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Noise Assisted High-gain Stabilization: Almost Surely or in Second Mean

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Abstract

For a linear control system with multiplicative white noise, we develop (asymptotic) formulas for the dependence of almost sure and second mean exponential growth rates on a high gain parameter k. We show that if the diffusion matrix is skew-symmetric so that the noise enters in a purely skew-symmetric way then the function g, where $g(p)/p$ denotes the exponential growth rate of the pth mean, converges to a straight line, uniformly for $p \in [0,2]$, as $k \to \infty$. We use these formulas to show that the feedback control system in Stratonovich form is high-gain stabilizable even if the zero dynamics are unstable, provided that the noise is strong enough. This contrasts with the noise free case where we need the zero dynamics to be exponentially stable.

We then consider a class of systems where the diffusion matrix is not skew-symmetric, and show that almost sure and p^{th} mean growth rates have different limiting behaviour as $k \to \infty$.

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1   Introduction

The dependence of dynamical properties of systems on parameters is central to many problems in control theory and in dynamical systems. For example, consider a linear single-input single-output control system of the form

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned} \tag{1}$$

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where $x \in \mathbf{R}^d$, A is a real $d \times d$-matrix, and $B, C^T \in \mathbf{R}^d$. A basic control design technique is to study the root locus for (1), i.e., the eigenvalues of $A - kBC$ as k varies. The root locus technique is used, for example, to show that if $CB > 0$ and if (A, B, C) is minimum phase, then high gain feedback control $u = -ky$ is stabilizing in the sense that the eigenvalues of $A - kBC$ are in the left half-plane for all k sufficiently large.

The parametric dependence of dynamical properties for linear stochastic differential equations (LSDEs) has been investigated with great intensity during the last decades. For a survey on approaches which are closest in spirit to the present work, as well as for further references, see Wihstutz [10]. We also note that for the investigation of the bifurcation behaviour of a noisy Duffing-van der Pol-oscillator (see Arnold [1] Chapter 9) a detailed study of the dependence on parameters of Lyapunov exponents for a noisy damped harmonic oscillator has been performed by Imkeller and Lederer [7, 8].

We want to investigate dynamical properties of certain LSDEs obtained by applying proportional feedback. More precisely, we consider a noisy version of (1) with proportional output feedback, i.e.,

$$dx = (A - kBC)x\,dt + \sum_{j=1}^{m} A_j x * dW_j(t)\,. \tag{2}$$

We are interested in the dependence of exponential growth rates for (2) on k, and, in particular, on asymptotic formulas, valid for large k. Whilst for deterministic systems this can be derived by studying the dependence of eigenvalues on parameters, for LSDEs the situation is more complicated. Indeed, there are several competing notions of growth rates (for example, in p^{th} mean or almost surely). Furthermore, the growth rates depend on which notion of solution of (2) is adopted – Itô versus Stratonovich.

The paper is organised as follows. In Section 2 we recall some basic facts about LSDEs and their exponential growth rates, specifically Lyapunov exponents and moment exponents. In Section 3 we consider almost sure growth rates, and we characterize under which conditions (2) can be stabilized almost surely by high gain feedback. This means that the leading Lyapunov exponent becomes negative for all sufficiently high gain. We restrict ourselves to the simplest possible non-trivial case, which is the case of a 2×2-system. To calculate the almost sure growth rate, i.e., the leading Lyapunov exponent, we use the Furstenberg-Khasminskii formula, which yields a closed, albeit complicated, formula. In subsection 3.1 we first investigate the case where A is symmetric. In this case, certain integrals in the Furstenberg-Khasminskii formula can be computed in terms of hypergeometric functions. In subsection 3.2 we consider the case of a general system matrix A, invoking the results for the symmetric case together with some slightly involved comparison arguments. These results are in a spirit of those obtained by Imkeller and Lederer [7, 8], who obtained explicit formulas for Lyapunov exponents of certain linear two-dimensional systems arising from the linearization of the Duffing-van der Pol-oscillator around zero.
In Section 4 we consider the 2nd mean growth rates and characterize under which conditions the system can be stabilized in second mean by high gain feedback. Here we use

the classical technique of considering the norm induced by a positive definite matrix P (thus defining a Lyapunov function), applying the Itô formula to obtain estimates for the exponential growth of second moments, and then choosing the matrix P in an appropriate way. As a result we obtain that when the noise enters the system in a purely skew-symmetric way, then for high gain the exponential growth rate of the second moment approaches twice the Lyapunov exponent, the almost sure exponential growth rate. When the noise is entering non skew-symmetrically, the growth rates have different limiting behaviour as $k \to \infty$.

2 Notions of growth rates for linear SDEs

To introduce the notions of growth rate for (2), consider a general linear stochastic differential equation

$$dx = Ax\,dt + \sum_{j=1}^{m} A_j x * dW_j(t)\,, \tag{3}$$

where A and A_j are $d \times d$-matrices, W_j are independent standard Wiener processes, $1 \le j \le m$, and $*$ stands for an interpretation of (3) either as an Itô or as a Stratonovich equation. Denote by $x(t, x_0)$ denote the solution of (3) at time t, with initial condition x_0 at time $t = 0$.

There are several non-equivalent notions of exponential growth rates for (3). The leading *Lyapunov exponent* of (3) is defined as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \log \Big(\sup_{\|x_0\|=1} \|x(t; x_0)\| \Big) \qquad P\text{-a.\,s.,} \tag{4}$$

where $\| \cdot \|$ is any norm on $\mathbf{R}^d$. Existence of the almost sure limit and the fact that the limit is constant a.\,s. follows from the subadditive ergodic theorem of Kingman [9]. The LSDE (3) is said to be *almost surely exponentially stable* if $\lambda < 0$.

The exponential growth rate in the pth mean is $g(p)/p$, where $g(p)$ is given by

$$g(p) = \lim_{t \to \infty} \frac{1}{t} \log E \Big(\sup_{\|x_0\|=1} \|x(t; x_0)\|^p \Big)\,. \tag{5}$$

In (5), E denotes expectation. The function $p \mapsto g(p)$ is a real analytic function, convex, and $g'(0) = \lambda$, provided certain non-degeneracy conditions are satisfied, see Arnold, Oeljeklaus and Pardoux [5]. Consequently,

$$\frac{g(p)}{p} \ge \frac{g(q)}{q} \ge g'(0) = \lambda \tag{6}$$

for $p \ge q > 0$. In particular, the growth rate in p^{th} mean is greater than or equal to the almost sure growth rate for $p > 0$. We are interested in the dependence of the

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growth rates given by (4) and (5) for the LSDE (2) as we vary the feedback gain k. For example: How does the feedback gain influence the difference between $g(2)$ and 2λ?

As mentioned earlier in the introduction, the dependence of growth rates of LSDEs on parameters has been of interest. One of the earliest results of this type, of relevance here, goes back to Arnold, Crauel and Wihstutz [2]. Consider a Stratonovich equation of the form

$$dx = Ax\,dt + \sum_{j=1}^{m} u_j B_j x \circ dW_j(t)\,, \tag{7}$$

where B_j, $1 \le j \le m$, form a basis for the $d(d-1)/2$-dimensional linear space of skew symmetric $d \times d$-matrices and, as before, the W_j are independent standard Wiener processes. In [2], the following limiting behaviour for the leading Lyapunov exponent is obtained:

$$\lambda \to \frac{1}{d}\operatorname{tr} A = \frac{1}{d}\sum_{i=1}^{d} a_{ii} \qquad \text{as} \quad \min u_j \to \infty\,.$$

This result has been subsequently generalized by Arnold, Eizenberg and Wihstutz [3] to allow for weaker conditions on the (B_j). One consequence of this result is the surprising observation that we can 'stabilize' (7) by high-intensity noise if $\operatorname{tr} A < 0$. So, given a deterministic system $\dot{x} = Ax$ with $\operatorname{tr} A < 0$, it suffices to agitate the system by noise as in (7), and increase the intensity u until almost sure stability is achieved.

Whilst high intensity noise would seem impractical, this result motivates us to determine if noise can enhance more traditional stabilization by proportional feedback.

Second mean feedback stabilization of a general controlled Itô LSDE

$$\begin{cases} dx &= (Ax + Bu_0)\,dt + \displaystyle\sum_{j=1}^{m}(A_j + u_j B_j)x\,dW_j(t) \\ y &= Cx \end{cases} \tag{8}$$

has been considered by Damm [6]. Here $u_0, \ldots, u_m$ are controls, and A, A_j, B, B_j and C are matrices of suitable dimensions. This problem is quite involved. It turns out that stabilization in second mean is equivalent to the existence of positive definite solutions to certain generalized linear matrix inequalities. We are not aware of any similar results for almost sure stabilization. Rather than pursuing this general feedback problem we limit our interest to the parametric dependence of the growth rates λ and $g(p)$ in a set-up less general than (8). We assume that the noise enters in a "non-systematic" way in the sense that it does not contribute systematically to the almost sure exponential growth rates of the system. More precisely, we impose the condition that the noise enters so that the leading Lyapunov exponent of the Stratonovich equation (2) is bounded in σ, where A is an arbitrary $d \times d$ matrix. This implies that the matrices $(A_j - \frac{\operatorname{tr} A_j}{d}\operatorname{Id})$, $1 \le j \le m$, are skew-symmetric with respect to some basis of $\mathbf{R}^d$, see Arnold, Oeljeklaus, and Pardoux [5]. We therefore assume that the matrices A_j in (2) are skew-symmetric, $1 \le j \le m$.

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Furthermore, we restrict attention to the 2×2 case of system (2). This is the simplest non-trivial case and allows for a neat and clear formulation of the main results. Higher dimensional cases would also be possible, but the technical conditions are considerably more involved, and we currently do not have a complete picture.

3 Almost sure exponential growth rates

As already mentioned, we will investigate the simplest case possible. That is, we investigate the dependence of the almost sure exponential growth rate, i. e., the Lyapunov exponent, on the parameters k and σ for the Stratonovich equation

$$dx = \begin{pmatrix} a - k & b \\ c & d \end{pmatrix} x \, dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \circ dW(t). \tag{9}$$

In (9), $a, b, c, d \in \mathbf{R}$ are fixed parameters. The drift matrix $\begin{pmatrix} a-k & b \\ c & d \end{pmatrix}$ in (9) arises via a change of coordinates from (1) in the case $u = -ky$, and $CB > 0$, if the dimension is 2. We show that $\lim_{k \to \infty} \lambda_{k,\sigma} = \frac{1}{2}(2d - \sigma^2)$, so that (9) is high-gain almost sure exponentially stabilizable if and only if $d < \frac{1}{2}\sigma^2$.

3.1 Remark Without high-gain k, we have from Arnold, Crauel and Wihstutz [2] that $\lim_{\sigma \to \infty} \lambda = \frac{1}{2}(a + d)$, and so almost sure stability if mixing of the negative trace by the noise is strong enough.
With high-gain, but no noise, high-gain stabilization is possible only if the zero-dynamics are exponentially stable, i. e., $d < 0$, whilst the noise term has a stabilizing effect and with high enough gain k the intensity of the noise only has to be strong enough to overcome the influence of possibly 'unstable zero dynamics'.

Whilst the results are simply stated, their derivation, as with analogous results obtained by Imkeller and Lederer [7, 8], is quite involved. To proceed we first recall some details about the well known method for calculating Lyapunov exponents of a linear SDE

$$dv = Ax \, dt + \sum_{j=1}^{m} A_j x \circ dW_j(t) \tag{10}$$

via the Furstenberg-Khasminskii formula. Here A, A_j are $d \times d$-matrices, and W_j are independent Wiener processes, $1 \le j \le m$, and the equation is interpreted in the Stratonovich sense.

Projection of (10) from $\mathbf{R}^d \setminus \{0\}$ onto the unit sphere $S^{d-1} = \{v \in \mathbf{R}^d : |v| = 1\}$ by $x \mapsto x/|x| =: s$ gives the (non-linear) SDE

$$ds = g_A(s) \, dt + \sum_{j=1}^{m} g_{A_j}(s) \circ dW_j(t) \tag{11}$$

on S^{d-1}, where $g_A(s) = As - (s, As)s$. The SDE (11) defines a random dynamical system on S^{d-1}. Associated to every Lyapunov exponent there exists an invariant measure for this random dynamical system, which is supported by the Oseledets space associated with this Lyapunov exponent; for details see Arnold [1], Chapters 3 and 4. Furthermore, the maximal Lyapunov exponent (almost sure exponential growth rate) is given by

$$\lambda = \int_{S^{d-1}} \left((s, As) + \sum_{j=1}^{m} \left(\frac{1}{2}((A_j + A_j^*)s, A_j s) - (s, A_j s)^2 \right) \right) dp(s), \qquad (12)$$

where p is a (suitable) invariant measure for the Markov semigroup induced by (11). Equation (12) is the Furstenberg-Khasminskii formula in the form it takes for a linear system induced by the linear SDE (10). In particular, if (11) is sufficiently non-degenerate, then there exists a unique invariant Markov measure p, and this measure has a smooth density with respect to the Lebesgue measure on S^{d-1}, which we denote by p again. Non-degeneracy means that a certain hypo-ellipticity condition is satisfied, whose precise form is not of interest here. The density p is given as a suitably normalized solution of the associated Fokker-Planck equation. See Arnold [1], or Imkeller and Lederer [7, 8].

In our particular case we have the linear Stratonovich SDE

$$dx = \begin{pmatrix} a - k & b \\ c & d \end{pmatrix} x \, dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \circ dW(t),$$

and (11) becomes

$$ds = g_A(s) \, dt + g_B(s) \circ dW(t)$$

with $g_A(s) = As - (s, As)s$,

$$\begin{aligned}
(s, As) &= (a - k)s_1^2 + (b + c)s_1 s_2 + d s_2^2, \\
g_A(s) &= \begin{pmatrix} (a - k)s_1 + b s_2 - (a - k)s_1^3 - (b + c)s_1^2 s_2 - d s_2^2 s_1 \\ c s_1 + d s_2 - (a - k)s_1^2 s_2 - (b + c)s_1 s_2^2 - d s_2^3 \end{pmatrix}, \\
g_B(s) &= \sigma \begin{pmatrix} -s_2 \\ s_1 \end{pmatrix},
\end{aligned}$$

and $s = (s_1, s_2)^T$. In polar coordinates, $s = (\cos \varphi, \sin \varphi)^T$ with $\varphi \in [-\pi/2, \pi/2]$, this becomes

$$\begin{aligned}
d\varphi &= \frac{ds_2}{\cos \varphi} \\
&= \frac{c \cos \varphi + d \sin \varphi - (a - k) \cos^2 \varphi \sin \varphi - (b + c) \cos \varphi \sin^2 \varphi - d \sin^3 \varphi}{\cos \varphi} \, dt \\
&\quad + \sigma \, dW(t) \\
&= \left(c \cos^2 \varphi + (d - a + k) \cos \varphi \sin \varphi - b \sin^2 \varphi \right) dt + \sigma \, dW(t).
\end{aligned}$$

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To determine the invariant measure for the associated Markov semigroup we first note that the SDE for the angle φ is elliptic. Consequently, there is a unique invariant measure for the Markov semigroup, and this invariant measure has a C^∞ density $\varphi \mapsto p_{k,\sigma}(\varphi)$, which is a solution of the Fokker-Planck equation. In the present case the Fokker-Planck equation results in the ordinary differential equation

$$-(\frac{1}{2}\sigma^2 p)'' + \big(c\cos^2\varphi + (d - a + k)\cos\varphi\sin\varphi - b\sin^2\varphi\big)p' = 0$$

with periodic boundary conditions on $[-\pi/2, \pi/2)$. This gives

$$p' = \frac{2}{\sigma^2}(c + (d - a + k)\cos\varphi\sin\varphi - (b + c)\sin^2\varphi)p + \gamma$$

where γ has to be chosen such that p is periodic. Rewriting in 2φ-terms gives

$$
\begin{aligned}
p' &= \frac{2}{\sigma^2}\Big(c + \frac{1}{2}(d - a + k)\sin 2\varphi - \frac{1}{2}(b + c)(1 - \cos 2\varphi)\Big)p + \gamma \\
&= \frac{1}{\sigma^2}\Big((c - b) + (d - a + k)\sin 2\varphi + (b + c)\cos 2\varphi\Big)p + \gamma \, .
\end{aligned}
$$

Using standard trigonometric identities and setting

$$R(\varphi, \eta) = \frac{1}{\sigma^2}\Big((c - b)(\varphi - \eta) + \frac{1}{2}(d - a + k)(\cos 2\eta - \cos 2\varphi) + \frac{1}{2}(b + c)(\sin 2\varphi - \sin 2\eta)\Big)$$

we obtain that the general, still to be normalized, solution p, is given by

$$p(\varphi) = e^{R(\varphi, -\frac{\pi}{2})} p(-\tfrac{\pi}{2}) + \gamma \int_{-\pi/2}^{\varphi} e^{R(\varphi, \eta)} \, d\eta \, . \tag{13}$$

In (13) γ has to be chosen such that $p(\frac{\pi}{2}) = p(-\frac{\pi}{2})$. For the leading Lyapunov exponent we then obtain

$$\lambda = \int_{-\pi/2}^{\pi/2} \Big((a - k)\cos^2\varphi + (b + c)\cos\varphi\sin\varphi + d\sin^2\varphi\Big) p(\varphi) \, d\varphi \tag{14}$$

3.2 Remark We will use the explicit formula for the leading Lyapunov exponent $\lambda = \lambda_{k,\sigma}$ given by (14), invoking the density $p = p(k)$ given explicitly by (13). This is not possible for higher dimensional LSDE. For $d > 2$ one might invoke the more systematic approach as described, for example, by Wihstutz [10]. This involves putting $\varepsilon = k^{-\alpha}$ for suitable $\alpha > 0$, and denoting the generator induced by the projected SDE (11) by L_ε, to obtain an expansion of $\lambda = \lambda(\varepsilon)$ around $\varepsilon = 0$ by expanding the expression $\lambda(\varepsilon) = \int q_\varepsilon \, p_\varepsilon$ given by (12), where p_ε is determined by $L_\varepsilon p_\varepsilon = 0$ (which is the Fokker-Planck equation). Performing this asymptotic expansion formally for the two-dimensional case considered here yields the limiting behaviour $\lambda = \lambda_{k,\sigma^2} + o(1)$ for k large in a more intuitive manner than the direct approach which we adopt. However, it would seem from the developments in [10] that justification of this more systematic asymptotic expansion approach needs arguments which are at least as complicated as those used in the direct approach.

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3.1 The case of a symmetric drift matrix

We first treat the case $b = c$. This means that the drift matrix A in (9) is symmetric. Then $\varphi \mapsto R(\varphi, \eta)$ is periodic for every η, and therefore (13) yields that periodicity of p holds if and only if $\gamma = 0$. This gives

$$p(\varphi) = \exp\left[-\frac{1}{2\sigma^2}\Big((d - a + k)(1 + \cos 2\varphi) - 2b\sin 2\varphi\Big)\right]p(-\pi/2)$$

In order to make p the density of a probability measure we have to choose

$$p(-\tfrac{\pi}{2}) = \left(\int_{-\pi/2}^{\pi/2} \exp\left[-\frac{1}{2\sigma^2}\Big((d - a + k)(1 + \cos 2\varphi) - 2b\sin 2\varphi\Big)\right]d\varphi\right)^{-1}.$$

Note that $p(-\tfrac{\pi}{2}) > 0$. Having determined the dependence of the invariant density p on k and σ, we now turn to the calculation of the leading Lyapunov exponent as given by (14), which here takes the form

$$\lambda = \int_{-\pi/2}^{\pi/2} \Big((a - k)\cos^2\varphi + 2b\cos\varphi\sin\varphi + d\sin^2\varphi\Big)\, p(\varphi)\, d\varphi. \tag{15}$$

We first ignore the normalizing factor $p(-\tfrac{\pi}{2})$ and consider

$$I := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \Big((a - k)\cos^2\varphi + 2b\cos\varphi\sin\varphi + d\sin^2\varphi\Big)$$

$$\times \exp\left(-\frac{1}{2\sigma^2}\big((d - a + k)(1 + \cos 2\varphi) - 2b\sin 2\varphi\big)\right)d\varphi.$$

Defining $R_k = \sqrt{(k + d - a)^2 + (2b)^2}$ and ψ_k via $\tan\psi_k = \dfrac{k + d - a}{2b}$, and using the formula

$$s\cos 2\varphi + t\sin 2\varphi = R\sin(2\varphi + \psi)$$

with $R = \sqrt{s^2 + t^2}$ and $\tan\psi = s/t$, we obtain

$$I = \exp\left(\frac{a - k - d}{2\sigma^2}\right)\int_{-\pi/2}^{\pi/2}\Big[(a - k)\cos^2\varphi + 2b\cos\varphi\sin\varphi + d\sin^2\varphi\Big]$$

$$\times \exp\left(-\frac{1}{2\sigma^2}R_k\sin(2\varphi + \psi_k)\right)d\varphi.$$

Rearranging

$$(a - k)\cos^2\varphi + 2b\cos\varphi\sin\varphi + d\sin^2\varphi$$
$$= (a - k - d)\cos^2\varphi + 2b\cos\varphi\sin\varphi + d$$
$$= \frac{a - k - d}{2}(\cos 2\varphi + 1) + b\sin^2\varphi + d$$
$$= \frac{1}{2}\Big[(a - k - d)\cos 2\varphi + 2b\sin 2\varphi + a + d - k\Big]$$

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and using periodicity of sine and cosine together with the identities $\sin(\beta + \pi/2) = \cos\beta$ and $\cos(\beta + \pi/2) = -\sin\beta$ we obtain

$$
\begin{aligned}
I &= \frac{1}{2}\exp\!\left(\frac{a-k-d}{2\sigma^2}\right) \\
&\quad \times \int_{-\pi/2}^{\pi/2}\Big[(a-k-d)\cos(2\varphi-\psi_k) + 2b\sin(2\varphi-\psi_k) + a + d - k\Big] \\
&\qquad \times \exp\!\left(-\frac{R_k}{2\sigma^2}\sin 2\varphi\right) d\varphi \\
&= \frac{1}{2}\exp\!\left(\frac{a-k-d}{2\sigma^2}\right) \\
&\quad \times \int_{-\pi/2}^{\pi/2}\Big[-(a-k-d)\sin(2\varphi-\psi_k) + 2b\cos(2\varphi-\psi_k) + a + d - k\Big] \\
&\qquad \times \exp\!\left(-\frac{R_k}{2\sigma^2}\cos 2\varphi\right) d\varphi \\
&= \frac{1}{2}\exp\!\left(\frac{a-k-d}{2\sigma^2}\right)\exp\!\left(\frac{R_k}{2\sigma^2}\right) \\
&\quad \times \int_{-\pi/2}^{\pi/2}\Big[(k+d-a)\big(\sin 2\varphi\cos\psi_k - \cos 2\varphi\sin\psi_k\big) \\
&\qquad\quad + 2b\big(\cos 2\varphi\cos\psi_k + \sin 2\varphi + \sin 2\varphi\sin\psi_k\big) + a + d - k\Big] \\
&\qquad \times \exp\!\left(-\frac{R_k}{\sigma^2}\cos^2\varphi\right) d\varphi\,.
\end{aligned}
$$

Integrating over a symmetric interval around zero we only keep integrals over even functions. This gives

$$
\begin{aligned}
I &= \exp\!\left(\frac{a-k-d+R_k}{2\sigma^2}\right) \\
&\quad \times \int_{0}^{\pi/2}\Big[(a-d-k)\cos 2\varphi\sin\psi_k + 2b\cos 2\varphi\cos\psi_k + a + d - k\Big] \\
&\qquad \times \exp\!\left(-\frac{R_k}{\sigma^2}\cos^2\varphi\right) d\varphi\,.
\end{aligned}
$$

Now $\cos\psi_k = 2b/R_k$ and $\sin\psi_k = (k+d-a)/R_k$, so

$$
\begin{aligned}
I &= \exp\!\left(\frac{a-k-d+R_k}{2\sigma^2}\right) \\
&\quad \times \int_{o}^{\pi/2}\left[\frac{4b^2-(k+d-a)^2}{R_k}\cos 2\varphi + a + d - k\right]\exp\!\left(-\frac{R_k}{\sigma^2}\cos^2\varphi\right) d\varphi \\
&= \exp\!\left(\frac{a-k-d+R_k}{2\sigma^2}\right) \\
&\quad \times \int_{0}^{\pi/2}\left[a + d - k + \frac{(k+d-a)^2-4b^2}{R_k} - \frac{(k+d-a)^2-4b^2}{R_k}2\cos^2\varphi\right] \\
&\qquad \times \exp\!\left(-\frac{R_k}{\sigma^2}\cos^2\varphi\right) d\varphi\,.
\end{aligned}
$$

From the theory of hypergeometric functions we know that

$$H := \int_0^{\pi/2} \exp\Big(-\frac{R_k}{\sigma^2}\cos^2\varphi\Big)\, d\varphi \;=\; \frac{\pi}{2}\mathcal{H}([\tfrac{1}{2}],[1],\frac{R_k}{\sigma^2})$$

and

$$K := \int_0^{\pi/2} \exp\Big(-\frac{R_k}{\sigma^2}\cos^2\varphi\Big)\cos^2\varphi\, d\varphi \;=\; \frac{\pi}{4}\mathcal{H}([\tfrac{3}{2}],[2],\frac{R_k}{\sigma^2})\,,$$

where $\mathcal{H}$ denotes hypergeometric functions. This gives

$$
\begin{aligned}
I \;=\;\; & \exp\Big(\frac{a-k-d+R_k}{2\sigma^2}\Big)\\
& \times\left(\Big[a+d-k+\frac{(k+d-a)^2-4b^2}{\sqrt{(k+d-a)^2+4b^2}}\Big]H - 2\Big[\frac{(k+d-a)^2-4b^2}{\sqrt{(k+d-a)^2+4b^2}}\Big]K\right).
\end{aligned}
$$

Using similar calculations we obtain for the normalizing factor

$$p(-\tfrac{\pi}{2}) = \pi \exp\Big(\frac{a-d-k+R_k}{2\sigma^2}\Big)\mathcal{H}([\tfrac{1}{2}],[1],\frac{R_k}{\sigma^2}) = 2\exp\Big(\frac{a-d-k+R_k}{2\sigma^2}\Big)H\,.$$

Now

$$
\begin{aligned}
a+d-k+\frac{(k+d-a)^2-4b^2}{\sqrt{(k+d-a)^2+4b^2}} &=\; 2d+e_1(k)\,,\\
\frac{(k+d-a)^2-4b^2}{\sqrt{(k+d-a)^2+4b^2}} &=\; k+e_2(k)\,,\\
H &=\; \frac{\pi}{2}\Big(\frac{\sigma^2}{R_k}\Big)^{1/2}+e_3(k)\,,\\
K &=\; \frac{\pi}{4}\Big(\frac{\sigma^2}{R_k}\Big)^{3/2}+e_4(k)\,,\quad \text{and}\\
R_k = \sqrt{(k+d-a)^2+(2b)^2} &=\; k+e_5(k)\,,
\end{aligned}
$$

where $|e_1(k)|\le Mk^{-1}$, $|e_2(k)|\le M$, $|e_3(k)|\le Mk^{-3/2}$, $|e_4(k)|\le Mk^{-5/2}$, and $|e_5(k)|\le Mk^{-1/2}$ for a suitable constant M. In the following we will denote constants independent of k by M without always noting when their value changes. This gives

$$I = \pi\Big(d-\frac{\sigma^2}{2}\Big)\exp\Big(\frac{a-d}{2\sigma^2}\Big)\frac{\sqrt{\sigma^2}}{k^{1/2}}+e_6(k) \tag{16}$$

and, introducing the notion J for later reference,

$$J := p(-\tfrac{\pi}{2}) = \pi\exp\Big(\frac{a-d}{2\sigma^2}\Big)\frac{\sqrt{\sigma^2}}{k^{1/2}}+e_7(k)\,, \tag{17}$$

where $|e_6(k)|\le Mk^{-3/2}$ and $|e_7(k)|\le Mk^{-3/2}$.

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From (15) we obtain
$$\lambda = \lambda_{k,\sigma^2} = \frac{I}{J} = d - \frac{\sigma^2}{2} + e_8(k)$$
with $|e_8(k)| \leq Mk^{-1}$.

This proves the following proposition.

3.3 Proposition *In the case of a symmetric drift matrix, i. e., in the case $b = c$, the leading Lyapunov exponent $\lambda = \lambda_{k,\sigma}$ of the LSDE (9) satisfies*
$$\lambda_{k,\sigma} = d - \frac{\sigma^2}{2} + O(k^{-1})$$
for k large. In particular,
$$\lim_{k \to \infty} \lambda_{k,\sigma} = d - \frac{\sigma^2}{2} \,.$$

3.2   The case of a general drift matrix

Having assumed $b = c$ in the drift matrix for the calculations in the previous subsection, we now use these calculations to obtain a result for the case $b \neq c$.

3.4 Theorem   *The Lyapunov exponent $\lambda = \lambda_{k,\sigma}$ of the linear Stratonovich SDE (9) satisfies*
$$\lambda_{k,\sigma} = d - \frac{\sigma^2}{2} + O(k^{-1})$$
for k large. In particular,
$$\lim_{k \to \infty} \lambda_{k,\sigma} = d - \frac{\sigma^2}{2}$$
for every a, b, c, d.

PROOF   Having established the result for the case $b = c$ in the previous section, we proceed by showing that the general case is close to the symmetric case for k sufficiently large.
First note that (13), (14), $p(-\frac{\pi}{2}) = p(\frac{\pi}{2})$, and $\int p(\varphi)\, d\varphi = 1$ gives $\lambda = I/J$ with
$$I = \int_{-\pi/2}^{\pi/2} q(\varphi)m(\varphi)g(\varphi)\, d\varphi \qquad \text{and} \quad J = \int_{-\pi/2}^{\pi/2} m(\varphi)g(\varphi)\, d\varphi \,,$$
where
$$
\begin{aligned}
q(\varphi) &= (a - k)\cos^2\varphi + (b + c)\cos\varphi\sin\varphi + d\sin^2\varphi \\
&= (a - d - k)\cos^2\varphi + (b + c)\cos\varphi\sin\varphi + d \,, \\
m(\varphi) &= \exp\Big(\frac{1}{2\sigma^2}[(b + c)\sin 2\varphi - (d - a + k)(\cos 2\varphi + 1)]\Big) \,, \\
g(\varphi) &= \exp\big(\frac{c - b}{\sigma^2}(\varphi + \frac{\pi}{2})\big) + \Big[1 - \exp\big(\frac{c - b}{\sigma^2}\pi\big)\Big]\Big[\exp\big(\frac{c - b}{\sigma^2}(\varphi - \frac{\pi}{2})\big)\Big]f(\varphi)
\end{aligned}
$$

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and f is given by

$$f(\varphi) = \frac{\displaystyle\int_{-\pi/2}^{\varphi} \exp\left(\frac{1}{2\sigma^2}\left[-2(c-b)\eta + (d-a+k)\cos 2\eta - (b+c)\sin 2\eta\right]\right)\,d\eta}{\displaystyle\int_{-\pi/2}^{\pi/2} \exp\left(\frac{1}{2\sigma^2}\left[-2(c-b)\eta + (d-a+k)\cos 2\eta - (b+c)\sin 2\eta\right]\right)\,d\eta}\,.$$

In the symmetric case we had $g(\varphi) \equiv 1$ on $[-\pi/2, \pi/2]$. We will show that I and J are close to the corresponding integrals with $g \equiv 1$. Whilst the integrals I and J look at first glance rather intractable, there are several simplifications if k is large. In particular, for k large we have $f(\varphi) \approx 0$ for $-\pi/2 \le \varphi \le -\pi/4$, and $f(\varphi) \approx 1$ for $\pi/4 \le \varphi \le \pi/2$. We will make this more precise below.

First note that $0 \le f \le 1$, hence there exists $M > 0$, independent of k, such that $|g(\varphi)| \le M$ for all φ. Next consider the behaviour of $m(\varphi)$ for large k. Here the relevant term is $-(d-a+k)(\cos 2\varphi + 1)$. Note that $\cos 2\varphi + 1 \ge 0$ for all φ. Define intervals I_k and J_k close to $-\pi/2$ and $\pi/2$, respectively, such that $1 + \cos 2\varphi \ge \dfrac{1}{\sqrt{k}}$ for all $\varphi \notin I_k \cup J_k$ by

$$I_k = [-\pi/2, -\cos^{-1}\frac{1}{\sqrt{2\sqrt{k}}}] \qquad \text{and} \quad J_k = [\cos^{-1}\frac{1}{\sqrt{2\sqrt{k}}}, \pi/2]\,.$$

For $\varphi \notin I_k \cup J_k$ then

$$-(d-a+k)(1+\cos 2\varphi) \le -\frac{\sqrt{k}}{3}$$

for k large. Noting that, for k large,

$$|q(\varphi)| = |(a-k)\cos^2\varphi + (b+c)\cos\varphi\sin\varphi + d\sin^2\varphi| \le 2k\,,$$

we obtain

$$\left|\int_{[-\pi/2,\pi/2]\setminus(I_k\cup J_k)} q(\varphi)m(\varphi)g(\varphi)\,d\varphi\right| \le k\,M\,\exp\left(-\frac{\sqrt{k}}{3}\right)$$

for some $M > 0$. Since $k\exp\left(-\frac{\sqrt{k}}{3}\right) \le \exp\left(-\frac{\sqrt{k}}{4}\right)$ for k large we thus obtain

$$\left|\int_{[-\pi/2,\pi/2]\setminus(I_k\cup J_k)} q(\varphi)m(\varphi)g(\varphi)\,d\varphi\right| \le M\exp\left(-\frac{\sqrt{k}}{4}\right)\,,$$

and so

$$\left|\int_{-\pi/2}^{\pi/2} q(\varphi)m(\varphi)g(\varphi)\,d\varphi - \int_{I_k\cup J_k} q(\varphi)m(\varphi)g(\varphi)\,d\varphi\right| \le M\exp\left(-\frac{\sqrt{k}}{4}\right)\,. \qquad (18)$$

By similar arguments one obtains

$$\left|\int_{-\pi/2}^{\pi/2} q(\varphi)m(\varphi)\,d\varphi - \int_{I_k\cup J_k} q(\varphi)m(\varphi)\,d\varphi\right| \le M\exp\left(-\frac{\sqrt{k}}{4}\right)\,. \qquad (19)$$

It follows from (18) and (19) that we only need to deal with the difference between I and $\int q(\varphi)m(\varphi)\,d\varphi$ on $I_k \cup J_k$.

Loosely speaking we have $g(\varphi) \approx 1$ on I_k and

$$g(\varphi) \approx \exp\left[\frac{c-b}{\sigma^2}\pi\right] + \left(1 - \exp\left[\frac{c-b}{\sigma^2}\pi\right]\right) = 1$$

on J_k, both uniformly in k.

To be more precise, consider first $f(\varphi)$ for $\varphi \in I_k$. For $\varphi \in I_k$ and k sufficiently large we have $\cos 2\varphi \leq -1/2$, and for $\varphi \in [-\pi/6, \pi/6]$ we have $\cos 2\varphi \geq 1/2$. It follows that, for $\varphi \in I_k$,

$$
\begin{aligned}
0 \leq f(\varphi) &\leq \frac{\int_{I_k} \exp\left(\frac{1}{2\sigma^2}\left[-2(c-b)\eta + (d-a+k)\cos 2\eta - (b+c)\sin 2\eta\right]\right)\,d\eta}{\int_{-\pi/6}^{\pi/6} \exp\left(\frac{1}{2\sigma^2}\left[-2(c-b)\eta + (d-a+k)\cos 2\eta - (b+c)\sin 2\eta\right]\right)\,d\eta} \\
&\leq M\exp(-k),
\end{aligned}
$$

whence

$$\left| g(\varphi) - \exp\left(\frac{c-b}{\sigma^2}(\varphi + \tfrac{\pi}{2})\right)\right| \leq M\exp(-k).$$

Expanding $\varphi \mapsto \exp\left(\frac{c-b}{\sigma^2}(\varphi + \frac{\pi}{2})\right)$ in Taylor series around $-\pi/2$ we obtain

$$g(\varphi) = 1 + \frac{c-b}{\sigma^2}(\varphi + \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^3 + e_1(k) \quad (20)$$

with $|e_1(k)| \leq Mk^{-1}$, for all $\varphi \in I_k$.

Similarly for $\varphi \in J_k$ we have

$$
\begin{aligned}
1 \geq f(\varphi) &= 1 - \frac{\int_{\varphi}^{\pi/2} \exp\left(\frac{1}{2\sigma^2}\left[-2(c-b)\eta + (d-a+k)\cos 2\eta - (b+c)\sin 2\eta\right]\right)\,d\eta}{\int_{-\pi/2}^{\pi/2} \exp\left(\frac{1}{2\sigma^2}\left[-2(c-b)\eta + (d-a+k)\cos 2\eta - (b+c)\sin 2\eta\right]\right)\,d\eta} \\
&\geq 1 - M\exp(-k),
\end{aligned}
$$

and expanding the exp-term in $\varphi \mapsto g(\varphi)$ in Taylor series around $\pi/2$ gives

$$g(\varphi) = 1 + \frac{c-b}{\sigma^2}(\varphi - \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^3 + e_2(k) \quad (21)$$

with $|e_2(k)| \leq Mk^{-1}$, for all $\varphi \in J_k$.

Based on these estimates and expansions we would expect, loosely speaking, that

$$\int_{I_k \cup J_k} q(\varphi)m(\varphi)g(\varphi)\,d\varphi \approx \int_{I_k \cup J_k} q(\varphi)m(\varphi)\,d\varphi.$$

Actually, this is not as simple as it looks. The problem is that the right hand side is $O(1/\sqrt{k})$, and so we have to be careful with arguments based on any "errors" being

small for $k \to \infty$.

We proceed more carefully: On I_k we obtain, invoking (20),

$$q(\varphi)m(\varphi)g(\varphi)$$
$$= q(\varphi)m(\varphi)$$
$$\times \left[1 + \frac{c-b}{\sigma^2}(\varphi + \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^3 + e_1(k) \right].$$

So

$$q(\varphi)m(\varphi)g(\varphi) - q(\varphi)m(\varphi)$$
$$= q(\varphi)\exp\left(\frac{b+c}{2\sigma^2}\sin 2\varphi\right)\exp\left(-\frac{k+d-a}{\sigma^2}\cos^2\varphi\right)$$
$$\times \left[\frac{c-b}{\sigma^2}(\varphi + \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^3 + e_1(k) \right].$$

Similarly on J_k we obtain from (21)

$$q(\varphi)m(\varphi)g(\varphi) - q(\varphi)m(\varphi)$$
$$= q(\varphi)\exp\left(\frac{b+c}{2\sigma^2}\sin 2\varphi\right)\exp\left(-\frac{k+d-a}{\sigma^2}\cos^2\varphi\right)$$
$$\times \left[\frac{c-b}{\sigma^2}(\varphi - \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^3 + e_2(k) \right].$$

We consider the contributions of the terms $\left((a-d-k)\cos^2\varphi\right)$ and $\left((b+c)\cos\varphi\sin\varphi+d\right)$ in $q(\varphi)$ separately. The term $\left((a-d-k)\cos^2\varphi\right)$ contributes

$$\int_{I_k} (a-d-k)\cos^2\varphi \, \exp\left(\frac{b+c}{2\sigma^2}\sin 2\varphi\right) \, \exp\left(-\frac{k+d-a}{\sigma^2}\cos^2\varphi\right)$$
$$\times \left[\frac{c-b}{\sigma^2}(\varphi + \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi + \pi/2)\right)^3 + e_1(k) \right] d\varphi$$

to $\int_{I_k}\left(q(\varphi)m(\varphi)g(\varphi) - q(\varphi)m(\varphi)\right)d\varphi$. The corresponding contribution from J_k is

$$\int_{J_k} (a-d-k)\cos^2\varphi \, \exp\left(\frac{b+c}{2\sigma^2}\sin 2\varphi\right) \, \exp\left(-\frac{k+d-a}{\sigma^2}\cos^2\varphi\right)$$
$$\times \left[\frac{c-b}{\sigma^2}(\varphi - \pi/2) + \frac{1}{2}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^2 + \frac{1}{6}\left(\frac{c-b}{\sigma^2}(\varphi - \pi/2)\right)^3 + e_2(k) \right] d\varphi.$$

Taking Taylor expansions of $\varphi \mapsto \sin 2\varphi$ and $\varphi \mapsto \cos^2\varphi$ around $-\pi/2$ and $\pi/2$, respectively, changing variables, and combining the contributions of the $(a-d-k)\cos^2\varphi$-terms from I_k and J_k we arrive at an error term of the form

$$\int_0^{\frac{1}{\sqrt{2\sqrt{k}}}} (a-d-k)\, x^2 \, \exp\left(-\frac{k+d-a}{\sigma^2}x^2\right)$$
$$\times \left[\left(\exp\left(-\frac{b+c}{\sigma^2}x\right) - \exp\left(\frac{b+c}{\sigma^2}x\right)\right)\left(\frac{c-b}{\sigma^2}x + \frac{1}{6}\left(\frac{c-b}{\sigma^2}\right)^3\right) \right. \tag{22}$$
$$\left. + \frac{1}{2}\left(\exp\left(-\frac{b+c}{\sigma^2}x\right) + \exp\left(\frac{b+c}{\sigma^2}x\right)\right)\left(\frac{c-b}{\sigma^2}x\right)^2 \right] dx \; + \; e_3(k)$$

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with $|e_3(k)| \leq Mk^{-3/2}$. Here we used that, after the change of variables around $\pi/2$ and around $-\pi/2$, respectively, $\cos^2 \varphi$ gives a term of the order $x^2 + O(x^4)$. Using integration by parts one verifies that the error terms $e_1(k)$ and $e_2(k)$ contribute an $O(k^{-3/2})$ term, and the $O(x^4)$ term from $\cos^2 \varphi$ contributes an $O(k^{-2})$ term.

Identifying the exponential terms inside the square brackets as $-2\sinh(\frac{b+c}{\sigma^2}x)$ and $\cosh(\frac{b+c}{\sigma^2}x)$, and expanding $\sinh x$ and $\cosh x$, respectively, around $x = 0$, (22) becomes

$$ Mk \int_0^{\frac{1}{\sqrt{2\sqrt{k}}}} x^4 \exp\left(-\frac{k+d-a}{\sigma^2}x^2\right) dx + O(k^{-3/2}) $$

for k large. Again using integration by parts this can be seen to grow not faster than $Mk^{-3/2}$ for $k \to \infty$ (recall that M is a variable constant).

Using similar calculations in which we again need to combine contributions from I_k and J_k we can show that the term $\big((b+c)\cos\varphi\sin\varphi + d\big)$ in $q(\varphi)$ contributes to the integral $\int_{I_k \cup J_k} \big(q(\varphi)m(\varphi)g(\varphi) - q(\varphi)m(\varphi)\big)\,d\varphi$ an error term bounded in magnitude by $Mk^{-3/2}$.

Collecting the above estimates we obtain

$$ \left| \int_{I_k \cup J_k} q(\varphi)m(\varphi)g(\varphi)\,d\varphi - \int_{I_k \cup J_k} q(\varphi)m(\varphi)\,d\varphi \right| \;\leq\; Mk^{-3/2} \qquad \text{and} $$

$$ \left| \int_{I_k \cup J_k} m(\varphi)g(\varphi)\,d\varphi - \int_{I_k \cup J_k} m(\varphi)\,d\varphi \right| \;\leq\; Mk^{-3/2} $$

for some positive M and k sufficiently large. In view of (18) and (19) this implies

$$ \left| \int_{-\pi/2}^{\pi/2} q(\varphi)m(\varphi)g(\varphi)\,d\varphi - \int_{-\pi/2}^{\pi/2} q(\varphi)m(\varphi)\,d\varphi \right| \;\leq\; Mk^{-3/2} \qquad \text{and} \qquad (23) $$

$$ \left| \int_{-\pi/2}^{\pi/2} m(\varphi)g(\varphi)\,d\varphi - \int_{-\pi/2}^{\pi/2} m(\varphi)\,d\varphi \right| \;\leq\; Mk^{-3/2}\,. \qquad (24) $$

Now put

$$ I_{\text{sym}} = \int_{-\pi/2}^{\pi/2} q(\varphi)m(\varphi)\,d\varphi \qquad \text{and} \quad J_{\text{sym}} = \int_{-\pi/2}^{\pi/2} m(\varphi)\,d\varphi\,; $$

the subscript 'sym' refers to the symmetric case treated in section 3.1. From (16) and (17) we obtain

$$ I_{\text{sym}} \;=\; \pi\Big(d - \frac{\sigma^2}{2}\Big)\exp\Big(\frac{a-d}{2\sigma^2}\Big)\frac{\sqrt{\sigma^2}}{k^{1/2}} \;+\; \eta_1(k) $$

$$ J_{\text{sym}} \;=\; \pi\exp\Big(\frac{a-d}{2\sigma^2}\Big)\frac{\sqrt{\sigma^2}}{k^{1/2}} \;+\; \eta_2(k) $$

with both $|\eta_j(k)| \leq Mk^{-3/2}$, $j = 1, 2$, where we note that the derivation of (16) and (17) goes through without changes if $2b$ is replaced by $b + c$.

Rewriting (23) and (24) with this notation gives

$$ I = I_{\text{sym}} + e_4(k) \qquad \text{and} \quad J = J_{\text{sym}} + e_5(k)\,, $$

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where $|e_j(k)| \leq Mk^{-\frac{3}{2}}$, $j = 4, 5$. So finally, we obtain

$$\lambda_{k,\sigma} = \frac{I}{J} = \frac{I_{\text{sym}} + e_4(k)}{J_{\text{sym}} + e_5(k)} = d - \frac{\sigma^2}{2} + e_6(k),$$

where, due to the $k^{-1/2}$-term in I_{sym} and J_{sym}, respectively, $|e_6(k)| \leq Mk^{-1}$. This proves the claim. $\qquad \square$

4 Exponential growth rates in second mean

In the calculation of Lyapunov exponents in the previous section we adopted a Stratonovich interpretation. To calculate the exponential growth rate of the second mean, i.e., $g(2)$, it is more suitable to work with an Itô interpretation. So in order to compare results between the two notions of growth rate we need to transform (9) from Stratonovich to Itô form. A general linear Stratonovich SDE

$$dx = Ax\,dt + \sum_{j=1}^{m} A_j x \circ dW_j(t)$$

is equivalent to the Itô LSDE

$$dx = \left(A + \tfrac{1}{2} \sum_{j=1}^{m} A_j^2\right) x\,dt + \sum_{j=1}^{m} A_j x\,dW_j(t).$$

In our case this transforms (9) to

$$dx = \begin{pmatrix} a - k - \tfrac{1}{2}\sigma^2 & b \\ c & d - \tfrac{1}{2}\sigma^2 \end{pmatrix} x\,dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x\,dW(t).$$

For an Itô equation

$$dx = A_0 x\,dt + A_1 x\,dW(t),$$

$g(2)$ coincides with the smallest $\eta \in \mathbf{R}$ such that there exists a symmetric, positive definite matrix P with

$$A_0^T P + PA_0 + A_1^T PA_1 \leq \eta P. \tag{25}$$

Indeed this follows from (25) by adopting $\|x\| = \langle x, Px \rangle^{1/2}$ as a norm in the definition of $g(2)$ and by using the Itô formula applied to $V(t) = \langle x(t), Px(t) \rangle$.

4.1 Proposition *The exponential growth rate of the second mean of the linear Stratonovich SDE (9), which is $g(2)/2$ with $g(2) = g_{k,\sigma}(2)$ given by (5), satisfies*

$$g_{k,\sigma}(2) = (2d - \sigma^2) + O(k^{-1}),$$

hence, for σ fixed,

$$\lim_{k \to \infty} g_{k,\sigma}(2) = 2d - \sigma^2$$

for every a, b, c, d.

PROOF From (6) we already know that $g_{k,\sigma}(2) \geq 2\lambda_{k,\sigma}$ for every k, σ, hence Theorem 3.4 yields

$$g_{k,\sigma}(2) \geq (2d - \sigma^2) + O(k^{-1}) \tag{26}$$

for $k \to \infty$. It remains to show that $g_{k,\sigma}(2) \leq (2d - \sigma^2) + O(k^{-1})$, i.e., that there exist η and $P > 0$ such that

$$\begin{bmatrix} a-k-\frac{1}{2}\sigma^2 & c \\ b & d-\frac{1}{2}\sigma^2 \end{bmatrix} P + P \begin{bmatrix} a-k-\frac{1}{2}\sigma^2 & b \\ c & d-\frac{1}{2}\sigma^2 \end{bmatrix} + \sigma^2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} P \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} < \eta P \tag{27}$$

and $\eta \leq (2d - \sigma^2) + O(k^{-1})$. Without loss of generality we choose $P = \begin{bmatrix} 1 & q \\ q & p \end{bmatrix}$.
Denoting by $Q(P)$ the left hand side of (27) we obtain

$$Q(P) = \begin{pmatrix} 2a - 2k - (1-p)\sigma^2 + 2cq & (a - k + d - 2\sigma^2)q + b + cp \\ (a - k + d - 2\sigma^2)q + b + cp & 2dp + (1-p)\sigma^2 + 2bq \end{pmatrix}.$$

Choosing $p = p(k) = 1 + \frac{k}{\sigma^2}$ we obtain

$$Q(P) = \begin{pmatrix} 2a - k + 2cq & (a + d - k - 2\sigma^2)q + b + c(1 + \frac{k}{\sigma^2}) \\ (a + d - k - 2\sigma^2)q + b + c(1 + \frac{k}{\sigma^2}) & (2d - \sigma^2 + \frac{\sigma^2}{p} + \frac{2bq}{p})p \end{pmatrix},$$

where we left p unresolved in the lower right entry. Now choose, for k sufficiently large, $q = q(k) = \frac{b+cp}{k+2\sigma^2-a-d}$. For this choice of $P = P(k)$ we obtain

$$Q(P) = \begin{pmatrix} 2a - k + 2cq & 0 \\ 0 & ((2d - \sigma^2) + \frac{\sigma^2}{p} + \frac{2bq}{p})p \end{pmatrix} \leq \left((2d - \sigma^2) + O(k^{-1})\right) P. \tag{28}$$

Combining (26) and (28) we conclude that for $k \to \infty$ the second mean exponent $g_{k,\sigma}(2)$ satisfies

$$g_{k,\sigma}(2) = 2d - \sigma^2 + O(k^{-1})$$

as required. □

Until now we have assumed that the noise enters in a skew-symmetric way. For the following slightly more general case the arguments carry over in a rather straightforward manner.

4.2 Theorem *Consider the linear Stratonovich SDE*

$$dx = \begin{pmatrix} a - k & b \\ c & d \end{pmatrix} x \, dt + \sigma \begin{pmatrix} \gamma & -1 \\ 1 & \gamma \end{pmatrix} x \circ dW(t) \tag{29}$$

with $\gamma \in \mathbf{R}$. Then:

(i) *The almost sure exponential growth rate of* (29), *the Lyapunov exponent* $\lambda_{k,\sigma}$, *is independent of* γ. *In particular,*

$$\lambda_{k,\sigma} = d - \frac{\sigma^2}{2} + O(k^{-1}) \qquad \text{and therefore} \qquad \lim_{k\to\infty} \lambda_{k,\sigma} = d - \frac{\sigma^2}{2}$$

for every $\gamma \in \mathbf{R}$;

(ii) *The exponential growth rate of the* 2nd *mean,* $g_{k,\sigma,\gamma}(2)/2$, *satisfies*

$$g_{k,\sigma,\gamma}(2) = 2d + 2\gamma^2 - \sigma^2 + O(k^{-1})$$

for large k. *In particular,*

$$\lim_{k\to\infty} g_{k,\sigma,\gamma}(2) = 2d + 2\gamma^2 - \sigma^2 \, .$$

NOTE The theorem can be rephrased as: $\dfrac{g(2)}{2} = \lambda + \gamma^2$ asymptotically.

PROOF It is straightforward to see that the leading Lyapunov exponent of (29) is independent of γ. Indeed, since $\gamma \operatorname{Id}$ commutes with all matrices, solutions of (29) and (9) can be transformed into each other as follows. If $x(t; x_0)$ is a solution of (9), then $y(t; x_0) = x(t; x_0)e^{\gamma\sigma W(t)}$ is a solution of (29). Since $\lim W(t)/t = 0$ a. s. for $t \to \infty$, the Lyapunov exponents of (29) are the same as those of (9). Consequently, application of Theorem 3.4 to (29) for $\gamma = 0$ proves (i).

Concerning the exponential growth rate of the 2nd mean, denote by I the identity matrix, and put $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Transforming the Stratonovich SDE $dx = A_0 x\, dt + A_1 x \circ dW(t)$ with $A_1 = \gamma I + \sigma J$ to Itô form gives

$$dx = (A_0 + \tfrac{1}{2}A_1^2)x\, dt + \sigma A_1 x\, dW(t) \, ,$$

where $A_1^2 = (\gamma^2 - \sigma^2)I + 2\gamma\sigma J$. In order to determine $g(2)$ we have to find the smallest $\eta \in \mathbf{R}$ for which there exists a positive definite P such that

$$\left(A_0 + \tfrac{1}{2}(\gamma^2 - \sigma^2)I + \gamma\sigma I\right)^T P + P\left(A_0 + \tfrac{1}{2}(\gamma^2 - \sigma^2)I + \gamma\sigma I\right) + \left(\gamma I + \sigma J\right)^T P\left(\gamma I + \sigma J\right) \le \eta P \, . \tag{30}$$

Now the left hand side of (30) equals

$$\left(A_0 + (\gamma^2 - \frac{\sigma^2}{2})I + 2\gamma\sigma J\right)^T P + P\left(A_0 + (\gamma^2 - \frac{\sigma^2}{2})I + 2\gamma\sigma J\right) + \sigma^2 J^T P J \, ,$$

which is simply (27) for the Itô LSDE

$$dx = \left(A_0 + (\gamma^2 - \frac{1}{2}\sigma^2)I + \gamma\sigma J\right)x\, dt + \sigma J x\, dW(t) \, ,$$

corresponding to the Stratonovich equation

$$dx = (A_0 + \gamma^2 I + \gamma\sigma J)x\, dt + \sigma J x \circ dW(t) \, . \tag{31}$$

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Since (31) has the same form as (9), and, in particular,

$$A_0 + \gamma^2 I + \gamma \sigma J = \begin{pmatrix} \tilde{a} - k & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}$$

with $\tilde{d} = d + \gamma^2$, it follows by applying Proposition 4.1 that

$$g(2) = 2d + 2\gamma^2 - \sigma^2 + O(k^{-1})$$

for $k \to \infty$, proving (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.3 Remark For the case of a skew-symmetric diffusion matrix we see from the calculations of λ and $g(2)$ that whilst

$$g(2) > 2\lambda$$

(the inequality is strict here, see Arnold, Oeljeklaus and Pardoux [5]), in the limit, for high-gain k tending to infinity,

$$\lim_{k \to \infty} g_{k,\sigma}(2) = 2 \lim_{k \to \infty} \lambda_{k,\sigma} = 2d - \sigma^2 \,.$$

This means that the functions $p \mapsto g_{k,\sigma}(p)$ converge with k to infinity to the linear $p \mapsto (d - \frac{1}{2}\sigma^2)p$, uniformly in $p \in [0,2]$, for every $\sigma > 0$. So the high-gain feedback leads to a degeneracy in $p \mapsto g(p)$.

4.4 Remark In fact, a similar degeneracy occurs in the case of the high-intensity noise problem of Arnold, Crauel and Wihstutz [2]. Recall that for the Stratonovich equation

$$dx = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x \, dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \circ dW(t)$$

we have

$$\lim_{\sigma \to \infty} \lambda = \frac{1}{2}(a + d) \,.$$

Transforming into the equivalent Itô equation

$$dx = \begin{pmatrix} a - \frac{1}{2}\sigma^2 & b \\ c & d - \frac{1}{2}\sigma^2 \end{pmatrix} x \, dt + \sigma \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \, dW(t) \,.$$

and invoking the characterization of $g(2)$ given around (25) one obtains

$$\lim_{\sigma \to \infty} g(2) = (a + d) \,.$$

To see this, we again need to find η so that

$$\tilde{Q}(P) := \begin{pmatrix} 2a - (1-p)\sigma^2 + 2cq & (a+d-\sigma^2)q + b + cp \\ (a+d-\sigma^2)q + b + cp & 2dp + (1-p)\sigma^2 + 2bq \end{pmatrix} \leq \eta \begin{pmatrix} 1 & q \\ q & p \end{pmatrix} \,.$$

Put $p = 1 + \dfrac{d-a}{\sigma^2}$. Then we need

$$
\begin{pmatrix} a+d+2cq & (a+d-2\sigma^2)q + b + c(1+\frac{d-a}{\sigma^2}) \\ (a+d-2\sigma^2)q + b + c(1+\frac{d-a}{\sigma^2}) & a+d-d\frac{(a-d)}{\sigma^2} + 2bq \end{pmatrix} \le \eta \begin{pmatrix} 1 & q \\ q & p \end{pmatrix}.
$$

Choosing

$$
q = \frac{b + c(1 + (\frac{d-a}{\sigma^2}))}{2\sigma^2 - a - d} = O(\frac{1}{\sigma^2}),
$$

for $2\sigma^2 > a + d$, gives

$$
\tilde{Q}(P) = \begin{pmatrix} a+d+O(\frac{1}{\sigma^2}) & 0 \\ 0 & a+d+O(\frac{1}{\sigma^2}) \end{pmatrix} \le \left((a+d) + O(\frac{1}{\sigma^2}) \right) \begin{pmatrix} 1 & q \\ q & p \end{pmatrix},
$$

so that $g_\sigma(2) \le (a+d) + O(\sigma^{-2})$. Since $g_\sigma(2) \ge 2\lambda_\sigma$ and

$$
\lim_{\sigma \to \infty} \lambda_\sigma = (a+d)/2,
$$

we conclude that

$$
\lim_{\sigma \to \infty} g_\sigma(2) = a + d.
$$

4.5 Remark In the case of a more general diffusion matrix considered in Theorem 4.2 the degeneration described in Remark 4.3 does not occur. In fact, here noise assisted high-gain stabilization may take place with respect to almost sure stability, but not with respect to second mean stability. In particular, with a diffusion matrix of the form $\sigma \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ we have convergence of the almost sure exponential growth rate $\lambda_{k,\sigma}$ to $d - \sigma^2/2$ for $k \to \infty$, whereas $g_{k,\sigma}(2)/2$ converges to $d + \sigma^2/2$ for $k \to \infty$.

5 Concluding remarks

We have considered the dependence of growth rates on the feedback gain k for the simplest case of an LSDE arising from proportional feedback applied to second order, relative degree one control system. We have obtained explicit formulas for the Lyapunov exponents and asymptotics of the growth rates valid for large enough k. We have shown, in particular, that in case of a purely skew-symmetric noise the asymptotic dependence on k for k large is the same whether we consider λ, the leading Lyapunov exponent, or whether we consider the growth rate in p^{th} mean for any $p \in [0, 2]$. This contrasts with the situation where for any fixed k we would have that

$$
\lambda < \frac{g(p)}{p} < \frac{g(q)}{q}
$$

for all $p < q$.

As an application of these asymptotic estimates we see that the Stratonovich equation (9) is high-gain stabilizable (almost surely or in p^{th} mean, $0 < p \le 2$) if, and only if,

$$\sigma^2 > 2d \,.$$

In particular, for $\sigma \ne 0$ we can allow $d > 0$, i.e., unstable zero dynamics, whereas if $\sigma = 0$ we need $d < 0$ i.e., the zero dynamics to be exponentially stable. We see in this simple example that the noise has a *stabilizing effect*.

When the noise enters in a certain non skew-symmetric way then the same comments apply to almost sure growth rates, so the noise is still *stabilizing with respect to almost sure stability*. However, the same noise is *destabilizing with respect to stability in* 2nd *mean*.

6   Acknowledgments

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