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# SYSTEMS OF CONTROLLED FUNCTIONAL DIFFERENTIAL EQUATIONS AND ADAPTIVE TRACKING*

A. ILCHMANN†, E. P. RYAN‡, AND C. J. SANGWIN§

**Abstract.** An adaptive servomechanism is developed in the context of the problem of approximate or practical tracking (with prescribed asymptotic accuracy), by the system output, of any admissible reference signal (absolutely continuous and bounded with essentially bounded derivative) for every member of a class of controlled dynamical systems modelled by functional differential equations.

**Key words.** adaptive control, nonlinear systems, functional differential equations, practical tracking, universal servomechanism

**AMS subject classifications.** 93C23, 93C10, 93C40, 34K20

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## 1. Introduction.

A servomechanism problem is addressed in the context of a class of controlled dynamical systems having the interconnected structure shown in the dashed box in Figure 1. In particular, the aim is the development of an adaptive servomechanism which, for every system of the underlying class, ensures practical tracking (in the sense that prespecified asymptotic tracking accuracy, quantified by $\lambda > 0$, is assured), by the system output, of an arbitrary reference signal assumed to be locally absolutely continuous and bounded with essentially bounded derivative. (We denote by $\mathcal{R}$ the class of such functions and remark that bounded globally Lipschitz functions form an easily recognized subclass.) The system consists of the interconnection of two blocks: The dynamic block $\Sigma_1$, which can be influenced directly by the system input/control $u$ (an $\mathbb{R}^M$-valued function), is also driven by the output $w$ from the dynamic block $\Sigma_2$. Viewed abstractly, the block $\Sigma_2$ can be considered as a causal operator which maps the system output $y$ (an $\mathbb{R}^M$-valued function) to $w$ (an internal quantity, unavailable for feedback purposes).

In essence, the underlying system class $\mathcal{S}$ consists of infinite-dimensional nonlinear $M$-input $u$, $M$-output $y$ systems $(p, f, g, T)$, given by a controlled nonlinear functional differential equation of the form

$$\dot{y}(t) = f(p(t), (Ty)(t)) + g(p(t), (Ty)(t), u(t)), \quad y|_{[-h,0]} = y^0 \in C([-h, 0]; \mathbb{R}^M), \tag{1.1}$$

where, loosely speaking, $h \geq 0$ quantifies the "memory" of the system, $p$ may be thought of as a (bounded) disturbance term, and $T$ is a nonlinear causal operator. While a full description of the system class $\mathcal{S}$ is postponed to section 3, we remark here that diverse phenomena are incorporated within the class including, for example, diffusion processes, delays (both point and distributed), and hysteretic effects.

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†Institute of Mathematics, Technical University Ilmenau, Weimarer Straße 25, 98693 Ilmenau, Germany (ilchmann@mathematik.tu-ilmenau.de).

‡Department of Mathematical Sciences, University of Bath, Claverton Down, Bath BA2 7AY, UK (epr@maths.bath.ac.uk).

§School of Mathematics and Statistics, University of Birmingham, Edgbaston, Birmingham B15 2TT, UK (sangwinc@for.mat.bham.ac.uk).

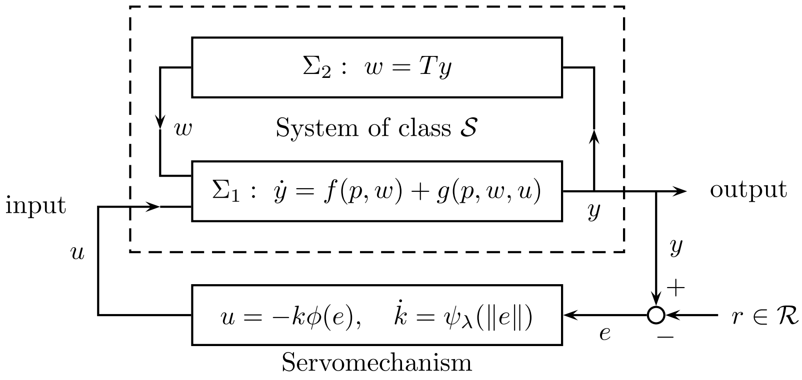


FIG. 1. $(\mathcal{R}, \mathcal{S})$*-universal* $\lambda$*-servomechanism.*

Furthermore, we remark that results pertaining to adaptive control of functional differential equations are also contained in [3], wherein both the underlying class of systems and the analytic framework differs in an essential manner from those of the present paper; restricted to a problem of adaptive *stabilization*, related results are also reported in [19], with the fundamental distinction that, in [19], *discontinuous* stabilizing feedback strategies are developed within an analytic framework of differential inclusions.

The control objective is to determine an $(\mathcal{R}, \mathcal{S})$-universal $\lambda$-servomechanism: specifically, to determine continuous functions $\phi : \mathbb{R}^M \to \mathbb{R}^M$ and $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ (parameterized by $\lambda > 0$) such that, for each system of class $\mathcal{S}$ and every reference signal $r \in \mathcal{R}$, the control

$$u(t) = -k(t)\phi(y(t) - r(t)), \quad \dot{k}(t) = \psi_\lambda(\|y(t) - r(t)\|), \quad k|_{[-h,0]} = k^0 \tag{1.2}$$

applied to (1.1) ensures (i) convergence of the controller gain, and (ii) tracking of $r(\cdot)$ with asymptotic accuracy quantified by $\lambda > 0$, in the sense that $\max\{\|y(t) - r(t)\| - \lambda, 0\} \to 0$ as $t \to \infty$. See Figure 1.

Given $\lambda > 0$, $r \in \mathcal{R}$ and writing

$$F : (t, w, y, k) \mapsto (f(p(t), w) + g(p(t), w, -k\phi(y - r(t))),\ \psi_\lambda(\|y - r(t)\|)), \tag{1.3}$$

we see that analysis of the behavior of a system $(p, f, g, T) \in \mathcal{S}$ under control (1.2) constitutes a study of an initial-value problem of the form

$$\dot{x}(t) = F(t, (\widehat{T}x)(t)), \quad x|_{[-h,0]} = x^0 := (y^0, k^0) \in C([-h, 0]; \mathbb{R}^N), \tag{1.4}$$

where $N = M+1$, $x(t) = (y(t), k(t))$, and $\widehat{T}$ is an operator defined on $C([-h, \infty); \mathbb{R}^N)$ by

$$(\widehat{T}x)(t) = (\widehat{T}(y, k))(t) := ((Ty)(t), y(t), k(t)). \tag{1.5}$$

The contribution of this paper is threefold in theme: First, we provide an existence theory for initial-value problems of the general form (1.4) under relatively mild hypotheses on $F$ and $\widehat{T}$; second, and within the framework of the first theme, we develop a universal servomechanism[1] for a class of nonlinear, infinite-dimensional systems; third, we elucidate the hypotheses on the right-hand side $\psi_\lambda$ of the gain adaptation equation in (1.2) under which the tracking objective is achievable. In the very specific context of the linear systems of section 2.2 below we will show that $\psi_\lambda : [0,\infty) \to [0,\infty)$ may be chosen as any continuous function with the properties $\psi_\lambda^{-1}(0) = [0,\lambda]$ and $\liminf_{s\to\infty} \psi_\lambda(s) \neq 0$. (In particular, $\psi_\lambda$ may be chosen to be a bounded function; one such choice is given by $\psi_\lambda(s) = \max\{s-\lambda, 0\}/s$ for $s > 0$ with $\psi_\lambda(0) := 0$.) This ensures that the gain $k$ can exhibit at most linear growth, a feature with attendant practical advantages.

We close this section with some remarks on notation. For $I \subset \mathbb{R}$ an interval $C(I;\mathbb{R}^N)$ (respectively, $AC_{\mathrm{loc}}(I;\mathbb{R}^N)$) denotes the set of continuous (respectively, locally absolutely continuous) functions $I \to \mathbb{R}^N$; $L^\infty_{\mathrm{loc}}(I;\mathbb{R}^N)$ denotes the space of measurable locally essentially bounded functions $I \to \mathbb{R}^N$. For $x : I \to \mathbb{R}^N$, the restriction of $x$ to $J \subset I$ is denoted by $x|_J$. The open ball of radius $r > 0$, centered at $c \in \mathbb{R}^N$, is written as $\mathbb{B}_r(c)$. For $\lambda > 0$, $d_\lambda$ denotes the Euclidean distance function for $[-\lambda, \lambda]$ given by

$$d_\lambda(\xi) := \max\{0, |\xi| - \lambda\}. \tag{1.6}$$

$\mathcal{R}$ denotes the space of bounded functions in $AC_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^M)$ with essentially bounded derivative; when equipped with the norm $\|\cdot\|_{1,\infty}$ given by $\|r\|_{1,\infty} = \sup_{t\in\mathbb{R}} \|r(t)\| + \text{ess-sup}_{t\in\mathbb{R}}\|\dot r(t)\|$, $\mathcal{R}$ can be identified as the Sobolev space $W^{1,\infty}(\mathbb{R};\mathbb{R}^M)$. We write $\mathbb{R}_+ := [0,\infty)$. $\mathcal{K}$ denotes the class of continuous, strictly increasing functions $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$; the subclass of *unbounded* class $\mathcal{K}$ functions is denoted $\mathcal{K}_\infty$. $\mathcal{KL}$ is the class of functions $\gamma : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that for each $t \in \mathbb{R}_+$, $\gamma(\cdot, t)$ is of class $\mathcal{K}$ and for each $s \in \mathbb{R}_+$, $\gamma(s,\cdot)$ is decreasing with $\gamma(s,t) \to 0$ as $t \to \infty$.

**2. Functional differential equations.** The focus of this section is the development of an existence theory, for initial-value problems of the form (1.4), of sufficient generality to accommodate the analysis of dynamic behavior of the adaptively controlled systems of later sections. While the literature is rich in existence results for functional differential equations (see, for example, [4]), we are unaware of a result directly applicable to the particular class of equations which form the focus of the present paper. For this reason, and to make the present paper self-contained, we provide an appropriate result in Theorem 2.3 below (with proof in the appendix). First, we make precise the class of admissible operators $\widehat{T}$ in (1.4).

DEFINITION 2.1 (the operator class $\mathcal{T}_h^{N,K}$). *For $h \geq 0$ and $N, K \in \mathbb{N}$, let $\mathcal{T}_h^{N,K}$ denote the space of operators $T : C([-h,\infty);\mathbb{R}^N) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^K)$ with the following properties.*

1. *For every $\delta > 0$ and every bounded interval $I \subset \mathbb{R}_+$, there exists $\Delta > 0$ such that, for all $x \in C([-h,\infty);\mathbb{R}^N)$,*

$$\sup_{t\in[-h,\infty)} \|x(t)\| < \delta \quad \Longrightarrow \quad \|(Tx)(t)\| < \Delta \quad \textit{for almost all (a.a.) } t \in I\,.$$

2. *For all $t \in \mathbb{R}_+$, the following hold:*
   (a) *for all $x, \xi \in C([-h,\infty);\mathbb{R}^N)$,*

$$x(\cdot) \equiv \xi(\cdot) \textit{ on } [-h,t] \quad \Longrightarrow \quad (Tx)(s) = (T\xi)(s) \quad \textit{for a.a. } s \in [0,t];$$

[1]The servomechanism can also tolerate disturbances on the output measurement in a sense to be described in section 3.

(b) *for all continuous* $\zeta : [-h,t] \to \mathbb{R}^N$, *there exist* $\tau, \delta, c > 0$ *such that, for all* $x, \xi \in C([-h,\infty);\mathbb{R}^N)$ *with* $x|_{[-h,t]} = \zeta = \xi|_{[-h,t]}$ *and* $x(s), \xi(s) \in \mathbb{B}_\delta(\zeta(t))$ *for all* $s \in [t, t+\tau]$,

$$\operatorname*{ess\,sup}_{s\in[t,t+\tau]} \|(Tx)(s) - (T\xi)(s)\| \;\le\; c \sup_{s\in[t,t+\tau]} \|x(s) - \xi(s)\| .$$

*Remark* 2.2. (i) The essence of property 1 of Definition 2.1 is a "bounded-input, *locally* bounded-output" assumption.

(ii) Property 2(a) is an assumption of causality.

(iii) Property 2(b) is a technical assumption on $T$ of a "locally Lipschitz" nature.

(iv) Let $T \in \mathcal{T}_h^{N,K}$ and $t \ge 0$. Given $x \in C([-h,t);\mathbb{R}^N)$, let $x^e$ denote an arbitrary extension of $x$ to $C([-h,\infty);\mathbb{R}^N)$. By virtue of property 2(a), $Tx^e|_{[0,t)}$ is uniquely determined by the function $x$, in the sense that the former is independent of the extension $x^e$ chosen for the latter. Expanding on this observation, we will adopt the following notational convention: For $s \in [0,t)$, we simply write $(Tx)(s)$ in place of $(Tx^e)(s)$, where $x^e \in C([-h,\infty);\mathbb{R}^N)$ is any continuous extension of $x$.

(v) For $\omega \in \mathbb{R}$, let $S_\omega$ denote the shift operator on functions $\mathbb{R} \to \mathbb{R}^M$ given by $(S_\omega x)(t) := x(t+\omega)$ for all $t \in \mathbb{R}$. Then

$$T \in \mathcal{T}_h^{N,K} \quad \Longrightarrow \quad TS_{-\omega} \in \mathcal{T}_{h+\omega}^{N,K} \quad \text{for all } \ \omega \ge 0. \tag{2.1}$$

(vi) Let $T_1, T_2 \in \mathcal{T}_h^{N,K}$ and $\tau_1, \tau_2 \in \mathbb{R}$. Then the operator $\tau_1 T_1 + \tau_2 T_2$, defined by $(\tau_1 T_1 + \tau_2 T_2)(y)(t) := \tau_1 (T_1 y)(t) + \tau_2 (T_2 y)(t)$, is also of class $\mathcal{T}_h^{N,K}$.

(vii) The class $\mathcal{T}_h^{N,N}$ differs from class $\mathcal{T}_h^{N}$ of [19, Definition 4] only insofar as operators of the former class have range $C([-h,\infty);\mathbb{R}^N)$ while operators of the latter class have domain $L^\infty_{\text{loc}}(\mathbb{R};\mathbb{R}^N)$.

**2.1. An existence theorem.** Consider the initial-value problem

$$\dot x(t) = F(t, (\widehat{T}x)(t)), \quad x|_{[-h,0]} = x^0 \in C([-h,0];\mathbb{R}^N), \tag{2.2}$$

where $\widehat{T}$ is a causal operator of class $\mathcal{T}_h^{N,K}$ and $F : [-h,\infty) \times \mathbb{R}^K \to \mathbb{R}^N$ is a Carathéodory function. (Specifically, (i) for almost all $t \in \mathbb{R}$, $F(t,\cdot)$ is continuous; (ii) for each fixed $w \in \mathbb{R}^K$, $F(\cdot, w)$ is measurable; (iii) for each compact $C \subset \mathbb{R}^K$ there exists $\kappa \in L^1_{\text{loc}}([-h,\infty);\mathbb{R}_+)$ such that

$$\|F(t,w)\| \le \kappa(t) \quad \text{for almost all } t \in [-h,\infty) \text{ and all } w \in C.)$$

By a solution of (2.2) on $[-h,\omega)$, we mean a function $x \in C([-h,\omega);\mathbb{R}^N)$, with $\omega \in (0,\infty]$ and $x|_{[-h,0]} = x^0$, such that $x|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (2.2) for almost all $t \in [0,\omega)$; $x$ is maximal if it has no right extension that is also a solution.

THEOREM 2.3. *Let* $N, K \in \mathbb{N}$, $\widehat{T} \in \mathcal{T}_h^{N,K}$, *and* $x^0 \in C([-h,0];\mathbb{R}^N)$. *Assume* $F : [-h,\infty) \times \mathbb{R}^K \to \mathbb{R}^N$ *is a Carathéodory function.*

*There exists a solution* $x : [-h,\omega) \to \mathbb{R}^N$ *of the initial-value problem* (2.2), *and every solution can be extended to a maximal solution; moreover, if* $F \in L^\infty_{\text{loc}}([-h,\infty) \times \mathbb{R}^K;\mathbb{R}^N)$ *and* $x : [-h,\omega) \to \mathbb{R}^N$ *is a bounded maximal solution, then* $\omega = \infty$.

*Proof.* For proof, see the appendix.

Next, we show that the operators of the class $\mathcal{T}_h^{N,K}$ encompass the input-output behavior of a diverse range of subsystems $\Sigma_2$ (see Figure 1).

**2.2. Linear systems.** *The finite-dimensional prototype.* Consider the well-studied class $\mathcal{L}$ of finite-dimensional, real, linear, minimum-phase, $M$-input ($u(t)$), $M$-output ($y(t)$) systems having high-frequency gain $B \in \mathbb{R}^{M\times M}$ with spectrum in the open right half complex plane. Under a suitable coordinate transformation (see, for example, [5, Proposition 2.1.2]), every system in $\mathcal{L}$ can be expressed in the form of two coupled subsystems

$$\left.\begin{aligned} \dot{y}(t) &= A_1 y(t) + A_2 z(t) + Bu(t), \quad & y(0) = y^0 \\ \dot{z}(t) &= A_3 y(t) + A_4 z(t), & z(0) = z^0 \end{aligned}\right\} \tag{2.3}$$

with $y(t), u(t) \in \mathbb{R}^M$, $z(t) \in \mathbb{R}^{N-M}$, and where $A_4$ has spectrum in the open left half complex plane. Introducing the linear operator $T$ given by

$$(Ty)(t) := A_1 y(t) + A_2 \int_0^t \exp(A_4(t-s))A_3 y(s)ds \tag{2.4}$$

and the function $p$ given by $p(t) := A_2 \exp(A_4 t)z^0$, then, with respect to an operator theoretic viewpoint, system (2.3) can be interpreted as

$$\dot{y}(t) = p(t) + (Ty)(t) + Bu(t), \quad y(0) = y^0. \tag{2.5}$$

With reference to Figure 1, (2.4) and (2.5) correspond to components $\Sigma_2$ and $\Sigma_1$ of the interconnected system.

*Regular linear systems with bounded observation operator.* The following example is adapted from [19] and extends the prototype linear class $\mathcal{L}$ to an infinite-dimensional setting by replacing the second of the differential equations (2.3) by an infinite-dimensional analogue on a Hilbert space $X$. Let $\mathbf{G}$ denote the transfer function of a regular (in the sense of [22]) linear system with state space $X$, with generating operators $(A, B, C, D)$, and with $\mathbb{R}^M$-valued input and $\mathbb{R}^Q$-valued output. This means, in particular, that (i) $A$ generates a strongly continuous semigroup $\mathbf{S} = (\mathbf{S}_t)_{t\geq 0}$ of bounded linear operators on $X$, (ii) the control operator $B$ is a bounded linear operator from $\mathbb{R}^M$ to $X_{-1}$, (iii) the observation operator $C$ is a bounded linear operator from $X_1$ to $\mathbb{R}^Q$, and (iv) the feedthrough operator $D$ is a linear operator from $\mathbb{R}^M$ to $\mathbb{R}^Q$. Here $X_1$ denotes the space dom$(A)$ (the domain of $A$) endowed with the graph norm, and $X_{-1}$ denotes the completion of $X$ with respect to the norm $\|z\|_{-1} = \|(s_0 I - A)^{-1}z\|$, where $s_0$ is any fixed element of the resolvent set of $A$ and $\|\cdot\|$ denotes the norm on $X$. As a regular linear system, the transfer function $\mathbf{G}$ is holomorphic and bounded on every half-plane $\mathbb{C}_\alpha$ with $\alpha > \omega(\mathbf{S}) := \lim_{t\to\infty} t^{-1}\ln\|\mathbf{S}_t\|$. Moreover,

$$\lim_{s\to\infty,\ s\in\mathbb{R}} \mathbf{G}(s) = D.$$

The system is said to be exponentially stable if the semigroup $\mathbf{S}$ is exponentially stable—that is, if $\omega(\mathbf{S}) < 0$. Henceforth, we assume that the system is exponentially stable and, moreover, we assume that the observation operator $C$ can be extended to a bounded linear operator from $X$ to $\mathbb{R}^Q$; this extended operator is again denoted by $C$.

In terms of the generating operators $(A, B, C, D)$, the transfer function $\mathbf{G}$ is given by

$$\mathbf{G}(s) = C(sI - A)^{-1}B + D.$$

For any $z^0 \in X$ and input $y \in L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}^M)$, the state $z(\cdot)$ and the output $w(\cdot)$ of the regular system (with bounded observation operator) satisfy the equations

$$\dot z(t) = Az(t) + By(t)\,, \quad z(0) = z^0, \tag{2.6}$$
$$w(t) = Cz(t) + Dy(t) \tag{2.7}$$

for almost all $t \geq 0$. The derivative on the left-hand side of (2.6) has, of course, to be understood in $X_{-1}$. In other words, if we consider the initial-value problem (2.6) in the space $X_{-1}$, then for any $z^0 \in X$ and $y \in L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}^M)$, (2.6) has a unique strong solution given by the variation of parameters formula (see [16, Chapter 4, Theorem 2.9])

$$z(t) = \mathbf{S}_t z^0 + \int_0^t \mathbf{S}_{t-s} By(s)\, ds. \tag{2.8}$$

Restricting to continuous inputs, define the operator $T : C(\mathbb{R}_+;\mathbb{R}^M) \to L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}^Q)$ by

$$(Ty)(t) := C\int_0^t \mathbf{S}_{t-s} By(s)\; ds + Dy(t), \quad t \geq 0. \tag{2.9}$$

(We remark that the above operator is the infinite-dimensional counterpart of the operator (2.4) in the case of the finite-dimensional prototype.) By exponential stability of the semigroup $\mathbf{S}$, there then exist constants $c_1 > 0$ such that

$$\|z\|_{L^\infty(\mathbb{R}_+;X)} \leq c_1 \left[\|z^0\| + \|y\|_{L^\infty(\mathbb{R}_+;\mathbb{R}^M)}\right] \quad \text{for all } (z^0,y) \in X \times L^\infty(\mathbb{R}_+;\mathbb{R}^M). \tag{2.10}$$

Setting $h = 0$, we see that property 2(a) of Definition 2.1 holds and property 2(b) is a consequence of the linearity of $T$ and (2.10), in view of (2.10), and causality property 1 of Definition 2.1 also holds. Therefore, the operator $T$ is of class $\mathcal{T}_0^{M,Q}$.

**2.3. Nonlinear systems.** *Input-to-state stable (ISS) systems.* Let $Z : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}^L$ be locally Lipschitz with $Z(0,0) = 0$. For $y \in L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}^M)$, let $z(\cdot, z^0, y)$ denote the maximal solution of the initial-value problem

$$\dot z(t) = Z(z(t), y(t)), \quad z(0) = z^0 \in \mathbb{R}^L. \tag{2.11}$$

Assume that the system is input-to-state stable (ISS) [20]; that is, there exist functions $\theta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that, for all $(z^0, y) \in \mathbb{R}^L \times L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}^M)$,

$$\|z(t,z^0,y)\| \leq \theta(\|z^0\|, t) + \operatorname*{ess\,sup}_{s\in[0,t]} \gamma(\|y(s)\|) \quad \text{for all } t \geq 0. \tag{2.12}$$

Let $W : \mathbb{R}^L \to \mathbb{R}^Q$ be locally Lipschitz and such that there exists $c > 0$ such that $\|W(z)\| \leq c\|z\|$ for all $z \in \mathbb{R}^L$. Now consider system (2.11) with output $w$ given by

$$w(t) = W(z(t, z^0, y)).$$

Fix $z^0 \in \mathbb{R}^L$ arbitrarily. Again, restricting to continuous inputs, define the operator $T : C(\mathbb{R}_+;\mathbb{R}^M) \to L^\infty_{\rm loc}(\mathbb{R}_+;\mathbb{R}^Q)$ by

$$(Ty)(t) := W(z(t, z^0, y)), \quad t \geq 0. \tag{2.13}$$

In view of (2.12), property 1 of Definition 2.1 evidently holds; setting $h = 0$, we see that property 2(a) also holds. Arguing as in [19, section 3.2.3], via an application of

Gronwall's lemma, it can be shown that property 2(b) holds. Therefore, the operator $T$ is of class $\mathcal{T}_0^{M,Q}$. We note that, strictly speaking, the above construction yields a family of operators $T_{z^0}$ parameterized by the initial data $z^0$.

*Systems in input affine form.* A particular generalization of the prototype class $\mathcal{L}$ of linear, finite-dimensional, minimum-phase systems is the class of nonlinear systems in input affine form

$$\left.\begin{array}{ll} \dot{y}(t) = a(t, y(t), z(t)) + b(t, y(t), z(t))u(t), & y(0) = y^0 \\ \dot{z}(t) = c(t, y(t), z(t)), & z(0) = z^0 \end{array}\right\} \tag{2.14}$$

where $a : \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^L \to \mathbb{R}^M$, $b : \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^L \to \mathbb{R}^{M \times M}$, and $c : \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^L \to \mathbb{R}^L$ are Carathéodory functions and $(y_e, z_e, u_e)$ is an equilibrium ($(y_e, z_e, u_e) = (0, 0, 0)$ in the linear prototype) in the sense that

$$a(t, y_e, z_e) = 0, \quad b(t, y_e, z_e)u_e = 0, \quad c(t, y_e, z_e) = 0 \quad \text{for all } t \geq 0.$$

The problem of construction of a $\lambda$-servomechanism for such systems has been investigated in [1, 6]. There, the minimum-phase property of the linear prototype in (2.3) is replaced by the assumptions that $z_e$ is a global, uniformly exponentially stable equilibrium of

$$\dot{\eta}(t) = c(t, y_e, \eta(t)). \tag{2.15}$$

We assume that (i) for each compact set $C \subset \mathbb{R}^M \times \mathbb{R}^L$, there exists $\kappa \in L^1_{\text{loc}}(\mathbb{R}_+)$ such that $\|c(t, y, z) - c(t, \xi, \zeta)\| \leq \kappa(t)\|(y, z) - (\xi, \zeta)\|$ for almost all $t \in \mathbb{R}_+$ and all $(y, z), (\xi, \zeta) \in C$, and (ii) for some constant $c_0 > 0$,

$$\|c(t, y, z) - c(t, y_e, z)\| \leq c_0 \, [1 + \|y - y_e\|] \quad \text{for all } (t, y, z) \in \mathbb{R}_+ \times \mathbb{R}^M \times \mathbb{R}^L.$$

Considering the second equations of (2.14) in isolation, for $y \in L^\infty_{\text{loc}}(\mathbb{R}_+, \mathbb{R}^M)$ we denote by $z(\cdot, z^0, y)$ the unique solution of

$$\dot{z}(t) = c(t, y(t), z(t)) = c(t, y_e, z(t)) + [c(t, y(t), z(t)) - c(t, y_e, z(t))], \quad z(0) = z^0.$$

Invoking exponential stability of the equilibrium of (2.15) in conjunction with converse Lyapunov theory (details omitted here), we may conclude the existence of a constant $c_1 > 0$ such that, for each $(z^0, y) \in \mathbb{R}^L \times L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^M)$,

$$\|z(t, z^0, y)\| \leq c_1 [\|z^0\| + 1 + \operatorname*{ess\,sup}_{s \in [0,t]} \|y(s)\|] \quad \text{for all } t \geq 0. \tag{2.16}$$

Fix $z^0 \in \mathbb{R}^L$ arbitrarily. Define the operator $T : C(\mathbb{R}_+; \mathbb{R}^M) \to L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^L)$ by

$$(Ty)(t) := z(t, z^0, y), \quad t \geq 0. \tag{2.17}$$

In view of (2.16), property 1 of Definition 2.1 evidently holds; setting $h = 0$, we see that property 2(a) also holds. An application of Gronwall's lemma (analogous to that adopted in [19, section 3.2.3] in the context of ISS systems) yields property 2(b). Therefore, the operator $T$ is of class $\mathcal{T}_0^{M,L}$. As in the case of ISS systems, we remark that, strictly speaking, the above construction yields a family of operators $T_{z^0}$ parameterized by the initial data $z^0$.

*The general case.* Elaborating on the above two cases, consider the system

$$\dot{z}(t) = Z(t, z(t), y(t)), \quad z(0) = z^0 \in \mathbb{R}^L, \tag{2.18}$$

with input $y \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$ and output

$$w(t) = W(t, z(t)) \in \mathbb{R}^Q.$$

Assume that $W : \mathbb{R}_+ \times \mathbb{R}^L \to \mathbb{R}^Q$ and $Z : \mathbb{R}_+ \times \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}^L$ are Carathéodory functions and such that the following hold: (i) for some constant $c > 0$, $\|W(t,z)\| \le c\|z\|$ for almost all $t \ge 0$ and all $z \in \mathbb{R}^L$; (ii) for each compact set $C \subset \mathbb{R}^L \times \mathbb{R}^M$, there exists $\kappa \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ such that $\|Z(t,z,y) - Z(t,\zeta,\xi)\| \le \kappa(t)\|(z,y) - (\zeta,\xi)\|$ for almost all $t \in \mathbb{R}_+$ and all $(z,y),(\zeta,\xi) \in C$; and (iii) for each $(z^0,y) \in \mathbb{R}^L \times L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$, the unique maximal solution of initial-value problem (2.18) has interval of existence $\mathbb{R}_+$. (We denote the solution by $z(\cdot, z^0, y)$.) Furthermore, we assume the existence of a function $\gamma \in \mathcal{K}$ such that, for each $z^0 \in \mathbb{R}^L$, there exists a constant $c > 0$ such that, for all $y \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^M)$,

$$\|z(t,z^0,y)\| \le c[1 + \text{ess-sup}_{s\in[0,t]}\gamma(\|y(s)\|)] \quad \text{for all } t \ge 0 \tag{2.19}$$

(a weaker condition than the ISS inequality (2.12)). Fix $z^0 \in \mathbb{R}^L$ arbitrarily. Define the operator $T : C(\mathbb{R}_+;\mathbb{R}^M) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^Q)$ by

$$(Ty)(t) = W(t, z(t,z^0,y)), \quad t \ge 0.$$

Then this construction yields a family (parameterized by the initial data $z^0$) of operators $T$ of class $\mathcal{T}_0^{M,Q}$: This family subsumes the operators discussed in sections 2.2 and 2.3 above.

**2.4. Nonlinear delay elements.** Let $\mathcal{D}^{M,Q}$ denote the class of functions $\mathbb{R} \times \mathbb{R}^M \to \mathbb{R}^Q : (t,y) \mapsto \Psi(t,y)$ that are measurable in $t$ and locally Lipschitz in $y$ uniformly with respect to $t$. Precisely, (i) for each fixed $y$, $\Psi(\cdot,y)$ is measurable, and (ii) for every compact $C \subset \mathbb{R}^M$ there exists a constant $c$ such that

$$\text{for a.a. } t, \quad \|\Psi(t,y) - \Psi(t,z)\| \le c\|y - z\| \quad \text{for all } y,z \in C.$$

For $i = 0,\ldots,n$, let $\Psi_i \in \mathcal{D}^{M,Q}$ and $h_i \in \mathbb{R}_+$. Define $h := \max_i h_i$. For $y \in C([-h,\infty);\mathbb{R}^M)$, let

$$(Ty)(t) := \int_{-h_0}^0 \Psi_0(s, y(t+s))\, ds + \sum_{i=1}^n \Psi_i(t, y(t-h_i)), \quad t \ge 0. \tag{2.20}$$

The operator $T$, so defined, is of class $\mathcal{T}_h^{M,Q}$; for details, see [19].

**2.5. Hysteresis.** A general class of nonlinear operators $C(\mathbb{R}_+;\mathbb{R}) \to C(\mathbb{R}_+;\mathbb{R})$, which includes many physically motivated hysteretic effects, is defined via assumptions (N1)–(N8) of [11, section 3]. Assumption (N1) implies that property 2(a) of Definition 2.1 holds with $h = 0$. Assumption (N5) implies that property 2(b) of Definition 2.1 holds. Finally, (N8) implies that property 1 of Definition 2.1 holds. Therefore, the nonlinear operators considered in [11] are of class $\mathcal{T}_0^{1,1}$. Examples of such operators, including relay hysteresis, backlash hysteresis, elastic-plastic hysteresis, and Preisach operators, are detailed in [11, section 5]. By way of illustration, we briefly describe the first two of these examples.

*Relay hysteresis.* Let $a_1 < a_2$ and let $\rho_1 : [a_1,\infty) \to \mathbb{R}$, $\rho_2 : (-\infty, a_2] \to \mathbb{R}$ be continuous, globally Lipschitz, and satisfying $\rho_1(a_1) = \rho_2(a_1)$ and $\rho_1(a_2) = \rho_2(a_2)$. For a given input $y \in C(\mathbb{R}_+;\mathbb{R})$ to the hysteresis element, the output $w$ is such that

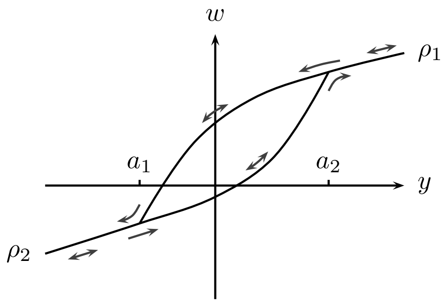


FIG. 2. *Relay hysteresis.*

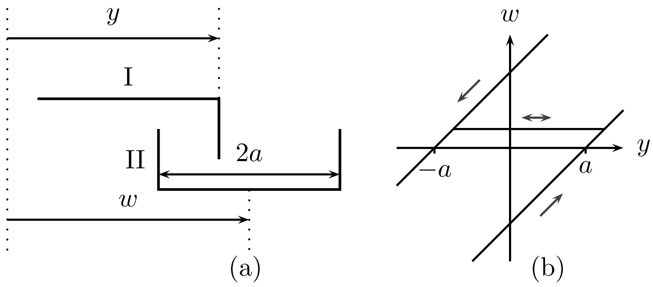


FIG. 3. *Backlash hysteresis.*

$(y(t), w(t)) \in \text{graph}(\rho_1) \cup \text{graph}(\rho_2)$ for all $t \in \mathbb{R}_+$: The value $w(t)$ of the output at $t \in \mathbb{R}_+$ is either $\rho_1(y(t))$ or $\rho_2(y(t))$, depending on which of the threshold values $a_2$ or $a_1$ was "last" attained by the input $y$. This situation is illustrated by Figure 2.

When suitably initialized, such a hysteresis element has the property that, to each input $y \in C(\mathbb{R}_+; \mathbb{R})$ there corresponds a unique output $w = Ty \in C(\mathbb{R}_+; \mathbb{R})$; the operator $T$, so defined, is of class $\mathcal{T}_0^{1,1}$. Full details may be found in [11, section 5]. (See also [12, 10].)

*Backlash hysteresis.* Next consider a one-dimensional mechanical link consisting of the two solid parts I and II, as shown in Figure 3(a), the displacements of which (with respect to some fixed datum) at time $t \geq 0$ are given by $y(t)$ and $w(t)$ with $|y(t) - w(t)| \leq a$ for all $t$, and $w(0) := y(0) + \xi$ for some prespecified $-a \leq \xi \leq a$.

Within the link there is mechanical play; that is to say, the position $w(t)$ of II remains constant as long as the position $y(t)$ of I remains within the interior of II. Thus, assuming the continuity of $y$, we have $\dot{w}(t) = 0$ whenever $|y(t) - w(t)| < a$. Given a continuous input $y \in C(\mathbb{R}_+; \mathbb{R})$, describing the evolution of the position of I, denote the corresponding position of II by $w = Ty$. The operator $T$ so defined (in effect we define a family $T_\xi$ of operators parameterized by the initial offset $\xi$) is known as *backlash* or *play* and is of class $\mathcal{T}_0^{1,1}$. Full details may be found in [11, section 5].

**3. Adaptive control.** We now focus on the adaptive control problem. The following subclass $\mathcal{J}$ of $\mathcal{K}$ functions will play an important role:

$$\mathcal{J} := \{\alpha \in \mathcal{K} | \text{ for each } \delta \in \mathbb{R}_+ \text{ there exists } \Delta \in \mathbb{R}_+ : \alpha(\delta\tau) \leq \Delta\alpha(\tau) \text{ for all } \tau \geq 0\}.$$

Furthermore, we define $\mathcal{J}_\infty := \mathcal{J} \cap \mathcal{K}_\infty$. For example, (a) for each $s > 0$, the function $\tau \mapsto \tau^s$ is of class $\mathcal{J}_\infty$, and (b) the function $\tau \mapsto \ln(1+\tau)$ is of class $\mathcal{J}_\infty$; its inverse $\tau \mapsto \exp(\tau) - 1$ is of class $\mathcal{K}_\infty$ but is *not* of class $\mathcal{J}$. In addition to their defining property, the ensuing properties of class $\mathcal{J}$ functions are readily established and will be freely invoked later in the analysis:

1. $\alpha, \beta \in \mathcal{J} \quad \Longrightarrow \quad \alpha \circ \beta \in \mathcal{J}$ and $\alpha + \beta \in \mathcal{J}$;
2. $\alpha \in \mathcal{J} \quad \Longrightarrow \quad \exists \Delta > 0 : \ \alpha(a+b) \le \Delta[\alpha(a) + \alpha(b)] \quad$ for all $\ a, b \in \mathbb{R}_+$.

We also record a property of $\mathcal{K}$ functions (and, a fortiori, a property of $\mathcal{J}$ functions):

3. Let $t > 0$, $I = [0,t]$, $\xi \in C(I;\mathbb{R}_+)$, and $\alpha \in \mathcal{K}$; then $\alpha(\max_{s\in I} \xi(s)) = \max_{s \in I} \alpha(\xi(s))$.

DEFINITION 3.1 (the system class). *Let $\alpha_f, \alpha_T \in \mathcal{J}$; then $\mathcal{S} = \mathcal{S}(\alpha_f, \alpha_T)$ denotes the class of $M$-input, $M$-output systems of the form* (1.1) *with the following properties (wherein $P, Q \in \mathbb{N}$ are arbitrary):*

1. $p \in L^\infty([-h,\infty);\mathbb{R}^P)$;
2. $f : \mathbb{R}^P \times \mathbb{R}^Q \to \mathbb{R}^M$ *is continuous and, for every compact set $C \subset \mathbb{R}^P$, there exists a constant $c_f \ge 0$ such that*
$$\|f(p,w)\| \le c_f\,[1 + \alpha_f(\|w\|)] \quad \textit{for all} \ \ (p,w) \in C \times \mathbb{R}^Q\,;$$
3. $g : \mathbb{R}^P \times \mathbb{R}^Q \times \mathbb{R}^M \to \mathbb{R}^M$ *is continuous and, for every compact set $C \subset \mathbb{R}^P$, there exists a positive definite, symmetric $G \in \mathbb{R}^{M\times M}$ such that*
$$\langle Gu, g(p,w,u)\rangle \ge \|u\|^2 \quad \textit{for all } (p,w,u) \in C \times \mathbb{R}^Q \times \mathbb{R}^M\,;$$
4. $T : C([-h,\infty);\mathbb{R}^M) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_+;\mathbb{R}^Q)$ *is of class $\mathcal{T}_h^{M,Q}$, and there exist $\alpha_T \in \mathcal{J}$ and constant $c_T \ge 0$ such that, for all $y \in C([-h,\infty);\mathbb{R}^M)$,*

$$\|(Ty)(t)\| \le c_T \left[1 + \max_{s\in[0,t]} \alpha_T(\|y(s)\|)\right] \quad \textit{for almost all} \ \ t \in \mathbb{R}_+\,. \tag{3.1}$$

For convenience, we denote a system of class $\mathcal{S}(\alpha_f, \alpha_T)$ by $(p,f,g,T) \in \mathcal{S}(\alpha_f,\alpha_T)$ and, whenever the functions $\alpha_f$ and $\alpha_T$ are contextually evident, we simply write $\mathcal{S}$ in place of $\mathcal{S}(\alpha_f,\alpha_T)$. We emphasize that, in the construction of an $(\mathcal{R},\mathcal{S})$-universal control strategy, only the (instantaneous) tracking error $e(t) = y(t) - r(t)$ is assumed to be available for feedback, and the only a priori structural information assumed is knowledge of the functions $\alpha_f$, $\alpha_T \in \mathcal{J}$. Some examples follow.

Assume $f$ has the polynomial form given by $\ f(p,w) := \sum_{i=0}^{l} p_i w^i$. Then property 2 of Definition 3.1 holds with $\alpha_f : s \mapsto s^m$ for $m \ge l$; if an upper bound for the degree $l$ of the polynomial is unknown, then the map $\alpha_f : s \mapsto \exp(s) - 1$ suffices.

If $g(p,w,u) = Bu$, as in the linear prototype (2.3), and $B \in \mathbb{R}^{M\times M}$ has spectrum in the open right half complex plane, then there exists a positive definite $G \in \mathbb{R}^{M\times M}$ satisfying $GB + B^TG = 2I$, whence property 3 of Definition 3.1.

Consider again the examples of operators in sections 2.2–2.5.

Let $T \in \mathcal{T}_h^{M,Q}$, given by (2.9), be the input-output operator of an exponentially stable regular linear system with $\mathbb{R}^M$-valued input and $\mathbb{R}^Q$-valued output. Then (2.10) and causality imply that (3.1) holds with the $\alpha_T \in \mathcal{J}$ given by $\alpha_T(s) = s$.

Let $T \in \mathcal{T}_h^{M,Q}$, given by (2.13), be the input-output operator of an ISS system with $\mathbb{R}^M$-valued input and $\mathbb{R}^Q$-valued output. If (2.12) holds for some function $\gamma$ of class $\mathcal{J}$, then (3.1) holds with $\alpha_T := \gamma$.

Let $\beta \in \mathcal{J}$, $h \in \mathbb{R}_+$, and $\Psi \in \mathcal{D}^{M,Q}$ (recall section 2.4), and assume that

$$\|\Psi(t,y)\| < \mu\,[1 + \beta(\|y\|)] \quad \text{for all } (t,y) \in \mathbb{R}_+ \times \mathbb{R}^M$$

for some $\mu \in \mathbb{R}_+$. Both the point delay given by $(Ty)(t) = \Psi(t, y(t-h))$ and the distributed delay given by $(Ty)(t) = \int_{-h}^{0} \Psi(s, y(t+s))\, ds$ are of class $\mathcal{T}_h^{M,Q}$, and (3.1) holds with $\alpha_T := \beta$.

Last, for the nonlinear operators of section 2.5, assumption (N8) of [11, section 3] asserts that such operators satisfy (3.1) with the $\alpha_T \in \mathcal{J}$ given by $\alpha_T(s) = s$.

**3.1. The servomechanism.** The servomechanism is designed as follows. Let $\alpha_f, \alpha_T \in \mathcal{J}$. Choose $\alpha \in \mathcal{J}_\infty$ with the property

$$\liminf_{s\to\infty} \frac{\alpha(s)}{s + \alpha_f(\alpha_T(s))} \neq 0. \tag{3.2}$$

For example, the choice $\alpha : s \mapsto s + \alpha_f(\alpha_T(s))$ suffices. For $\lambda > 0$, choose $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ to be a continuous function with the properties

$$\text{(i)} \quad \liminf_{s\to\infty} \frac{s\psi_\lambda(s)}{\alpha(s)} \neq 0 \quad \text{and} \quad \text{(ii)} \quad \psi_\lambda^{-1}(0) := \{s \mid \psi_\lambda(s) = 0\} = [0, \lambda]. \tag{3.3}$$

For example, the choice $\psi_\lambda$ given by $\psi_\lambda(s) := d_{\alpha(\lambda)}(\alpha(s))/s$ for $s > 0$, with $\psi_\lambda(0) := 0$, suffices.

Define the continuous function

$$\phi : \mathbb{R}^M \to \mathbb{R}^M, \quad e \mapsto \begin{cases} \alpha(\|e\|)\|e\|^{-1} e, & e \neq 0, \\ 0, & e = 0. \end{cases} \tag{3.4}$$

Writing $\mathcal{S} = \mathcal{S}(\alpha_f, \alpha_T)$, the next objective is to show that the strategy

$$u(t) = -k(t)\phi(e(t)), \quad \dot{k}(t) = \psi_\lambda(\|e(t)\|), \quad e(t) := y(t) - r(t) \tag{3.5}$$

is an $(\mathcal{R}, \mathcal{S})$-universal $\lambda$-servomechanism.

Theorem 3.2. *Let $\alpha_f, \alpha_T \in \mathcal{J}$. Choose $\alpha \in \mathcal{J}_\infty$ so that* (3.2) *holds and define the continuous $\phi : \mathbb{R}^M \to \mathbb{R}^M$ by* (3.4). *Let $\lambda > 0$ and let $\psi_\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous with properties* (3.3). *Then feedback strategy* (3.5) *is an $(\mathcal{R}, \mathcal{S})$-universal $\lambda$-servomechanism in the sense that for all $(p, f, g, T) \in \mathcal{S}(\alpha_f, \alpha_T)$, $r \in \mathcal{R}$, and $(y^0, k^0) \in C([-h, 0]; \mathbb{R}^{M+1})$ the feedback controlled initial-value problem*

$$\left.\begin{aligned} &\dot{y}(t) = f(p(t), (Ty)(t)) + g(p(t), (Ty)(t), -k(t)\phi(y(t) - r(t))) \\ &\dot{k}(t) = \psi_\lambda(\|y(t) - r(t)\|) \\ &(y, k)|_{[-h,0]} = (y^0, k^0) \end{aligned}\right\} \tag{3.6}$$

*has a solution. Every solution can be extended to a maximal solution and every maximal solution $(y, k) : [0, \omega) \to \mathbb{R}^{M+1}$ has the following properties:*

(i) *$(y, k)$ is bounded;*
(ii) *$\omega = \infty$;*
(iii) *$\lim_{t\to\infty} k(t)$ exists and is finite;*
(iv) *$\lim_{t\to\infty} d_\lambda(\|y(t) - r(t)\|) = 0$, with $d_\lambda$ as in* (1.6).

We preface the proof of Theorem 3.2 by a proposition. (Proof of the latter is straightforward and omitted here.)

Proposition 3.3. *Let $\xi \in AC_{loc}(\mathbb{R}_+; \mathbb{R}_+)$, $k \in C(\mathbb{R}_+; \mathbb{R}_+)$, $\beta \in \mathcal{K}$, and $c \geq 0$. If $k$ is monotonically nondecreasing and unbounded, and $\dot{\xi}(t) \leq c - k(t)\beta(\xi(t))$ for almost all $t \in \mathbb{R}_+$, then $\xi(t) \to 0$ as $t \to \infty$.*

*Proof of Theorem* 3.2. Write $N := M + 1$ and $K := Q + M + 1$. Define $F : [-h, \infty) \times \mathbb{R}^K \to \mathbb{R}^N$ by (1.3) and define $\widehat{T} : C([-h, \infty); \mathbb{R}^N) \to L^\infty_{\text{loc}}(\mathbb{R}_+; \mathbb{R}^K)$ by (1.5). Thus, the initial-value problem (3.6) is equivalent to (2.2). By the continuity of $f$, $g$, $\phi$, $\psi_\lambda$ and (essential) boundedness of $p$, it follows that $F$ is a Carathéodory function with the property that, for each $w \in \mathbb{R}^K$, $F(\cdot, w) \in L^\infty_{\text{loc}}([-h, \infty); \mathbb{R}^N)$. By assumption, $T \in \mathcal{T}_h^{M,Q}$ and so $\widehat{T} \in \mathcal{T}_h^{N,K}$. Therefore, by Theorem 2.3, (3.6) has a solution and every solution can be maximally extended. Moreover, every *bounded* maximal solution has interval of existence $[-h, \infty)$.

Let $(y, k) : [-h, \omega) \to \mathbb{R}^N$ be a maximal solution of (3.6). Writing $e := y - r$, we have

(3.7)
$$\left.\begin{aligned} \dot{e}(t) &= f(p(t), (T(e + r))(t)) \\ &\qquad + g(p(t), (T(e + r))(t)), -k(t)\phi(e(t))) - \dot{r}(t) \\ \dot{k}(t) &= \psi_\lambda(\|e(t)\|) \end{aligned}\right\} \quad \text{for a.a. } t \in [0, \omega).$$

By (essential) boundedness of $p$ and property 3 of Definition 3.1 of $g$, there exists a positive definite, symmetric $G$ such that

(3.8)
$$\langle Ge(t), g(p(t), (T(e+r))(t)), -k(t)\phi(e(t)))\rangle \le -k(t)\alpha(\|e(t)\|)\|e(t)\| \quad \text{for a.a. } t \in [0, \omega).$$

Define $c_0 := \sqrt{2\|G^{-1}\|}$ and $c_1 := \sqrt{2/\|G\|}$. For notational convenience, we introduce functions $V, W \in AC_{\text{loc}}([0, \omega); \mathbb{R}_+)$ given by

$$V(t) := \frac{1}{2}\langle Ge(t), e(t)\rangle \quad \text{and} \quad W(t) := \sqrt{V(t)}$$

with

$$c_0^{-1}\|e(t)\| \le W(t) \le c_1^{-1}\|e(t)\| \quad \text{for all } \ t \in [0, \omega). \tag{3.9}$$

By (3.7), (3.8) and properties of $f$, $g$, and $T$, together with (essential) boundedness of $p$, $r$, and $\dot{r}$, there exist constants $c_f, c_T > 0$ such that

(3.10)
$$\begin{aligned} \dot{V}(t) = \langle Ge(t), \dot{e}(t)\rangle \le c_f\|G\| \left[1 + \alpha_f\left(c_T + c_T \max_{s\in[0,t]} \alpha_T(\|e(s) + r(s)\|)\right)\right] \|e(t)\| \\ - k(t)\alpha(\|e(t)\|)\|e(t)\| + \|G\|\|r\|_{1,\infty}\|e(t)\| \quad \text{for a.a. } t \in [0, \omega). \end{aligned}$$

Invoking properties of $\mathcal{J}$ functions, we may conclude that, for some constant $c_2 > 0$,

(3.11)
$$\dot{V}(t) \le c_2 \left[1 + \max_{s\in[0,t]} \alpha_f(\alpha_T(\|e(s)\|))\right] \|e(t)\| - k(t)\alpha(\|e(t)\|)\|e(t)\| \quad \text{a.a. } t \in [0, \omega).$$

By (3.2) and the first of properties (3.3), there exist constants $\gamma > \|e(0)\|$, $c_\gamma$, $\tilde{c}_\gamma > 0$ such that

(3.12)
$$\alpha_f(\alpha_T(s)) \le c_\gamma\alpha(s) \quad \text{for all } \ s \ge \gamma \qquad \text{and} \qquad \psi_\lambda(s) \ge \frac{c_\gamma\alpha(s)}{\tilde{c}_\gamma s} \quad \text{for all } \ s \ge \gamma.$$

With a view to proving Theorem 3.2(i), we first show that $e$ is bounded. Seeking a contradiction, suppose that $e$ (equivalently, $W$) is unbounded. For each $n \in \mathbb{N}$, define

$$\tau_n := \inf\{t \in [0, \omega) \mid c_1W(t) = n + 1 + \gamma\}, \quad \sigma_n := \sup\{t \in [0, \tau_n] \mid c_1W(t) = n + \gamma\}.$$

Recalling that $\gamma > \|e(0)\| \geq c_1 W(0)$, this construction yields a sequence of disjoint intervals $(\sigma_n, \tau_n)$ such that

$$\left.\begin{array}{l} \max_{t\in[0,\tau_n]} c_1 W(t) = c_1 W(\tau_n) = n+1+\gamma \\ c_1 W(\sigma_n) = n+\gamma \\ c_1 W(t) \in (n+\gamma, n+1+\gamma) \quad \text{for all} \quad t \in (\sigma_n, \tau_n) \end{array}\right\} \quad \text{for all} \quad n \in \mathbb{N}.$$

Moreover, for all $n \in \mathbb{N}$,

$$\max_{s\in[0,t]} c_1 W(s) = \max_{s\in[\sigma_n,t]} c_1 W(s) \leq n+1+\gamma < 2n+2\gamma \leq 2c_1 W(t) \quad \text{for all} \quad t \in [\sigma_n, \tau_n],$$

which, together with (3.9) and properties of $\mathcal{J}$ functions, implies the existence of constants $c_3, c_4 > 0$ such that

$$\begin{aligned} \max_{s\in[0,t]} \alpha(\|e(s)\|) &\leq \max_{s\in[0,t]} \alpha(c_0 W(s)) \leq \alpha(2c_0 W(t)) \leq \alpha(2c_0 c^{-1}\|e(t)\|) \\ &\leq c_3\alpha(\|e(t)\|) \leq c_3\alpha(c_0 W(t)) \leq c_4\alpha(c_1 W(t)) \quad \text{for all} \quad t \in \cup_{n\in\mathbb{N}}[\sigma_n, \tau_n]. \end{aligned} \tag{3.13}$$

Noting that, for all $n \in \mathbb{N}$, $\alpha(\|e(t)\|) \geq \alpha(\gamma)$ for all $t \in [\sigma_n, \tau_n]$ and invoking (3.13) together with (3.9), (3.11), and (3.12), we may conclude the existence of constants $c_5, c_6 > 0$ such that

$$\dot{V}(t) \leq [c_5 - k(t)]\alpha(\|e(t)\|)\|e(t)\| \leq c_6\alpha(c_1 W(t))W(t) \quad \text{for all} \quad t \in \cup_{n\in\mathbb{N}}[\sigma_n, \tau_n]. \tag{3.14}$$

Our next task is to show that supposition of the unboundedness of $e$ implies the unboundedness of $k$. Invoking (3.12), (3.14), and (3.9) yields

$$\begin{aligned} 2\ln\left(\frac{n+1+\gamma}{1+\gamma}\right) &= \ln V(\tau_n) - \ln V(\sigma_1) = \sum_{j=1}^{n}[\ln V(\tau_j) - \ln V(\sigma_j)] = \sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j} \frac{\dot{V}(t)}{V(t)}\,dt \\ &\leq c_6 \sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j} \frac{\alpha(c_1 W(t))}{W(t)}\,dt \leq c_6 c_0 \sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j} \frac{\alpha(\|e(t)\|)}{\|e(t)\|}\,dt. \end{aligned} \tag{3.15}$$

By construction of $(\sigma_n, \tau_n)$ we have

$$\gamma < \|e(t)\| \qquad \text{if } t \in (\sigma_j, \tau_j).$$

Hence substituting the second inequality of (3.12) into (3.15) yields

$$2\ln\left(\frac{n+1+\gamma}{1+\gamma}\right) \leq c_6 c_0 \frac{\tilde{c}_\gamma}{c_\gamma} \sum_{j=1}^{n}\int_{\sigma_j}^{\tau_j} \psi_\lambda(\|e(t)\|)\,dt \leq c_6 c_0 \frac{\tilde{c}_\gamma}{c_\gamma} k(\tau_n) \quad \text{for all } n \in \mathbb{N},$$

and so $k(t) \to \infty$ as $t \uparrow \omega$. Let $n^* \in \mathbb{N}$ be such that $k(\sigma_{n^*}) \geq 2c_5$. By the first inequality in (3.14),

$$\dot{V}(t) \leq -c_5\alpha(\|e(t)\|)\|e(t)\| < 0 \quad \text{for a.a. } t \in [\sigma_{n^*}, \tau_{n^*}],$$

which contradicts the fact that $V(\tau_{n^*}) = W^2(\tau_{n^*}) > W^2(\sigma_{n^*}) = V(\sigma_{n^*})$. Therefore, $e$ is bounded.

By the boundedness of $e$ and continuity of $\psi_\lambda$, it follows that $\dot{k}$ is bounded, and so $k$ is bounded on every compact subinterval of $[0, \omega)$. Therefore $\omega = \infty$.

Next, we prove the boundedness of $k$. By the boundedness of $e$ and (3.11), there exists a constant $c_9 > 0$ such that

$$\dot{V}(t) \leq c_9 - k(t)\beta(V(t)) \quad \text{for a.a. } t \in [0, \infty),$$

where $\beta \in \mathcal{K}$ is given by $\beta(s) = \alpha\,(c_1\sqrt{s})\,c_1\sqrt{s}$. Seeking a contradiction, suppose $k$ is unbounded. Then $k(t) \uparrow \infty$ as $t \to \infty$ and so, by Proposition 3.3, $V(t) \to 0$ as $t \to \infty$. Therefore, there exists $\tau \in [0, \infty)$ such that $\|e(t)\| < \lambda$ for all $t \in [\tau, \infty)$ and so $\dot{k}(t) = 0$ for all $t \in [\tau, \infty)$, which again contradicts the supposition of the unboundedness of $k$.

We have now established Theorem 3.2(i) and (ii). Assertion (iii) follows by the boundedness and monotonicity of $k$. By the boundedness of $e$ and $\dot{e}$ (see (3.6)), it follows that $t \mapsto e(t)$ is uniformly continuous. By the continuity of $\psi_\lambda(\|\cdot\|)$, we see that $\psi_\lambda(\|e(\cdot)\|)$ is also uniformly continuous. By the boundedness of $k$, $\int_0^\infty \psi_\lambda(\|e(t)\|)dt < \infty$. By Barbălat's lemma [2], we conclude that $\psi_\lambda(\|e(t)\|) \to 0$ as $t \to \infty$, whence, recalling that $\psi_\lambda^{-1}(0) = [0, \lambda]$, we have assertion (iv). $\square$

**3.2. Discussion.** Theorem 3.2 also holds in the situation wherein the output measurement is subject to an additive disturbance term $\eta$, in which case the control and gain adaptation become

$$u(t) = -k(t)\phi(y(t) - r(t) + \eta(t)), \quad \dot{k}(t) = \psi_\lambda(\|y(t) - r(t) + \eta(t)\|), \quad k|_{[-h,0]} = k^0.$$

If the disturbance $\eta$ is of class $\mathcal{R}$, then, by Theorem 3.2, $\lim_{t\to\infty} d_\lambda(\|y(t) + \eta(t) - r(t)\|) = 0$. Thus, from a strictly analytical viewpoint, in the presence of output disturbances of class $\mathcal{R}$, the disturbance-free analysis is immediately applicable to replacing the reference signal $r$ by the signal $r - \eta =: \hat{r} \in \mathcal{R}$. Even though the reference signal $r$ and disturbance signal $\eta$ are assumed to be of the same class $\mathcal{R}$, in practice these signals might be distinguished by their respective spectra ($\eta$ typically having "high-frequency" content). Moreover, from a practical viewpoint, one might reasonably expect that the disturbance $\eta$ is "small"; if an a priori bound on the magnitude of the disturbance is available, then $\lambda$ should be chosen to be commensurate with such a bound.

We remark on the flexibility of choice in the controller functions $\alpha \in \mathcal{J}_\infty$ and $\psi_\lambda$ (continuous), which are required only to satisfy (3.2) and (3.3). In essence, (3.2) reflects the reasonable requirement that the "strength" of the controller nonlinearity $\alpha$ should be capable of counteracting the potentially destabilizing effects of the (unknown) system nonlinearities; condition (3.3)(i) translates to a requirement that the gain adaptation function $\psi_\lambda$ should be commensurate (in the sense of (3.3)(i)) with the strength of the function $\alpha$. Next, we illustrate by example that the latter condition is also reasonable.

Consider the scalar nonlinear system

$$\dot{y}(t) = a|y(t)|^\epsilon y(t) + u(t), \quad y(0) = y^0 \in \mathbb{R}, \tag{3.16}$$

with $a \in \mathbb{R}$ and $\epsilon > 0$. The choice $\alpha : s \mapsto s^{1+\epsilon}$ implies that (3.2) holds. For $\lambda > 0$, the choice

$$\psi_\lambda : s \mapsto s^\epsilon \min\{d_\lambda(s), 1\} \tag{3.17}$$

implies that (3.3) holds. Therefore, by Theorem 3.2, the control

$$\begin{aligned} u(t) &= -k(t)|y(t) - r(t)|^\epsilon (y(t) - r(t)), \\ \dot{k}(t) &= |y(t) - r(t)|^\epsilon \min\{d_\lambda(|y(t) - r(t)|), 1\}, \quad k(0) = k^0, \end{aligned}$$

ensures that, for every $r \in \mathcal{R}$, the tracking objective is achieved with asymptotic accuracy quantified by $\lambda > 0$.

Now assume that $\epsilon > 0$ is "small." We will investigate the consequences of replacing the above choice of $\psi_\lambda$ (for which (3.3)(i) holds) by the simpler function $s \mapsto \min\{d_\lambda(s), 1\}$ (equivalent to setting $\epsilon = 0$ in (3.17) and for which (3.3)(i) fails to hold). Taking $r = 0$, $a = 1$, $y^0 > 0$, and (for simplicity) $k^0 = 0$, a straightforward calculation reveals that the control objective is not achievable by the control

$$u(t) = -k(t)|y(t)|^\epsilon y(t), \quad \dot{k}(t) = \min\{d_\lambda(s), 1\}, \quad k(0) = 0.$$

In particular, the feedback-controlled initial-value problem can exhibit finite-time "blow-up" of its solution: Specifically, for each $y^0 > (2/\epsilon)^{1/\epsilon}$, the solution of the feedback-controlled system is such that $y(t) \uparrow \infty$ as $t \uparrow T$ with $T \in (0, T^*)$, where $T^* := 1 - \sqrt{1 - (2/\epsilon)(y^0)^{-\epsilon}} < 1$.

Now consider again linear systems, such as the motivating class $\mathcal{L}$ of finite-dimensional, linear, minimum-phase systems described in section 2.2, and let $\mathcal{R}$ be the space of bounded absolutely continuous functions $\mathbb{R} \to \mathbb{R}^M$ with essentially bounded derivative. As is well known (see, for example, [7]), the following output feedback strategy (a variant of the seminal results in [23, 15, 13, 14]) is an $(\mathcal{R}, \mathcal{L})$-universal $\lambda$-servomechanism in the sense that, for each system of class $\mathcal{L}$ and reference signal $r \in \mathcal{R}$, the strategy ensures (i) boundedness of the state, (ii) convergence of the controller gain, and (iii) output tracking with prescribed accuracy $\lambda$ (in the sense that $d_\lambda(\|e(t)\|) \to 0$ as $t \to \infty$, where $e(t) := y(t) - r(t)$ is the tracking error):

$$u(t) = -k(t)e(t), \qquad \dot{k}(t) = d_\lambda^2(\|e(t)\|), \quad k(0) = k^0. \tag{3.18}$$

Generalizations of this strategy to nonlinear finite-dimensional settings are reported in, for example, [7, 17, 6, 18, 24]; applications to biotechnological processes are contained in [8, 9].

Each of $\alpha_f$ and $\alpha_T$ can be taken to be the identity map $id : s \mapsto s$, and so $\mathcal{L} \subset \mathcal{S}(id, id)$. In this context, $\alpha : s \mapsto s$ and $\psi_\lambda : s \mapsto d_\lambda^2(s)$ are allowable choices, in which case we recover (3.18). Note that the latter choice for $\psi_\lambda$, being quadratic in nature, implies that the controller gain $k(\cdot)$ can exhibit rapid growth whenever the tracking error is large. Such behavior may be undesirable from a practical viewpoint.

A very simple but admissible alternative choice of a *bounded* function $\psi_\lambda$ is $s \mapsto \min\{d_\lambda(s), 1\}$. This choice ensures that $k$ exhibits at most linear growth and the overall control strategy (3.5) reduces to

$$u(t) = -k(t)(y(t) - r(t)), \qquad \dot{k}(t) = \min\{d_\lambda(\|y(t) - r(t)\|), 1\}, \qquad k|_{[-h,0]} = k^0. \tag{3.19}$$

Theorem 3.2 ensures that this control achieves the tracking objective, with prespecified asymptotic error bound $\lambda > 0$, not only for the motivating finite-dimensional class $\mathcal{L}$, but also for general interconnections of linear systems of the form in Figure 1, encompassing those cases where $\Sigma_2$ corresponds to linear delay elements (both pointwise and distributed) or to an exponentially stable infinite-dimensional regular linear system (such as a diffusion process), or linear combinations of these.

**Appendix. Proof of Theorem 2.3.** (i) By property 2(b) of Definition 2.1 there exist $\tau > 0$, $\delta > 0$, and $c > 0$ such that, for all $x, \xi \in C([-h, \infty); \mathbb{R}^N)$ with $x|_{[-h,0]} = x^0 = \xi|_{[-h,0]}$ and $x(t), \xi(t) \in \mathbb{B}_\delta(x^0(0))$ for all $t \in [0, \tau]$,

$$\operatorname*{ess\,sup}_{t \in [0,\tau]} \|(\widehat{T}x)(t) - (\widehat{T}\xi)(t)\| \le c \sup_{t \in [0,\tau]} \|x(t) - \xi(t)\|.$$

By property 1 of Definition 2.1 of $\widehat{T}$, there exists $\Delta > 0$ such that for all $x \in C([-h,\infty);\mathbb{R}^N)$,

$$\sup_{t\in[-h,\infty)} \|x(t)\| < \delta^* := \delta + \|x^0\|_\infty \quad \Longrightarrow \quad \|(\widehat{T}x)(t)\| < \Delta \quad \text{for almost all} \quad t \in [0,\tau].$$

Since $F$ is a Carathéodory function, there exists integrable $\gamma : [0,\tau] \to \mathbb{R}$ such that

$$\|F(t,w)\| \le \gamma(t) \quad \text{for all} \quad (t,w) \in [0,\tau] \times \mathbb{B}_\Delta(0). \tag{A.1}$$

Define $\Gamma : [-h,\tau] \to \mathbb{R}_+$ by

$$\Gamma(t) := \begin{cases} 0, & t \in [-h,0), \\ \int_0^t \gamma(s)ds, & t \in [0,\tau], \end{cases}$$

and let $0 < \beta < \tau$ be such that $\Gamma(\beta) < \delta$.

Next, we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of continuous functions $[-h,\beta] \to \mathbb{R}^N$ as follows. Let $n \in \mathbb{N}$. For $i = 1,\dots,n$, define $x_n^i : [-h, i\beta/n] \to \mathbb{R}^N$ by the recursive formula:

$$i = 1: \quad x_n^1(t) := \begin{cases} x^0(t), & t \in [-h,0], \\ x^0(0), & t \in (0,\beta/n], \end{cases}$$

$$i > 1: \quad x_n^i(t) := \begin{cases} x_n^{i-1}(t), & t \in [-h,(i-1)\beta/n], \\ x^0(0) + \int_0^{t-(\beta/n)} F(s,(\widehat{T}x_n^{i-1})(s))ds, & t \in ((i-1)\beta/n, i\beta/n]. \end{cases}$$

Observe that if $i \in \{1,\dots,n-1\}$ and $\|x_n^i(t)\| < \delta^*$ for all $t \in [-h,(i\beta)/n]$, then (a) $\|x_n^{i+1}(t)\| < \delta^*$ for all $t \in [-h,(i\beta)/n]$, and (b) $\|(\widehat{T}x_n^i)(t)\| < \Delta$ for all $t \in [0,(i\beta)/n]$, which, in turn, implies for all $t \in (i\beta/n,(i+1)\beta/n]$

$$\|x_n^{i+1}(t) - x^0(0)\| \le \int_0^{t-\beta/n} \|F(s,(\widehat{T}x_n^i)(s))\|ds \le \int_0^{t-\beta/n} \gamma(s)ds = \Gamma(t-\beta/n) < \delta.$$

Noting that $\|x_n^1(t)\| \le \|x^0\|_\infty < \delta^*$ for all $t \in [-h,\beta/n]$, we may now infer (by induction on $i$) that

$$\|x_n^i(t)\| < \delta^* \quad \text{for all} \quad i \in \{1,\dots,n\}, \quad t \in [-h, i\beta/n].$$

For notational convenience, we write $x_n := x_n^n$. By causality of $\widehat{T}$, the sequence $\{x_n\}_{n\in\mathbb{N}}$ so constructed has the property that, for each $n \in \mathbb{N}$,

$$x_n(t) = \begin{cases} x^0(t), & t \in [-h,0], \\ x^0(0), & t \in (0,\beta/n], \\ x^0(0) + \int_0^{t-(\beta/n)} F(s,(\widehat{T}x_n)(s))ds, & t \in (\beta/n,\beta]. \end{cases} \tag{A.2}$$

Moreover, for all $n \in \mathbb{N}$, $\|x_n(t)\| < \delta^*$ for all $t \in [-h,\beta]$, and so the sequence $\{x_n\}_{n\in\mathbb{N}}$ is uniformly bounded.

Next we prove that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is equicontinuous. Let $\epsilon > 0$. On the closed interval $[0,\beta]$, $\Gamma$ is uniformly continuous, and so there exists some $\bar{\delta} > 0$ such that

$$t,s \in [0,\beta] \quad \text{with} \quad |t-s| < \bar{\delta} \implies |\Gamma(t) - \Gamma(s)| < \epsilon. \tag{A.3}$$

Let $n \in \mathbb{N}$, $s, t \in [0, \beta]$ with $|t - s| < \bar{\delta}$. Without loss of generality, we assume that $s \le t$. We consider three exhaustive cases.

First, if $0 \le s \le t \le \beta/n$, then $\|x_n(t) - x_n(s)\| = 0$. Second, if $0 < s \le \beta/n \le t \le \beta$, then $t - \beta/n < \bar{\delta}$, and so

$$\|x_n(t) - x_n(s)\| = \|x_n(t) - x^0(0)\| \le \Gamma(t - \beta/n) < \epsilon.$$

Third, if $\beta/n \le s \le t \le \beta$, then

$$\|x_n(t) - x_n(s)\| \le |\Gamma(t - \beta/n) - \Gamma(s - \beta/n)| < \epsilon.$$

Recalling that $x_n|_{[-h,0]} = x^0$ for all $n$, we conclude that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is equicontinuous. By the Arzelà–Ascoli theorem and extracting a subsequence if necessary, we may assume that the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges uniformly on $[-h, \beta]$ to a continuous limit which we denote by $x$. Clearly $x|_{[-h,0]} = x^0$.

By property 2(b) of Definition 2.1, $\lim_{n\to\infty}(\widehat{T}x_n)(t) = (\widehat{T}x)(t)$ for almost all $t \in [0, \beta]$ and so, by the continuity of the function $F(t, \cdot)$,

$$\lim_{n\to\infty} F(t, (\widehat{T}x_n)(t)) = F(t, (\widehat{T}x)(t)) \quad \text{for a.a.} \quad t \in [0, \beta].$$

Noting that $\|(\widehat{T}x_n)(s)\| < \Delta$ for all $s \in [0, \beta]$, and also invoking (A.1), we next have $\|F(s, (\widehat{T}x)(s))\| \le \gamma(s)$ for all $s \in [0, \beta]$ and all $n \in \mathbb{N}$. Therefore,

$$\text{(A.4)} \qquad \lim_{n\to\infty} \int_{t-\beta/n}^{t} F(s, (\widehat{T}x_n)(s))ds = 0 \quad \text{for all } t \in (0, \beta]$$

and, by the Lebesgue dominated convergence theorem,

$$\text{(A.5)} \qquad \lim_{n\to\infty} \int_0^t F(s, (\widehat{T}x_n)(s))ds = \int_0^t F(s, (\widehat{T}x)(s))ds \quad \text{for all } t \in [0, \beta].$$

By (A.2), (A.4), and (A.5), it follows that

$$x(t) = \begin{cases} x^0(t), & t \in [-h, 0], \\ x^0(0) + \int_0^t F(s, (\widehat{T}x)(s))ds, & t \in (0, \beta], \end{cases}$$

and so $x$ is a solution of the initial-value problem.

(ii) Let $x : [-h, \omega) \to \mathbb{R}^N$ be a solution of (2.2). Define

$$\mathcal{A} := \left\{(\rho, \xi) |\ \omega \le \rho \le \infty,\ \xi : [-h, \rho) \to \mathbb{R}^N \text{ is a solution of (2.2) with } \xi|_{[-h,\omega)} = x\right\}.$$

On this nonempty set define a partial order $\preceq$ by

$$(\rho_1, \xi_1) \preceq (\rho_2, \xi_2) \quad \Longleftrightarrow \quad \rho_1 \le \rho_2 \text{ and } \xi_1(t) = \xi_2(t) \text{ for all } t \in [-h, \rho_1).$$

Let $\mathcal{O}$ be a totally ordered subset of $\mathcal{A}$. Let $P := \sup\{\rho | (\rho, \xi) \in \mathcal{O}\}$ and let $\Xi : [-h, P) \to \mathbb{R}^M$ be defined by the property that, for every $(\rho, \xi) \in \mathcal{O}$, $\Xi|_{[0,\rho)} = \xi$. Then $(P, \Xi)$ is in $\mathcal{A}$ and is an upper bound for $\mathcal{O}$. By Zorn's lemma, it follows that $\mathcal{A}$ contains at least one maximal element.

(iii) Assume that $x \in C([-h, \omega); \mathbb{R}^N)$ is a bounded maximal solution of (2.2) and that $F \in L^\infty_{\text{loc}}([-h, \infty) \times \mathbb{R}^K; \mathbb{R}^N)$. Seeking a contradiction, suppose $\omega < \infty$. By the boundedness of $x$, together with property 1 of Definition 2.1 of $\widehat{T}$, it follows that

$\dot{x}(\cdot)$ is essentially bounded. Therefore, $x$ is uniformly continuous and so extends to a continuous function $x : [-h, \omega] \to \mathbb{R}^N$. Now consider the initial-value problem

$$\dot{v}(t) = S_\omega F\big(t, (\widehat{T} S_{-\omega} v)(t)\big), \qquad v|_{[-(h+\omega),0]} = S_\omega x. \tag{A.6}$$

By (2.1) and the above existence result, the initial-value problem (A.6) has a solution $\tilde{v} : [-(h+\omega), \tau) \to \mathbb{R}^N$, $\tau > 0$. It follows that $\tilde{x} = S_{-\omega}\tilde{v} : [-h, \omega+\tau) \to \mathbb{R}^N$ is a solution of the original initial-value problem (2.2) and is a proper right extension of the solution $x$. This contradicts the maximality of $x$. Therefore, $\omega = \infty$. ∎

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