ON THE λ-EQUATIONS FOR MATCHING CONTROL LAWS [*]

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Abstract. We discuss matching control laws for underactuated systems. We previously showed that this class of matching control laws is completely charactarized by a linear system of first order partial differential equations for one set of variables (λ) followed by a linear system of first order PDEs for the second set of variables ($\widehat{g}$, $\widehat{V}$). Here we derive a new first order system of partial differential equations that encodes all compatibility conditions for the λ-equations. We give four examples illustrating different features of matching control laws. The last example is a system with two unactuated degrees of freedom that admits only basic solutions to the matching equations. There are systems with many matching control laws where only basic solutions are potentially useful. We introduce a rank condition indicating when this is likely to be the case.

Key words. nonlinear control, matching control laws, λ-equations, stabilization

AMS subject classifications. 93C10, 93D15

1. Introduction . Effective procedures for designing control laws are very important in nonlinear control theory. Explicit analytic formulae for control laws play a role similar to explicit solutions to differential equations. Such formulae exist in only a few special cases, but those that exist serve as simple models to develop and test more general techniques.

In this paper we discuss a class of full state feedback control laws for underactuated systems. In [5] we showed that this class of matching control laws is completely charactarized by a linear system of first order partial differential equations for one set of variables (λ) followed by a linear system of first order PDEs for the second set of variables ($\widehat{g}$, $\widehat{V}$). These equations always have a simple family of solutions which we call basic solutions. The system of equations for the first set of variables (λ-equations) is overdetermined. Here we derive a new first order system of partial differential equations that encodes all compatibility conditions for the λ-equations (we call these the ν-equations). If only one degree of freedom is unactuated, the solutions to all these systems of PDEs can be completely analyzed. It is often possible to get explicit formulae for the solutions to these equations. We also provide an example of a system with two unactuated degrees of freedom that has only basic solutions. There are systems with many matching control laws where only basic solutions are potentially useful. We write down a rank condition indicating when this is likely to be the case.

During the last few years several researchers have investigated control laws in which the closed loop system assumes a certain structure. Numerous papers have been written on this subject, see [1] - [13] and the references therein. The control laws that form the subject of this present paper are described by equations (2.4) and (2.6). These equations were independently derived in [10] and [5]. The λ-equations were first introduced in paper [5]. Even though the initial matching equations of [10] and [5] form a highly nonlinear system of PDEs, introduction of the λ variables triangulates the system. The system is triangulated in the sense that all solutions are

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obtained by first solving first order linear equations for λ and then solving first order linear equations for the remaining variables.

This paper is organized as follows. Section 2 reviews matching control laws and the λ-equations, and introduces the ν-equations. Section 3 specializes to systems with one unactuated degree of freedom. Sections 4, 5 and 6 contain examples illustrating three different features of matching control laws. The rank condition appears at the end of Section 5. Later we apply it in Section 6. In Section 7 we describe the final example of a system with two unactuated degrees of freedom. We show that this system has only basic matching control laws.

2. Matching equations. We use the following notation.
- n is the number of the degrees of freedom of the mechanical system
- $x = (x^1, \ldots, x^n)$ are configuration variables denoting the position of the system, and $\dot{x} = (\dot{x}^1, \ldots, \dot{x}^n)$ are the corresponding velocities
- $g_{ij}(x)$ is the mass matrix
- $V(x)$ is the potential energy
- $C_i(x, \dot{x})$ are the dissipation terms
- $u_i(x, \dot{x})$ are the control inputs

Let $m \leq n$ be the number of unactuated degrees of freedom. We will assume that degrees of freedom numbered 1 through m are unactuated and use indices a, b, $\ldots$ to indicate unactuated degrees of freedom. The indices i, j, $\ldots$ will run from 1 to n. We adopt the convention of summation over the repeated indices .

Given this, the equations of motion of the system are

$$g_{rj}\ddot{x}^j + [j\,k,\,r]\,\dot{x}^j\,\dot{x}^k + C_r + \frac{\partial V}{\partial x^r} = u_r\,, \quad r = 1, \ldots, n, \qquad (2.1)$$

where $[i\,j,\,k]$ is the Christoffel symbol of the first kind,

$$[i\,j,\,k] = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right) \qquad (2.2)$$

Our assumtion that the first m degrees of freedom are not actuated means that

$$u_1 = \cdots = u_m = 0. \qquad (2.3)$$

We are looking for control laws u_i such that the closed loop system can be written in the form

$$\widehat{g}_{rj}\ddot{x}^j + \widehat{[j\,k,\,r]}\,\dot{x}^j\,\dot{x}^k + \widehat{C}_r + \frac{\partial\widehat{V}}{\partial x^r} = 0\,, \quad r = 1, \ldots, n,$$

where $\widehat{[i\,j,\,k]}$ is defined as in (2.2) with $\widehat{g}$ in place of g. Such a control law will be given by

$$u_\ell = \left([j\,k,\,\ell] - g_{\ell i}\,\widehat{g}^{ir}\,\widehat{[j\,k,\,r]}\right)\dot{x}^j\,\dot{x}^k + \left(C_\ell - g_{\ell i}\,\widehat{g}^{ij}\,\widehat{C}_j\right)$$

$$+ \left(\frac{\partial V}{\partial x^\ell} - g_{\ell i}\,\widehat{g}^{ij}\,\frac{\partial\widehat{V}}{\partial x^j}\right), \qquad \ell = 1, \ldots, n, \qquad (2.4)$$

Condition (2.3) translates into

$$\left([j\,k,\,a] - g_{ai}\,\widehat{g}^{ir}\,\widehat{[j\,k,\,r]}\right)\dot{x}^j\,\dot{x}^k + \left(C_a - g_{ai}\,\widehat{g}^{ij}\,\widehat{C}_j\right)$$

$$+ \left(\frac{\partial V}{\partial x^a} - g_{ai}\,\widehat{g}^{ij}\,\frac{\partial\widehat{V}}{\partial x^j}\right) = 0\,, \qquad a = 1, \ldots, m, \qquad (2.5)$$

In order to satisfy these equations it is sufficient to have

$$
\begin{aligned}
g_{ai}\,\widehat{g}^{ir}\,\widehat{[j\,k,\,r]} &= [j\,k,\,a] \\
g_{ai}\,\widehat{g}^{ir}\,\widehat{C}_r &= C_a \\
g_{ai}\,\widehat{g}^{ir}\,\frac{\partial\widehat{V}}{\partial x^r} &= \frac{\partial V}{\partial x^a}
\end{aligned}
\tag{2.6}
$$

These are the *matching equations*, see [5], [10]. Following [5], introduce variables λ_a^j relating the unknown mass matrix $\widehat{g}$ to the original mass matrix g,

$$
\lambda_a^r \;=\; g_{ai}\,\widehat{g}^{ir}\,.
\tag{2.7}
$$

Using λ_a^j, the matching equations take the form

$$
\begin{aligned}
\lambda_a^r\,\widehat{[j\,k,\,r]} &= [j\,k,\,a] \\
\lambda_a^j\,\widehat{C}_j &= C_a \\
\lambda_a^j\,\frac{\partial\widehat{V}}{\partial x^j} &= \frac{\partial V}{\partial x^a}
\end{aligned}
\tag{2.8}
$$

THEOREM 2.1. *The following equations are equivalent to the matching equations .*

λ-equations:

$$
\frac{\partial}{\partial x^k}\,(\,g_{ai}\,\lambda_b^i\,) \;-\; [k\,a,\,i]\,\lambda_b^i \;-\; [k\,b,\,i]\,\lambda_a^i \;=\; 0, \qquad
\begin{aligned}
k &= 1,\ldots,n \\
a,b &= 1,\ldots,m
\end{aligned}
\tag{2.9}
$$

$\widehat{g}$-equations:

$$
\lambda_a^\ell\,\frac{\partial\widehat{g}_{ij}}{\partial x^\ell} \;+\; \frac{\partial\lambda_a^\ell}{\partial x^i}\cdot\widehat{g}_{\ell j} \;+\; \frac{\partial\lambda_a^\ell}{\partial x^j}\cdot\widehat{g}_{\ell i} \;=\; \frac{\partial g_{ij}}{\partial x^a}, \qquad
\begin{aligned}
a &= 1,\ldots,m \\
i,j &= 1,\ldots,n
\end{aligned}
\tag{2.10}
$$

$\widehat{V}$-equations:

$$
\lambda_a^j\,\frac{\partial\widehat{V}}{\partial x^j} \;=\; \frac{\partial V}{\partial x^a}
\tag{2.11}
$$

$\widehat{C}$-equations:

$$
\lambda_a^j\,\widehat{C}_j \;=\; C_a
\tag{2.12}
$$

For the proof see [5], [4].

REMARK 2.2. *These equations always have a set of solutions of the form*

$$
\lambda_a^k \;=\; \varkappa\,\delta_a^k, \qquad
\widehat{g} \;=\; \frac{1}{\varkappa}\,g + g^o, \qquad
\widehat{V} \;=\; \frac{1}{\varkappa}\,V + V^o, \qquad
\widehat{C}_j \;=\; \frac{1}{\varkappa}\,C_j
$$

with $\varkappa \neq 0$ any constant, V^o arbitrary function of the variables x^ℓ, $\ell = m+1,\ldots,n$, and g^o any symmetric matrix valued function of the variables x^ℓ such that $g_{ia}^o = 0$. We will call these solutions basic.

The λ-equations are a system of $\frac{1}{2}\,m(m+1)\cdot n$ equations for $n\cdot m$ unknowns. It is not surprising that there are extra compatibility conditions. By viewing system (2.9) in the correct way, we are able to write down the compatibility conditions. Denote

$$\nu_{ab} \,=\, g_{ai}\,\lambda_b^i\,. \tag{2.13}$$

Because the matrix g_{ij} is assumed to be non-degenerate, the matrix comprised of its m first rows has rank m. This implies that m^2 out of $m\cdot n$ λ's can be expressed as linear combinations of ν's, i.e.,

$$\lambda_b^\beta \,=\, h^{\beta a}\,\nu_{ab}\,.$$

Substituting this in the λ-equations, we obtain

$$\partial_k\,\nu_{ab} \,-\, [a\,k,\,\beta]\,h^{\beta d}\,\nu_{db} \,-\, [b\,k,\,\beta]\,h^{\beta d}\,\nu_{da} \,=\, [a\,k,\,\rho]\,\lambda_b^\rho \,+\, [b\,k,\,\rho]\,\lambda_a^\rho\,, \tag{2.14}$$

where index ρ varies over the remaining $(n-m)$ indices. We will view system (2.14) of $\frac{1}{2}\,m(m+1)\cdot n$ equations as a linear algebraic system for the $m\,(n-m)$ variables λ_a^ρ,

$$\left[A_{(k,a,b)}\right]_\rho^c\,\lambda_c^\rho \,=\, F_{(k,a,b)}\,. \tag{2.15}$$

We know that this system has at least one solution by Remark 2.2. Thus, the rank of the matrix A is at most $m\cdot n$. In order for system (2.15) to have a solution, the vector $F_{(k,a,b)}$ must be perpendicular to the kernel of the transposed matrix, $A^\star$,

$$F\,\perp\,\ker A^\star\,. \tag{2.16}$$

Let the kernel of the matrix $A^\star$ be generated by the vectors ξ_r. The orthogonality condition (2.16) then takes the form of the following system of linear first order partial differential equations for ν_{ab}:

$$\xi_r^{(k,a,b)}(x)\,\left[\,\partial_k\,\nu_{ab} \,-\, [a\,k,\,\beta]\,h^{\beta d}\,\nu_{db} \,-\, [b\,k,\,\beta]\,h^{\beta d}\,\nu_{da}\,\right] \,=\, 0\,. \tag{2.17}$$

THEOREM 2.3. *The general solution to the λ-equations is given by any set of λ_a^ρ solving the algebraic system (2.15), and $\lambda_b^\beta \,=\, h^{\beta a}\,\nu_{ab}$, where ν_{ab} is any solution to equations (2.17).*

In general, if $m > 1$, system (2.16) may be quite complicated and we do not have a satisfactory description of its solutions.

3. Systems with one unactuated degree of freedom. If only one degree of freedom is unactuated, we do have a reasonable description of all solutions to system (2.17). Assume, for simplicity, that $g_{11}(x) > 0$. Then, after rescaling x^1 if necessary, we will have $g_{11}(x) \,=\, 1$. More precisely, from the very beginning we could use, instead of $(x^1, x^2, \ldots, x^n)$, the coordinates $(z^1, z^2, \ldots, z^n)$ which are related to x as follows:

$$\frac{\partial z^1}{\partial x^1} \,=\, \sqrt{g_{11}(x)}, \qquad z^2 \,=\, x^2,\quad \ldots,\, z^n \,=\, x^n$$

In z coordinates the mass matrix is $\tilde{g}_{ij}(z) \,=\, g_{k\ell}(x)\,\frac{\partial x^k}{\partial z^i}\,\frac{\partial x^\ell}{\partial z^j}$ and, hence, $\tilde{g}_{11}(z) = 1$. On the other hand, the structure of the equations of motion (2.1) does not change because of their tensorial form, and the condition $u_1 = 0$ remains the same, again,

because $\tilde{u}_1 = u_k \frac{\partial x^k}{\partial z^1} = u_1 \sqrt{g_{11}(x)}$. Thus, we assume that the coordinates are chosen appropriately, and $g_{11}(x) = 1$.

In the case of one unactuated degree of freedom one is solving for λ_1^i. The λ-equation reads

$$\frac{\partial \nu}{\partial x^k} = 2 \,[k\, 1,\, i]\, \lambda_1^i\,, \tag{3.1}$$

where $\nu = g_{1i}\,\lambda_1^i$. Notice that $[k\, 1,\, 1] = 0$. View the equations (3.1) as a system of linear algebraic equations for the variables λ_1^ρ, $\rho = 2,\ldots,n$. In order for this system of n equations in $(n-1)$ unknowns to have a solution, the vector

$$v = \begin{pmatrix} \partial_1\, \nu \\ \ldots \\ \partial_n \nu \end{pmatrix}$$

must be perpendicular to the kernel of the matrix

$$A^\star = \begin{pmatrix} [1\,1,\, 2] & \ldots & [n\, 1,\, 2] \\ \ldots & \ldots & \ldots \\ [1\,1,\, n] & \ldots & [n\, 1,\, n] \end{pmatrix} \quad .$$

Let the kernel of $A^\star$ be generated by the vectors $\xi_r = (\xi_r^1,\ldots,\xi_r^n)$. The orthogonality condition for v translates into the system of equations

$$X_r(\nu) \equiv \xi_r^1(x)\,\frac{\partial \nu}{\partial x^1} + \ldots + \xi_r^n(x)\,\frac{\partial \nu}{\partial x^n} = 0. \tag{3.2}$$

The standard procedure to solve such a system of equations is to complete the system into an involutive system by adding equations $[X_r,\, X_s](\nu) = 0$, $[[X_r,\, X_s],\, X_t](\nu) = 0$, $\ldots$, where $[\eta^i\,\partial_i,\, \zeta^j\,\partial_j] = (\eta^i \partial_i(\zeta^k) - \zeta^i\,\partial_i(\eta^k))\,\partial_k$ is the commutator of vector-fields. Recall that a system of equations

$$Y_1(\nu) = 0,\ldots,\, Y_K(\nu) = 0$$

is involutive if $[Y_p,\, Y_q] = f_{pq}^r(x)\,Y_r$.

Thus we have proved the following result.

THEOREM 3.1.   *With one unactuated degree of freedom there is a coordinate system such that the ν-equations, (2.17), become a homogeneous linear system of equations for one unknown function. This system, (3.2), can be completed into an involutive system.*

REMARK 3.2. *Note that we want to preserve the relationship (2.7), i.e.,*

$$\Xi_i \equiv g_{1i} - \lambda_1^j\,\widehat{g}_{ji} = 0.$$

Using only the $\widehat{g}$-equations, one computes

$$
\begin{aligned}
0 &= \lambda_1^j\,\{\,\lambda_1^\ell\,\frac{\partial \widehat{g}_{ij}}{\partial x^\ell} + \frac{\partial \lambda_a^\ell}{\partial x^i}\cdot\widehat{g}_{\ell j} + \frac{\partial \lambda_a^\ell}{\partial x^j}\cdot\widehat{g}_{\ell i} - \frac{\partial g_{ij}}{\partial x^1}\,\} \\
&= \lambda_1^\ell\,\frac{\partial\,\Xi_i}{\partial x^\ell} + \frac{\partial \lambda_1^\ell}{\partial x^i}\,\Xi_\ell + \frac{\partial \lambda_1^\ell}{\partial x^i}\,g_{\ell 1} + \lambda_1^\ell\,\frac{\partial g_{i1}}{\partial x^\ell} - \lambda_1^j\,\frac{\partial g_{ij}}{\partial x^1} \\
&= \lambda_1^\ell\,\frac{\partial\,\Xi_i}{\partial x^\ell} + \frac{\partial \lambda_1^\ell}{\partial x^i}\,\Xi_\ell + \frac{\partial}{\partial x^i}\,(\lambda_1^\ell\,g_{\ell 1}) - \lambda_1^\ell\,\left(\frac{\partial g_{\ell 1}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^1} - \frac{\partial g_{i1}}{\partial x^\ell}\right)
\end{aligned}
$$

Now invoke the λ-equation to obtain

$$\lambda_1^\ell \, \frac{\partial \, \Xi_i}{\partial x^\ell} \; + \; \frac{\partial \lambda_1^\ell}{\partial x^i} \, \Xi_\ell \; = 0.$$

We see that equality (2.7) holds locally provided it holds on a hypersurface transverse to λ_1^i. See also [5, Proposition 1.4]

Given any non-zero solution of the λ-equations, there is a local coordinate system $y^1, \ldots, y^n$ such that

$$\lambda_1^i(x) \, \frac{\partial y^j}{\partial x^i} \; = \; \delta_1^j. \tag{3.3}$$

Let G and $\widehat{G}$ represent g and $\widehat{g}$ in the y-coordinates. The equation (2.7) then reads

$$G_{ij}(y) \, \frac{\partial y^i}{\partial x^1} \, \frac{\partial y^j}{\partial x^r} \; = \; \widehat{G}_{1k} \, \frac{\partial y^k}{\partial x^r}. \tag{3.4}$$

The $\widehat{g}$-equations read

$$\frac{\partial \widehat{G}_{ij}}{\partial y^1} \; = \; \frac{\partial g_{k\ell}}{\partial x^1} \, \frac{\partial x^k}{\partial y^i} \, \frac{\partial x^\ell}{\partial y^j} \, , \tag{3.5}$$

and the $\widehat{V}$-equations become

$$\frac{\partial \widehat{V}}{\partial y^1} \; = \; \frac{\partial V}{\partial x^1} \, .$$

It is easy to see that the following result holds.

THEOREM 3.3. *Given any non-zero solution to the λ-equation, there is a unique solution to the $\widehat{g}$- and $\widehat{V}$-equations with initial data prescribed at $y^1 = 0$.*

REMARK 3.4. *Note that equation(3.4) gives directly*

$$\widehat{G}_{k1} \; = \; G_{ki} \, \frac{\partial y^i}{\partial x^1} \, ,$$

and so one only needs to solve (3.5) for $n(n-1)/2$ quantities $\widehat{G}_{ij}$, $\quad 2 \le i, j \le n$.

4. Example 1: Inverted pendulum in a vertical plane. As the first example we consider the inverted pendulum restricted to a vertical plane with horizontal and vertical actuation of the base, see Figure 1.
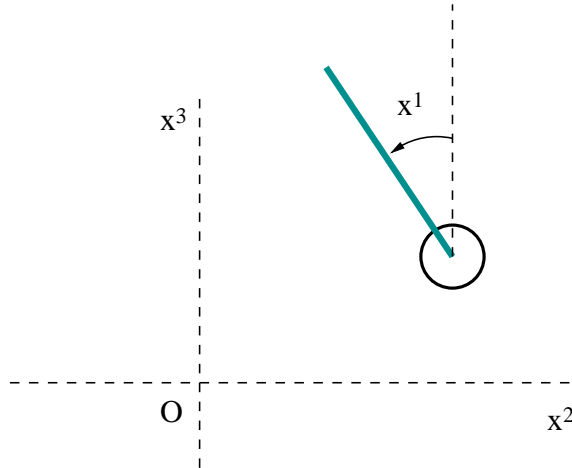

Figure 1

After rescaling units, the mass matrix and potential energy are given by

$$g = \begin{pmatrix} 1 & -a\,\cos(x^1) & -a\,\sin(x^1) \\ -a\,\cos(x^1) & 1 & 0 \\ -a\,\sin(x^1) & 0 & 1 \end{pmatrix}$$

$$V = b\,x^3 + \cos(x^1)$$

Since only x^1 is unactuated, we will simplify notation and use λ^i to denote λ_1^i. The λ-equations (2.9) are

$$\partial_1\,\nu = 2a\,\sin(x^1)\,\lambda_1^2 - 2a\,\cos(x^1)\,\lambda_1^3, \quad \partial_2\,\nu = 0, \quad \partial_3\,\nu = 0$$

with $\nu = \lambda^1 - a\,\cos(x^1)\,\lambda^2 - a\,\sin(x^1)\,\lambda^3$. It is not difficult to see that the general solution to these equations is

$$\lambda^1 = \nu(x^1) + \frac{1}{2}\,\cot(x^1)\,\partial_1\nu(x^1) + a\frac{\lambda^3(x^1,x^2,x^3)}{\sin(x^1)}$$

$$\lambda^2 = \frac{1}{2a\,\sin(x^1)}\,\partial_1\nu(x^1) + \cot(x^1)\,\lambda^3(x^1,x^2,x^3)$$

$$\lambda^3 = \lambda^3(x^1,x^2,x^3)\,,$$

where $\nu(x^1)$, $\lambda^3(x^1,x^2,x^3)$ are arbitrary. In order to obtain a managable explicit solution to the matching equations, we will choose

$$\nu(x^1) = a\,\mu_0\,\sin^2(x^1) + \sigma_0 - a\,\mu_0, \qquad \lambda^3 = 0$$

with free parameters σ_0 and μ_0. Then

$$\lambda^1 = \sigma_0\,, \qquad \lambda^2 = \mu_0\,\cos(x^1).$$

The coordinates

$$y^1 = \frac{1}{\sigma_0}\,x^1\,, \quad y^2 = x^2 - \mu_0\,\sin(x^1)\,, \quad y^3 = x^3$$

satisfy (3.3). Following Remark 3.4, we need to solve the $\widehat{g}$-equations only for $\widehat{g}_{22}$, $\widehat{g}_{23}$, and $\widehat{g}_{33}$. These equations are

$$\frac{\partial}{\partial y^1}\,\widehat{g}_{22} = \frac{\partial}{\partial y^1}\,\widehat{g}_{23} = \frac{\partial}{\partial y^1}\,\widehat{g}_{33} = 0.$$

Clearly,

$$\widehat{g}_{22} = \widehat{g}_{22}(y^2,y^3) = \widehat{g}_{22}(x^2 - \mu_0\,\sin(x^1),\ x^3)$$
$$\widehat{g}_{23} = \widehat{g}_{23}(x^2 - \mu_0\,\sin(x^1),\ x^3)$$
$$\widehat{g}_{33} = \widehat{g}_{33}(x^2 - \mu_0\,\sin(x^1),\ x^3)$$

From $g = \widehat{g}\,\lambda$, we obtain the rest of $\widehat{g}_{ij}$ as:

$$\widehat{g}_{11} = \frac{1}{\sigma_0} + \frac{a\mu_0}{\sigma_0^2}\,\cos^2(x^1) + \frac{\mu_0^2}{\sigma_0^2}\,\cos^2(x^1)\,\widehat{g}_{22}$$

$$\widehat{g}_{12} = -\frac{a}{\sigma_0} - \frac{\mu_0}{\sigma_0}\,\cos(x^1)\,\widehat{g}_{22}$$

$$\widehat{g}_{13} = -\frac{a}{\sigma_0}\,\sin(x^1) - \frac{\mu_0}{\sigma_0}\,\cos(x^1)\,\widehat{g}_{23}$$

The $\widehat{V}$-equation yields

$$\widehat{V} \;=\; \frac{1}{\sigma_0}\;\cos(x^1) \;+\; w(y^2, y^3)\,.$$

The $\widehat{C}$-equation reads $\lambda^j\,\widehat{C}_j \;=\; 0$. One solution is

$$\widehat{C} \;=\; -\,\sigma_0\,R(x)\,
\begin{pmatrix}
\frac{\mu_0^2}{\sigma_0^2}\cos^2(x^1) & -\frac{\mu_0}{\sigma_0}\cos(x^1) & -\frac{\mu_0}{\sigma_0}\cos(x^1) \\
-\frac{\mu_0}{\sigma_0}\cos(x^1) & 1 & 1 \\
-\frac{\mu_0}{\sigma_0}\cos(x^1) & 1 & 1
\end{pmatrix}\cdot
\begin{pmatrix}
\dot{x}^1 \\ \dot{x}^2 \\ \dot{x}^3
\end{pmatrix}$$

The resulting control law can be obtained explicitly from equation (2.4). The expression is too long to be included in this paper.

PROPOSITION 4.1. *If the functions $\widehat{g}_{22}(y^2, y^3)$, $\widehat{g}_{23}(y^2, y^3)$, $\widehat{g}_{33}(y^2, y^3)$, $w(y^2, y^3)$, and $R(x)$, and the parameters μ_0 and σ_0 are chosen so that*

$$\widehat{g}_{22}(0) > 0\,, \quad \widehat{g}_{23}(0) = 0\,, \quad \widehat{g}_{33}(0) = 1\,,$$

$$\partial_{y^2}^2 w(0) > 0, \quad \partial_{y^2}\partial_{y^3} w(0) = 0, \quad \partial_{y^3}^2 w(0) > 0, \quad R(0) > 0,$$

$$\sigma_0 < 0, \quad \widehat{g}_{22}(0)\,\mu_0^2 + a\,\mu_0 + \sigma_0 > 0\,, \quad \widehat{g}_{22}(0)\,(a\,\mu_0 - \sigma_0) + a^2 < 0\,,$$

then $x = \dot{x} = 0$ is a locally asymptoitcally stable equilibrium of the closed loop system.

5. Example 2: Inverted pendulum cart on a seesaw . In the previous example the kernel of the matrix $A^\star$, (2.15), was two-dimensional. Generically, for systems with one unactuated degree of freedom the dimension of the kernel will be 1. The following example illustrates this situation. The inverted pendulum cart on a seesaw is shown in Figure 2. There are several interesting ways to actuate this system. We will consider the case with actuated cart and pendulum, and unactuated seesaw.
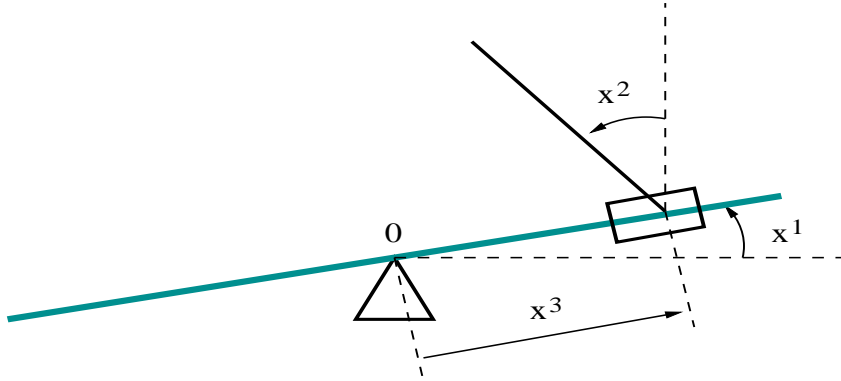

Figure 2

The rescaled mass matrix and potential energy of the system are given by

$$g \;=\;
\begin{pmatrix}
b + (x^3)^2 & a\,x^3\,\sin(x^1 - x^2) & 0 \\
a\,x^3\,\sin(x^1 - x^2) & 1 & -a\,\cos(x^1 - x^2) \\
0 & -a\,\cos(x^1 - x^2) & 1
\end{pmatrix}$$

and

$$V \;=\; x^3\,\sin(x^2) \;+\; a\,\cos(x^1)\,.$$

The theory in section 3 was presented with special coordinates so that $g_{11} = 1$. However, in practice this is not necessary.

As before, we write λ^i for λ^i_1 and ν for $g_{1j} \lambda^j$. The λ-equations are

$$\partial_1 \nu = 2a\, x^3 \cos(x^1 - x^2)\, \lambda^2 - 2\, x^3\, \lambda^3$$
$$\partial_2 \nu = 0$$
$$\partial_3 \nu = 2a \sin(x^1 - x^2)\, \lambda^2 + 2\, x^3\, \lambda^1$$

Hence, $\nu = \nu(x^1, x^3)$. Plug in $\lambda^1 = (\nu - g_{12}\, \lambda^2 - g_{13}\, \lambda^3)/g_{11}$ and solve for λ^2 and λ^3:

$$\lambda^1 = \frac{1}{2b} \left(2\nu - x^3 \partial_3\, \nu \right)$$

$$\lambda^2 = \frac{1}{2ab \sin(x^1 - x^2)} \left(- 2\, x^3\, \nu + (b + (x^3)^2)\, \partial_3\, \nu \right)$$

$$\lambda^3 = \frac{1}{2b\, x^3 \sin(x^1 - x^2)} \left(- 2(x^3)^2 \cos(x^1 - x^2)\, \nu \right.$$

$$\left. + x^3\, (b + (x^3)^2) \cos(x^1 - x^2)\, \partial_3 \nu - b \sin(x^1 - x^2)\, \partial_2 \nu \right)$$

Notice, that λ^2 and λ^3 blow up as x approaches 0 unless $\nu = \kappa\, (b + (x^3)^2)$. Since $g = \widehat{g}\, \lambda$, one must have $\det \widehat{g} \to 0$ as $x \to 0$, i.e., $\widehat{g}$ degenerates at $x = 0$. This means that $\widehat{H}(x, \dot{x}) = \frac{1}{2} \widehat{g}_{ij}\, \dot{x}^i\, \dot{x}^j + \widehat{V}$ cannot serve as a Lyapunov function unless $\nu = \kappa\, (b + (x^3)^2)$. This ν corresponds exactly to the basic solutions of the matching equations from Remark 2.2. This illustrates the following general principle.

REMARK 5.1. *If $(x_0, 0)$ is the desired equilibrium of a system and*

$$rank\, A^\star(x_0) < \limsup_{x \to x_0} rank\, A^\star(x), \tag{5.1}$$

then only basic solutions of the matching equations should be tested to produce a stabilizing control law from (2.4).

6. Example 3: inverted pendulum cart on a roller coaster. Consider a cart with inverted pendulum on a roller coaster. Special cases of this mechanical system include the inverted pendulum on a rotor arm, the inverted pendulum on a verticle disk, and the inverted pendulum cart on an incline. By assuming that the size of the base of the cart is relatively small we may neglect the inertia of the base of the cart. It is therefore sufficient to model the cart with one point mass for the base, and one point mass a fixed distance away for the pendulum. The pendulum joint will be unactuated.

The configuration of the system may be described by a position and an angle. Assume that the shape of the roller coaster is given as a curve $x(s)$ in $\mathbb{R}^3$ parametrized by arc length, s, from a fixed point. Assume that the pendulum is always in the plane containing the tangent vector, $\tau(s)$, and the vertical direction, e_3. Let ϕ be the angle between the pendulum and e_3. By rescaling mass, length, and time, we will write

$$g = \begin{pmatrix} 1 & b \sin(\alpha - \phi) \\ b \sin(\alpha - \phi) & \left(1 + k(s)^2 \sin^2 \phi \left(\frac{\sin^2 \alpha - n_3^2}{\sin^4 \alpha} \right) \right) \end{pmatrix}$$
$$V = a\, x^3 + \cos \phi,$$

where a and b are positive parameters, $0 < b < 1$, and x^3 is the vertical component of $x(s)$. The (unit) tangent vector to the curve is $\tau(s) = \frac{x'(s)}{|x'(s)|}$, where $'$ stands for the derivative with respect to s. The curvature of the curve is $k(s) = |\tau'(s)|$. Denote by $n(s)$ the principal normal to the curve. Recall that $\tau'(s) = k(s) \, n(s)$. In the formula above n_3 is the vertical component of the principal normal, and α is the angle between τ and the vertical direction. Index 1 corresponds to ϕ, index 2 corresponds to s. The unactuated degree of freedom corresponds to the ϕ variable. The λ-equations (3.1) then read as follows:

$$\partial_1 \nu = -2\, b \, \cos(\alpha - \phi) \, \lambda_1^2$$
$$\partial_2 \nu = k(s)^2 \, \sin(2\,\phi) \, \Big(\frac{\sin^2 \alpha - n_3^2}{\sin^4 \alpha}\Big) \, \lambda_1^2 \,. \tag{6.1}$$

The orthogonality equation (3.2) then, obviously, is

$$k(s)^2 \, \sin(2\,\phi) \, \Big(\frac{\sin^2 \alpha - n_3^2}{\sin^4 \alpha}\Big) \, \frac{\partial \nu}{\partial \phi} + 2\, b \, \cos(\alpha - \phi) \, \frac{\partial \nu}{\partial s} = 0 \,. \tag{6.2}$$

It is not clear if all solutions to this equation can be written explicitly for a general curve. We consider here two particular cases when this is possible. The first case is when $\sin^2 \alpha = n_3^2$. This occurs exactly when the roller coaster lies in one vertical plane. The second case is when $\alpha(s)$ is constant. This occurs when the track is constantly inclined.

6.1. Case 1: $\sin^2 \alpha = n_3^2$. Note that this case includes the interesting examples of an inverted pendulum on a vertical disk and an inverted pendulum cart on an incline.

As is readily seen from (6.2), the general solution of (6.2) in this case is $\nu = \nu(\phi)$, an arbitrary function. Then

$$\lambda_1^2 = -\frac{1}{2\, b \, \cos(\alpha - \phi)} \, \frac{\partial \nu}{\partial \phi}$$

and

$$\lambda_1^1 = \nu(\phi) + \frac{1}{2} \, \tan(\alpha - \phi) \, \frac{\partial \nu}{\partial \phi}.$$

This is a general solution of the λ-equation. From here one must solve the $\widehat{g}$- and $\widehat{V}$-equations. For special choices of $\alpha(s)$ and/or $\nu(\phi)$ these equations have explicit closed form solutions.

6.2. Case 2: $\alpha(s) = \alpha_0$. Examples with $\alpha(s)$ constant include an inverted pendulum cart traveling on any path in a horizontal plane, a cart on a vertically oriented circular helix, or any constantly inclined track.

Since $\frac{d\alpha}{ds} = -k(s)\frac{n_3(s)}{\sin(\alpha)}$, if $\alpha(s) = \alpha_0$, then we have $k(s)n_3(s) = 0$. To solve equation (6.2), we introduce new coordinates

$$z^1 = \beta(s)$$
$$z^2 = \beta(s) + \cos(\alpha_0) \ln|\csc \phi + \cot \phi| - \sin(\alpha_0) \ln|\sec \phi + \tan \phi|,$$

where

$$\beta(s) = \int_0^s \frac{k^2(p)}{b \sin^2(\alpha_0)} \, dp \,.$$

Equation (6.2) is equivalent to the fact that ν is an arbitrary function of z^2. Thus, from (6.1) and the definition of ν we find

$$\lambda_1^1 = \nu(z^2) - \frac{\sin(\alpha_0 - \phi)}{\sin(2\phi)} \frac{d\nu}{dz^2}; \qquad \lambda_1^2 = \frac{1}{b\sin(2\phi)} \frac{d\nu}{dz^2}.$$

6.3. End of the roller coaster example. From the computations in case 1 and case 2 one can see that the general solution to the matching equations for the cart on a roller coaster will be fairly complicated. However, we can show that any *linear* control law is the first order germ (linearization) of some matching control law. In fact, the only requirement for this is that there is no rank drop at the equilibrium, i.e.,

$$\operatorname{rank} A^\star(x_0) = \limsup_{x \to x_0} \operatorname{rank} A^\star(x). \tag{6.3}$$

We assume that the dissipative term at the equilibrium satisfies the following natural assumptions:

$$C_\ell(x_0, 0) = 0, \qquad \frac{\partial}{\partial x^i}\bigg|_{(x_0, 0)} C_\ell = 0.$$

LEMMA 6.1. *If condition (6.3) is satisfied for a two degree of freedom system, then the first order germs of matching control laws at $(x_0, 0)$ exhaust all linear control laws for which the closed loop system has an equilibrium at $(x_0, 0)$.* Proof. Given a linear control input

$$u_j^{\mathrm{lin}} = v_j + a_{ij}(x^i - x_0^i) + b_{ij}\dot{x}^i$$

with $u_1^{\mathrm{lin}} = 0$, we will find a matching control law with the same germ. From the general expression (2.4) for the matching control law, we see that the first order germ is

$$u_j^{\mathrm{germ}} = (V_j - g_{j\ell}\,\widehat{g}^{\ell r}\,\widehat{V}_r) + (V_{jr} - g_{j\ell}\,\widehat{g}^{\ell i}\,\widehat{V}_{ir})(x^r - x_0^r) + (C_{ji} - g_{j\ell}\,\widehat{g}^{\ell r}\,\widehat{C}_{ri})\dot{x}^i,$$

where

$$V_j = \frac{\partial V}{\partial x^j}\bigg|_{x_0}, \quad V_{jr} = \frac{\partial^2 V}{\partial x^j \partial x^r}\bigg|_{x_0}, \quad C_{ji} = \frac{\partial C_j}{\partial \dot{x}^i}\bigg|_{(x_0, 0)},$$

and $\widehat{V}_j$, $\widehat{V}_{jr}$, and $\widehat{C}_{ji}$ are defined similarly. Equating like terms gives

$$\widehat{V}_{\ell j} = \widehat{g}_{\ell i}\,g^{ir}(V_{rj} - a_{rj}), \quad \widehat{C}_{\ell j} = \widehat{g}_{\ell i}\,g^{ir}(C_{rj} - b_{rj}), \quad \widehat{V}_\ell = \widehat{g}_{\ell i}\,g^{ir}(V_r - v_r) = 0. \tag{6.4}$$

One can see that $\widehat{V}_j$, $\widehat{V}_{jr}$, and $\widehat{C}_{ji}$ are specified once $\widehat{g}_{ij}(x_0)$ is known. Moreover, there exists a non-degenerate symmetric $\widehat{g}_{ij}(x_0)$ such that the resulting $\widehat{V}_{\ell j}$ will be symmetric, see [1, Lemma 1]. To conclude the argument, we now show that any non-degenerate symmetric $\widehat{g}_{ij}(x_0)$ arises as a zero order germ of a solution to $\widehat{g}$-equation. Also, any $\widehat{V}_\ell$, $\widehat{V}_{\ell j}$ satisfying $\widehat{V}_\ell = \widehat{g}_{\ell i}\,g^{ir}(V_r - v_r)$ and $\widehat{V}_{\ell j} = \widehat{g}_{\ell i}\,g^{ir}(V_{rj} - a_{rj})$ arises as a solution to the $\widehat{V}$-equation.

Given a non-degenerate $\widehat{g}_{ij}(x_0)$, define the non-degenerate $\lambda_i^j(x_0) = g_{ik}(x_0)\,\widehat{g}^{kj}(x_0)$. Set $\nu_0 = g_{11}(x_0)\,\lambda_1^1(x_0) + g_{12}(x_0)\,\lambda_1^2(x_0)$. The λ-equations in this case are

$$
\begin{aligned}
\partial_1\nu - 2\,[1\,1,\,2]\,\frac{1}{g_{11}}\,\nu &= 2\,[1\,1,\,2]\,\lambda_1^2 \\
\partial_2\nu - 2\,[1\,2,\,1]\,\frac{1}{g_{11}}\,\nu &= 2\,[2\,1,\,2]\,\lambda_1^2
\end{aligned}
\tag{6.5}
$$

By the rank condition, we know that the rank of $A^\star$ in the neighborhood of x_0 is either identically 0 or identically 1. If this rank is 0, then ν can be any constant times g_{11}. We simply choose $\nu(x) = (\nu_0/g_{11}(x_0))\,g_{11}(x)$. Any solution to the algebraic equation $\nu(x) = g_{11}(x)\,\lambda_1^1(x) + g_{12}(x)\,\lambda_1^2(x)$ is a solution to the λ-equation. If the rank of $A^\star$ is 1, then the orthogonality condition, (2.16), is

$$
[2\,1,\,2]\,\partial_1\nu - [1\,1,\,2]\,\partial_2\nu + 2\,([1\,1,\,2]\,[1\,2,\,1] - [2\,1,\,2]\,[1\,1,\,2]\,)\,\frac{1}{g_{11}}\,\nu = 0\,.
\tag{6.6}
$$

At $x = x_0$, either ∂_1 or ∂_2 is not parallel to the vector $[2\,1,\,2]\,\partial_1 - [1\,1,\,2]\,\partial_2$. Assume it is ∂_1. Then initial conditions to equation (6.6) can be specified along the line $x^2 = x_0^2$. In particular, we can choose the initial values so that

$$
\left(\partial_1\nu - 2\,[1\,1,\,2]\,\frac{1}{g_{11}}\,\nu\right)\Big|_{x=x_0} = 2\,[1\,1,\,2]\Big|_{x=x_0}\,\lambda_1^2(x_0)\,,\qquad \nu(x_0) = \nu_0\,.
\tag{6.7}
$$

The second equation in (6.5),

$$
\left(\partial_2\nu - 2\,[1\,2,\,1]\,\frac{1}{g_{11}}\,\nu\right)\Big|_{x=x_0} = 2\,[2\,1,\,2]\Big|_{x=x_0}\,\lambda_1^2(x_0)\,,
$$

will be satisfied automatically since the rank of $A^\star$ is 1. Let $\nu(x)$ be a solution of (6.6) with initial condition, $\nu(x^1, 0)$, satisfying (6.7). Now, one solves (6.5) for $\lambda_1^2(x)$ and then $\nu(x) = g_{11}(x)\,\lambda_1^1(x) + g_{12}(x)\,\lambda_1^2(x)$ for $\lambda_1^1(x)$.

Now that the λ-equations are solved, we turn to $\widehat{g}$-equations. These equations take the form

$$
\frac{\partial}{\partial y^1}\,\widehat{g} + R\widehat{g} = S\,,
$$

where $\frac{\partial}{\partial y^1} = \lambda_1^1\partial_1 + \lambda_1^2\partial_2$. The initial conditions can be set on any line transverse to $\frac{\partial}{\partial y^1}$, in particular, along the line $\lambda_1^1(x_0)\,(x^1 - x_0^1) + \lambda_1^2(x_0)\,(x^2 - x_0^2) = 0$. On this line set $\lambda_2^i(x) = \lambda_2^i(x_0)$, and $\widehat{g}(x) = g(x)\cdot(\lambda(x_0))^{-1}$. The solution to equation (2.10) with this initial data then has the desired value at $x = x_0$.

It remains to show that the $\widehat{V}$-equation has a solution such that $\widehat{V}_\ell = 0$ and

$$
\lambda_r^\ell(x_0)\,\widehat{V}_{\ell j} = V_{rj} - a_{rj}\,,
\tag{6.8}
$$

where $\lambda_r^\ell(x_0) = g_{rk}(x_0)\,\widehat{g}^{k\ell}(x_0)$. Since $\lambda_r^\ell(x_0)$ is non-degenerate, either $\lambda_1^1(x_0) \neq 0$ or $\lambda_1^2(x_0) \neq 0$. Consider the case with $\lambda_1^2(x_0) \neq 0$. In this case the line $x^2 = x_0^2$ is non-characteristic for the $\widehat{V}$-equation

$$
\lambda_1^1\,\partial_1\,\widehat{V} + \lambda_1^2\,\partial_2\,\widehat{V} = \partial_1 V\,.
\tag{6.9}
$$

Pick the initial value, $\widehat{V}\big|_{x^2=x_0^2}$, so that

$$\widehat{V}_1 = 0, \quad \widehat{V}_{11} = \left. \frac{\lambda_2^2\, V_{11} - \lambda_1^2\, (V_{12} - a_{12})}{\lambda_1^1\, \lambda_2^2 - \lambda_2^1\, \lambda_1^2} \right|_{x=x_0}$$

and solve equation (6.9). Since $x = x_0$ is an equilibrium, $V_1 = 0$ and $(V_2 - v_2) = 0$. From the differential equation, (6.9), we see that $\widehat{V}_2 = 0$. Differentiating equation (6.9) with respect to x^1 and x^2, we see that $W_{ij} = \widehat{V}_{ij}$ satisfies

$$\begin{aligned}
\lambda_1^1(x_0)\, W_{11} + \lambda_1^2(x_0)\, W_{12} &= V_{11} \\
\lambda_1^1(x_0)\, W_{12} + \lambda_1^2(x_0)\, W_{22} &= V_{12} \,.
\end{aligned} \tag{6.10}$$

By construction,

$$\lambda_2^1(x_0)\, W_{11} + \lambda_2^2(x_0)\, W_{12} = V_{12} - a_{12} \,. \tag{6.11}$$

Notice that (6.4) implies that $W_{\ell j} = \widehat{g}_{\ell i}\, g^{ir}\, (V_{rj} - a_{rj})$ also satisfy equations (6.10) and (6.11). Since the solution to the algebraic system (6.10), (6.11) is unique, we conclude that equation (6.8) is valid, as required.

7. Example 4: A double pendulum on a wheel. Our next example is the system with two unactuated degrees of freedom depicted in Figure 3. Only joint **A** is actuated.

After rescaling, the entries g_{ij} of the mass matrix are

$$g_{ij} = m_{ij}\, \cos(x^i - x^j)$$

and the potential energy is

$$V = a_1\, \cos(x^1) + a_2\, \cos(x^2) + a_3\, \cos(x^3) \,.$$

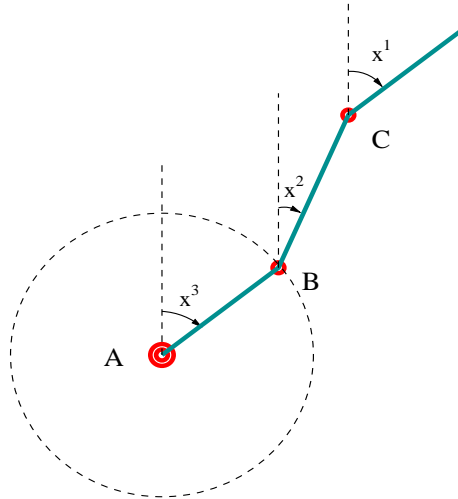The parameters $m_{ij} = m_{ji}$ and a_j are positive.

Figure 3

There are six unknown λ_a^i. Define $\nu_{ab} = g_{ai} \lambda_b^i$. Note that we must have $\nu_{12} = \nu_{21}$. The computations in this section were performed using Maple. The λ-equations (2.9) are

$$\partial_1 \nu_{11} = -2 m_{12} \sin(x_1 - x_2) \lambda_1^2 - 2 m_{13} \sin(x_1 - x_2) \lambda_1^3$$
$$\partial_2 \nu_{11} = 0$$
$$\partial_3 \nu_{11} = 0$$

$$\partial_1 \nu_{22} = 0$$
$$\partial_2 \nu_{22} = +2 m_{12} \sin(x_1 - x_2) \lambda_2^1 - 2 m_{23} \sin(x_2 - x_3) \lambda_2^3$$
$$\partial_3 \nu_{22} = 0$$

$$\partial_1 \nu_{12} = - m_{12} \sin(x_1 - x_2) \lambda_2^2 - m_{13} \sin(x_1 - x_3) \lambda_2^3$$
$$\partial_2 \nu_{12} = + m_{12} \sin(x_1 - x_2) \lambda_1^1 - m_{23} \sin(x_2 - x_3) \lambda_1^3$$
$$\partial_3 \nu_{12} = 0$$

The second step is to express λ_1^1, λ_2^1, λ_1^2, and λ_2^2, in terms of ν_{11}, ν_{12}, and ν_{22},
After substitution into the above equations we obtain

$$\partial_1 \nu_{11} = D_{1,1}^1 \nu_{11} + D_{1,1}^2 \nu_{12} + B_{1,1}^1 \lambda_1^3$$
$$\partial_2 \nu_{11} = 0$$
$$\partial_3 \nu_{11} = 0$$
$$\partial_1 \nu_{22} = 0$$
$$\partial_2 \nu_{22} = D_{3,2}^2 \nu_{12} + D_{3,2}^3 \nu_{22} + B_{3,2}^2 \lambda_2^3 \qquad (7.1)$$
$$\partial_3 \nu_{22} = 0$$
$$\partial_1 \nu_{12} = D_{2,1}^2 \nu_{12} + D_{2,1}^3 \nu_{22} + B_{2,1}^2 \lambda_2^3$$
$$\partial_2 \nu_{12} = D_{2,2}^1 \nu_{11} + D_{2,2}^2 \nu_{12} + B_{2,2}^1 \lambda_1^3$$
$$\partial_3 \nu_{12} = 0$$

Here the $D_{i,j}^k$ and $B_{i,j}^k$ are explicit expressions involving x. In Section 2 we described a general procedure to obtain compatibility conditions for this system. In this particular case, however, we use a different tactic: we compute and compare the mixed derivatives of ν_{ab}. The first set of equations we obtain is

$$\partial_3 \partial_1 \nu_{12} = K_{11} \partial_3 \lambda_2^3 + K_{12} \lambda_2^3 = 0$$
$$\partial_3 \partial_2 \nu_{22} = K_{21} \partial_3 \lambda_2^3 + K_{22} \lambda_2^3 = 0$$

Direct computation shows that $\det(K_{ij}) \neq 0$. Hence, $\lambda_2^3 = 0$. Similarly,

$$\partial_3 \partial_1 \nu_{11} = L_{11} \partial_3 \lambda_1^3 + L_{12} \lambda_1^3 = 0$$
$$\partial_3 \partial_2 \nu_{12} = L_{21} \partial_3 \lambda_1^3 + L_{22} \lambda_1^3 = 0$$

and $\det(L_{ij}) \neq 0$. Hence, $\lambda_1^3 = 0$.
Next, we substitute $\lambda_1^3 = \lambda_2^3 = 0$ into (7.1) and solve for ν_{11}, ν_{12}, ν_{22}. This gives

$$\nu_{11} = \text{const}, \qquad \nu_{22} = \frac{m_{22}}{m_{11}} \nu_{11}, \qquad \nu_{12} = \frac{m_{12}}{m_{11}} \cos(x^2 - x^1) \nu_{11}$$

Returning to λ-equations we see that

$$\lambda_1^1 \,=\, \lambda_2^2 \,=\, \frac{1}{m_{11}}\, \nu_{11}\,, \qquad \lambda_2^1 \,=\, \lambda_1^2 \,=\, 0\,.$$

Our computation shows that the only solutions of the matching equations are the basic solutions defined in Remark 2.2.

REFERENCES

[1] F. ANDREEV, D. AUCKLY, S. GOSAVI, L. KAPITANSKI, A. KELKAR, AND W. WHITE, *Matching linear system, and ball and beam*, (2000), submitted; archived at math.OC/0003177

[2] F. ANDREEV, D. AUCKLY, L. KAPITANSKI, A. KELKAR, AND W. WHITE, *Matching control laws for a ball and beam system*, Proc. of IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, 2000, pp. 161-162.

[3] F. ANDREEV, D. AUCKLY, L. KAPITANSKI, A. KELKAR, AND W. WHITE, *Matching and digital control implementation for underactuated systems*, Proc. of ACC, Chicago, Illinois, June 2000, pp. 3934-3938.

[4] D. AUCKLY AND L. KAPITANSKI, *Mathematical problems in the control of underactuated systems*, In: Nonlinear dynamics and renormalization group, Eds. I. M. Sigal and C. Sulem, CRM Proceedings and Lecture Notes, vol. 27, (2001) pp. 29-40, American Mathematical Society, Providence, RI

[5] D. AUCKLY, L. KAPITANSKI, AND W. WHITE, *Control of nonlinear underactuated systems*, Comm. Pure Appl. Math, vol 53 number 3, (2000), pp.354-369.

[6] G. BLANKENSTEIN, R. ORTEGA, A. J. VAN DER SCHAFT, *The matching conditions of controlled Lagrangians and interconnection and damping assignment passivity based control*. Preprint (2001)

[7] A. M. BLOCH, D. CHANG, N. LEONARD AND J. E. MARSDEN, *Controlled Lagrangians and the stabilization of mechanical systems II: Potential shaping*, IEEE Trans. Automat. Control, (2001), to appear.

[8] A. BLOCH, N. LEONARD, AND J. MARSDEN, *Stabilization of mechanical systems using controlled Lagrangians*, Proc. of the 36th IEEE Conference on Decision and Control, vol 36, (1997), pp. 2356-2361.

[9] A. M. BLOCH, N. LEONARD AND J. E. MARSDEN, *Controlled Lagrangians and the stabilization of mechanical systems I: The first matching theorem*, IEEE Trans. Automat. Control, 45 (2000), 2253-2270.

[10] J. HAMBERG, *General matching conditions in the in the theory of controlled Lagrangians*. In.: Proc. of IEEE CDC vol. 38, (1999), pp. 2519-2523.

[11] J. HAMBERG, *Controlled Lagrangians, symmetries and conditions for strong matching*, Proc. of IFAC Workshop on Lagrangian and Hamiltonian methods for nonlinear control, 2000, pp. 62-67.

[12] J. HAMBERG, *Simplified conditions for matching and for generalized matching in the theory of controlled Lagrangians*. In. Proc. of ACC , Chicago, Illinois, June 2000, pp. 3918-3923.

[13] D. V. ZENKOV, A. M. BLOCH, N. E. LEONARD, AND J. E. MARSDEN, *Matching and stabilization of low-dimensional nonholonomic systems*. Proc. CDC 39 (2000), 1289-1295