On Reflecting Boundary Problem for Optimal Control

Oana-Silvia Serea

ON REFLECTING BOUNDARY PROBLEM FOR OPTIMAL CONTROL*

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Abstract. This paper deals with Mayer's problem for controlled systems with reflection on the boundary of a closed subset K. The main result is the characterization of the *possibly discontinuous* value function in terms of a unique solution *in a suitable sense* to a partial differential equation of Hamilton–Jacobi–Bellman type.

Key words. control of variational inequality, boundary reflection, viscosity solutions

1. Introduction. We investigate the Mayer control problem:

$$\text{Minimize } g(x(T)) \tag{1}$$

for a given $T > 0$ over all absolutely continuous solutions of the following differential variational inequality:

$$\begin{cases} \text{(i) } x'(t) \in f(x(t), u(t)) - N_K(x(t)) \text{ for almost all } t \geq t_0, \\ \text{(ii) } x(t) \in K \text{ for all } t \geq t_0, \; x(t_0) = x_0, \text{ and} \\ \; u(\cdot) : [0, \infty) \to U \text{ is a measurable function,} \end{cases} \tag{2}$$

where $N_K(x)$ is the normal cone to K at $x \in K$ (see Definition 1).

Here K is a nonempty closed subset of $\mathbb{R}^N$, $g : K \to \mathbb{R}$ and f is a function from $K \times U$ into $\mathbb{R}^N$.

If $\mathcal{U}(t_0)$ is the set of measurable controls on $[t_0, \infty)$ with values in U, the value function corresponding to the optimal control problem (1), (2) is given by

$$V(t_0, x_0) = \inf_{u(\cdot) \in \mathcal{U}(t_0)} g(x(T; t_0, x_0, u(\cdot)) \text{ for all } (t_0, x_0) \in [0, T] \times K, \tag{3}$$

where $x(\cdot; t_0, x_0, u(\cdot))$ denotes the solution of (2) starting from (t_0, x_0).

By the very definition it is easy to see that the value function is finite on its domain $[0, T] \times K$, if and only if (2) has solutions. This explains the choice of the form of the right-hand side of the differential inclusion (2). We notice that $N_K(x) = \{0\}$ whenever $x \in \mathrm{int}K$; f is modified only on the boundary of K, so (2) is a problem with reflection at the boundary. We shall show that this reflection allows us to obtain the existence of solutions to (2) (see section 1).

Our main purpose in this paper is to characterize the value function (3) by an equation of Hamilton–Jacobi type.

Of course, the characterization is based on a suitable definition for the notion of viscosity solutions of a Hamilton–Jacobi–Bellman inequality (HJBI) that we will introduce below.

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More precisely, we will prove that the value function V is the unique solution to

$$\text{(HJBI)} \qquad \begin{cases} \frac{\partial V}{\partial t}(t,x) + H(x, \frac{\partial V}{\partial x}(t,x)) - \left\langle \frac{\partial V}{\partial x}(t,x),\ N_K(x) \right\rangle \ni 0 \\ \qquad \text{if } (t,x) \in [0,T) \times K, \\ \quad \text{with the condition } V(T,x) = g(x) \text{ if } x \in K, \end{cases}$$

where $H(x,p) := \min_{u \in U} \langle f(x,u), p \rangle$.

If the boundary of K, $\partial K \in C^1$, and K is the closure of an open set, we will show that V is a viscosity solution of the following Hamilton–Jacobi equation with Neumann-type boundary condition in the sense of [13]:

$$(4) \qquad \begin{cases} \frac{\partial V}{\partial t}(t,x) + H(x, \frac{\partial V}{\partial x}(t,x)) = 0 \text{ if } (t,x) \in [0,T) \times K, \\ \frac{\partial V}{\partial n}(t,x) = 0 \text{ if } (t,x) \in [0,T) \times \partial K, \\ \text{with the condition } V(T,x) = g(x),\ x \in K, \end{cases}$$

where $n(x)$ is the unit outward normal to K at $x \in \partial K$.

It is well known that the value function for the Skorokhod control problem (see [3], [13]) with a smooth K is a viscosity solution of (4). The Skorokhod problem for a smooth K has been considered and solved by Lions [13] and Lions and Snitzman [14]. Another study was made by Tanaka in [18] when K is convex with normal reflection. By a different approach, this problem was considered like a viability problem for a differential inclusion by Frankowska in [8]. Note that the notion of solutions of the Skorokhod problem is not the same as the notion of solutions to (2) that we use in this paper, but for the smooth case the two control problems lead to the same Hamilton–Jacobi equation (4).

Our second interest is to establish that the two following systems,

$$(5) \qquad \begin{cases} \text{(i) } x'(t) \in F(x(t)) - N_K(x(t)) \text{ for almost all } t \geq t_0, \\ \text{(ii) } x(t) \in K \text{ for all } t \in [t_0, \infty),\ t_0 \geq 0,\ x(t_0) = x_0 \in K \end{cases}$$

and

$$(6) \qquad \begin{cases} \text{(i) } x'(t) \in \Pi_{\overline{co}T_K(x(t))} F(x(t)) \text{ for almost all } t \geq t_0, \\ \text{(ii) } x(t) \in K \text{ for all } t \in [t_0, \infty),\ t_0 \geq 0,\ x(t_0) = x_0 \in K, \end{cases}$$

have the same set of solutions.

Here K is compact, $F : K \to \mathbb{R}^N$ is a set valued map and $\overline{co}A$ is the closed convex hull of a set A.

In general, the map $x \to N_K(x)$ has no easy continuity properties and so the right side of the differential inclusion is (2). For this reason the set of solutions to (5) or (6) may be empty. So it is necessary to find regularity hypotheses for K in order to provide existence and eventually uniqueness results for (5) or (6).

These kind of results for a general map F can be applied, in particular, when $F(\cdot) = f(\cdot, U)$, allowing us to obtain properties of the set of solutions to (2).

Our main contribution here is the fact that by introducing the projection on the closed convex hull of $T_K(x)$ in (6) we succeed in treating the case where the set K is only compact, improving the already known equivalence and existence results of [2] where K is supposed to be sleek.

Existence and equivalence results for (5) and (6) are established by Henry [11] for a convex set. The convexity assumption on the set K, has been relaxed by Cornet in [6], who merely required the tangential regularity. We also refer to Thibault [19] for the case of a closed set K for an existence result of viable solution, but the reflection is made using the Clarke normal cone. Note that in [19] the set K may depend on t.

We note that the boundary reflection control problem was not yet well studied for nonsmooth K. We also succeed in generalizing some existence and equivalence results of [2] for the systems (5) and (6).

Let us explain how this paper is organized.

In the first section we introduce some preliminaries and we study the systems (5), (6).

In the second section we prove that the value function is a viscosity solution of (HJBI) in the sense of Definition 3, and a uniqueness result for the solutions of this partial differential inequality is also established.

In the third section we study the case of discontinuous and only bounded value functions for our control problem. Our main result says that V is the unique generalized solution to the corresponding (HJBI) for arbitrary discontinuous terminal cost g.

The fourth section concerns existence and uniqueness results of l.s.c. solutions to (HJBI) in the sense of Definition 16.

The last section is an appendix with technical proofs of our claims.

2. Preliminaries.

2.1. Definitions, assumptions, and notations. We assume that $f : K \times U \to \mathbb{R}$ is continuous and satisfies

$$(\mathrm{H}_f) \qquad \begin{cases} \|f(x,u)\| \le a(1+\|x\|), \\ \|f(x,u) - f(y,u)\| \le c_1 \|x-y\| \text{ for all } x, y \in K, u \in U, \\ \text{the set } f(x,U) \text{ is convex,} \end{cases}$$

where $c_1, a > 0$ are constants; U is a compact metric space.

We recall the notions of tangent and normal cones.

Definition 1. *For $x \in K$, we define by*

$$T_K(x) = \left\{ v \in \mathbb{R}^N | \lim_{h \to 0^+} \inf d_K(x+hv)/h = 0 \right\}$$

the tangent cone to K at x and by

$$N_K(x) = T_K(x)^- = \{ p \in \mathbb{R}^N | \langle p,\ v \rangle \le 0 \textit{ for all } v \in T_K(x) \}$$

the normal cone to K at x.

Recall that $T_K(x)$ is a closed cone and $N_K(x)$ is a closed convex cone.

Let us describe some classes of sets which will be used in the following sections.

Definition 2. *A closed set $K \subset \mathbb{R}^N$ is called proximal retract if there exists a neighborhood I of K such that the projection $\Pi_K(\cdot)$ is single-valued in I, with $\Pi_K(x) := \{z \in K \mid \|x-z\| = \inf_{y \in K} \|x-y\|\}$ for all $x \in \mathbb{R}$.*

We will describe some of the properties of such sets. This will be the key for the proof of the existence and uniqueness results concerning (5) and (HJBI). The class of proximal retracts includes closed, convex subsets of $\mathbb{R}^N$ and submanifolds of $\mathbb{R}^N$ of class $C^{1,1}$. Another class of proximal retracts is the class of weakly convex sets (see [8] for the definition and the geometrical interpretation). A complete characterization of proximal retract sets is made in [17] (see Theorem 4.1, p. 5245). In particular, such sets have the property that there exists $\rho > 0$ such that every nonzero normal "can be realized" by a ball with a radius equal to ρ. This characterization says, in particular, that only "exterior" corners are allowed.

So, if K is proximal retract, then from Theorem 4.1 in [17], Lemma 4.2 and Theorem 2.2 in [6] we have the following:

- There exist $r, c > 0$ such that the application $x \to N_K(x) \cap B(0,r) + cx$ is monotone[1] on K. This monotonicity property, which is equivalent to Definition 2, is very important because it allows us to establish the uniqueness of solutions to (2).
- The set K is sleek, i.e., the map $x \to T_K(x)$ is l.s.c.
- For all $x \in K$, $T_K(x) = C_K(x)$, where $C_K(x)$ denotes Clarke's tangent cone.[2]

Note that the class of sleek sets is larger then the class of proximal retracts.

2.2. Viscosity solutions. To describe the value function as a unique solution to the corresponding HJBI, we introduce the following definition of solutions to (HJBI).

Definition 3. *A viscosity supersolution of* (HJBI) *is an l.s.c. function* $\psi : (0,T) \times K \to \mathbb{R}$ *such that*

$$\begin{gathered}
\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg\min\,(\psi - \phi)\,, \\
\text{if } x_0 \in \mathrm{int} K,\ \frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) \leq 0 \\
\text{and if } x_0 \in \partial K,\ \text{there exists } y_0 \in N_K(x_0) \text{ such that} \\
\frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) - \left\langle y_0,\ \frac{\partial \phi}{\partial x}(t_0, x_0)\right\rangle \leq 0
\end{gathered}$$

and a viscosity subsolution of (HJBI) *is a u.s.c. function* $\varphi : (0,T) \times K \to \mathbb{R}$ *such that*

$$\begin{gathered}
\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg\max\,(\varphi - \phi)\,, \\
\text{if } x_0 \in \mathrm{int} K,\ \frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) \geq 0 \\
\text{and if } x_0 \in \partial K,\ \text{there exists } z_0 \in N_K(x_0) \text{ such that} \\
\frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) - \left\langle z_0,\ \frac{\partial \phi}{\partial x}(t_0, x_0)\right\rangle \geq 0.
\end{gathered}$$

A viscosity solution of (HJBI) *is a function which is both subsolution and supersolution.*

It is clear that a viscosity solution is a continuous function because it is simultaneously u.s.c. and l.s.c.

Remark 4. A motivation for our definition of (HJBI) is the fact that, when (t_0, x_0) is a differentiability point of V, we have in the usual sense

$$\frac{\partial V}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial V}{\partial x}(t_0, x_0)\right) - \left\langle N_K(x_0),\ \frac{\partial V}{\partial x}(t_0, x_0)\right\rangle \ni 0. \tag{7}$$

[1] Recall that a set valued map $G : K \to \mathbb{R}^N$ is monotone if $\langle y_1 - y_2, x_1 - x_2 \rangle \geq 0$ for all $y_i \in G(x_i)$, $i \in \{1, 2\}$.

[2] $C_K(x) = \{v \mid \lim_{h \to 0+, K \ni x' \to x} d_K(x' + hv)/h = 0\}$. This tangent cone is always convex.

Indeed, there exists $\lambda \in [0,1]$ such that

$$0 = \lambda \left(\frac{\partial V}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial V}{\partial x}(t_0, x_0)\right) - \left\langle y_0, \frac{\partial V}{\partial x}(t_0, x_0)\right\rangle \right)$$
$$+ (1-\lambda) \left(\frac{\partial V}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial V}{\partial x}(t_0, x_0)\right) - \left\langle z_0, \frac{\partial V}{\partial x}(t_0, x_0)\right\rangle \right),$$

and because $N_K(x_0)$ is convex, (7) is verified.

It is quite natural to obtain an equation of the form (7), namely a partial differential inequality. The motivation lies in the fact that for a smooth set, the reflection is channeled in a fixed direction, given by the outward normal. For nonsmooth sets the outward normal will be replaced with the normal cone which, in general, contains many directions.

Note that this definition contains those given by Lions in [13], when the boundary of K, $\partial K \in C^1$.

2.3. Control systems with reflection on the boundary of a constraint set. In this section we study the differential inequalities (5) and (6) by explaining the method which we use in order to get a boundary reflection for closed sets K. This allows us to give some applications to the properties of solutions to the controlled system (2).

We consider a closed set K, a set valued map $F : K \to \mathbb{R}^N$, and the following differential inclusion:

$$\begin{cases} \text{(i) } x'(t) \in F(x(t)) \text{ for almost all } t \geq t_0, \\ \text{(ii) } x(t_0) = x_0 \in K,\ t_0 \geq 0. \end{cases} \tag{8}$$

The equation (6) appears naturally if we want a given closed set to become a viability[3] domain of a new system which is "as close as possible" to the original dynamic system (8).

Indeed, when the necessary and sufficient condition for the existence of viable solutions

$$F(x) \cap T_K(x) \neq \emptyset \text{ for all } x \in K$$

is not satisfied, the natural way to solve the above problem is to introduce the projected problem (6).

We note that $\Pi_{\overline{co}T_K(x)} F(x) = F(x)$ whenever $x \in \text{int}K$; F is modified only on the boundary of K, so (6) is a problem of reflection at the boundary. Moreover, the application $x \to \Pi_{\overline{co}T_K(x)} F(x)$ has no easy continuity properties, but, thanks to the properties of the projection on a convex cone, it is possible to prove that the solutions to (5) and (6) coincide. We do not make any assumption on the regularity of the set K, improving already known results of [2] where the set K is sleek.

It is easier to find sufficient conditions for the set K in order to obtain continuity properties of the right-hand side of (5). So, for the study of existence and uniqueness of solutions we consider (5). We have the following proposition.

Proposition 5. (i) *Suppose that K is closed and F is a set valued map. Then the sets of absolutely continuous solutions to* (5) *and* (6) *are equal.*

[3]Recall that a solution $x(\cdot)$ to (8) is called viable in K if $x(t) \in K$ for all $t \geq 0$. The set K is a viability domain for (8) if for all $x_0 \in K$ there exists a solution to (8) which is viable in K.

Moreover if F is a Marchaud map[4] *and K is bounded and sleek, then*

(ii) *for every $(t_0, x_0) \in [0, \infty) \times K$ there exists a solution of* (5) *or equivalently of* (6).

(iii) *the restriction of the map $(t_0, x_0) \in [0, T] \times K \to S_F(t_0, x_0)$ to a compact set C is compact into $[0, \infty) \times K \times W^{1,1}(0, \infty\ ; K)e^{-bt}$ for all b with $b > a$. Here $S_F(t_0, x_0)$ denotes the set of solutions to* (5) *starting from (t_0, x_0).*

Before giving the proof, we note that the first part of the above proposition is a generalization of Theorem 10.1.1 in [2] where the set K is supposed to be sleek; here K is only bounded. The second part recalls well-known existence and compactness results (see [1] and [2]).

Proof. (i) Using Proposition 0.6.4 from [1] we deduce that

$$\Pi_{\overline{co}T_K(x)} F(x) \subset F(x) - N_K(x) \text{ for all } x \in K,$$

and, consequently, a solution to (6) is also a solution to (5).

Conversely, if $x(t) \in K$ for all $t \geq t_0$, we have

$$\lim_{h \to 0} \frac{x(t+h) - x(t)}{h} \in T_K(x(t)) \text{ and } \lim_{h \to 0} \frac{x(t) - x(t-h)}{h} \in -T_K(x(t)) \text{ for a.e. } t \geq t_0,$$

so $x'(t) \in T_K(x(t)) \cap -T_K(x(t)) \subset N_K(x(t))^{\perp}$ a.e. $t \geq t_0$.

Let $t \geq t_0$ be a derivability point of $x(\cdot)$, and let $x'(t) = f(t) - p(t)$ with $f(t) \in F(x(t))$ and $p(t) \in N_K(x(t))$.

The above arguments say that $\langle x'(t) - f(t), x'(t) \rangle = 0$.

Thus, $x'(t) \in \Pi_{\overline{co}T_K(x(t))} f(t) \subset \Pi_{\overline{co}T_K(x(t))} F(x(t))$ for a.e. $t \geq t_0$.

(ii) We prove now the existence of a solution to the differential inequality (5). If K is sleek, then the map $x \to N_K(x)$ has a closed graph.

For all $x \in K$, we set $H(x) = F(x) - a(1 + ||x||)B \cap N_K(x)$, where B denotes the unit ball of $\mathbb{R}^N$.

Because the map $x \to a(1 + ||x||)B \cap N_K(x)$ is Marchaud, H is also Marchaud. Hence by Theorem 2.1.3 in [1] the existence of solutions of (5) follows.

Let us prove that the closed subset K is a viability domain for the differential inclusion (5).

Indeed, using the equality $I - \Pi_{N_K(x)} = \Pi_{T_K(x)}$, we have that for any $x \in K$ and $f \in F(x)$, $f - \Pi_{N_K(x)} f \in (F(x) - N_K(x)) \cap T_K(x)$.

Using the estimation $||\Pi_{N_K(x)} f|| \leq ||f|| \leq a(1 + ||x||)$, we get that $\Pi_{N_K(x)} f \in a(1 + ||x||)B \cap N_K(x)$ and consequently $f - \Pi_{N_K(x)} f \in H(x) \cap T_K(x)$.

The above arguments say that H satisfies the hypotheses of viability theorem 4.2.1 of [1], and since $H(x) \subset F(x) - N_K(x)$, the second part ensues.

(iii) See Theorem 2.2.1 in [1]. $\square$

Now, let us begin a short study of the optimal control problem with reflected trajectories. From now on, we consider that the set valued map F is given by the equality

$$F(x) = f(x, U) = \{f(x, u), u \in U\} \text{ for all } x \in K.$$

[4]A set valued map F from $\mathbb{R}^N$ onto $\mathbb{R}^N$ is called Marchaud map if F is u.s.c. with nonempty compact convex values and has a linear growth.

We denote by $S_f(t_0, x_0)$ the set of absolutely continuous solutions to

$$(9) \qquad \begin{cases} \text{(i) } x'(t) \in f(x(t), u(t)) - N_K(x(t)) \text{ for almost all } t \geq t_0, \\ \text{(ii) } x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \in K \text{ for all } u(\cdot) \in \mathcal{U}(t_0), \end{cases}$$

and by $S_F(t_0, x_0)$ the set of absolutely continuous solutions to

$$(10) \qquad \begin{cases} \text{(i) } x'(t) \in F(x(t)) - N_K(x(t)) \text{ for almost all } t \geq t_0, \\ \text{(ii) } x(t) \in K \text{ for all } t \geq t_0,\ x(t_0) = x_0 \in K. \end{cases}$$

We will prove now that (9) and (10) are equivalent. In this paper, we will use one of these systems to simplify our proofs.

PROPOSITION 6. *Suppose that K is a compact sleek set and (H_f) holds.*

(i) *If $x(\cdot)$ is a solution to* (10) *starting from* $(t_0, x_0) \in [0, T] \times K$, *then there exists $u(\cdot) \in \mathcal{U}(t_0)$ such that $x(\cdot)$ is equal to $x(\cdot; t_0, x_0, u(\cdot))$, the solution of* (9).

(ii) *As a direct consequence of* (i),

$$S_F(t_0, x_0) = S_f(t_0, x_0) \text{ for all } (t_0, x_0) \in [0, \infty) \times K.$$

Proof. We essentially use the fact that K is sleek (which implies that the application $x \to N_K(x)$ has a closed graph) and Theorem 1.14.1 from [1]. Consider $\Phi(t) := \{v \in U |\ x'(t) \in f(x(t), v) - N_K(x(t))\}$ for a.e. $t \geq t_0$. We can prove that the multivalued function Φ has a measurable selection which gives our measurable control $u(\cdot) \in \mathcal{U}(t_0)$. $\square$

Moreover, with an easy computation, using the fact that K is proximal retract and Gronwall's inequality, we obtain the following estimation.

LEMMA 7. *Assume that (H_f) holds true and K is a bounded proximal retract. Then for $x_0(\cdot) \in S_f(t_0, x_0)$, $x_1(\cdot) \in S_f(t_1, x_1)$ with fixed $u(\cdot) \in \mathcal{U}(t_0)$ and for $t \geq t_1 \geq t_0$, there exists $C > 0$, a constant depending on t, such that*

$$\|x_0(t; t_0, x_0, u(\cdot)) - x_1(t; t_1, x_1, u(\cdot))\| \leq C(\|x_0 - x_1\| + |t_0 - t_1|).$$

We omit the proof of Lemma 7 because it is an easy adaptation of Lemma 4.4, p. 143, proved in [6]. As a direct consequence of the above estimation we obtain the following.

COROLLARY 8. *Assume that (H_f) holds true and K is a bounded proximal retract. Then for fixed $u(\cdot) \in \mathcal{U}(t_0)$ there exists an unique solution of* (2).

2.4. The optimal control problem. First, we give some standard results concerning the regularity of V without proof. Later we shall prove the existence and the uniqueness of viscosity solutions of (HJBI).

LEMMA 9. *Suppose that (H_f) holds true and K is a compact proximal retract. Then we have the following:*

(i) *(Existence of an optimal control.) If g is l.s.c., then V is l.s.c. and there exists an optimal trajectory starting from each point $(t_0, x_0) \in [0, T] \times K$, i.e., there exists $\bar{x}(\cdot) \in S_F(t_0, x_0)$ such that*

$$V(t_0, x_0) = g(\bar{x}(T; t_0, x_0, \bar{u}(\cdot))) \text{ for all } (t_0, x_0) \in [0, T] \times K.$$

(ii) *If g is a Lipschitz function, then V is locally Lipschitz and bounded.*

Next we give the Bellman dynamic programming.

PROPOSITION 10 (dynamic programming principle). *Let $g : K \to \mathbb{R}$ be a bounded function, K a compact proximal retract, and suppose that (H_f) holds. Then, for all $(t_0, x_0) \in [0, T] \times K$ we have*

$$(11) \qquad V(t_0, x_0) = \inf_{x \in S_F(t_0, x_0)} V(t_0 + h, x(t_0 + h)) \text{ with } h > 0 \text{ small enough.}$$

3. The Hamilton–Jacobi partial differential variational inequality.

3.1. Existence result. The aim of this section is to provide an existence and a comparison result for viscosity solutions to a partial differential inequality with a kind of boundary conditions for nonsmooth sets, which generalizes first order Hamilton–Jacobi equations with Neumann conditions for smooth sets.

Using the dynamic programming principle we prove that the value function for the control problem (1), (2) is the viscosity solution of (HJBI) in the sense of Definition 3.

PROPOSITION 11. *If K is a compact proximal retract, g a Lipschitz function, and (H_f) holds true, then V is a locally Lipschitz viscosity solution of* (HJBI) *with the final condition $V(T, x) = g(x)$ for all $x \in K$.*

This theorem can be considered as an existence result of solutions to (HJBI).

Proof. First we prove that V is a supersolution.

We consider $(t_0, x_0) \in \arg\min(V - \psi), \psi \in C^1$, with

$$V(t_0, x_0) = \psi(t_0, x_0) \text{ and } V(t, x) \geq \psi(t, x)$$

in a neighborhood of (t_0, x_0).

For all $h > 0$ small enough, there exists $x_h(\cdot) \in S_F(t_0, x_0)$ such that

$$\psi(t_0, x_0) + h^2 = V(t_0, x_0) + h^2 \geq V(t_0 + h, x_h(t_0 + h)) \geq \psi(t_0 + h, x_h(t_0 + h)).$$

For a subset A of $\mathbb{R}^N$ we denote by $B(A, \varepsilon) = \{x \in \mathbb{R}^N | \inf_{y \in A} ||y - x|| \leq \varepsilon\}$. $B(A, \varepsilon)$ denotes the neighborhood of the set A with a radius equal to $\varepsilon > 0$.

Let M be a bound of F on K. Using the Lipschitz property of $F(\cdot)$ and the upper semicontinuity of $N_K(\cdot) \cap B(0, M)$, we have that, for all $\varepsilon > 0$, there exists an $h > 0$ small enough such that the following inclusions hold:

$$\begin{aligned}
\frac{1}{h}(x_h(t_0 + h) - x_0) &\in \frac{1}{h} \int_{t_0}^{t_0+h} (F(x_h(s)) - N_K(x_h(s)) \cap B(0, M))ds \\
&\subset \frac{1}{h} \int_{t_0}^{t_0+h} (F(x_0))ds + B(0, 1)\frac{1}{h} \int_{t_0}^{t_0+h} L||x_h(s) - x_0||ds \\
&\quad - \frac{1}{h} \int_{t_0}^{t_0+h} B(N_K(x_0) \cap B(0, M), \varepsilon)ds \\
&= F(x_0) + B(0, 1)\frac{1}{h} \int_{t_0}^{t_0+h} L||x_h(s) - x_0||ds - B(N_K(x_0) \cap B(0, M), \varepsilon).
\end{aligned}$$

Hence for all $\varepsilon > 0$, there exists a sequence h_n such that $\lim_{n \to \infty} h_n = 0$ and

$$\lim_n \frac{1}{h_n}(x_{h_n}(t_0 + h_n) - x_0) \in F(x_0) - B(N_K(x_0) \cap B(0, M), \varepsilon).$$

Letting $\varepsilon \to 0$ we obtain that

$$\lim_n \frac{1}{h_n}(x_{h_n}(t_0 + h_n) - x_0) \in F(x_0) - N_K(x_0) \cap B(0, M). \tag{12}$$

Moreover,

$$\begin{aligned}
&\lim_n \left(\frac{1}{h_n}[\psi(t_0 + h_n, x_{h_n}(t_0 + h_n; t_0, x_0, u(\cdot))) - \psi(t_0, x_0)] - h_n \right) \qquad (13) \\
&= \lim_n \left(\frac{1}{h_n} \left[\psi \left(t_0 + h_n, x_0 + h_n \left(\frac{1}{h_n}(x_{h_n}(t_0 + h_n) - x_0) \right) - \psi(t_0, x_0) \right] - h_n \right).
\end{aligned}$$

Using (12) and (13) we have the following.

First case ($x_0 \in \mathrm{int}K$). Then $N_K(x_0) = \{0\}$ and there exists $u \in U$ such that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \left\langle \frac{\partial \psi}{\partial x}(t_0, x_0),\ f(x_0, u) \right\rangle \leq 0,$$

and consequently

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \inf_{u \in U} \left\langle \frac{\partial \psi}{\partial x}(t_0, x_0), f(x_0, u) \right\rangle \leq 0.$$

Second case ($x_0 \in \partial K$). Then $\{0\} \subset N_K(x_0)$ and there exist $u \in U$, $y_u \in N_K(x_0)$ such that:

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \left\langle \frac{\partial \psi}{\partial x}(t_0, x_0),\ f(x_0, u) - y_u \right\rangle \leq 0.$$

So, there exists $w_0 = y_u \in N_K(x_0) \cap B(0, M)$ such that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \inf_{u \in U} \left\langle \frac{\partial \psi}{\partial x}(t_0, x_0),\ f(x_0, u) \right\rangle - \left\langle w_0,\ \frac{\partial \psi}{\partial x}(t_0, x_0) \right\rangle \leq 0$$

and V is a supersolution.

The proof of the fact that V is subsolution is similar and we omit it. □

3.2. Uniqueness result. This section concerns the uniqueness of the viscosity solutions of (HJBI). The importance of this result leads us to treat it separately. Moreover, the characterization of the value function as the unique solution of (HJBI) ensues.

THEOREM 12 (uniqueness result in the Lipschitz case). *Assume that (H_f) holds true. Let K be a compact proximal retract and g be a Lipschitz function. Then there exists at most one uniformly continuous viscosity solution of* (HJBI) *which satisfies the final condition $V(T, x) = g(x)$ for all $x \in K$.*

The proof can be adapted from Evans [7]. We only underline that the difference to Evans' proof is due to the monotonicity of the multivalued function $x \to N_K(x) \cap B(0, M) + cx$.

4. The discontinuous case. In this section we investigate the value function V when $g : K \to \mathbb{R}$ is supposed to be bounded. In this case the value is only a bounded function. A natural question is how to use the viscosity theory to describe V. Here we establish a relation between the value and the viscosity sub or supersolutions of (HJBI). This kind of problem has been studied for the Bolza problem in [15], [16].

The main point of this section is to prove the following.

THEOREM 13. *Suppose that K is a proximal retract and (H_f) holds.*

(i) *If g is bounded, then for every $(t, x) \in [0, T] \times K$*

$$V(t, x) = \inf\{\psi(t, x) |\ \psi \textit{ l.s.c. supersolution of } \text{(HJBI)};\ \psi(T, \cdot) \geq g(\cdot)\} \textit{ and}$$
$$V(t, x) = \sup\{\varphi(t, x) |\ \varphi \textit{ u.s.c. subsolution of } \text{(HJBI)};\ \varphi(T, \cdot) \leq g(\cdot)\}.$$

(ii) *If g is l.s.c., then*

$$V = \min\{\psi \mid \psi \textit{ l.s.c. supersolution of } \text{(HJBI)};\ \psi(T, \cdot) \geq g(\cdot)\}.$$

(iii) *If g is u.s.c., then*

$$V = \max\{\varphi \mid \varphi \textit{ u.s.c. subsolution of } (\text{HJBI}) \ ; \ \varphi(T,\cdot) \le g(\cdot)\}.$$

Before giving the proof, we note that the above theorem allows us to get, in particular, a stronger uniqueness result. More precisely, if ψ is an l.s.c. supersolution and φ is a u.s.c. subsolution of (HJBI) satisfying $\psi(T,\cdot) \ge \varphi(T,\cdot)$ on K, then $\psi \ge \varphi$ on $[0,T] \times K$.

Proof. (i) Let ψ be an l.s.c. supersolution of (HJBI) with $\psi(T,\cdot) \ge g(\cdot)$. We want to prove that $V \le \psi$ on $[0,T] \times K$. To do this we use the following lemma proved in the appendix.

Lemma 14. *Assume that* (H_f) *holds true, K is a compact proximal retract, and $\psi : (0,T) \times K \to \mathbb{R}$ is an l.s.c. viscosity supersolution of* (HJBI). *Then for every $(t_0, x_0) \in (0,T) \times K$ there exists a solution $x(\cdot; t_0, x_0, u(\cdot))$ of* (2) *such that*

$$\psi(t, x(t)) \le \psi(t_0, x_0) \textit{ for all } t \in [t_0, T]. \tag{14}$$

So we obtain that there exists an $x(\cdot) \in S_F(t_0, x_0)$ satisfying (14). Hence we have $V(t_0, x_0) \le g(x(T)) \le \psi(T, x(T)) \le \psi(t_0, x_0)$.

Using the very definition of the value function, for all $\varepsilon > 0$ there exists $u_\varepsilon(\cdot) \in \mathcal{U}(t_0)$ such that $g\left(x\left(T; t_0, x_0, u_\varepsilon(\cdot)\right)\right) < V(t_0, x_0) + \varepsilon$.

For $M_1 > \sup_{x \in K} g(x)$ we define $l_\epsilon : \mathbb{R}^N \to \mathbb{R}$ by the following formula:

$$l_\epsilon(x) = \begin{cases} g\left(x\left(T; t_0, x_0, u_\varepsilon(\cdot)\right)\right) \text{ if } x = x\left(T; t_0, x_0, u_\varepsilon(\cdot)\right), \\ M_1, \text{ if } x \ne x\left(T; t_0, x_0, u_\varepsilon(\cdot)\right). \end{cases}$$

Obviously l_ε is l.s.c. so V_{l_ε}, the value function of the control problem with g replaced by l_ε, is a l.s.c. supersolution of (HJBI) and $V_{l_\epsilon}(T,\cdot) = l_\epsilon(\cdot) \ge g(\cdot)$.

We also have $V_{l_\epsilon}(t_0, x_0) = g\left(x\left(T; t_0, x_0, u_\varepsilon(\cdot)\right)\right) \le V(t_0, x_0) + \varepsilon$. By the definition of the infimum we obtain

$$V(t_0, x_0) = \inf\{\psi(t_0, x_0) \mid \psi \text{ l.s.c. supersolution of (HJBI)}; \ \psi(T,\cdot) \ge g(\cdot)\}.$$

Now let us prove the second relation. Let $(t_0, x_0) \in (0,T) \times K$. We denote by $A(t_0, x_0) := \{x(T) | \ x(\cdot) \in S_F(t_0, x_0)\}$. By Proposition 5 $A(t_0, x_0)$ is a compact set. We define $h : \mathbb{R}^N \to \mathbb{R}$ by

$$h(y) = \begin{cases} V(t_0, x_0) \text{ if } y \in A(t_0, x_0), \\ m \text{ if } y \in \mathbb{R}^N \setminus A(t_0, x_0), \end{cases}$$

where $m = \inf_{x \in K} g(x)$. So, h is u.s.c. because $A(t_0, x_0)$ is closed.

Obviously we have that $V_h(t_0, x_0) = V(t_0, x_0)$ and $V_h(T,\cdot) \le h(\cdot) \le g(\cdot)$.

Moreover, V_h is (see Lemma 21 in the appendix) a u.s.c. subsolution for (HJBI).

Now, to complete the proof of (i) we use the definition of the supremum and the following lemma proved in the appendix.

Lemma 15. *Assume that* (H_f) *holds true, K is a compact proximal retract, and $\varphi : (0,T) \times K \to \mathbb{R}$ is a u.s.c. viscosity subsolution of* (HJBI) *such that $\varphi(T,x) \le g(x)$ for all $x \in K$. Then $V(t,x) \ge \varphi(t,x)$ for every $(t,x) \in (0,T) \times K$.*

The proofs of (ii) and (iii) are direct consequences of Lemma 14, Lemma 15, and Lemma 21. ∎

5. On l.s.c. solutions of HJBI with reflection on smooth sets. If $g : \mathbb{R}^N \to \mathbb{R}$ is an l.s.c. function, then V is also l.s.c. In [5], [8] a modification of the concept of viscosity solutions for semicontinuous functions was proposed. This approach is based on a construction of "touching from one side" functions, which is usual for viscosity solutions theory.

We suppose that K is a $C^{1,1}$ submanifold with boundary. If we denote by $n(x)$ the unit outward normal to K at $x \in \partial K$, the normal cone $N_K(x)$ is generated by $n(x)$, i.e., $N_K(x) = [0, \infty)n(x)$ for $x \in \partial K$ and $N_K(x) = \{0\}$ for $x \in \mathrm{int}K$.

We propose a definition for l.s.c. solutions to the HJBI of Barron–Jensen–Frankowska type.

DEFINITION 16. *A viscosity l.s.c. solution of* (HJBI) *is a function* $\psi : [0,T] \times K \to \mathbb{R}$ *such that*

$$\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg\min (\psi - \phi),$$
$$\text{if } (t_0, x_0) \in [0,T) \times \mathrm{int}K, \text{ we have } \frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) \leq 0;$$
$$\text{if } (t_0, x_0) \in (0,T] \times \mathrm{int}K, \text{ we have } \frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) \geq 0;$$
$$\text{if } (t_0, x_0) \in [0,T) \times \partial K, \text{ then there exists } u \in U \text{ such that}$$
$$\frac{\partial \phi}{\partial t}(t_0, x_0) + \left\langle (f(x_0, u) - \Pi_{N_K(x)} f(x_0, u)), \frac{\partial \phi}{\partial x}(t_0, x_0) \right\rangle \leq 0;$$
$$\text{if } (t_0, x_0) \in (0,T] \times \partial K \text{ and } \min_{u \in U} \langle f(x_0, u), n(x_0) \rangle > 0, \text{ then for all } u \in U,$$
$$\frac{\partial \phi}{\partial t}(t_0, x_0) + \left\langle (f(x_0, u) - \Pi_{N_K(x)} f(x_0, u)), \frac{\partial \phi}{\partial x}(t_0, x_0) \right\rangle \geq 0.$$

Note that in $\mathrm{int}K$ the equation is satisfied in the Barron–Jensen–Frankowska sense (see [5], [8]).

We obtain the following uniqueness results.

PROPOSITION 17. *Suppose that K is a $C^{1,1}$ submanifold with boundary and for any $u \in U$ and for all $x_0 \in K$ we have $\langle f(x_0, u), n(x_0) \rangle < 0$. If g is l.s.c. and (H_f) holds true, then the value function V is the unique l.s.c. viscosity solution of* (HJBI) *which verifies the final condition $V(T, x) = g(x)$ for all $x \in K$, and for all $(t, x) \in (0,T] \times \partial K$ we have*

$$\lim \inf_{\substack{(t', x') \to (t, x) \\ x \in intK}} V(t', x') = V(t, x).$$

The proof is similar to the proof of Theorem 2.3 in [8].

PROPOSITION 18. *Suppose that K is a $C^{1,1}$ submanifold with boundary and for any $u \in U$ and for all $x_0 \in K$ we have $\langle f(x_0, u), n(x_0) \rangle > 0$. If g is l.s.c. and (H_f) holds true, then the value function V is the unique l.s.c. viscosity solution of* (HJBI) *which verifies the final condition $V(T, x) = g(x)$ for all $x \in K$, and for all $(t, x) \in (0,T] \times \partial K$ we have*

$$\lim \inf_{\substack{(t', x') \to (t, x) \\ x \in intK}} V(t', x') = V(t, x).$$

Proof. Step 1. V satisfies Definition 16. The proof of the first inequality is similar to the proof of the fact that V is an l.s.c. supersolution of (HJBI).

For proving the second inequality we observe that for all $x_0 \in \partial K$ and $u \in U$,

$$\Phi_u(x_0) := f(x_0,u) - \Pi_{N_K(x)} f(x_0,u) \in \Pi_{T_K(x)} f(x_0,u) = \Pi_{\partial T_K(x)} f(x_0,u).$$

Consequently, $-f(x_0,u) + \Pi_{N_K(x)} f(x_0,u) \in -\Pi_{\partial T_K(x)} f(x_0,u)$.

As K is a $\mathrm{C}^{1,1}$ submanifold and for all $x_0 \in K$, $\min_{u\in U}\langle f(x_0,u), n(x_0)\rangle > 0$, $\Phi_u(\cdot)$ is a Lipschitz application on ∂K. Moreover, ∂K is locally invariant (see [2, viability theorem 3.2.4] by $\Phi_u(\cdot)$ and by $-\Phi_u(\cdot)$ (because $\partial T_K(x) = -\partial T_K(x)$).

Now let $(t_0,x_0) \in \arg\min (V-\phi)$, $\phi \in C^1$. We have two cases.

First case ($x_0 \in \partial K$). For a fixed constant control $u \in U$, we consider the solution of

$$\begin{cases} x'(t) = -f(x(t),u) + \Pi_{N_K(x(t))} f(x(t),u), \\ \qquad\qquad x(t_0) = x_0, \end{cases}$$

which stays in ∂K because of the invariance properties of $\Phi_u(\cdot)$. Using the dynamic programming principle we get $V(t_0,x_0) \geq V(t_0-h, x(t_0-h))$ with $h>0$ small enough. So, $\phi(t_0,x_0) \geq \phi(t_0-h,x(t_0-h))$ with $h>0$ small enough. Recall that $\phi \in C^1$ and consequently

$$\frac{\partial \phi}{\partial t}(t_0,x_0) + \left\langle (f(x_0,u) - \Pi_{N_K(x)} f(x_0,u)), \frac{\partial \phi}{\partial x}(t_0,x_0) \right\rangle \geq 0.$$

Second case ($x_0 \in \mathrm{int} K$). $N_K(x_0) = \{0\}$ and for all $u \in U$, because f is a Lipschitz application, there exists $B(x_0;r_u)$, $r_u > 0$, such that the solution to

$$\begin{cases} x'(t) = -f(x(t),u), \\ \quad x(t_0) = x_0 \end{cases}$$

stays in $B(x_0;r_u)$. Using the dynamic programming principle, for $h>0$ small enough $V(t_0,x_0) \geq V(t_0-h,x(t_0-h))$ so $\phi(t_0,x_0) \geq \phi(t_0-h,x(t_0-h))$. Because $\phi \in C^1$ we obtain $\frac{\partial \phi}{\partial t}(t_0,x_0) + \langle (f(x_0,u), \frac{\partial \phi}{\partial x}(t_0,x_0)\rangle \geq 0$. This allows us to say that V is an l.s.c. solution of (HJBI).

Step 2 (uniqueness). Now let us prove that V is the unique l.s.c. solution of (HJBI). Let W be an l.s.c. solution of (HJBI) with $W(T,x) = g(x)$ for all $x \in K$. We have already proved (see Theorem 13) that $W \geq V$.

For the reverse inequality we consider $(t_0,x_0) \in (0,T) \times K$ and $x(\cdot) \in S_F(t_0,x_0)$. There exists $u(\cdot) \in \mathcal{U}(t_0)$ such that $x(\cdot) = x(\cdot;t_0,x_0,u(\cdot))$. We have two cases.

First case ($x(T) \in \mathrm{int}K$). For a fixed $u(\cdot) \in \mathcal{U}(t_0)$, ∂K is invariant by $\Phi_{u(.)}(\cdot)$ and $\Phi_{u(.)}(\cdot)$ is Lipschitz in the second variable, so we have that $x([t_0,T]) \subset \mathrm{int}K$.

By the measurable viability theorem (see Theorem 4.7 in [10], [2]) *Epi*(W) is viable for the dynamics given by $(t,x,y) \to (-1, -f(x(t),u(t)), 0)$. For the solution starting from $(T, x(T), W(T,x(T)))$, we have for all $t \in [t_0,T]$, $W(T-t, x(T-t)) \leq W(T,x(T))$, so $W(t_0,x_0) \leq W(T,x(T)) = g(x(T))$.

Second case ($x(T) \in \partial K$). Denote by τ the first time with the property $x(\tau) \in \partial K$. Using invariance properties of $\Phi_{u(.)}(\cdot)$ and because $\Phi_{u(.)}(\cdot)$ is Lipschitz in the second variable, we obtain that $x([t_0,\tau)) \subset \mathrm{int}K$ and $x([\tau,T]) \subset \partial K$.

As in the above case, we apply the measurable viability theorem (see Theorem 4.7 in [10], [2]) to *Epi*(W), on the one hand, to $[\tau,T]$ for the dynamics given by $(t,x,y) \to (-1, -f(x(t),u(t)) + \Pi_{N_K(x(t))} f(x(t),u(t)), 0)$ for the solution starting from $(T, x(T), W(T,x(T))$ and, on the other hand, to $[t_0,\tau_n)$ for the dynamics given by

$(t,x,y) \to (-1,-f(x(t),u(t)),0)$ for the solution starting from $(\tau, x(\tau), W(\tau, x(\tau)) = \lim_n (\tau_n, x_n, W(\tau, x(\tau)), x_n \in \text{int}K$.

We have $W(t_0,x_0) \le W(\tau,x(\tau))$ and $W(\tau,x(\tau)) \le W(T,x(T)) = g(x(T))$. Consequently, by definition of the value function $W(t_0,x_0) \le V(t_0,x_0)$. $\square$

We note that here we can obtain uniqueness for l.s.c. solutions only in two (extremal) cases, where the vector field $f(x,u)$ is pointing only outside of the domain or only inside. For the intermediate situation it seems that we cannot obtain uniqueness (see the counterexample given below). The lack of uniqueness can be a consequence of the fact that in the intermediate situation we lose the Lipschitz regularity of $\Phi_u(\cdot)$ in ∂K and the idea of the above proof will fail.

Counterexample. Now we will show that a uniqueness result is not possible using our definition without imposing boundary properties on our dynamics as we did in the above propositions. We do this by giving a counterexample.

Let $K = [0,1] \subset \mathbb{R}$. For a dynamics given by $f = 0$ and $g = 1$ the value function is $V(t,x) = 1$ for all $(t,x) \in [0,1] \times [0,1]$. Moreover V is an l.s.c. solution of HJBI in the sense of our definition. Define

$$u(t,x) = \begin{cases} 1 \text{ if } (t,x) \in [0,1] \times (0,1], \\ 0 \text{ if (t,x)} \in [0,1] \times \{0\}. \end{cases}$$

It is easy to verify that u is also an l.s.c. solution for the HJBI and we do not have uniqueness because $\langle f(x_0), n(x_0) \rangle = 0$ for all $x_0 \in \partial K$.

For another definition of the discontinuous solution Ley [12] obtained a counterexample proving that there is no uniqueness to HJB with a notion of the solution in the Ishii–Barles–Perthame sense.

6. Appendix. Let us give the proof of Lemma 14 and Lemma 15. We shall use the following classical viability theorem and the fact that the definition of super and subsolutions to (HJBI) can be written equivalently in terms of subdifferentials. (See [15] to get formulations of viscosity solutions in terms of subdifferentials of the PDE associated to the Mayer control problem with $K = R^N$.)

THEOREM 19 (see [2, viability theorem 3.2.4]). *Assume that G is a Marchaud map and let $D \subset \mathbb{R}^N$ be closed. If for every $z \in D$ we have*

$$\text{(15)} \qquad \textit{for all } p \in N_D(z), \quad \min_{y \in G(z)} \langle y, p \rangle \le 0,$$

then for every $x_0 \in D$, $t_0 < T$, there exists a solution $x(\cdot)$ to the Cauchy problem $x'(s) \in G(x(s))$, $x(t_0) = x_0$ such that $x(t) \in D$ for all $t \in [t_0, T]$.

Now we give the proof of Lemma 14 and Lemma 15.

Proof of Lemma 14. Fix $t_0 \in (0,T)$. We set

$$D_\psi = \text{cl}(\{(t,x,r) \ : \ t \in (0,T], \ x \in K, \ r \ge \psi(t,x)\}) \cup [T,\infty) \times K \times \mathbb{R},$$

$$\tilde{F}(t,x,r) = \begin{cases} 0 \text{ if } t < 0, \\ \frac{t}{t_0}(1, F(x) - N_K(x) \cap B(0,M), 0) \text{ if } t \in [0,t_0], \\ (1, F(x) - N_K(x) \cap B(0,M), 0) \text{ if } t \in [t_0,T], \\ (1, F(x) - N_K(x) \cap B(0,M), 0) \text{ if } t > T, \end{cases}$$

where cl denote the closure and M is a bound of F on K. We show that (15) holds true for $\tilde{F}$ and D_ψ.

First case ($x_0 \in \text{int}K$). Let $z_0 = (s_0, x_0, r_0 := \psi(t_0,x_0)) \in D_\psi$. If $s_0 = 0$, then $\tilde{F} = 0$. Obviously (15) holds.

If $s_0 \geq T$ and $(p_s, p_x, p_r) \in N_{D_\psi}(s_0, x_0, r_0)$, then $p_s \leq 0$, $p_x = 0$, $p_r = 0$. Hence (15) holds.

It remains to consider the case $s_0 \in (0, T)$. We have $N_{D_\psi}(s_0, x_0, r_0) \subset N_{D_\psi}(s_0, x_0, \psi(s_0, x_0))$. Let $(p_s, p_x, p_r) \in N_{D_\psi}(s_0, x_0, \psi(s_0, x_0))$.

If $p_r < 0$, then $(p_s/-p_r,\ p_x/-p_r) \in \partial_-\psi(s_0, x_0)$ (see Proposition 4.1 in [9]).

Since ψ is a supersolution of (HJBI) there exists $y_0 \in N_K(x_0)$ such that

$$\frac{p_s}{-p_r} + \min_{z \in F(x_0)} \left\langle z,\ \frac{p_x}{-p_r} \right\rangle - \left\langle y_0,\ \frac{p_x}{-p_r} \right\rangle \leq 0 \quad \text{and}$$

$$\frac{p_s}{-p_r} + \min_{z \in \{F(x_0) - N_K(x_0) \cap B(0,M)\}} \left\langle z,\ \frac{p_x}{-p_r} \right\rangle \leq 0.$$

Hence $\min_{\tilde{y} \in \tilde{F}(s_0, x_0, r_0)} \langle \tilde{y}, (p_s, p_x, p_r) \rangle \leq 0$.

Now we consider the case $p_r = 0$. By a Rockafellar's lemma (see, for instance, Lemma 4.2 in [9]) there exists a sequence $s_n \to s_0$, $x_n \to x_0$, and $p_{s_n} \to p_s$, $p_{x_n} \to p_x$, $p_{r_n} \to 0$, $p_{r_n} < 0$ such that $(p_{s_n}, p_{x_n}, p_{r_n}) \in N_{D_\psi}(p_{s_n}, p_{x_n}, p_{r_n})$. Since $p_{r_n} < 0$ we obtain from the previous case that

$$\min_{\tilde{y}_n \in \tilde{F}(s_n, x_n, r_n)} \langle \tilde{y}, (p_{s_n}, p_{x_n}, p_{r_n}) \rangle \leq 0.$$

We get $\min_{\tilde{y} \in \tilde{F}(s_0, x_0, r_0)} \langle \tilde{y}, (p_s, p_x, p_r) \rangle \leq 0$, because $\tilde{F}$ is Marchaud.

Second case ($x_0 \in \partial K$). Let $z_0 = (s_0, x_0, r_0 := \psi(t_0, x_0)) \in D_\psi$. If $s_0 = 0$, then $\tilde{F} = 0$. Obviously (15) holds true.

If $s_0 \geq T$ and $(p_s, p_x, p_r) \in N_{D_\psi}(s_0, x_0, r_0)$, then $p_s \leq 0$, $p_x \in N_K(x_0)$, $p_r = 0$. Hence $[F(x_0) - N_K(x_0)] \cap T_K(x_0) \neq \emptyset$ and (15) holds.

It remains to consider $s_0 \in (0, T)$, which is similar to the first case.

Finally we obtain $\min_{\tilde{y} \in \tilde{F}(s_0, x_0, r_0)} \langle \tilde{y}, (p_s, p_x, p_r) \rangle \leq 0$.

In view of the above theorem we have a solution $z(\cdot)$ to the Cauchy problem $z'(s) \in \tilde{F}(z(s))$, $z(t_0) = z_0$. Let $z(s) = (t(s), x(s), r(s))$.

By the definition of $\tilde{F}$ we have $t(s) = s$, $r(s) = r_0 = \psi(t_0, x_0)$. Hence, (14) holds true and the proof is completed. $\square$

Proof of Lemma 15. The proof is divided into several steps.

First step. We fix $t_0 \in (0, T)$, $u(\cdot) \in \mathcal{U}(t_0)$ such that $u(\cdot)$ is a continuous function. We set

$$D_\varphi = \mathrm{cl}(\{(t, x, r) \ :\ t \in (0, T],\ x \in K,\ r \leq \varphi(t, x)\}) \cup [T, \infty) \times K \times \mathbb{R},$$

$$G(t, x, r) = \begin{cases} 0 \text{ if } t < 0, \\ \frac{t}{t_0}(1, F(x, u(t)) - N_K(x) \cap B(0, M), 0) \text{ if } t \in [0, t_0], \\ (1, F(x, u(t)) - N_K(x) \cap B(0, M), 0) \text{ if } t \in [t_0, T], \\ (1, F(x, u(t)) - N_K(x) \cap B(0, M), 0) \text{ if } t > T, \end{cases}$$

and we want to prove that for every $(t_0, x_0, \varphi(t_0, x_0)) \in (0, T) \times K \times \mathbb{R}$ the solution $x(\cdot; t_0, x_0, u(\cdot))$ to (2) satisfies

$$\text{(16)} \qquad \varphi(t, x(t)) \geq \varphi(t_0, x_0) \text{ for all } t \in [t_0, T].$$

Since φ is a u.s.c. viscosity subsolution of (HJBI) we have

$$\begin{gathered}\text{for any } \phi \in C^1 \text{ and } (t_0, x_0) \in \arg\max(\varphi - \phi), \\ \text{there exists } z_0 \in N_K(x_0) \text{ such that} \\ \frac{\partial \phi}{\partial t}(t_0, x_0) + H\left(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right) - \left\langle z_0, \frac{\partial \phi}{\partial x}(t_0, x_0)\right\rangle \geq 0,\end{gathered}$$

$$\text{so } -\frac{\partial \phi}{\partial t}(t_0, x_0) + \min_{y_0 \in N_K(x_0) \cap B(0,M)} \left\{\left\langle (f(x_0, u(t_0)) - y_0), -\frac{\partial \phi}{\partial x}(t_0, x_0)\right\rangle\right\} \leq 0.$$

Using the same arguments as we used in the proof of the above lemma and because G is a Marchaud map, we obtain

$$\min_{y_0 \in N_K(x_0) \cap B(0,M)} \langle (f(x_0, u(t_0)) - y_0), (p_s, p_x, p_r)\rangle \leq 0.$$

So, for every $(p_t, p_x, p_r) \in N_{D_\varphi}(t_0, x_0, \varphi(t_0, x_0))$,

$$\min_{\tilde{y} \in G(t_0, x_0, \varphi(t_0, x_0))} \langle \tilde{y}, (p_s, p_x, p_r)\rangle \leq 0.$$

Using Theorem 19 we obtain that for every $(t_0, x_0, \varphi(t_0, x_0)) \in (0, T) \times K \times \mathbb{R}$ the solution $x(\cdot; t_0, x_0, u(\cdot))$ to (2) satisfies (16).

Second step. Fix $u(\cdot) \in \mathcal{U}(t_0)$. Then there exists a sequence $u_n(\cdot) \subset \mathcal{U}(t_0)$ of continuous functions such that $u_n(\cdot) \to u(\cdot)$ in $L^\infty(0, T; U)$.

For all $n \in \mathbb{N}$, $x_n(\cdot; t_0, x_0, u_n(\cdot))$ satisfies

$$x_n'(t) \in f(x_n(s), u_n(s))ds - \int_{t_0}^t N_K(x_n(s)) \cap B(0, M)ds, \text{ or equivalently}$$

$$x_n(t) \in x_0 + \int_{t_0}^t f(x_n(s), u_n(s))ds - \int_{t_0}^t N_K(x_n(s)) \cap B(0, M)ds. \tag{17}$$

As $B(0, M)$ is a compact set and the application $N_K(\cdot) \cap B(0, M)$ is u.s.c., there exists a measurable selection $y_n(\cdot) \in N_K(x_n(s)) \cap B(0, M)$. Moreover, for all $s \in [t_0, T]$ there exists

$$\lim_{n\to\infty} y_n(s) = y(s) \in N_K(x(s)) \cap B(0, M).$$

Hence, by the Lebesgue theorem we obtain that

$$\int_{t_0}^t y_n(s)ds \to \int_{t_0}^t y(s)ds \in \int_{t_0}^t N_K(x(s)) \cap B(0, M)ds \text{ a.e. } t \in [t_0, T].$$

Recall that the restriction of the application $(t_0, x_0) \in [0, T] \times K \to S_F(t_0, x_0)$ to a compact set C is compact into $[0, \infty) \times K \times W^{1,1}(0, \infty\ ; K)e^{-bt}$ for all b with $b > a$. Since $x_n(\cdot; t_0, x_0, u_n(\cdot)) \in S_F(t_0, x_0)$ there exists $x(\cdot) \in S_F(t_0, x_0)$ such that $\lim_{n\to\infty} x_n(\cdot) = x(\cdot)$ in $W^{1,1}(0, T\ ; K)$.

Passing to the limit in (17) we obtain that for almost all $t \in [t_0, T]$ $x(t) \in x_0 + \int_{t_0}^t f(x(s), u(s))ds - \int_{t_0}^t N_K(x(s)) \cap B(0, M))ds$ and consequently, $x(\cdot) = x(\cdot; t_0, x_0, u(\cdot))$.

Third step. Assume that $\bar{x}(\cdot; t_0, x_0, \bar{u}(\cdot))$ is an optimal trajectory for V, starting from $(t_0, x_0) \in [0, T] \times K$, i.e., $V(t_0, x_0) = g(\bar{x}(T; t_0, x_0, \bar{u}(\cdot))$.

Then there exists a sequence of continuous functions $u_n(\cdot)$, such that $u_n(\cdot) \to \bar{u}(\cdot)$ in $L^\infty(0, T; U)$ and consequently $x_n(\cdot; t_0, x_0, u_n(\cdot)) \to \bar{x}(\cdot; t_0, x_0, \bar{u}(\cdot))$ in $W^{1,1}(0, T; K)$.

Using the above arguments for every $n \in \mathbb{N}$, $\varphi(t, x_n(t; t_0, x_0, u_n(\cdot)) \geq \varphi(t_0, x_0)$ for all $t \in [t_0, T]$ and consequently, $\varphi(T, x_n(T; t_0, x_0, u_n(\cdot)) \geq \varphi(t_0, x_0)$.

As φ is u.s.c., we obtain by letting $n \to \infty$

$$\begin{aligned} V(t_0, x_0) &= g(\bar{x}(T; t_0, x_0, \bar{u}(\cdot))) \geq \varphi(T, (\bar{x}(T; t_0, x_0, \bar{u}(\cdot))) \\ &\geq \lim\sup_{n\to\infty} \varphi(T, x_n(T; t_0, x_0, u_n(\cdot)) \geq \varphi(t_0, x_0). \end{aligned}$$

Then $V \geq \varphi$ and the proof is complete. ☐

For the proof of Lemma 21 let us establish a stability result for (HJBI).

Lemma 20. *Assume that $H : K \times \mathbb{R}^N \to \mathbb{R}$ is a continuous Hamiltonian and let K be a compact proximal retract. If $w_n : (0, T) \times K \to \mathbb{R}$ is an increasing (decreasing) sequence of uniformly locally bounded l.s.c. (u.s.c.) supersolutions (subsolutions) of* (HJBI) *and w is a pointwise limit of w_n, then w is an l.s.c. (u.s.c.) supersolution (subsolution) of* (HJBI).

The proof of this lemma is adapted from [3]. The main difficulties and changes with Barles's proof are given by the regularity of the application $N_K(\cdot)$.

Lemma 21. *Assume that (H_f) holds. Let K be a compact proximal retract and g be a u.s.c. (respectively, l.s.c.) function. Then V is a u.s.c. subsolution (respectively, l.s.c. supersolution) of* (HJBI).

Proof. We define a sequence $g_n : K \to \mathbb{R}$ by

$$g_n(x) = \sup_{y\in K}(g(y) - n||x - y||).$$

The supconvolutions g_n are Lipschitz, $g_n(x) \geq g_{n+1}(x)$, and $\lim g_n(x) = g(x)$ for every $x \in K$. Using Proposition 11, V_{g_n} is a Lipschitz solution to (HJBI) with $V_{g_n}(T, \cdot) = g_n(\cdot)$ and $V_{g_n} \geq V$.

Denote $U(t, x) = \lim V_{g_n}(t, x)$. Then, using the above result, U is a u.s.c. subsolution of (HJBI) and $U(T, x) = g(x)$. Obviously $U \geq V$ and by Lemma 15 we have that $U = V$. So V is a u.s.c. subsolution for (HJBI).

The proof in the l.s.c. case is similar to the u.s.c. case. ☐

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