' which is to the immediate left of its suffix x''_1 is a b. Since x''_1 is the longest presuf of p'', it follows that $j = |x''_1| + 1$. By Lemma 6.8, $p''_\alpha[j']$ and hence $p''_\alpha[j]$ are to the left of the suffix x''_1 of p''_1. Since $|p'_g[j]$ is to the right of t_b, $|p'_g$ is aligned with or to the right of the suffix x''_1 of p''_1. The claim follows for this case.

Next, suppose $p''_\alpha = p''_1$. A mismatch occurs in Step 1 since p'_e is not the leftmost presuf pattern instance. Further, this mismatch occurs at some text character t_e to the right of $p''_\alpha[j]$ since no comparisons are made to the left of $p''_\alpha[j]$ in Step 1 in this case. Each comparison in Step 1 compares a text character with b. Since $p'_g[j]$ must be to the right of t_b, either $|p'_g$ is to the right of t_e or a b in p'_g overlaps t_e. However, p'_g will not survive in the latter case. The claim follows.

The lemma follows. ∎

Finally, we state the following theorem; the proof is similar to the proof of Theorem 4.17.

THEOREM 6.19. *There is a string-matching algorithm with a comparison complexity of* $n(1+\frac{8}{3(m+1)})$ *comparisons which uses* $O(m)$ *space and takes* $O(n+m)$ *time following preprocessing of the pattern*; *the preprocessing time is* $O(m^2)$.

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