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DIFFERENTIAL GAMES WITH ERGODIC PAYOFF*

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Abstract. We address a zero-sum differential game with ergodic payoff. We study this problem via the viscosity solutions of an associated Hamilton–Jacobi–Isaacs equation. Under certain condition, we establish the existence of a value and prove certain representation formulae.

Key words. differential games, value, Hamilton–Jacobi–Isaacs equation, viscosity solution

AMS subject classifications. 90D25, 90D26

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1. Introduction. In this article, we consider a general, nonlinear controlled dynamical system

$$\dot{x}(t) = b(x(t), u_1(t), u_2(t)) \tag{1.1}$$

with performance index $r(x(t), u_1(t), u_2(t))$, where u_1, u_2 are controls. Associated to this controlled dynamical system we can pose two kinds of problem—$\mathcal{H}_\infty$ control and the differential game. In $\mathcal{H}_\infty$ control, the performance index is referred to as the output response and u_2 as disturbance. A closed set $\mathcal{T}$ with respect to which the undisturbed system ($u_2 = 0$) is stable and a constant γ are given. The problem is to find a strategy $\alpha = \alpha[u_2]$ such that

$$\int_0^t |r(x(s), \alpha[u_2](s), u_2(s))|^2 \, ds \leq \gamma^2 \int_0^t |u_2(s)|^2 \, ds \tag{1.2}$$

for all $t \geq 0$ and all disturbances u_2. If we can find such a strategy, we say that the problem is solvable with disturbance attenuation level γ.

The other problem is the differential game problem. In this case, we call the performance index the running payoff function. There are two controllers or decision makers called players. Player 1 wishes to minimize the running payoff function on finite or infinite time horizon over his control variables $u_1(t)$, whereas Player 2 wishes to maximize the same over his control variables $u_2(t)$. Since the interests of the two players are conflicting, the basic problem is to resolve this conflict by arriving at solution that serves the interests of both players. In other words, we look for a min-max/max-min solution to this problem. For infinite horizon problems, one usually considers two payoff criteria: the discounted payoff criterion and the ergodic or averaged payoff criterion. These two payoff criteria are, in some sense, complementary to each other. The immediate future is far more important than the distant future in the discounted payoff criterion. Quite contrary to this, the finite time behavior of

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the system is irrelevant in the ergodic payoff criterion. It is the asymptotic behavior of the ergodic payoff that matters. Thus in the ergodic payoff criterion, one looks for some kind of stability or averaging mechanism taking place.

The differential game (in the sense of Elliott–Kalton) with discounted payoff criterion has been studied extensively in the literature; see [1] and the references therein. The basic idea is to show that the lower and upper value functions satisfy the dynamic programming principle (DPP) and thus they are viscosity solutions to corresponding Hamilton–Jacobi–Isaacs (HJI) equations. If the Isaacs minimax principle holds, then by a minimax theorem, one obtains that the differential game has value. This procedures does not seem to be applicable to the differential games with ergodic payoff. Thus in order to study the differential games with ergodic payoff, we need to approach the problem in a different way.

In the traditional approach to differential games, one first establishes the DPP, which in turn leads to the HJI equations. In this article, we follow a reverse approach which was used by Świech [18] to treat a stochastic differential game with a finite horizon payoff criterion (see also [17]). The main idea is to use the integration along the trajectories of the controlled dynamical system to study the HJI equations. Since the HJI equation in general does not admit classical solutions, we need to use the concept of viscosity solutions introduced by Crandall and Lions [4]. We show that if the HJI equation corresponding to ergodic payoff criterion has a viscosity solution, then the scalar quantity appearing in the HJI equation is the ergodic value for the differential game problem under certain stability assumption on the dynamics. Further, under a dissipativity assumption, we show that the HJI equation has a viscosity solution. The novelty of this approach is that it is quite simple and it can be used to prove the DPP.

There is a close connection between $\mathcal{H}_\infty$ control and differential games. A $\mathcal{H}_\infty$ control problem can be viewed as a differential game problem (see [3]). Using this observation, several authors have studied HJI equations and established DPP for the solutions; see [6], [9], [13], [15], [16], and the references therein. In [16] an $\mathcal{H}_\infty$ control problem is considered and studied using the viscosity solution techniques. Some representation formulas are proved for the viscosity solutions of the associated HJI equation. As a consequence, the author obtained the DPP for the viscosity solutions and established the value function to be the minimal viscosity solution under some nonnegativity assumptions and certain stability assumptions. In [9], [13], an analogous problem in the stochastic case is considered. Here the authors first obtained the DPP. The value function is again shown to be the minimal viscosity solution. The results in the deterministic case are obtained by letting the diffusion coefficient be zero. Further in [13], the uniqueness of the viscosity solution is established in a certain class of functions with some growth conditions. Thus the results in these articles are similar to ours. However, the assumptions considered there are different from the assumptions in this article. Note that in the mentioned articles, the ergodic value corresponding to the associated differential game turns out to be zero. Thus the results presented in this article can be seen as more general concerning differential games with ergodic payoff. We now describe our problem.

Let U_i, $i = 1, 2$, be given compact metric spaces. Let $\mathcal{A}_i$, $i = 1, 2$, denote the set of all measurable functions $u_i : [0, \infty) \to U_i$. The set $\mathcal{A}_i$ is called the set of all admissible controls for player i. Consider the d-dimensional controlled dynamical system $x(\cdot)$ described by

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & b(x(t), u_1(t), u_2(t)), \ t > 0, \\ x(0) & = & x, \end{array} \right. \tag{1.3}$$

where $b : \mathbf{R}^d \times U_1 \times U_2 \to \mathbf{R}^d$ and $u_i(\cdot) \in \mathcal{A}_i$. We assume that

(A1) b is continuous and there exists a constant $C_1 > 0$ such that for all $u_1 \in U_1$ and $u_2 \in U_2$

$$|b(x, u_1, u_2) - b(y, u_1, u_2)| \leq C_1 |x - y|.$$

Let $r : \mathbf{R}^d \times U_1 \times U_2 \to \mathbf{R}^d$ be the payoff function. We assume that

(A2) r is continuous and there exists a constant $C_2 > 0$ such that for all $u_1 \in U_1$ and $u_2 \in U_2$

$$|r(x, u_1, u_2) - r(y, u_1, u_2)| \leq C_2 |x - y|.$$

Let Γ denote the set of all maps $\alpha : \mathcal{A}_2 \to \mathcal{A}_1$ that are nonanticipative in the sense that for any $t > 0$ and $u_2, \tilde{u}_2 \in U_2$, $u_2(s) = \tilde{u}_2(s)$ for all $s \leq t$ implies $\alpha[u_2](s) = \alpha[\tilde{u}_2](s)$ for all $s \leq t$. Similarly, Δ is defined to be the set of all maps from $\mathcal{A}_1$ to $\mathcal{A}_2$ that are nonanticipative.

Let

$$\rho^+(x) := \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \limsup_{T \to \infty} \frac{1}{T} \int_0^T r(x(t), \beta[u_1](s), u_1(s))\, ds,$$
$$\rho^-(x) := \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \limsup_{T \to \infty} \frac{1}{T} \int_0^T r(x(t), u_2(s), \alpha[u_2](s))\, ds.$$

The functions $\rho^+(x), \rho^-(x)$ are called the upper and lower ergodic value functions associated with the differential game. If $\rho^+(x) = \rho^-(x) = \rho$, a constant for all x, we say that the differential game with ergodic payoff criterion has a value.

The rest of the paper is organized as follows. In section 2, we prove that if the associated HJI equation has a viscosity solution (ρ, w), then the upper and lower values coincide with ρ, and thus the differential game has value. We then prove some more representation formulas for the ergodic value. We also prove DPP for viscosity solution and a partial uniqueness result for viscosity solutions. In section 3, we show the existence of a viscosity solution to the HJI equation in two ways under a suitable assumption. Section 4 contains some concluding remarks.

2. Viscosity solutions and ergodic value.

Consider the following HJI equations

$$\rho = \inf_{u_1 \in U_1} \sup_{u_2 \in U_2} \{b(x, u_1, u_2) \cdot Dw(x) + r(x, u_1, u_2)\}, \ x \in \mathbf{R}^d \tag{2.1}$$

and

$$\rho = \sup_{u_2 \in U_2} \inf_{u_1 \in U_1} \{b(x, u_1, u_2) \cdot Dw(x) + r(x, u_1, u_2)\}, \ x \in \mathbf{R}^d. \tag{2.2}$$

Definition 2.1. *A viscosity subsolution of* (2.1) *is a pair* (ρ, w), *where* ρ *is a real number and* $w(\cdot)$ *is an upper semicontinuous function such that for* $x \in \mathbf{R}^d$ *and a smooth function* ϕ, *we have*

$$\rho \leq \inf_{u_1 \in U_1} \sup_{u_2 \in U_2} \{b(x, u_1, u_2) \cdot D\phi(x) + r(x, u_1, u_2)\}$$

whenever $w - \phi$ has a local maximum at x. A pair (ρ, w) of a real number ρ and a lower semicontinuous function $w(\cdot)$ is said to be a viscosity supersolution of (2.1) *if for $x \in \mathbf{R}^d$ and a smooth function ϕ, we have*

$$\rho \geq \inf_{u_1 \in U_1} \sup_{u_2 \in U_2} \{b(x, u_1, u_2) \cdot D\phi(x) + r(x, u_1, u_2)\}$$

whenever $w - \phi$ has a local minimum at x. A viscosity solution of (2.1) *is a pair (ρ, w) that is both viscosity sub- and supersolution of* (2.1). *Similarly, a viscosity solution of* (2.2) *is defined.*

We now proceed to prove the main result of this section, which provides estimates for ρ^+ in terms of viscosity sub- and supersolutions of (2.1) and, similarly, for ρ^-, in terms of viscosity sub- and supersolutions of (2.2). We prove this result under the following additional assumption:

(A3) For each $x \in \mathbf{R}^d$, there is a constant $M = M(x) > 0$ such that $|x(t)| < M$ for all $t \geq 0$, where $x(\cdot)$ is the solution of (1.3) under any pair of admissible controls $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$.

Remark 2.2. Since for any $t, s \geq 0$,

$$x(t) - x(s) = \int_s^t b(x(\tau), u_1(\tau), u_2(\tau))\, d\tau$$

and $|x(\tau)| \leq M$ by assumption (A3), we can find a constant $C > 0$ such that

$$|x(t) - x(s)| \leq C|t - s|.$$

Thus under assumptions (A1) and (A3), the solutions of (1.3) are globally Lipschitz continuous.

We now state and prove the main result of this section. Throughout the section, we assume (A1)–(A3).

THEOREM 2.3. (i) *Let (ρ, w) be a viscosity subsolution of* (2.1). *Then*

$$\rho \leq \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \liminf_{T \to \infty} \frac{1}{T} \int_0^T r(x(s), u_1(s), \beta[u_1](s))\, ds. \tag{2.3}$$

(ii) *Let (ρ, w) be a viscosity supersolution of* (2.1). *Then*

$$\rho \geq \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \limsup_{T \to \infty} \frac{1}{T} \int_0^T r(x(s), u_1(s), \beta[u_1](s))\, ds. \tag{2.4}$$

(iii) *Let (ρ, w) be a viscosity subsolution of* (2.2). *Then*

$$\rho \leq \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \liminf_{T \to \infty} \frac{1}{T} \int_0^T r(x(s), \alpha[u_2](s), u_2(s))\, ds. \tag{2.5}$$

(iv) *Let (ρ, w) be a viscosity supersolution of* (2.2). *Then*

$$\rho \geq \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \limsup_{T \to \infty} \frac{1}{T} \int_0^T r(x(s), \alpha[u_2](s), u_2(s))\, ds. \tag{2.6}$$

Proof. We prove (iii) and (iv); (i) and (ii) can be proved similarly.

Let (ρ, w) be a viscosity subsolution of (2.2). Assume that w is $C^{1,1}$ (i.e., w is differentiable with bounded and Lipschitz derivatives). Let K be the common

Lipschitz constant associated with w, Dw. Then (ρ, w) satisfies (2.2) in the classical sense. In particular, for any $\epsilon > 0$ and any $x \in \mathbf{R}^d$,

$$\rho - \epsilon < \sup_{u_2 \in U_2} \inf_{u_1 \in U_1} (b(x, u_1, u_2) \cdot Dw(x) + r(x, u_1, u_2)). \tag{2.7}$$

Set

$$\Lambda(x, u_2) = \inf_{u_1 \in U_1} (b(x, u_1, u_2) \cdot Dw(x) + r(x, u_1, u_2)).$$

Then it is easy to note that Λ is uniformly continuous on $\mathbf{R}^d \times U_2$. Since U_2 is separable, we can find a sequence $\{u_2^i\}$ in U_2 and a family of balls $\{B_{r_i}(x_i)\}$ covering $\mathbf{R}^d$ such that

$$\rho - \epsilon < \Lambda(x, u_2^i) \text{ for all } x \in B_{r_i}(x_i) \text{ and } i.$$

Note that here the sequence $\{u_2^i\}$ can be chosen to be finite since U_2 is compact. In that case, the sequence of balls $\{B_{r_i}(x_i)\}$ should be replaced by a finite family of open sets.

Define, $\psi : \mathbf{R}^d \to U_2$ by

$$\psi(x) = u_2^k \text{ if } x \in B_{r_k}(x_k) \setminus \bigcup_{i=1}^{k-1} B_{r_i}(x_i).$$

Then ψ is a Borel map and $\rho - \epsilon < \Lambda(x, \psi(x))$ $\forall x \in \mathbf{R}^d$. We make the following claims.

Claim A. For $x \in \mathbf{R}^d$, $m > 0$, there exists $\beta^m \in \Delta$ such that

$$(\rho - \epsilon)N - C\frac{N}{m} - \int_0^N r(x(s), u_1(s), \beta^m[u_1](s))\, ds \le w(x(N)) - w(x)$$

for any positive integer N, where $x(\cdot)$ is the solution of (1.3) with the initial condition $x(0) = x$ under controls $(u_1(\cdot), \beta^m[u_1](\cdot))$ and C is a constant depending on K, C_1, C_2 but not on x, N, and m.

Claim B. For each $\alpha \in \Gamma$, we can find $\tilde{u}_1(\cdot) \in \mathcal{A}_1$ and $\tilde{u}_2(\cdot) \in \mathcal{A}_2$ such that

$$\beta^m[\tilde{u}_1](\cdot) = \tilde{u}_2(\cdot) \text{ and } \alpha[\tilde{u}_2](\cdot) = \tilde{u}_1(\cdot). \tag{2.8}$$

Assuming the claims to be true, we complete the proof of (2.5). Divide the inequality in Claim A by N, and let $N \to \infty$ to obtain

$$(\rho - \epsilon) \le C\frac{1}{m} + \liminf_{N \to \infty} \frac{1}{N} \int_0^N r(x(s), u_1(s), \beta^m[u_1](s))\, ds. \tag{2.9}$$

Using (2.8) in (2.9), we deduce

$$(\rho - \epsilon) \le C\frac{1}{m} + \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \liminf_{N \to \infty} \frac{1}{N} \int_0^N r(x(s), \alpha[u_2](s), u_2(s))\, ds.$$

Letting $m \to \infty$, we obtain

$$(\rho - \epsilon) \le \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \liminf_{N \to \infty} \frac{1}{N} \int_0^N r(x(s), \alpha[u_2](s), u_2(s))\, ds.$$

We now need to replace the limit along the integers by the limit along any real sequence. For this, choose any sequence $T_n \to \infty$. Then

$$\begin{aligned}\frac{1}{T_n}\int_o^{T_n} & r(x(s),\alpha[u_2](s),u_2(s))\,ds \\ &= \frac{1}{T_n}\int_0^{[T_n]} r(x(s),\alpha[u_2](s),u_2(s))\,ds + \frac{1}{T_n}\int_{[T_n]}^{T_n} r(x(s),\alpha[u_2](s),u_2(s))\,ds.\end{aligned}$$

Using (A3), we note that the second term on the right-hand side of the above equality vanishes as $n \to \infty$. Note also the fact that

$$\left|\frac{1}{T_n}\int_0^{[T_n]} r(x(s),\alpha[u_2](s),u_2(s))\,ds - \frac{1}{[T_n]}\int_0^{[T_n]} r(x(s),\alpha[u_2](s),u_2(s))\,ds\right| \to 0$$

as $n \to \infty$. Thus

$$\lim_{n\to\infty}\frac{1}{T_n}\int_o^{T_n} r(x(s),\alpha[u_2](s),u_2(s))\,ds = \lim_{n\to\infty}\frac{1}{[T_n]}\int_o^{[T_n]} r(x(s),\alpha[u_2](s),u_2(s))\,ds.$$

Since this is true for any sequence (T_n) tending to ∞, we obtain

$$(\rho-\epsilon) \le \inf_{\alpha\in\Gamma}\sup_{u_2(\cdot)\in\mathcal{A}_2}\liminf_{T\to\infty}\frac{1}{T}\int_0^T r(x(s),\alpha[u_2](s),u_2(s))\,ds.$$

This proves (2.5) under the assumption that w is $C^{1,1}$. We now turn to the general case. Let w_ϵ be the sup-convolution of w, i.e.,

$$w_\epsilon(y) = \sup_{|z|\le M+2}\left\{w(z) - \frac{|z-y|^2}{2\epsilon}\right\}.$$

Then w_ϵ converges to w uniformly as $\epsilon \to 0$ on $B_{M+1} := \bar{B}(0, M+1)$, and w_ϵ are Lipschitz continuous and satisfy a.e. on B_{M+1}

$$\rho \le \inf_{u_2\in U_2}\sup_{u_1\in U_1}\{b(y,u_1,u_2)\cdot Dw_\epsilon(y) + r(y,u_1,u_2)\} + \sigma_1(\epsilon)$$

for some modulus σ_1 (see [10], [11]). For each $\delta > 0$, let w_ϵ^δ be a smooth approximation of w_ϵ such that $w_\epsilon^\delta, Dw_\epsilon^\delta$ are smooth and they converge to w_ϵ, Dw_ϵ uniformly on compact sets, respectively, and they all have the same Lipschitz constant. Now, using these facts, we can find another modulus σ_2 such that

$$\rho \le \inf_{u_2\in U_2}\sup_{u_1\in U_1}\{b(y,u_1,u_2)\cdot Dw_\epsilon^\delta(y) + r(y,u_1,u_2)\} + \sigma_1(\epsilon) + \sigma_2(\delta) \tag{2.10}$$

on $B_{M+1/2}$. Note that σ_2 may depend on ϵ and x. Observe that while proving (2.5), we have used the smoothness of w only in B_M. Thus we can use the above arguments with w_ϵ^δ and (2.10) to conclude

$$\rho \le \inf_{\alpha\in\Gamma}\sup_{u_2(\cdot)\in\mathcal{A}_2}\liminf_{T\to\infty}\frac{1}{T}\int_o^T r(x(s),\alpha[u_2](s),u_2(s))\;ds + \sigma_1(\epsilon) + \sigma_2(\delta),$$

where $x(\cdot)$ is the solution of (1.3) with the initial condition $x(0) = x$ under the controls $(\alpha[u_2](\cdot), u_2(\cdot))$. Now letting δ and then ϵ to 0, we obtain (2.5). This completes the proof of part (iii). We now proceed to prove the claims.

Proof of Claim A. Let $t = \frac{1}{m}$. Define

$$u_2^m(s) = \psi(x) \text{ for } s \in [0, t).$$

We extend the definition of $(u_2^m(\cdot), x(\cdot))$ to $[0, (i+1)\,t)$ assuming that it has been defined on $[0, i\,t)$ as follows. Let $x(\cdot)$ be the solution (2.1) with initial value x and controls $(u_1(\cdot), u_2^m(\cdot))$ in the interval $[0, it)$. Set

$$u_2^m(s) = \psi(x((i\,t)^-)) \text{ for } s \in [i\,t, (i+1)\,t).$$

Note that $x((it)^-)$ exists since $X(\cdot)$ is Lipschitz continuous and bounded. This defines $u_2^m(\cdot)$ on $\mathbf{R}$.

Let $x(.)$ be the solution of (1.3) with initial value $x(0) = x$ and controls $(u_1(\cdot), u_2^m(\cdot)\,)$. Then,

$$\begin{aligned}
w(x((i+1)t)) - w(x(it))& \\
&= \int_{it}^{(i+1)t} Dw(x(s)) \cdot b(x(s), u_1(s), u_2^m(s))\, ds \\
&= \int_{it}^{(i+1)t} (Dw(x(s)) - Dw(x(it))) \cdot b(x(s), u_1(s), u_2^m(s))\, ds \\
&\quad + \int_{it}^{(i+1)t} Dw(x(it)) \cdot (b(x(s), u_1(s), u_2^m(s)) - b(x(it), u_1(s), u_2^m(s)))\, ds \\
&\quad + \int_{it}^{(i+1)t} (Dw(x(it)) \cdot b(x(it), u_1(s), u_2^m(s)) + r(x(it), u_1(s), u_2^m(s)))\, ds \\
&\quad + \int_{it}^{(i+1)t} (r(x(s), u_1(s), u_2^m(s)) - r(x(it), u_1(s), u_2^m(s)))\, ds \\
&\quad - \int_{it}^{(i+1)t} r(x(s), u_1(s), u_2^m(s))\, ds.
\end{aligned}$$

Note that w, Dw, b are all Lipschitz along the trajectory $x(\cdot)$ and they are bounded by assumptions (A1) and (A3). Using these facts in the above together with the definition of ψ, we obtain,

$$\begin{aligned}
w(x((i+1)t)) - w(x(it)) &\geq -C \int_{it}^{(i+1)t} (s - i\,t)\, ds + (\rho - \epsilon) \int_{it}^{(i+1)t} ds \\
&\quad - \int_{it}^{(i+1)t} r(x(s), u_1(s), u_2^m(s))\, ds \\
&= -C\,t^2 + (\rho - \epsilon)t - \int_{it}^{(i+1)t} r(x(s), u_1(s), u_2^m(s))\, ds
\end{aligned}$$

for a constant $C > 0$ which will depend only on x and other Lipschitz constants. Now define a strategy $\beta^m \in \Delta$ by $\beta^m[u_1](\cdot) = u_2^m(\cdot)$ for $u_1(\cdot) \in \mathcal{A}_1$. Note that $\beta^m[u_1](\cdot)$ on

$[it,(i+1)t)$ depends only on $[0,it)$. Adding these inequalities for $i=0,\ldots,Nm-1$, we get the inequality stated in Claim A.

Proof of Claim B. We define such controls inductively. Let $\tilde{u}_2(\cdot)=\psi(x)$ on $[0,t)$. Define $\tilde{u}_1|_{[0,t)}=\alpha[\tilde{u}_2]|_{[0,t)}$. Having known $\tilde{u}_1(\cdot)$ and $\tilde{u}_2(\cdot)$ on $[0,it)$, we define $\tilde{u}_2(\cdot)$ on $[it,(i+1)t)$ by $\tilde{u}_2(s)=\psi(x(it)^-)$, where $x(\cdot)$ satisfies

$$\dot{x}(s)=b(x(s),\tilde{u}_1(s),\tilde{u}_2(s)),\ s\in[0,it)$$

and $x(0)=x$. It is now easy to check (2.8). This completes the proof of Claim B.

We now prove part (iv). Let w be a viscosity supersolution of (2.2) and assume $w\in C^{1,1}$. The proof for general w follows from an argument as in that of (iii). One has for any $\epsilon>0$ and any $x\in\mathbf{R}^d$,

$$\sup_{u_2\in U_2}\inf_{u_1\in U_1}(b(x,u_1,u_2)\cdot Dw(x)+r(x,u_1,u_2))<\rho+\epsilon.$$

Set

$$\Lambda(x,u_1,u_2)=(b(x,u_1,u_2)\cdot Dw(x)+r(x,u_1,u_2)).$$

By the uniform continuity of Λ, we can find a countable family $B_{r_i}(x_i)\times B_{r_i}(u_2^i)$ covering $\mathbf{R}^d$ and a sequence $u_1^i\in U_1$ such that

$$\Lambda(x,u_1^i,u_2)<\rho+\epsilon\quad\forall\ (x,u_2)\in B_{r_i}(x_i)\times B_{r_i}(u_2^i).$$

Define a map $\psi:\mathbf{R}^d\times U_2\to U_1$ by

$$\psi(x,u_2)=u_1^k \text{ if } (x,u_2)\in B_{r_k}(x_k)\times B_{r_k}(u_2^k)\setminus\bigcup_{i=1}^{k-1}B_{r_i}(x_i)\times B_{r_i}(u_2^i).$$

Then ψ is Borel measurable and

$$\Lambda(x,\psi(x,u_2),u_2)<\rho+\epsilon\quad\forall(x,u_2).$$

Claim C. For each integer $m>0$, there exists $\alpha^m\in\Gamma$ such that

$$\int_0^N r(x(s),\alpha^m[u_2](s),u_2(s))\,ds+w(x(N))-w(x)\le(\rho+\epsilon)N+C\frac{N}{m}$$

for all positive integers N and $u_2(\cdot)\in\mathcal{A}_2$, where $x(\cdot)$ is the solution of (1.3) with the initial condition $x(0)=x$ under controls $(\alpha^m[u_2](\cdot),u_2(\cdot))$ and C is a constant independent of N and m.

Assuming that the claim is true, we see, on dividing by N and letting $N\to\infty$,

$$\limsup_{N\to\infty}\frac{1}{N}\int_0^N r(x(s),\alpha^m[u_2](s),u_2(s))\ ds\le(\rho+\epsilon)+\frac{C}{m},$$

which implies

$$\inf_{\alpha\in\Gamma}\sup_{u_2(\cdot)\in U_2}\limsup_{N\to\infty}\frac{1}{N}\int_0^N r(x(s),\alpha[u_2](s),u_2(s))\ ds\le\rho.$$

From this one can deduce (iv).

Proof of Claim C. Let $t = 1/m$. Define $\alpha^m[u_2](s)) = \psi(x, u_2(s))$ for $s \in [0, t)$. Assuming that we have defined $\alpha^m[u_2](\cdot), x(\cdot)$ on $[0, it)$, we extend its definition to $[0, (i+1)t)$ as follows. Let $x(\cdot)$ satisfy (1.3) in $(0, it)$ with the initial condition $x(0) = x$ under the controls $(\alpha^m[u_2](\cdot), u_2(\cdot))$. Then define $\alpha^m[u_2](s) = \psi(x((it)^-, u_2(s)))$ for $s \in [it, (i+1)t)$. This defines $\alpha^m \in \Gamma$.

Now let $x(\cdot)$ denote the solution of (1.3) with the initial condition $x(0) = x$ under the controls $(\alpha^m[u_2](\cdot), u_2(\cdot))$. Then, for any i, as in Claim A, we can show that

$$w(x((i+1)t)) - w(x(it)) \le C\, t^2 + (\rho + \epsilon)t - \int_{it}^{(i+1)t} r(x(s), \alpha^m[u_2](s), u_2(s))\, ds.$$

Summing over i from 0 to $Nm - 1$, we obtain Claim C. □

As an immediate consequence of the theorem, we obtain the following comparison principle.

COROLLARY 2.4. *Assume that* (ρ, w), $(\bar\rho, \bar w)$ *are viscosity sub- and supersolutions of* (2.1) *(or* (2.2)*). Then,* $\rho \le \bar\rho$.

Proof. We prove for the case of (2.1). The proof of (2.2) follows similarly. By parts (i) and (ii) of Theorem 2.3, we have

$$\rho \le \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \liminf_{T \to \infty} \frac{1}{T} \int_0^T r(x(s), u_1(s), \beta[u_1](s))\ ds$$

and

$$\bar\rho \ge \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \limsup_{T \to \infty} \frac{1}{T} \int_0^T r(x(s), u_1(s), \beta[u_1](s))\ ds.$$

Hence $\rho \le \bar\rho$. □

Remark 2.5. In this corollary, we have not assumed any growth on w and $\bar w$. If w and $\bar w$ are given to be bounded, then one can give a very simple proof of this comparison principle using comparison principle for stationary HJI equations (see [12]).

Note that under assumptions (A1)–(A3), if (2.1) has a viscosity solution (ρ, w), then $\rho = \rho^+$, and if (2.2) has a viscosity solution $(\bar\rho, w)$, then $\rho = \rho^-$, using Theorem 2.3. Thus if the Isaacs minimax condition holds, i.e., for any $x, p \in \mathbf{R}^d$, if we have

$$\inf_{u_2 \in U_2} \sup_{u_1 \in U_1} \{b(x, u_1, u_2) \cdot p + r(x, u_1, u_2)\} = \sup_{u_1 \in U_1} \inf_{u_2 \in U_2} \{b(x, u_1, u_2) \cdot p + r(x, u_1, u_2)\},$$

then, using Fan's minimax theorem [8] we can deduce the following result. We omit the details.

THEOREM 2.6. *Assume that the Isaacs minimax condition holds. Assume that* (ρ, w) *is a viscosity solution of* (2.1) *or equivalently of* (2.2). *Then* $\rho = \rho^+(x) = \rho^-(x)$ *for all* $x \in \mathbf{R}^d$.

By interchanging the roles of taking limits as $T \to \infty$ and taking infimum and supremum over controls in the proof of the Theorem 2.3, we obtain the following result.

THEOREM 2.7. *Let* (ρ, w) *be a viscosity solution of* (2.1). *Then*

$$\rho = \lim_{T \to \infty} \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \frac{1}{T} \int_0^T r(x(s), u_1(s), \beta[u_1](s))\ ds.$$

Similarly, if $(\bar{\rho}, \bar{w})$ *is a viscosity solution of* (2.2)*, then*

$$\bar{\rho} = \lim_{T\to\infty} \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot)\in\mathcal{A}_2} \frac{1}{T} \int_0^T r(x(s), \alpha[u_2](s), u_2(s))\ ds.$$

Remark 2.8. Let $w^+(T,x)$ and $w^-(T,x)$ denote the upper and lower value functions of the finite horizon problem with horizon T, dynamics (1.3), payoff function r, and zero terminal cost; i.e., they are defined as follows:

$$w^+(T,x) := \sup_{\beta\in\Delta} \inf_{u_1(\cdot)\in\mathcal{A}_1} \int_0^T r(s, x(s), u_1(s), \beta[u_1](s))\ ds$$

and

$$w^-(T,x) := \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot)\in\mathcal{A}_2} \int_0^T r(s, x(s), \alpha[u_2](s), u_2(s))\ ds,$$

where $x(\cdot)$ is solution of (1.3) with the initial condition $x(0) = x$ under respective controls. Then the conclusion of the above theorem can be restated as

$$\rho = \lim_{T\to\infty} \frac{w^+(T,x)}{T} \text{ and } \bar{\rho} = \lim_{T\to\infty} \frac{w^-(T,x)}{T}.$$

This can be seen as the longtime behavior of viscosity solutions of HJI equations corresponding to differential games on finite horizon. We refer to [2], [14] for the study of longtime behavior of viscosity solutions of Hamilton–Jacobi equations.

We now give another representation formula for ρ in terms of the discounted value of the differential game. Let w_λ denote the upper value of the differential game on an infinite horizon with discount factor $\lambda > 0$, i.e.,

$$w_\lambda(x) = \sup_{\beta\in\Delta} \inf_{u_1(\cdot)\in\mathcal{A}_1} \int_0^\infty e^{-\lambda s} r(x(s), u_1(s), \beta[u_1](s))\ ds;$$

then

$$\rho = \lim_{\lambda\to 0} \lambda w_\lambda(x).$$

An analogous statement holds for the lower value function. This is the content of our next result. We closely follow the arguments in the proof of Theorem 2.3.

THEOREM 2.9. (i) *Let* (ρ, w) *be a viscosity solution of* (2.1)*. Then*

$$\rho = \lim_{\lambda\to 0} \sup_{\beta\in\Delta} \inf_{u_1(\cdot)\in\mathcal{A}_1} \lambda \int_0^\infty e^{-\lambda s} r(x(s), u_1(s), \beta[u_1](s))\ ds.$$

(ii) *Similarly, if* (ρ, w) *is a viscosity solution of* (2.2)*, then*

$$\rho = \lim_{\lambda\to 0} \inf_{\alpha\in\Gamma} \sup_{u_2(\cdot)\in\mathcal{A}_2} \lambda \int_0^\infty e^{-\lambda s} r(x(s), \alpha[u_2](s), u_2(s))\ ds.$$

Proof. We prove only (ii); (i) can be proved in an analogous way. Again we prove this under the additional assumption that w is $C^{1,1}$. The proof of the general case can be done as before.

Fix x. Let $\beta^m \in \Delta$ be as in the proof of Theorem 2.3. Let $u_1(\cdot) \in \mathcal{A}_1$. Let $x(\cdot)$ denote the solution of (1.3) with the initial condition $x(0) = x$ under the controls $(u_1(\cdot), \beta^m[u_1](\cdot))$. Then for a.e. s,

$$\frac{d}{ds} e^{-\lambda s} w(x(s)) = e^{-\lambda s} b(x(s), u_1(s), \bar{u}_2) \cdot Dw(x(s)) - \lambda e^{-\lambda s} w(x(s)).$$

Now following the arguments in the proof of Claim A of Theorem 2.3, we obtain

$$\begin{aligned}
&e^{-\lambda(i+1)t} w(x((i+1)t)) - e^{-\lambda it} w(x(it)) \\
&\qquad = \int_{it}^{(i+1)t} e^{-\lambda s} Dw(x(s)) \cdot b(x(s), u_1(s), \beta^m[u_1](s))\, ds \\
&\qquad \geq -C \int_{it}^{(i+1)t} e^{-\lambda s} (s - i\,t)\, ds + (\rho - \epsilon) \int_{it}^{(i+1)t} e^{-\lambda s}\ ds \\
&\qquad\quad - \int_{it}^{(i+1)t} e^{-\lambda s} r(x(s), u_1(s), \beta^m[u_1](s))\, ds \\
&\qquad \geq -C\, t \frac{1}{\lambda} \left[e^{-\lambda it} - e^{-\lambda(i+1)t} \right] + (\rho - \epsilon) \frac{1}{\lambda} \left[e^{-\lambda it} - e^{-\lambda(i+1)t} \right] \\
&\qquad\quad - \int_{it}^{(i+1)t} e^{-\lambda s} r(x(s), u_1(s), \beta^m[u_1](s))\, ds.
\end{aligned}$$

Adding these inequalities for $i = 0, \ldots, Nm - 1$, and multiplying by λ, we get

$$\begin{aligned}
\lambda e^{-\lambda N} w(x(N)) - \lambda w(x) \geq C \frac{1}{m} [1 - e^{-\lambda N}] + (\rho - \epsilon)[1 - e^{-\lambda N}] \\
- \lambda \int_0^N e^{-\lambda s} r(x(s), u_1(s), \beta^m[u_1](s))\ ds.
\end{aligned}$$

Now letting $N \to \infty$, we obtain

$$\rho - \epsilon + \lambda w(x_0) \leq \lambda \int_0^\infty e^{-\lambda s} r(x(s), u_1(s), \beta^m[u_1](s))\ ds - C \frac{1}{m}.$$

Using (2.8), we get

$$\rho - \epsilon + \lambda w(x_0) \leq \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \lambda \int_0^\infty e^{-\lambda s} r(x(s), \alpha[u_2](s), u_2(s))\ ds.$$

Now taking limit as $\lambda \to 0$ and then $\epsilon \to 0$, we get

$$\rho \leq \liminf_{\lambda \to 0} \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \lambda \int_0^\infty e^{-\lambda s} r(x(s), \alpha[u_2](s), u_2(s))\ ds.$$

Similarly, we can obtain

$$\rho \geq \limsup_{\lambda \to 0} \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \lambda \int_0^\infty e^{-\lambda s} r(x(s), \alpha[u_2](s), u_2(s))\ ds.$$

This completes part (ii). ◻

Remark 2.10. If (ρ, w) is a viscosity subsolution of (2.2), then note that the following result holds:

$$\rho \leq \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \liminf_{\lambda \to 0} \lambda \int_0^\infty e^{-\lambda s} r(x(s), \alpha[u_2](s), u_2(s)) \, ds.$$

Similar statements hold for the other cases.

We now present a dynamic programming principle for the viscosity solutions of (2.1) and (2.2).

THEOREM 2.11. (i) *Let* (ρ, w) *be a viscosity solution of* (2.1). *Then for any* $T > 0$,

$$w(x) = \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \left[\int_0^T r(x(s), u_1(s), \beta[u_1](s)) \, ds + w(x(T)) \right] - \rho T.$$

(ii) *Let* (ρ, w) *be a viscosity solution of* (2.2). *Then for any* $T > 0$,

$$w(x) = \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \left[\int_0^T r(x(s), \alpha[u_2](s), u_2(s)) \, ds + w(x(T)) \right] - \rho T.$$

Proof. We prove (ii); (i) can be proved analogously. Let $T > 0$ and m a positive integer. Take $t = T/m$. As in Claim C, we obtain $\alpha^m(.)$, given ϵ, $u_2(.)$, such that

$$w(x(T)) - w(x) \leq -\int_0^T r(x(s), u_1(s), u_2(s)) \, ds + (\rho + \epsilon)T - C\frac{T^2}{m}.$$

Therefore

$$w(x) \geq \inf_{\alpha \in \Gamma} \sup_{u_2(\cdot) \in \mathcal{A}_2} \left(\int_0^T r(x(s), \alpha[u_2](s), u_2(s)) \, ds + w(x(T)) \right) - \rho T.$$

We can prove the other inequality similarly. ◻

We now turn our attention to the uniqueness of w. Define a set Z as follows: $z \in Z$ if $z = \lim_{t_n \to \infty} x(t_n)$, where $t_n \to \infty$ and $x(\cdot)$ is a solution of (1.3) with an initial condition $x(0) = x_0$ for some $x_0 \in \mathbf{R}^d$ under some controls $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$. Then Z is nonempty under assumption (A3). We now show that if (ρ, w_1) and (ρ, w_2) are two viscosity solutions of (2.1) such that $w_1 \equiv w_2$ on Z, then $w_1 \equiv w_2$.

THEOREM 2.12. *Let* (ρ, w_1) *and* (ρ, w_2) *be two viscosity solutions of* (2.1) *such that* $w_1 \equiv w_2$ *on* Z. *Then* $w_1 \equiv w_2$. *An analogous result holds for* (2.2).

Proof. We prove this for the case when w_1, w_2 are $C^{1,1}$. The general case follows similarly as in the proof of Theorem 2.3. Let m be a positive integer. Let α^m be as in Claim C when we take $w = w_2$, and let β^m be as in Claim A when we take $w = w_1$. Taking α^m as α in (2.9), we obtain $\tilde{u}_1(\cdot) \in \mathcal{A}_1$ and $\tilde{u}_2(\cdot) \in \mathcal{A}_2$ such that

$$\beta^m[\tilde{u}_1](\cdot) = \tilde{u}_2(\cdot) \text{ and } \alpha^m[\tilde{u}_2](\cdot) = \tilde{u}_1(\cdot).$$

Using this, we obtain

$$w_1(x(N)) - w_1(x) \geq -\int_0^N r(x(s), \alpha^m[\tilde{u}_2](s), \tilde{u}_2(s)) \, ds + (\rho - \epsilon)N - C\frac{N}{m}$$

and

$$w_2(x(N)) - w_2(x) \leq -\int_0^N r(x(s), \alpha^m[\tilde{u}_2](s), \tilde{u}_2(s))\, ds + (\rho - \epsilon)N + C\frac{N}{m}.$$

From these two inequalities, we obtain

$$w_1(x) - w_2(x) \leq w_1(x(N)) - w_2(x(N)) + 2C\frac{N}{m}. \tag{2.11}$$

Using the compactness and equi-Lipschitz continuity of trajectories, we get a trajectory $\bar{x}(\cdot)$ such that $x(\cdot) \to \bar{x}(\cdot)$ as $m \to \infty$. (Note that $x(\cdot)$ above depends on m.) Now from (2.11) we obtain by letting $m \to \infty$

$$w_1(x) - w_2(x) \leq w_1(\bar{x}(N)) - w_2(\bar{x}(N)).$$

Now letting $N \to \infty$, we see that

$$w_1(x) - w_2(x) \leq 0.$$

Similarly, we can prove

$$w_2(x) - w_1(x) \leq 0.$$

Thus $w_1 \equiv w_2$. ∎

Remark 2.13. The uniqueness result in [13] is established under certain growth conditions on the solutions. Here we have obtained similar results without any such condition. Our uniqueness result, however, is not complete. We have shown that if two solutions coincide on the set Z, then they are identical. In view of this, it would be interesting to investigate the structure of Z.

3. Existence results. In the previous section, we studied some representation formulas related to the viscosity solutions of (2.1) and (2.2). We now study the existence of viscosity solutions to (2.1) and (2.2). We refer to [9] for analogoues results. Here we present two simple proofs of the existence result.

To this end we make the following assumption.

(A4) There exists a constant $C_3 > 0$ such that for all $x, y \in \mathbf{R}^d$ and $(u_1, u_2) \in U_1 \times U_2$,

$$\langle b(x, u_1, u_2) - b(y, u_1, u_2),\ x - y\rangle \leq -C_3|x - y|^2.$$

Remark 3.1. (i) Let $(u_1(\cdot), u_2(\cdot)) \in \mathcal{A}_1 \times \mathcal{A}_2$. Let $x(\cdot)$ and $y(\cdot)$ denote the solutions of (1.3) with the initial conditions $x(0) = x$ and $y(0) = y$, respectively, under these controls. Then using (A4), we get

$$\frac{d}{dt}|x(t) - y(t)|^2 \leq -C_3|x(t) - y(t)|^2.$$

Now using Gronwall's inequality, we obtain

$$|x(t) - y(t)| \leq |x - y|e^{-C_4 t}$$

for a constant $C_4 > 0$.

(ii) Using Gronwall's inequality, it is easy to see that (A1) and (A4) together imply (A3).

We now give some examples where (A4) holds.

Example 3.2. (i) *Let U_1, U_2 be subsets of $\mathbf{R}^m$ and $\mathbf{R}^q$, respectively, for some m and q. Let b be given by*

$$b(x, u_1, u_2) = Bx + C_1 u_1 + C_2 u_2 + b_1(x, u_1, u_2),$$

where B is a $d \times d$ matrix, C_1 a $d \times m$ matrix, C_2 a $d \times q$ matrix, and $b_1 : \mathbf{R}^d \times U_1 \times U_2 \to \mathbf{R}^d$. We assume the following:

$$\exists \alpha > 0 \text{ such that } \langle Bx, x \rangle \leq -\alpha |x|^2$$

and

$$|b_1(x, u_1, u_2) - b(y, u_1, u_2)| \leq \alpha_1 |x - y| \text{ for some } \alpha_1 < \alpha.$$

Under these assumptions, it is easy to verify that (A4) *is satisfied.*

(ii) *Let U_1, U_2 be as above and let b be given by*

$$b(x, u_1, u_2) = A + B_1 u_1 + B_2 u_2 + \bar{b}(x),$$

where A is a $d \times d$ matrix, B_1 a $d \times m$ matrix, and B_2 a $d \times q$ matrix. Assume that there are matrices C_1, C_2 of orders $d \times m$ and $d \times q$, respectively, such that $A + B_1 C_1 + B_2 C_2$ is stable. Further assume that $\bar{b}$ is bounded and Lipschitz continuous. Then (A4) *is satisfied.*

We now prove the existence via the vanishing limit in the discounted payoff criterion.

THEOREM 3.3. *Assume* (A1), (A2), *and* (A4). *Let w_λ be the unique viscosity solution in the class of linear growth functions of*

$$\lambda w_\lambda(x) = \inf_{u_1(\cdot) \in \mathcal{A}_1} \sup_{u_2(\cdot) \in \mathcal{A}_2} \left(b(x, u_1, u_2) \cdot D w_\lambda(x) + r(x, u_1, u_2) \right). \tag{3.1}$$

Then $\lambda w_\lambda(x) \to \rho$, a constant as $\lambda \to 0$. Also for any $\bar{x} \in \mathbf{R}^d$, $w_\lambda(\cdot) - w_\lambda(\bar{x})$ converges uniformly on compact sets to a continuous function $w(\cdot)$. Thus (ρ, w) is a viscosity solution of (2.1) *for any $\bar{x} \in \mathbf{R}^d$. Moreover, $\rho = \rho^+(x)$ for all $x \in \mathbf{R}^d$. An analogous result holds for the existence of a viscosity solution to* (2.2).

Proof. Using standard results in differential games and viscosity solutions [1], we have

$$w_\lambda(x) = \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \int_0^\infty e^{-\lambda t} r(x(t), u_1(t), \beta[u_1](t)) \, dt.$$

Let $u_1(\cdot) \in \mathcal{A}_1$ and $u_2(\cdot) \in \mathcal{A}_2$. Then using Remark 3.1(i), we see that

$$\left| \int_0^\infty e^{-\lambda s} r(x(s), u_1(s), u_2(s)) \, ds - \int_0^\infty e^{-\lambda s} r(y(s), u_1(s), u_2(s)) \, ds \right| \leq \frac{1}{C_4 + \lambda} |x - y|,$$

where $x(\cdot), y(\cdot)$ are solutions of (1.3) with initial conditions $x(0) = x$ and $y(0) = y$, respectively, under the controls $(u_1(\cdot), u_2(\cdot))$. Using this fact, it is easy to note that w_λ is Lipschitz continuous where the Lipschitz constant is independent of λ. Therefore by Ascoli-Arzela's theorem for a fixed $\bar{x}$, $w_\lambda(x) - w_\lambda(\bar{x})$ converges locally uniformly to a continuous function $w(x)$ and $\lambda w_\lambda(x)$ converges to a constant ρ. By the stability of viscosity solutions, we note that (ρ, w) is a viscosity solution of (2.1). Now by Theorem 2.6, $\rho = \rho^+(x)$ for all $x \in \mathbf{R}^d$. ∎

We now turn our attention to the increasing horizon limit case. Let $T > 0$ and w_0 be any Lipschitz continuous function. Now consider the HJI equation in $(0, T) \times \mathbf{R}^d$,

$$
\left\{ \begin{array}{lcl} w_t(t,x) & = & \inf\limits_{u_1 \in U_1} \sup\limits_{u_2 \in U_2} \{b(x,u_1,u_2) \cdot Dw(t,x) + r(x,u_1,u_2)\}, \\ w(0,x) & = & w_0(x). \end{array} \right. \tag{3.2}
$$

Then we have the following theorem.

THEOREM 3.4. *Assume* (A1), (A2), *and* (A4). *Let $w(t,x)$ be the unique viscosity solution of* (3.2) *in the class of linear growth functions. Then $\frac{w(T,x)}{T} \to \rho$, a constant, and $w(T,x) - \rho T$ converges locally uniformly to a continuous function $w_\infty(x)$ such that (ρ, w_∞) is a viscosity solution of* (2.1). *Moreover, $\rho = \rho^+(x)$ for all $x \in \mathbf{R}^d$. An analogous results holds for* (2.2).

Proof. Using standard results in differential games and viscosity solutions [7], we have the following representation formula for $w(t,x)$:

$$
w(T,x) = \sup_{\beta \in \Delta} \inf_{u_1(\cdot) \in \mathcal{A}_1} \left[\int_0^T r(x(s), u_1(s), \beta[u_1](s)) \, ds + w_0(x(T)) \right].
$$

As in above theorem, using Remark 3.1(i), we can show that

$$
|w(T,x) - w(T,y)| \le \frac{1 - e^{-C_{10} T}}{C_{10}} |x - y|.
$$

Using Ascoli-Arzela's theorem, it is easy to see that $\frac{w(T,x)}{T} \to \rho$, a constant $w(T,x) - \rho T \to w_\infty(x)$ locally uniformly to a continuous function $w_\infty(x)$. We now need to show that (ρ, w_∞) is a viscosity solution of (2.1). Let

$$
w^\epsilon(t,x) = w(t/\epsilon, x) \text{ for } t \in [0,1].
$$

Then $w^\epsilon(t,x) - \rho \frac{t}{\epsilon} \to w_\infty(t,x)$ locally uniformly as $\epsilon \to 0$. Now it is easy to see that w^ϵ is viscosity solution of

$$
\left\{ \begin{array}{lcl} \epsilon w^\epsilon_t(t,x) & = & \inf_{u_1 \in U_1} \sup_{u_2 \in U_2} \{b(x,u_1,u_2) \cdot Dw^\epsilon(t,x) + r(x,u_1,u_2)\}, \\ w(0,x) & = & w_0(x) \end{array} \right.
$$

in $(0,1) \times \mathbf{R}^d$. Using the stability of viscosity solutions [5], we get that (ρ, w_∞) is a viscosity solution of (2.1). This completes the proof. ☐

4. Conclusions. In this paper, we have studied a zero sum differential game with ergodic payoff. We have identified the scalar appearing in the HJI equation as the ergodic value. Under a dissipativity-type condition, we have also established the existence of a viscosity solution to HJI equations. We have carried out two asymptotics, namely, we have shown that the ergodic value is the vanishing limit of the discounted value. At the same time, the ergodic value is also the time averaged limit of the finite horizon value. Finally we wish to mention that although we have identified the scalar appearing in the HJI equation as the ergodic value, we have not been able to establish the uniqueness (in some sense) of the solution of the HJI equation. We have obtained only a partial uniqueness result. Thus the uniqueness issue and the existence of viscosity solution to HJI equations under (A3) alone still remain problems that need further investigation.

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