A New Approach to Adaptive Nonlinear Regulation *

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Abstract

This paper shows how the theory of adaptive observers can be effectively used in the design internal models for nonlinear output regulation. The main result obtained in this way is a new method for the synthesis of adaptive internal models which substantially enhances the existing theory of adaptive output regulation, by allowing nonlinear internal models and more general classes of controlled plants.

Keywords: Adaptive Observers, Internal Model, Regulation, Tracking, Nonlinear Control.

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1 Introduction

The problem of controlling the output of a system so as to achieve asymptotic tracking of prescribed trajectories and/or asymptotic rejection of disturbances is a central problem in control theory. There are essentially three different possibilities to approach the problem: tracking by dynamic inversion, adaptive tracking, tracking via internal models. Tracking by dynamic inversion consists in computing a precise initial state and a precise control input (or equivalently a reference trajectory of the state), such that, if the system is accordingly initialized and driven, its output exactly reproduces the reference signal. The computation of such control input, though, requires "perfect knowledge" of the entire trajectory to be tracked as well as "perfect knowledge" the model of the controlled plant. Thus, this type of approach is not suited in the presence of large uncertainties on plant parameters as well as on the reference signal. Adaptive tracking can successfully handle parameter uncertainties, but it still presupposes the knowledge of the entire trajectory which is to be tracked (to be used in the design of the adaptation algorithm) and therefore this approach is not suited in the problem of tracking unknown trajectories. Internal-model-based tracking on the other hand, is able to handle simultaneously uncertainties in plant parameters as well as in the trajectory which is to be tracked. It has been proven that, if the trajectory to be tracked belongs to the set of all trajectories generated by some fixed dynamical system, a controller which incorporates an internal model of such a system is able to secure asymptotic decay to zero of the tracking error for every possible trajectory in this set and does it robustly with respect to parameter uncertainties. This is in sharp contrast with the two approaches mentioned above, where in lieu of the assumption that a signal is within a class of signals generated by an exogenous system, one instead needs to assume complete knowledge of the past, present and future time history of the trajectory to be tracked. It is for this reason that the internal-model-based approach seems to be the best suited in problems of tracking of unknown reference trajectories or rejecting unknown disturbances.

A generalized problem of tracking and asymptotic disturbance rejection is usually cast as follows. A nonlinear system is given, modelled by equations of the form

$$\dot{x} = f(x, u, v) y = k(x, v) e = h(x, v)$$

with state x, control input u, measured output y, regulated output e. In this system, v is an exogenous input, which represents actual *disturbances* as well as *commands* to be followed, and it is assumed that, as a function time, v(t) can be seen as generated by a separate autonomous dynamical system, called the *exosystem*. Generally speaking, the problem of tracking and asymptotic disturbance rejection (sometimes also referred to as the generalized *servomechanism problem* or the *output regulation problem*) is to design a controller so as to obtain a closed-loop system in which:

• all trajectories are bounded, and

• the regulated output e(t) asymptotically decays to 0 as $t \to \infty$.

The peculiar aspect of this design problem is the characterization of the class of all possible exogenous inputs (disturbances as well commands) as the set of all possible solutions of a fixed (finite-dimensional) differential equation. This can be seen as an intermediate choice sitting between two extremes: the (pessimistic) case in which the design is required to obtain certain goals in the presence of the *worst possible* exogenous input and the (optimistic) case in which the controller is assumed to have access to the exogenous input v. In this design problem, the controller does not have access to the exogenous input in real time, but the latter is restricted to range over a "finite dimensional" set of functions (such as the set of solutions of a fixed differential equation). The vector v may include constant uncertain parameters, which have a trivial dynamics and hence can be viewed as solutions of a (trivial) differential equation. In other words, in this setting, any source of uncertainty (about an actual disturbance affecting the system, about an actual trajectory to be tracked, about any unknown constant parameter in the plant or about any unknown constant parameter in the exosystem itself) is treated as uncertainty in the initial condition of a fixed autonomous finite dimensional dynamical system, which is then seen as source of all possible, constant as well time-varying, uncertainties.

For linear multivariable systems this problem was addressed in very elegant geometric terms by Davison, Francis, Wonham [6, 8, 7] and others. A nonlinear enhancement of this theory, which uses a combination of geometry and nonlinear dynamical systems theory, was presented in [13, 11, 10, 5] in the context of solving the problem near an equilibrium, in the presence of exogenous signals which were produced by a Poisson stable system. In particular, Huang showed how, by appropriately designing the internal model, the controlled output could be steered to zero in spite of plant parameter uncertainties, thus extending to the nonlinear setting one of the most remarkable features of internal-model-based design for linear systems. Under suitable hypotheses, the (local) design methods presented in these works have been extended [14, 12, 19] to the case of arbitrary large (but compact) sets of initial data. A substantial limitation of classical internal model-based control (for linear as well as nonlinear systems) is the sensitivity to parameters uncertainties in the exosystem. This limitation, though, was later addressed and solved under convenient hypotheses in the paper [20], where the possibility of using techniques of adaptive control to cope with unknown parameters in the exosystem was successfully demonstrated.

In the recent paper [2], the problem in question has been posed in more general terms, not tied, as all previous contributions were, to the existence of a privileged equilibrium point about which the (local as well semi-global) analysis was conducted. The more general foundations laid in this way make it possible to overcome certain restrictions of the earlier theory, notably the assumption that the controlled plant has an asymptotically stable zerodynamics, which is replaced by the substantially weaker hypothesis that the latter possess a compact attractor. Another major enhancement of this newer approach is a systematic method for the design of nonlinear internal models (see [3]). The presence of parametric uncertainties in the exosystem, however, is not explicitly addressed in these works.

The purpose of the present paper is to show how the problem of handling *parametric*

uncertainties in the exosystem can be successfully addressed by means of a new approach which reposes, on one hand, on the general non-equilibrium theory developed in [2] and, on the other hand, on the theory of adaptive observers for nonlinear system pioneered in [1] and [17]. The result obtained in this way is a totally new method for the synthesis of adaptive internal models which substantially extends the adaptive regulation theory presented in [20], by allowing nonlinear internal models and more general classes of controlled plants.

2 Output regulation and limit sets

The purpose of output regulation is to obtain a closed-loop system in which all trajectories with initial conditions in a fixed (but otherwise arbitrary) compact set are bounded and the regulated output converges to zero as time tends to infinity. As shown in [2], intimately associated with this problem is the notion of *limit set* of a given bounded set of initial conditions. For convenience of the reader, the notion in question is summarized as follows.

Consider an *autonomous* ordinary differential equation

$$\dot{x} = f(x) \tag{1}$$

in which $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, with f(x) a locally Lipschitz function. Let

$$\phi: (t,x) \mapsto \phi(t,x)$$

define the flow of (1). Suppose the flow is forward complete. The ω -limit set of a subset $B \subset \mathbb{R}^n$, written $\omega(B)$, is the totality of all points $x \in \mathbb{R}^n$ for which there exists a sequence of pairs (x_k, t_k) , with $x_k \in B$ and $t_k \to \infty$ as $k \to \infty$, such that

$$\lim_{k \to \infty} \phi(t_k, x_k) = x \,.$$

In case $B = \{x_0\}$ the set thus defined, $\omega(x_0)$, is precisely the ω -limit set, as defined by Birkhoff, of the point x_0 . Note that, in general

$$\bigcup_{x_0 \in B} \omega(x_0) \subset \omega(B) \, ,$$

but the equality may not hold.

It is well-known that $\phi(t, x_0)$, if is bounded in positive time, the set $\omega(x_0)$ is non-empty, compact, invariant, and

$$\lim_{t \to \infty} \operatorname{dist}(\phi(t, x_0), \omega(x_0)) = 0.$$

If B is not just the singleton $\{x_0\}$, the following more general property holds. Recall that a set A is said to *uniformly attract* a set B under the flow of (1) if for every $\varepsilon > 0$ there exists a time \bar{t} such that

$$\operatorname{dist}(\phi(t, x), A) \leq \varepsilon$$
, for all $t \geq \overline{t}$ and for all $x \in B$.

Then the following holds (see [9, page 8]).

Lemma 1 If B is a nonempty bounded set for which there is a compact set J which uniformly attracts B (thus, in particular, if B is any nonempty bounded set whose positive orbit has a bounded closure), then $\omega(B)$ is nonempty, compact, invariant and uniformly attracts B. Moreover, if $\omega(B) \in int(B)$, then $\omega(B)$ is stable in the sense of Lyapunov.

3 Class of systems and main assumptions

In this paper we discuss the design of output regulators for nonlinear systems modelled by equations of the form

$$\dot{z} = f_0(\varrho, w, z) + f_1(\varrho, w, z, e_1)e_1
\dot{e}_1 = e_2
\vdots
\dot{e}_{r-1} = e_r
\dot{e}_r = q(\varrho, w, z, e_1, \dots, e_r) + u
e = e_1
y = col(e_1, \dots, e_r),$$
(2)

with state $(z, e_1, \ldots, e_r) \in \mathbb{R}^n \times \mathbb{R}^r$, control input $u \in \mathbb{R}$, regulated output $e \in \mathbb{R}$, measured output $y \in \mathbb{R}^r$, in which the exogenous (disturbance) input $w \in \mathbb{R}^s$ is generated by an exosystem

$$\dot{w} = s(\varrho, w) \,. \tag{3}$$

In this model, $\rho \in \mathbb{R}^p$ is a vector of *constant* uncertain parameters, ranging over a fixed compact set P. The vector ρ is the aggregate of a finite set of uncertain parameters affecting the controlled plant and another, possibly different, set of uncertain parameters affecting the exosystem. These parameters may be regarded as "trivial components" of an "augmented" exogenous input, but for the sake of clarity, and also consistency with some of the earlier literature, their role will be kept separate. Occasionally, throughout the paper, the "augmented" exosystem

$$\begin{array}{lll}
\dot{\varrho} &=& 0\\
\dot{w} &=& s(\varrho, w)
\end{array}$$
(4)

will be rewritten in more compact form as

$$\dot{\mathbf{w}} = \mathbf{s}(\mathbf{w}), \qquad (5)$$

where $\mathbf{w} = \operatorname{col}(\varrho, w)$.

The functions $f_0(\cdot), f_1(\cdot), q(\cdot), s(\cdot)$ in (2) and (4) are assumed to be at least continuously differentiable. The initial conditions of (2) range on a set $Z \times E$, in which Z is a fixed *compact* subset of \mathbb{R}^n and $E = \{(e_1, \ldots, e_r) \in \mathbb{R}^r : |e_i| \leq c\}$, with c a fixed number. The initial conditions of the exosystem (5) range on a *compact* subset \mathbf{W} of $\mathbb{R}^p \times \mathbb{R}^s$. In this framework the problem of output regulation is to design an output feedback regulator of the form

$$\dot{\zeta} = \varphi(\zeta, y)$$

 $u = \gamma(\zeta, y)$

such that for all initial conditions $\mathbf{w}(0) \in \mathbf{W}$ and $(z(0), e_1(0), \ldots, e_r(0)) \in Z \times E$ the trajectories of the closed-loop system are bounded and $\lim_{t\to\infty} e(t) = 0$.

Augmenting (2) with (4) yields a system which, viewing u as input and e as output, has relative degree r. The associated "augmented" zero dynamics, which is forced by the control

$$c(\varrho, w, z) = -q(\varrho, w, z, 0, \dots, 0), \qquad (6)$$

is given by

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w) \tag{7}$$

$$\dot{z} = f_0(\varrho, w, z).$$

Occasionally, throughout the paper, we will find it convenient to rewrite the latter in more compact form as

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \,, \tag{8}$$

having set $\mathbf{z} = \operatorname{col}(\varrho, w, z)$. Accordingly, we set $\mathbf{Z} = \mathbf{W} \times Z$ and, with a mild abuse of notation, we replace $c(\varrho, w, z)$ by $c(\mathbf{z})$ in (6).

In what follows, we retain three of the basic assumptions that were introduced in [2] and express certain properties of the augmented zero dynamics (7). The assumptions in question are the following ones:

Assumption (i): the set **W** is a differential submanifold (with boundary) of $\mathbb{R}^p \times \mathbb{R}^s$, and **W** is invariant for (5). \triangleleft

Assumption (ii): there exists a compact subset \mathcal{Z} of $\mathbf{W} \times \mathbb{R}^n$ which contains the positive orbit of the set \mathbf{Z} under the flow of (8), and $\omega(\mathbf{Z})$ is a differential submanifold (with boundary) of $\mathbf{W} \times \mathbb{R}^n$. Moreover there exists a number $d_1 > 0$ such that

$$\mathbf{z} \in \mathbf{W} \times \mathbb{R}^n$$
, $\operatorname{dist}(\mathbf{z}, \omega(\mathbf{Z})) \le d_1 \implies \mathbf{z} \in \mathbf{Z}$.

As a remark on the above hypotheses, note that, since the positive orbit of the set \mathbf{Z} under the flow of (8) is bounded, the set $\omega(\mathbf{Z})$, namely the ω -limit set of \mathbf{Z} under the flow of (8), is a nonempty, compact and invariant subset of $\mathbf{W} \times \mathbb{R}^n$ which uniformly attracts all trajectories of (8) with initial conditions in \mathbf{Z} . It can also be shown (as in [2]) that for every $\mathbf{w} \in \mathbf{W}$ there is $z \in \mathbb{R}^n$ such that $(\mathbf{w}, z) \in \omega(\mathbf{Z})$. In what follows, for convenience, the set $\omega(\mathbf{Z})$ will be simply denoted as \mathcal{A}_0 .

The last condition in assumption (ii) implies that \mathcal{A}_0 is stable in the sense of Lyapunov. The next hypothesis, which will be used in the last part of the paper, is that the set \mathcal{A}_0 is locally exponentially attractive.

Assumption (iii): There exist $M \ge 1$, a > 0 and $d_2 \le d_1$ such that

$$\mathbf{z}_0 \in \mathbf{W} \times \mathbb{R}^n$$
, $\operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0) \le d_2 \implies \operatorname{dist}(\mathbf{z}(t, \mathbf{z}_0), \mathcal{A}_0) \le M e^{-at} \operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0)$

in which $\mathbf{z}(t, \mathbf{z}_0)$ denotes the solution of (8) passing through \mathbf{z}_0 at time t = 0.

The results presented in [2], as essentially all previous results on output regulation, relied upon the hypothesis that the set of all "feed-forward inputs capable to secure perfect tracking" (that, is, the set of inputs of the form $u(t) = c(\mathbf{z}(t))$, with $\mathbf{z}(t)$ a trajectory of the restriction of (8) to \mathcal{A}_0) could be seen as a subset of the set of outputs of a suitable linear system. The system in question was used to construct a (linear, as a matter of fact) internal model. This assumption was weakened in [3], where a general method for the construction of fully nonlinear internal models was presented, but the method in question did not allow for the presence of uncertain parameters in the exosystem. In this paper we introduce a different kind of hypothesis, leading to a somewhat more restricted class of internal models, but which – in return – allows for uncertain parameters in the exosystem.

Assumption (iv): there exist a positive integer d, a C^1 map

$$\begin{aligned} \tau & : & \mathcal{Z} & \to & \mathbb{R}^d \\ & & \mathbf{z} & \mapsto & \tau(\mathbf{z}) \end{aligned}$$

a C^0 map

$$\begin{array}{rccc} \theta & : & P & \to & \mathbb{R}^q \\ & \varrho & \mapsto & \theta(\varrho) \, , \end{array}$$

an observable pair $(A, C) \in \mathbb{R}^{d \times d} \times \mathbb{R}^{1 \times d}$, and two C^1 maps $\phi : \mathbb{R} \to \mathbb{R}^d$ and $\Omega : \mathbb{R} \to \mathbb{R}^{d \times q}$ such that the following identities (which we call *immersion property*)

$$\frac{\partial \tau}{\partial \mathbf{z}} \mathbf{f}_0(\mathbf{z}) = A \,\tau(\mathbf{z}) + \phi(C \tau(\mathbf{z})) + \Omega(C \tau(\mathbf{z})) \,\theta(\varrho) \tag{9}$$

$$c(\mathbf{z}) = C\,\tau(\mathbf{z})\tag{10}$$

hold for all $\mathbf{z} \in \mathcal{A}_0, \ \varrho \in P. \triangleleft$

Remark. Without loss of generality (see [16, page 208]), we can assume throughout that the matrices A and C in (11) have the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Furthermore, note that since the maps $\Omega(\cdot)$ and $\phi(\cdot)$ are continuously differentiable and the relations (9) – (10) are supposed to hold over the compact set \mathcal{A}_0 , it can be assumed without loss of generality that functions $\phi(\cdot)$ and $\Omega(\cdot)$ have compact support. This being the case, the functions in question can be assumed *globally Lipschitz*, i.e. there exist L_{ϕ} and L_{Ω} such that

$$|\phi(s_1) - \phi(s_2)| \le L_{\phi}|s_1 - s_2|, \qquad |\Omega(s_1) - \Omega(s_2)| \le L_{\Omega}|s_1 - s_2|,$$

for all s_1, s_2 . \triangleleft

Remark. Note that Assumption (iv) can be rephrased by saying that for each initial condition $\mathbf{z}(0) \in \mathcal{A}_0$ of (8), there is a pair $\xi(0), \theta$ such that the control input $u(t) = c(\mathbf{z}(\mathbf{t}))$ (which is the unique input capable of keeping e(t) identically at zero) can be seen as output of a system of the form

$$\dot{\xi} = A\xi + \phi(y) + \Omega(y)\theta$$

$$\dot{\theta} = 0$$

$$y = C\xi. \quad \triangleleft$$
(11)

In the remaining part of this section we show that there is no loss of generality in addressing the simpler case in which the relative degree of (2) is r = 1. As a matter of fact consider the change of variable

$$e_r \mapsto \tilde{e} := e_r + g^{r-1}a_0e_1 + g^{r-2}a_1e_2 + \ldots + ga_{r-2}e_{r-1}$$

where g is a *positive* design parameter and a_i , i = 0, ..., r - 2, are such that all roots of the polynomial $\lambda^{r-1} + a_{r-2}\lambda^{r-1} + ... + a_1\lambda + a_0 = 0$ have negative real part. This changes system (2) into a system of the form

in which

$$\tilde{z} = \operatorname{col}(z, e_1, \dots, e_{r-1})$$

$$\tilde{f}_0(\varrho, w, \tilde{z}) = \begin{pmatrix} f_0(\varrho, w, z) + f_1(\varrho, w, z, e_1)e_1 \\ e_2 \\ & \ddots \\ e_{r-1} \\ -g^{r-1}a_0e_1 - g^{r-2}a_1e_2 - \dots - ga_{r-2}e_{r-1} \end{pmatrix} \qquad \tilde{f}_1(\varrho, w, \tilde{z}, \tilde{e}) = \begin{pmatrix} 0 \\ 0 \\ \ddots \\ 0 \\ 1 \end{pmatrix}$$

and

$$\tilde{q}(\varrho, w, \tilde{z}, \tilde{e}, g) = q(\varrho, w, z, e_1, \dots, e_r) - g^{r-1}a_0e_2 - \dots - g^2a_{r-3}e_{r-1} - ga_{r-2}[\tilde{e} - g^{r-1}a_0e_1 - g^{r-2}a_1e_2 - \dots - ga_{r-2}e_{r-1}].$$
(13)

Let the initial conditions of (12) range on a set of the form $Z \times Z_e \times \tilde{E}$, in which $Z_e = \{(e_1, \ldots, e_{r-1} : |e_i| \le c\}$ and $\tilde{E} = \{\tilde{e} : |\tilde{e}| \le \tilde{c}\}$ with

$$\tilde{c} \ge (1 + g^{r-1}a_0 + g^{r-2}a_1 + \ldots + ga_{r-2})c$$

(note the dependence on the choice of the a_i 's and of g).

Let system (12) be augmented with (4) and consider a regulation problem with *regulated* output \tilde{e} and *measured* output $\tilde{y} = \tilde{e}$. The system, viewed as a system with input u and output \tilde{e} , has relative degree 1 and its zero dynamics, forced by the control

$$\tilde{c}(\varrho, w, \tilde{z}) = -\tilde{q}(\varrho, w, \tilde{z}, 0, g), \qquad (14)$$

is given by

$$\dot{\varrho} = 0
\dot{w} = s(\varrho, w)
\dot{\tilde{z}} = \tilde{f}_0(\varrho, w, \tilde{z}).$$
(15)

Consistently with the notation used for (7), the latter can be rewritten in more succinct form as \tilde{z} (0)

$$\tilde{\mathbf{z}} = \mathbf{f}_0(\tilde{\mathbf{z}}) \quad \text{with} \quad \tilde{\mathbf{z}} = \operatorname{col}(\varrho, w, \tilde{z}).$$
 (16)

Suppose that a controller of the form

$$\dot{\zeta} = \varphi(\zeta, \tilde{y})
\mu = \gamma(\zeta, \tilde{y})$$
(17)

has been found which solves the problem of output regulation thus defined. Then, it is immediate to realize that the controller

$$\dot{\zeta} = \varphi(\zeta, e_r + g^{r-1} a_0 e_1 + g^{r-2} a_1 e_2 + \dots + g a_{r-2} e_{r-1}) u = \gamma(\zeta, e_r + g^{r-1} a_0 e_1 + g^{r-2} a_1 e_2 + \dots + g a_{r-2} e_{r-1})$$
(18)

solves the problem of output regulation for the original plant (2). To this end note, first of all, that (18) is an admissible controller for (2), because it is driven only by the components e_1, \ldots, e_r of the measured output y of (2). Trivially, the composition of (2) with (18) differs from the composition of (12) with (17) only by a linear change of coordinates, and for any initial state of (2) in $Z \times E$, the corresponding initial state of (12) is in $Z \times Z_e \times \tilde{E}$. Thus all trajectories of (2), controlled by (18), with initial conditions in $Z \times E$ are bounded. The trajectories in question are such that $\lim_{t\to\infty} \tilde{e}(t) = 0$. But since

$$\dot{e}_1 = e_2 \vdots \dot{e}_{r-1} = -(g^{r-1}a_0e_1 + g^{r-2}a_1e_2 + \dots + ga_{r-2}e_{r-1}) + \tilde{e}$$

and the a_i 's are coefficients of a Hurwitz polynomial, it is readily concluded that also $\lim_{t\to\infty} e_1(t) = 0$. Therefore (18) solves the problem of output regulation for the system (2).

In the light of these considerations, what is left to show in order to prove the desired claim (namely the fact that there is no loss of generality in addressing the problem for systems having relative degree 1) is that the zero dynamics (15) and the associated map (14) inherit, from (7) and (6), the appropriate properties which make the solution of the problem of output regulation possible. Specifically, we will prove that if (7) and (6) satisfy assumptions (i)-(iv) above, then (15) and (14) satisfy an *identical* set of assumptions, *provided that* the parameter g is chosen sufficiently large. This is formalized in the next Lemma.

Lemma 2 Suppose that assumptions (i)-(iv) hold for (7) and (6). Set $\tilde{\mathbf{Z}} = \mathbf{W} \times Z \times Z_e$. Then there exists $g^* > 0$ such that for all fixed $g \ge g^*$ the following hold:

(ii)' there exists a compact subset $\tilde{\mathcal{Z}}$ of $\mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^{r-1}$ which contains the positive orbit of the set $\tilde{\mathbf{Z}}$ under the flow of (16), and $\tilde{\mathcal{A}}_0 := \omega(\tilde{\mathbf{Z}})$ is a differential submanifold (with boundary) of $\mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^{r-1}$. Moreover there exists a number $\tilde{d}_1 > 0$ such that

$$\tilde{\mathbf{z}} \in \mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^{r-1}, \quad \operatorname{dist}(\tilde{\mathbf{z}}, \tilde{\mathcal{A}}_0) \le \tilde{d}_1 \qquad \Rightarrow \qquad \tilde{\mathbf{z}} \in \tilde{\mathbf{Z}}.$$

(iii)' there exist $\tilde{M} \ge 1$, $\tilde{a} > 0$ and $\tilde{d}_2 \le \tilde{d}_1$ such that

 $\tilde{\mathbf{z}}_0 \in \mathbf{W} \times \mathbb{R}^n \times \mathbb{R}^{r-1}, \quad \operatorname{dist}(\tilde{\mathbf{z}}_0, \tilde{\mathcal{A}}_0) \le \tilde{d}_2 \qquad \Rightarrow \qquad \operatorname{dist}(\tilde{\mathbf{z}}(t, \tilde{\mathbf{z}}_0), \tilde{\mathcal{A}}_0) \le \tilde{M} e^{-\tilde{a}t} \operatorname{dist}(\tilde{\mathbf{z}}_0, \tilde{\mathcal{A}}_0)$

in which $\tilde{\mathbf{z}}(t, \tilde{\mathbf{z}}_0)$ denotes the solution of (16) passing through $\tilde{\mathbf{z}}_0$ at time t = 0.

 $\tilde{\tau}$

(iv)' there exist a C^1 map

such that the immersion property¹

$$\frac{\partial \tilde{\tau}}{\partial \tilde{\mathbf{z}}} \,\tilde{\mathbf{f}}_0(\tilde{\mathbf{z}}) = A \tilde{\tau}(\tilde{\mathbf{z}}) + \phi(C \tilde{\tau}(\tilde{\mathbf{z}})) + \Omega(C \tilde{\tau}(\tilde{\mathbf{z}})) \,\theta(\varrho)$$
$$\tilde{c}(\tilde{\mathbf{z}}) = C \tilde{\tau}(\tilde{\mathbf{z}})$$

holds for all $\tilde{\mathbf{z}} \in \tilde{\mathcal{A}}_0$, and $\varrho \in P$.

Proof. Consider the change of variable

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{r-1} \end{pmatrix} := D_g^{-1} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{r-1} \end{pmatrix} \quad \text{with} \quad D_g = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & g & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g^{r-2} \end{pmatrix}$$

which transforms system (15) into

$$\dot{z} = f_0(\varrho, w, z) + f_1(\varrho, w, z, x_1) x_1$$

$$\dot{x} = gAx$$
(19)

¹As above, with a mild abuse of notation we rewrite $\tilde{c}(\varrho, w, z, e_1, \ldots, e_{r-1})$ as $\tilde{c}(\tilde{\mathbf{z}})$.

where A is a Hurwitz matrix. Note that if q > 1, which we can assume without loss of generality, $(e_1(0), \ldots, e_r(0)) \in E$ implies $x(0) \in Z_e$. System (19) augmented with (4) can be regarded as a particular case of system (59) of the Appendix, to which Lemma 7 applies. In particular by property (b) of the latter, there is a number $g^* > 0$ such that for all $g \ge g^*$ the positive orbit of $\mathbf{Z} \times Z_e$ under the flow of (4) – (19) is bounded. As a consequence, the ω -limit set \mathcal{A}_0 of $\mathbf{Z} \times Z_e$, is a nonempty, compact, invariant set which uniformly attracts $\mathbf{Z} \times Z_e$. We prove now that $\mathcal{A}_0 = \mathcal{A}_0 \times \{0\}$. To this end note first of all that $\mathcal{A}_0 \times \{0\}$ by construction is contained in \tilde{A}_0 . Moreover, x is necessarily 0 at any point of \mathcal{A}_0 . In fact suppose, by contradiction, that there is a point (\mathbf{z}, x) of \mathcal{A}_0 with $x \neq 0$. As g > 0and A is Hurwitz, it follows that the trajectory x(t) of (4) - (19) originating from (\mathbf{z}, x) is unbounded in backward time, which contradicts the fact that \mathcal{A}_0 is a compact invariant (in particular in backward time) set. Finally, since \mathcal{A}_0 is the ω -limit set of **Z** under the flow of (7), we can conclude that necessarily $\mathcal{A}_0 = \mathcal{A}_0 \times \{0\}$. This in particular proves claim (ii)'. Claim (iii)', namely exponential attractivity of \mathcal{A}_0 , is an easy consequence of property (a) of Lemma 7 and of the fact that the lower subsystem of (19) is exponentially stable. To prove claim (iv)', note that (6), (13), (14) imply $\tilde{c}(\tilde{\mathbf{z}})|_{\mathcal{Z}\times\{0\}} = c(\mathbf{z})$. From this claim (iv)' immediately follows by assumption (iv), taking as $\tilde{\tau}(\tilde{\mathbf{z}})$ any differentiable function such that $\tilde{\tau}(\tilde{\mathbf{z}})|_{\mathcal{A}_0 \times \{0\}} = \tau(\mathbf{z})|_{\mathcal{A}_0}$. This completes the proof. \triangleleft

Motivated by the previous considerations and result, in what follows we focus our attention on the case in which r = 1, i.e. on the special case in which system (2) is a system of the form

$$\dot{z} = f_0(\varrho, w, z) + f_1(\varrho, w, z, e_1)e_1
\dot{e}_1 = q(\varrho, w, z, e_1) + u
e = e_1
y = e_1$$
(20)

and we assume that assumptions (i) - (ii) - (iii) - (iv) hold.

4 The adaptive internal model

4.1 The structure of the regulator

The proposed regulator is a system of the form

$$u = \xi_{1} + v$$

$$\dot{\xi} = A\xi + \phi(\xi_{1}) + \Omega(\xi_{1})\hat{\theta} + H(X,\xi_{1})v - M(X)dzv_{\ell}(\hat{\theta})$$

$$\dot{\hat{\theta}} = \beta(X,\xi_{1})v - dzv_{\ell}(\hat{\theta})$$

$$\dot{X} = FX + G\Omega(\xi_{1})$$
(21)

in which ξ_1 denotes the first component of ξ , the matrix X is a $(d-1) \times q$ matrix, M(X) is a $d \times q$ matrix defined as

$$M(X) = \begin{pmatrix} 0\\ X \end{pmatrix} \,,$$

while the vectors $H(X, \xi_1), \beta(X, \xi_1)$ and the matrices F, G have the form described below. The function $dzv_{\ell}(\cdot)$ is defined as

$$dzv_{\ell}(col(s_1,\ldots,s_q)) = col(dz_{\ell}(s_1),\ldots,dz_{\ell}(s_q))$$

in which $dz_{\ell}(\cdot)$ is any continuously differentiable function satisfying

$$dz_{\ell}(x) = \begin{cases} 0 & \text{if } |x| \le \ell \\ x & \text{if } |x| \ge \ell + 1 \end{cases}$$
(22)

and the amplitude ℓ of the *dead-zone* is chosen so that

$$\ell > \max_{\varrho \in P} |\theta(\varrho)|$$
.

This controller can be viewed as a "copy" of (11), corrected by an "innovation term", augmented with an "adaptation law" for $\hat{\theta}$ and with a "filter" which generates the "auxiliary state" X. The additional input v, which is a "stabilizing control", will eventually be taken as v = -ky.

Following the theory of adaptive observers of [1] and [17], the functions $H(X, \xi_1), \beta(X, \xi_1)$ and the matrices F, G of (21) are chosen as follows. Define new variables

$$\widetilde{\theta} = \widehat{\theta} - \theta(\varrho)
\eta = \xi - M(X)\widetilde{\theta}.$$
(23)

(note that $\eta_1 = \xi_1$) and observe that, in the new variables, the second equation of (21) reads as follows (for convenience, we omit the arguments (X, ξ_1) in H and β and the argument Xin M)

$$\dot{\eta} = A(\eta + M\tilde{\theta}) + \phi(\xi_1) + \Omega(\xi_1)(\theta(\varrho) + \tilde{\theta}) + Hv - \dot{M}\tilde{\theta} - M\beta v$$

$$= A\eta + [AM + \Omega(\xi_1) - \dot{M}]\tilde{\theta} + [H - M\beta]v + \phi(\xi_1) + \Omega(\xi_1)\theta(\varrho).$$
(24)

The third equation, instead, becomes trivially

$$\tilde{\theta} = \beta v - \operatorname{dzv}_{\ell} \left(\tilde{\theta} + \theta(\varrho) \right).$$

The choices of $H(X,\xi_1), \beta(X,\xi_1)$ and of F, G are meant to simplify the terms

$$[AM + \Omega(\xi_1) - \dot{M}]\tilde{\theta} + [H - M\beta]v$$

in the expression (24). First of all, note that choosing

$$H = M\beta + K$$

with K a constant vector (whose expression will be determined later), the second term becomes equal to Kv. As for the first term, the idea is to impose that

$$[AM + \Omega(\xi_1) - \dot{M}]\tilde{\theta} = b\beta^{\mathrm{T}}\tilde{\theta}$$

in which b is a $d \times 1$ fixed vector. The identity in question holds if M satisfies

$$\dot{M} = (A - bCA)M + (I - bC)\Omega(\xi_1)$$

and β is taken as

$$\beta^{\mathrm{T}} = CAM + C\Omega(\xi_1) \,.$$

In this way, the second equation of (21) takes the simplified form

$$\dot{\eta} = A\eta + b\beta^{\mathrm{T}}\tilde{\theta} + Kv + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho), \qquad (25)$$

on which we will return later. To show that the required differential equation for M can be enforced, pick a column vector $b = col(1, b_2, ..., b_d)$. Then, bearing in mind the definition of M, it is easily realized that the required differential equation holds if the matrices F and Gin the differential equation for X have the form (see [17])

$$F = \begin{pmatrix} -b_2 & 1 & \cdots & 0 & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot\\ -b_{d-1} & 0 & \cdots & 0 & 1\\ -b_d & 0 & \cdots & 0 & 0 \end{pmatrix}, \qquad G = \begin{pmatrix} -b_2 & 1 & \cdots & 0 & 0 & 0\\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot\\ -b_{d-1} & 0 & \cdots & 0 & 1 & 0\\ -b_d & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}.$$
 (26)

In summary, the quantities $H(X, \xi_1), \beta(X, \xi_1), F, G$ which appear in the controller (21) are determined as follows: F and G are the matrices in (26), $\beta(X, \xi_1)$ is chosen as

$$\beta(X,\xi_1) = \left[CA\begin{pmatrix}0\\X\end{pmatrix} + C\Omega(\xi_1)\right]^{\mathrm{T}}$$
(27)

and $H(X,\xi_1)$ is chosen as

$$H(X,\xi_1) = \begin{pmatrix} 0\\ X \end{pmatrix} [CA\begin{pmatrix} 0\\ X \end{pmatrix} + C\Omega(\xi_1)]^{\mathrm{T}} + K.$$
(28)

The vector b, whose entries determine the choice of F and G and the parameter K, which appears in the expression of $H(X, \xi_1)$, will be chosen later.

The controller thus defined determines a closed loop system which, in the coordinates indicated above, can be written as (recall that $e_1 = e$)

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w)$$

$$\dot{z} = f_0(\varrho, w, z) + f_1(\varrho, w, z, e)e$$

$$\dot{e} = q(\varrho, w, z, e) + \eta_1 + v$$

$$\dot{\eta} = A\eta + b\beta^{\mathrm{T}}\tilde{\theta} + Kv + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho)$$

$$\dot{\tilde{\theta}} = \beta v - \mathrm{dzv}_{\ell} (\tilde{\theta} + \theta(\varrho))$$

$$\dot{X} = FX + G\Omega(\eta_1),$$
(29)

where β is a function of X and η_1 . This system, viewed as a system with input v and output e, has relative degree 1 and its zero dynamics are those of

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w)$$

$$\dot{z} = f_0(\varrho, w, z)$$

$$\dot{\eta} = A\eta - K[q(\varrho, w, z, 0) + \eta_1] + b\beta^{\mathrm{T}}\tilde{\theta} + \phi(\eta_1) + \Omega(\eta_1)\theta(\varrho)$$

$$\dot{\tilde{\theta}} = -\beta[q(\varrho, w, z, 0) + \eta_1] - \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho))$$

$$\dot{X} = FX + G\Omega(\eta_1).$$
(30)

The intuition suggests that if the latter have convenient asymptotic properties, in particular possess a locally exponentially stable compact attractor, an additional control of the form v = -ke, (with large k > 0) should be able to solve the problem of output regulation. Thus, in following subsection, the asymptotic properties of (30) will be studied.

4.2 Trajectories of (30) are bounded

In studying the asymptotic properties of this system, it is convenient to take advantage of the "immersion" assumption (iii) introduced above. Specifically, suppose that the initial conditions for ρ, w, z are taken in the set \mathbf{Z} , a subset of a set \mathcal{Z} which by hypothesis is positively invariant for the subsystem formed by the top three equations of (30). Thus, for any of such initial conditions and for any $t \geq 0$, the function $\tau(\rho, w(t), z(t))$ is well defined and it is legitimate to consider the change of variables

$$\chi = \eta - \tau(\varrho, w, z) \,.$$

This transforms system (30) in a system of the form (use here (9) and (10) which hold on $\mathcal{A}_0 \subset \mathcal{Z}$)

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w)$$

$$\dot{z} = f_0(\varrho, w, z)$$

$$\dot{\chi} = (A - KC)\chi_1 + b\beta^{\mathrm{T}}\tilde{\theta} + \Delta(\chi_1, \tau_1, \theta) + e(\varrho, w, z)$$

$$\dot{\tilde{\theta}} = -\beta\chi_1 - \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho))$$

$$\dot{X} = FX + G\Omega(\chi_1 + \tau_1),$$
(31)

in which

$$\Delta(\chi_1, \tau_1, \theta) = \phi(\chi_1 + \tau_1) - \phi(\chi_1) + [\Omega(\chi_1 + \tau_1) - \Omega(\chi_1)]\theta(\varrho)$$

is a term which vanishes at $\chi_1 = 0$ and

$$e(\varrho, w, z) = K(c(\varrho, w, z) - \tau_1(\varrho, w, z)) + A\tau(\varrho, w, z)) + \phi(\tau_1(\varrho, w, z)) + \Omega(\tau_1(\varrho, w, z))\theta(\varrho) - \frac{\partial \tau}{\partial z} f_0(\varrho, w, z) - \frac{\partial \tau}{\partial w} s(\varrho, w)$$
(32)

is a term vanishing on \mathcal{A}_0 . In particular note that, since $\phi(y)$ and $\Omega(y)$ can be taken to be globally Lipschitz and θ ranges over a compact set, there exists a number L such that

$$|\Delta(\chi_1, \tau_1, \theta)| \le L_{\phi} |\chi_1| + L_{\Omega} |\chi_1| |\theta| \le L |\chi_1|$$

for all χ_1, τ_1, θ .

The idea is now to choose the b_i 's and K so that system (31) has certain desirable asymptotic properties. To this end, let the b_i be such that the polynomial

$$p(\lambda) = \lambda^{d-1} + b_2 \lambda^{d-2} + \dots + b_{d-1} \lambda + b_d$$
(33)

has d-1 distinct roots with negative real part. As a consequence the matrix F in the bottom equation of (31) is Hurwitz (and has distinct eigenvalues). This, in view of the assumptions on the top three equations, suggests that the asymptotic properties of (31) are entirely determined by those of the fourth and fifth equation.

As indicated in [17, Theorem 2.1], the appropriate choice for K in (25) is

$$K = Ab + \lambda b \tag{34}$$

in which $\lambda > 0$. To see why this is the case note first of all that, using a little algebra, it is not difficult to prove the following.

Lemma 3 Choose K as in (34) and set

$$T = \begin{pmatrix} 1 & 0\\ \hat{b} & I \end{pmatrix}, \qquad \hat{b} = - \begin{pmatrix} b_2\\ \cdot\\ b_{d-1}\\ b_d \end{pmatrix}.$$

Then

$$T(A - KC)T^{-1} = \begin{pmatrix} -\lambda & \hat{c} \\ 0 & F \end{pmatrix}, \quad Tb = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad CT^{-1} = C,$$

in which $\hat{c} = (1 \quad 0 \quad \cdots \quad 0)$ and F is the matrix defined in (26).

¿From this fact, standard arguments can be invoked to claim boundedness of the trajectories of (31). In fact, the following result holds.

Lemma 4 Suppose assumptions (i), (ii), (iv) hold. There is a number λ^* such that, if $\lambda \geq \lambda^*$, all trajectories of (31) are bounded.

Proof. First of all, recall that, by assumption (ii), $(\varrho, w(t), z(t)) \in \mathbb{Z}$ for all $t \ge 0$, where \mathbb{Z} is a compact set. Thus, looking at the expression (32) of $e(\varrho, w, z)$, it is seen that there exist a number \overline{e} (depending on the design parameter λ because the latter appears in K) such that

$$|e(\varrho, w(t), z(t))| \le \bar{e} \qquad \forall t \ge 0.$$
(35)

Observe also that, so long as trajectories of (31) exist on some interval [0, T], |X(t)| is bounded, by a number which only depends on |X(0)| (because $|\Omega(\cdot)|$, having compact support, is bounded by some fixed number). As a consequence, also $|\beta(t)|$ is bounded, again by a number which only depends on |X(0)|. Thus, system (31) cannot have finite escape times.

This being the case, to prove the Lemma it remains to show that also χ and $\hat{\theta}$ are bounded. To this end, let χ be partitioned as $\chi = \operatorname{col}(\chi_1, \chi_2)$, in which χ_2 is a $(d-1) \times 1$ vector and change χ_2 into

$$\zeta = \hat{b}\chi_1 + \chi_2$$
 .

In this way, the fourth and fifth equations of system (31) are changed into

$$\dot{\chi}_{1} = -\lambda\chi_{1} + \hat{c}\zeta + \beta^{\mathrm{T}}\tilde{\theta} + C\Delta(\chi_{1},\tau_{1},\theta) + Ce(\varrho,w,z)$$

$$\dot{\zeta} = F\zeta + (\hat{b} \quad I) \Delta(\chi_{1},\tau_{1},\theta) + (\hat{b} \quad I) Ce(\varrho,w,z)$$

$$\dot{\tilde{\theta}} = -\beta\chi_{1} - \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho)).$$
(36)

With this in mind, choose for (36) the Lyapunov function

$$V(\chi_1, \zeta, \tilde{\theta}) = \chi_1^2 + \zeta^{\mathrm{T}} P \zeta + \tilde{\theta}^{\mathrm{T}} \tilde{\theta} , \qquad (37)$$

in which P is the positive definite solution of $PF + F^{T}P = -I$. This yields

$$\dot{V} = -2\lambda\chi_{1}^{2} + 2\chi_{1}\hat{c}\zeta + 2\chi_{1}\beta^{\mathrm{T}}\tilde{\theta} + 2\chi_{1}C\Delta(\chi_{1},\tau_{1},\theta) + 2\chi_{1}Ce(\varrho,w,z)
-|\zeta|^{2} + 2\zeta^{\mathrm{T}}P(\hat{b} \quad I)\Delta(\chi_{1},\tau_{1},\theta) + 2\zeta^{\mathrm{T}}P(\hat{b} \quad I)Ce(\varrho,w,z)
-2\tilde{\theta}^{\mathrm{T}}\beta\chi_{1} - 2\tilde{\theta}^{\mathrm{T}}\mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho))
\leq -2\lambda\chi_{1}^{2} - |\zeta|^{2} - 2\tilde{\theta}^{\mathrm{T}}\mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho)) + L_{1}|\chi_{1}|^{2} + L_{2}|\chi_{1}||\zeta|
+ L_{3}|\chi_{1}||e(\varrho,w,z)| + L_{4}|\zeta||e(\varrho,w,z)|$$
(38)

in which L_i , i = 1..., 4 are suitable positive constants. By completing the squares and using (35), we obtain

$$\dot{V} \le -(2\lambda - L_1 + \frac{1}{2}L_2^2)\chi_1^2 - \frac{1}{2}|\zeta|^2 - 2\tilde{\theta}^{\mathrm{T}}\mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho)) + L_3|\chi_1|\bar{e} + L_4|\zeta|\bar{e}.$$
(39)

Bearing in mind the definition (22) and the choice of ℓ , observe that

 $\tilde{\theta}^{\mathrm{T}} \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho)) \ge 0 \quad \text{for all } \tilde{\theta} \in \mathbb{R}^{q} \quad \text{and} \quad \varrho \in P.$ (40)

It is also easy to check that for any $\delta > \sqrt{q}(2\ell+1)$ there is a positive number c_1 such that

$$|\tilde{\theta}| \ge \delta \qquad \Rightarrow \qquad 2\tilde{\theta}^{\mathrm{T}} \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho)) \ge c_1 |\tilde{\theta}|^2 \qquad \text{for all } \tilde{\theta} \in \mathbb{R}^q \quad \text{and} \quad \varrho \in P \,.$$
 (41)

Pick a value of λ large enough so that $\overline{\lambda} := 2\lambda - L_1 - L_2^2/2 > 0$. Inequality (39), in view of property (41), yields

$$|\tilde{\theta}| \ge \delta \qquad \Rightarrow \qquad \dot{V} \le -c_2 |(\chi_1, \zeta, \tilde{\theta})|^2 + c_3 |(\chi_1, \zeta, \tilde{\theta})| \bar{e}$$

in which $c_2 = \min\{\overline{\lambda}, \frac{1}{2}, c_1\}$ and $c_3 = 3\max\{L_3, L_4\}$. From this, it is seen that

$$|\tilde{\theta}| \ge \delta$$
 and $|(\chi_1, \zeta, \tilde{\theta})| > \frac{c_3}{c_2} \bar{e} \implies \dot{V} < 0.$ (42)

Property (40), on the other hand, yields

$$\dot{V} \le -c_2 |(\chi_1, \zeta)|^2 + c_3 |(\chi_1, \zeta)| \bar{e}$$

from which it is seen that

$$|(\chi_1,\zeta)| > \frac{c_3}{c_2} \bar{e} \qquad \Rightarrow \qquad \dot{V} < 0.$$
(43)

We show now that a combination of (42) and (43) yields the desired result, namely the boundedness of $(\chi_1(t), \zeta(t), \tilde{\theta}(t))$. As a matter of fact set

$$r := \sqrt{\delta^2 + \left(\frac{c_3}{c_2}\,\bar{e}\right)^2}$$

and note that, since

$$|(\chi_1, \zeta, \tilde{\theta})| > r \qquad \Rightarrow \qquad |(\chi_1, \zeta, \tilde{\theta})| > \frac{c_3}{c_2}\bar{e}$$

and

$$|(\chi_1,\zeta,\tilde{\theta})| > r \qquad \Rightarrow \qquad |(\chi_1,\zeta)| > \frac{c_3}{c_2}\bar{e} \quad \text{or} \quad |\tilde{\theta}| > \delta,$$

relations (43) and (42) imply

$$|(\chi_1,\zeta,\tilde{\theta})| > r \qquad \Rightarrow \qquad \dot{V} < 0.$$

From this, bearing in mind the fact that $V(\chi_1, \zeta, \tilde{\theta})$ is a quadratic form, the result follows by standard arguments. \triangleleft

We can therefore draw the following conclusion about system (30). Let the initial conditions $\eta(0), \tilde{\theta}(0), X(0)$ be taken in fixed compact sets **H**, Θ , **X**.² Then, the positive orbit of the set

$$\mathbf{B} = \mathbf{Z} \times \mathbf{H} \times \Theta \times \mathbf{X}$$

under the flow of (30) is bounded. As a consequence $\omega(\mathbf{B})$, the ω -limit set of \mathbf{B} under the flow of (30), is a non-empty, compact and invariant set, which uniformly attracts all trajectories of (30) with initial conditions in \mathbf{B} .

²Recall that $\tilde{\theta}(t) = \hat{\theta}(t) - \theta(\varrho)$ and $\eta(t) = \xi(t) - \beta(X(t), \xi_1(t))\tilde{\theta}(t)$. Thus, to establish boundedness of trajectories when $\xi(0)$, $\hat{\theta}(0)$ and X(0) are taken in fixed compact sets it suffices to consider the case in which $\tilde{\theta}(0)$, $\eta(0)$ and X(0) are taken in fixed compact sets.

4.3 The limit set of (30)

We proceed now to investigate the structure of the set $\omega(\mathbf{B})$. To this end, we look at the equivalent system (31), we note that the three top equations are independent of the bottom ones and we rewrite them in compact form as in (8) (and consistently we rewrite the term $e(\varrho, w, z)$ as $e(\mathbf{z})$ and $\tau(z, w, \varrho)$ as $\tau(\mathbf{z})$). In particular, because of the special triangular structure of (31), we note that if $(\mathbf{z}, \chi, \tilde{\theta}, X)$ is a point of $\omega(\mathbf{B})$, necessarily \mathbf{z} is a point in the ω -limit set of \mathbf{Z} under the flow of (8), that is, \mathbf{z} is a point of \mathcal{A}_0 . This implies that on $\omega(\mathbf{B})$ we have $e(\mathbf{z}) = 0$ and thus system (31) simplifies as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\chi} &= (A - KC)\chi + b\beta^{\mathrm{T}}\tilde{\theta} + \Delta(\chi_1, \tau_1, \theta) \\ \dot{\tilde{\theta}} &= -\beta\chi_1 - \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho)) \\ \dot{\chi} &= FX + G\Omega(\chi_1 + \tau_1(\mathbf{z})). \end{aligned}$$
(44)

What we will be able to prove in the following is that on points of $\omega(\mathbf{B})$ necessarily $\chi = 0$, $\tilde{\theta} = 0$ and the value of X is entirely determined by the properties of the system

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \dot{X} = FX + G\Omega(\tau_1(\mathbf{z})),$$

$$(45)$$

in which $\tau_1(\mathbf{z})$ is the obvious abbreviated notation for $\tau_1(z, w, \varrho)$. To this end, though, an extra hypothesis is needed, which will be explained after having shown an interesting feature of the system in question.

Lemma 5 The graph of the map

$$\sigma : \mathcal{A}_0 \to \mathbb{R}^{(d-1) \times q}$$
$$\mathbf{z} \mapsto \int_{-\infty}^0 e^{-Fs} G\Omega(\tau_1(\mathbf{z}(s, \mathbf{z}))) ds$$

is invariant for (45).

Proof. Let $\mathbf{z}(t, \mathbf{z}_0)$ denote the solution of (8) passing through \mathbf{z}_0 at time t = 0 and note that, if $\mathbf{z}_0 \in \mathcal{A}_0$, then $\mathbf{z}(t, \mathbf{z}_0) \in \mathcal{A}_0$ for all t (thus, in particular, since \mathcal{A}_0 is compact, $|\mathbf{z}(t, \mathbf{z}_0)|$ is bounded by a number which depends only on \mathcal{A}_0). Then, since F is a Hurwitz matrix, the map $\sigma(\cdot)$ is well defined. As simple calculation shows that

$$\sigma(\mathbf{z}(t,\mathbf{z}_0)) = e^{Ft}\sigma(\mathbf{z}_0) + \int_0^t e^{F(t-s)}G\Omega(\tau_1(\mathbf{z}(s,\mathbf{z}_0)))ds.$$

This shows that

$$\operatorname{graph}(\sigma) = \{(\mathbf{z}, X) : \mathbf{z} \in \mathcal{A}_0, X = \sigma(\mathbf{z})\}$$

is invariant for (45). \triangleleft

Remark. Consider the restriction of (45) to $\mathcal{A}_0 \times \mathbb{R}^{(d-1)\times q}$. Since the graph of $\sigma(\cdot)$ is invariant for (45), changing X into $\tilde{X} = X - \sigma(\mathbf{z})$, yields

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$$

 $\dot{\tilde{X}} = F\tilde{X}$.

We see from this that the solution X(t) of (45) passing through X_0 at time t = 0 can be expressed as

$$X(t) = e^{Ft} [X_0 - \sigma(\mathbf{z}_0)] + \sigma(\mathbf{z}(t, \mathbf{z}_0)). \qquad \triangleleft$$
(46)

We introduce now an additional hypothesis, reminiscent of the classical hypothesis of *persistence of excitation*.

Assumption (v): Consider the map $\varphi : \mathcal{A}_0 \to \mathbb{R}^{q \times 1}$ defined as

$$\varphi: \mathbf{z} \mapsto \beta(\sigma(\mathbf{z}), \tau_1(\mathbf{z}))$$

It is assumed that for any initial condition $\mathbf{z}_0 \in \mathcal{A}_0$ the identity

$$\gamma^{\mathrm{T}}\varphi(\mathbf{z}(t,\mathbf{z}_0)) = 0, \quad \text{for all } t \in \mathbb{R}$$

implies $\gamma = 0. \triangleleft$

Remark. In other words, the assumption of "persistency of excitation", in the present context, is spelled as follows: for any initial condition $\mathbf{z}_0 \in \mathcal{A}_0$, the q outputs of the autonomous system

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \varphi = \beta(\sigma(\mathbf{z}), \tau_1(\mathbf{z}))$$

are linearly independent functions, on the entire time axis. \triangleleft

Under this hypothesis, the set $\omega(\mathbf{B})$ assumes a very simple structure. As a matter of fact, the following result holds.

Lemma 6 Suppose that, in addition to assumptions (i), (ii), (iv), also assumption (v) holds. Then the values of χ and $\tilde{\theta}$ on any point of $\omega(\mathbf{B})$ are necessarily zero.

Proof. By contradiction, suppose a point $\mathbf{p} = (\mathbf{z}, \chi_0, \tilde{\theta}_0, X)$ with either $\chi_0 \neq 0$ or $\tilde{\theta}_0 \neq 0$ is in $\omega(\mathbf{B})$. Since $\omega(\mathbf{B})$ is compact and invariant, in particular in backward time, the backward trajectory of (44) starting at this point is bounded. Along this trajectory, the function

$$V(t) := V(\chi_1(t), \zeta(t), \tilde{\theta}(t))$$

in (37) satisfies $V(t) \leq C$ for all $t \leq 0$, for some C > 0. Moreover, since $e(\mathbf{z}) = 0$ on $\omega(\mathbf{B})$, the same computations indicated in the proof of Lemma 4 show that

$$\dot{V}(t) \le -(2\lambda - L_1 - \frac{1}{2}L_2^2)|\chi_1(t)|^2 - \frac{1}{2}|\zeta(t)|^2 - 2\tilde{\theta}(t)dzv_\ell(\tilde{\theta}(t) + \theta(\varrho))$$

in which L_1 and L_2 are the same constants introduced in the proof of Lemma 4. From this, using property (40), it turns out that if $\lambda \geq \lambda^*$ (where λ^* is the same as in Lemma 4) then V(t) is non-increasing along trajectories. As consequence, since V(t) is bounded, that there must exist a finite number V_{α} such that

$$\lim_{t \to -\infty} V(t) = V_{\alpha}$$

The trajectory in question is attracted, in backward time, by its own α -limit set $\alpha(\mathbf{p})$, which, as it is well known, is nonempty, compact and invariant. Moreover, by definition, the function $V(\chi_1, \zeta, \tilde{\theta})$ has the same value V_{α} at any point of $\alpha(\mathbf{p})$.

Now, as in the classical proof of LaSalle's invariance principle, pick an initial condition $\hat{\mathbf{p}}$ in the set $\alpha(\mathbf{p})$ and consider the corresponding trajectory of (44), which remains in $\alpha(\mathbf{p})$ for all times. Along such trajectory, V(t) is constantly equal to V_{α} and hence

$$\chi_1(t) = 0, \qquad \zeta(t) = 0, \qquad \operatorname{dzv}_\ell(\hat{\theta}(t) + \theta(\varrho)) = 0 \qquad \text{for all } t \in \mathbb{R}.$$

Entering these constraints in (44), and observing that the vector b is nonzero, it is seen that necessarily

$$\begin{aligned} \theta^{1}\beta &= 0\\ \dot{\tilde{\theta}} &= 0\\ \dot{X} &= FX + G\Omega(\tau_{1}(\mathbf{z})) \end{aligned}$$

The second condition shows that $\tilde{\theta}(t)$ is a constant, say $\tilde{\theta}^*$, along such trajectory. The third condition, says that X(t) is a solution of

$$\dot{X} = FX + G\Omega(\tau_1(\mathbf{z}))$$

Now, since F is Hurwitz and has distinct eigenvalues (because so are the roots of the polynomial (33)), it is seen from (46) that X(t) is bounded for $t \leq 0$ only if $X(0) = \sigma(\mathbf{z}(0))$, where $\sigma(\cdot)$ is the map introduced in Lemma 5, in which case $X(t) = \sigma(\mathbf{z}(t))$. Since X(t) has to be bounded because $\alpha(\mathbf{p})$ is compact, it follows that X(t) is necessarily equal to $\sigma(\mathbf{z}(t))$. This being the case, bearing in mind the expression of β and the definition of the map $\varphi(\cdot)$, the first condition shows that necessarily

$$(\hat{\theta}^*)^{\mathrm{T}}\varphi(\mathbf{z}(t)) = 0, \quad \text{for all } t \in \mathbb{R}.$$

Thus, in view of the assumption of persistency of excitation, it follows that $\tilde{\theta}^* = 0$. It is seen in this way that $(\chi_1, \zeta, \tilde{\theta}) = (0, 0, 0)$ at any point of $\alpha(\mathbf{p})$, and this proves that $V_{\alpha} = 0$. But this is a contradiction, because V(t) is non-increasing along trajectories and V(0) is strictly positive, if either $\chi_0 \neq 0$ or $\tilde{\theta}_0 \neq 0$.

To complete the analysis, it remains to determine the values of X on points of $\omega(\mathbf{B})$. Knowing that $\chi_1 = 0$ on any of such points, it follows from the previous analysis and in particular from Lemma 5 that $X = \sigma(\mathbf{z})$. Altogether, bearing in mind how system (30) and system (44) are related, the following conclusion holds.

Proposition 1 Under the assumptions (i), (ii), (iv) and (v) the set $\omega(\mathbf{B})$ is the graph of a continuous map defined on \mathcal{A}_0 . Any point of $\omega(\mathbf{B})$ is a point $(\mathbf{z}, \eta, \tilde{\theta}, X)$ in which $\mathbf{z} \in \mathcal{A}_0$ and

$$\eta = \tau(\mathbf{z}), \qquad \theta = 0, \qquad X = \sigma(\mathbf{z}).$$

4.4 Exponential attractivity of the limit set of (30)

Finally, we prove that the set $\omega(\mathbf{B})$ is also locally exponentially attractive for the trajectories of the zero dynamics (30) of system (29), if so is the set \mathcal{A}_0 for the trajectories of (8). This fact is formalized in the next proposition.

Proposition 2 Suppose that, in addition to assumptions (i)-(ii)-(iv) and (v), also assumption (iii) holds. Then $\omega(\mathbf{B})$ is locally exponentially attractive for (30).

Proof. Consider again the equivalent system (31), let the compact notation $\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z})$ be used for the first three equations and let the variables (χ_1, ζ) , introduced in the proof of Lemma 4, replace χ . Let $\bar{\sigma} : \mathcal{Z} \to \mathbb{R}^{(d-1)\times q}$ be any continuously differentiable map which agrees on \mathcal{A}_0 with the map σ introduced in Lemma 5, and change X into $\tilde{X} = X - \sigma(\mathbf{z})$. In this way, the last equation of (31) is transformed into an equation of the form

$$\dot{\tilde{X}} = F\tilde{X} + Q(\mathbf{z}) + R(\chi_1, \mathbf{z})$$

in which

$$Q(\mathbf{z}) = F\bar{\sigma}(\mathbf{z}) + G\Omega(\tau_1(\mathbf{z})) - \frac{\partial\bar{\sigma}}{\partial\mathbf{z}}\mathbf{f}_0(\mathbf{z})$$

is a (matrix-valued) function vanishing on \mathcal{A}_0 while

$$R(\chi_1, \mathbf{z}) = G[\Omega(\chi_1 + \tau_1(\mathbf{z})) - \Omega(\chi_1)]$$

is vanishing for $\chi_1 = 0$ for all $\mathbf{z} \in \mathcal{Z}$. Let $\tilde{X}_i, Q_i(\mathbf{z}), R_i(\chi_1, \mathbf{z})$ denote the *i*-th columns of $\tilde{X}, Q(\mathbf{z}), R(\chi_1, \mathbf{z})$. Setting $\mathbf{x} = \operatorname{col}(\chi_1, \zeta, \tilde{\theta}, \tilde{X}_1, \ldots, \tilde{X}_q)$, system (31) can be conveniently rewritten as

$$\begin{aligned} \dot{\mathbf{z}} &= \mathbf{f}_0(\mathbf{z}) \\ \dot{\mathbf{x}} &= \mathbf{g}(\mathbf{z}, \mathbf{x}) + \nu(\mathbf{z}) \end{aligned}$$

in which

$$\nu(\mathbf{z}) = \begin{pmatrix} Ce(\mathbf{z}) \\ \left(\begin{array}{c} \hat{b} & I \end{array} \right) e(\mathbf{z}) \\ 0 \\ Q_1(\mathbf{z}) \\ \vdots \\ Q_q(\mathbf{z}) \end{pmatrix}$$

is a vector of functions vanishing on \mathcal{A}_0 . Observing that $\mathbf{g}(\mathbf{z}, 0) = 0$, set

$$\mathbf{A}(\mathbf{z}) = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}(\mathbf{z}, 0)$$

and consider the expansion

$$\mathbf{g}(\mathbf{z}, \mathbf{x}) = \mathbf{A}(\mathbf{z})\mathbf{x} + \mathbf{h}(\mathbf{z}, \mathbf{x})$$
.

The matrix $\mathbf{A}(\mathbf{z})$ is the matrix

$$\mathbf{A}(\mathbf{z}) = \begin{pmatrix} -\lambda + r_1(\mathbf{z}) & \hat{c} & \beta^{\mathrm{T}}(\sigma(\mathbf{z}), \tau_1(\mathbf{z})) & 0 & \cdots & 0\\ r_2(\mathbf{z}) & F & 0 & 0 & \cdots & 0\\ -\beta(\sigma(\mathbf{z}), \tau_1(\mathbf{z})) & 0 & 0 & 0 & \cdots & 0\\ r_{31}(\mathbf{z}) & 0 & 0 & F & \cdots & 0\\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots\\ r_{3q}(\mathbf{z}) & 0 & 0 & 0 & \cdots & F \end{pmatrix}$$

in which

$$r_1(\mathbf{z}) = C \begin{bmatrix} \frac{\partial \Delta}{\partial \chi_1} \end{bmatrix}_{\chi_1=0} \qquad r_2(\mathbf{z}) = \begin{pmatrix} \hat{b} & I \end{pmatrix} \begin{bmatrix} \frac{\partial \Delta}{\partial \chi_1} \end{bmatrix}_{\chi_1=0} \qquad r_{3i}(\mathbf{z}) = \begin{bmatrix} \frac{\partial R_i}{\partial \chi_1} \end{bmatrix}_{\chi_1=0}.$$

Moreover, by construction, the vector $\mathbf{h}(\mathbf{z}, \mathbf{x})$ is such that

$$\lim_{|\mathbf{x}|\to 0} \frac{|\mathbf{h}(\mathbf{z},\mathbf{x})|}{|\mathbf{x}|} = 0,$$

uniformly in **z** (as the latter ranges over a compact set). In this way, system (31) is rewritten as $\dot{\mathbf{r}} = \mathbf{r} \mathbf{r}$

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \dot{\mathbf{x}} = \mathbf{A}(\mathbf{z})\mathbf{x} + \mathbf{h}(\mathbf{z}, \mathbf{x}) + \nu(\mathbf{z}).$$

$$(47)$$

With this in mind, consider now the auxiliary system

$$\dot{\mathbf{z}} = \mathbf{f}_0(\mathbf{z}) \dot{\mathbf{y}} = \mathbf{A}(\mathbf{z})\mathbf{y}$$
(48)

with initial conditions $(\mathbf{z}(0), \mathbf{y}(0))$ in the compact set $\mathbf{Z} \times \mathbf{Y}$ where $\mathbf{Y} = {\mathbf{y} : |\mathbf{y}| \le c}$, with c > 1. Arguments identical to those used in the proof of Lemma 4 and Lemma 6 make it

possible to claim the existence of a $\lambda^* > 0$ such that for all $\lambda > \lambda^*$ the trajectories $(\mathbf{z}(t), \mathbf{y}(t))$ are bounded in positive time and that

$$\omega(\mathbf{Z} \times \mathbf{Y}) = \mathcal{A}_0 \times \{0\}.$$

As a matter of fact note that, by assumption, the trajectories $\mathbf{z}(t)$ are such that $\mathbf{z}(t) \in \mathcal{Z}$ for all $t \geq 0$. As far as the trajectories $\mathbf{y}(t)$ are concerned, consider the candidate Lyapunov function

$$U(\mathbf{y}) = V(\chi_1, \zeta, \tilde{\theta}) + \sum_{i=1}^q \tilde{X}_i^T P \tilde{X}_i$$

where $V(\chi_1, \zeta, \tilde{\theta})$ is the function defined in (37) and P is the positive definite solution of $PF + F^T P = -I$. The time derivative of $U(\mathbf{y}(t))$ along the solutions of (48) can be estimated as

$$\begin{split} \dot{U} &= -2(\lambda - r_{1}(\mathbf{z}))\chi_{1}^{2} + 2\hat{c}\chi_{1}\zeta + 2\chi_{1}\beta^{\mathrm{T}}(\sigma(\mathbf{z}),\tau_{1}(\mathbf{z}))\tilde{\theta} - |\zeta|^{2} + 2\zeta^{\mathrm{T}}Pr_{2}(\mathbf{z})\chi_{1} \\ &- 2\tilde{\theta}^{\mathrm{T}}\beta(\sigma(\mathbf{z}),\tau_{1}(\mathbf{z}))\chi_{1} - \sum_{i=1}^{q}|\tilde{X}_{i}|^{2} + \sum_{i=1}^{q}2\tilde{X}_{i}^{\mathrm{T}}Pr_{3i}(\mathbf{z})\chi_{1} \\ &\leq -2(\lambda - \bar{r}_{1})\chi_{1}^{2} - |\zeta|^{2} - \sum_{i=1}^{q}|\tilde{X}_{i}|^{2} \\ &+ 2\,\hat{c}\,|\chi_{1}|\,|\zeta| + 2\,\bar{r}_{2}\,|P|\,|\zeta|\,|\chi_{1}| + \sum_{i=1}^{q}2\,\bar{r}_{3}\,|P|\,|\tilde{X}_{i}|\,|\chi_{1}| \end{split}$$

in which $\bar{r}_i = \max_{\mathbf{z}\in\mathcal{Z}} |r_i(\mathbf{z})|$, i = 1, 2 and $\bar{r}_3 = \max_{\mathbf{z}\in\mathcal{Z},i=1,\dots,q} |r_{3i}(\mathbf{z})|$. Standard arguments can be used to show that a large value of λ renders \dot{U} non positive, from which boundedness of $\mathbf{y}(t)$ follows. Moreover the same arguments of the proof of Lemma 6 can be repeated to show that, under the condition of persistence of excitation expressed by Assumption (v), points on $\omega(\mathbf{Z} \times \mathbf{Y})$ of (48) are necessarily characterized by $\mathbf{y} = 0$, from which it follows that $\omega(\mathbf{Z} \times \mathbf{Y}) = \mathcal{A}_0 \times \{0\}$.

We show now that $\mathcal{A}_0 \times \{0\}$ is locally exponentially attractive for (48). To this end, let $\mathbf{z}(t, \mathbf{z}_0)$ and $\mathbf{y}(t, \mathbf{y}_0)$ denote the solution pair of (48) passing through \mathbf{z}_0 and, respectively, \mathbf{y}_0 at time t = 0. Recall (see section 2) that $\mathcal{A}_0 \times \{0\}$ attracts the set $\mathbf{Z} \times \mathbf{Y}$ uniformly. Therefore, since $|\mathbf{y}| \leq \text{dist}((\mathbf{z}, \mathbf{y}), \mathcal{A}_0 \times \{0\})$, for any $\varepsilon > 0$ there exists $T_{\varepsilon} > 0$ such that

$$|\mathbf{y}(t,\mathbf{y}_0)| \le \varepsilon$$
 for all $t \ge T_{\varepsilon}$ and all $(\mathbf{z}_0,\mathbf{y}_0) \in \mathbf{Z} \times \mathbf{Y}$. (49)

With this in mind, let δ be such that $\operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0) \leq \delta$ implies $\mathbf{z}(t, \mathbf{z}_0) \in \mathbf{Z}$ for all $t \geq 0$, which is always possible, since \mathcal{A}_0 is stable in the sense of Lyapunov for the upper equation of (48). Pick any \mathbf{z}_0 within a δ -distance from \mathcal{A}_0 and regard the bottom equation of (48) as a *time-varying linear* system

$$\dot{\mathbf{y}} = \mathbf{A}(\mathbf{z}(t, \mathbf{z}_0))\mathbf{y} \,. \tag{50}$$

Pick a pair $t \ge t_0 \ge 0$ and let $\Phi(t, t_0, \mathbf{z}_0)$ denote the associated state transition matrix (which, of course, depends on the pick of \mathbf{z}_0). By construction, the *i*-th column $\phi_i(t, t_0, \mathbf{z}_0)$ of $\Phi(t, t_0, \mathbf{z}_0)$ is the solution of (50) which satisfies $\phi_i(t_0, t_0, \mathbf{z}_0) = \mathbf{v}_i$, where \mathbf{v}_i is a vector in which all entries are zero but the *i*-th one, which is equal to 1. Consider now again (48) with initial conditions $\mathbf{z}(0) = \mathbf{z}(t_0, \mathbf{z}_0)$ and $\mathbf{y}(0) = \mathbf{v}_i$ (note that $(\mathbf{z}(t_0, \mathbf{z}_0), \mathbf{v}_i) \in \mathbf{Z} \times \mathbf{Y}$). Since (48) is time invariant, we observe that $\mathbf{y}(t, \mathbf{v}_i) = \phi_i(t + t_0, t_0, \mathbf{z}_0)$ for all $t \ge 0$. Thus, by appealing to (49), it is deduced that, for any ε , there exists T_{ε} such that

$$|\phi_i(t+t_0, t_0, \mathbf{z}_0)| \leq \varepsilon \quad \text{for all } t \geq T_{\varepsilon}$$

and all \mathbf{z}_0 , so long as that dist $(\mathbf{z}_0, \mathcal{A}_0) \leq \delta$. This, in turn, by standard results (see e.g. [18, page 92], implies the existence of positive numbers M and a (independent of \mathbf{z}_0) such that

$$|\Phi(t, t_0, \mathbf{z}_0)| \le M e^{-a(t-t_0)} \quad \text{for all } t \ge t_0 \ge 0,$$
 (51)

and all \mathbf{z}_0 , so long as that $\operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0) \leq \delta$.

By a classical converse Lyapunov theorem (see Theorem 3.12 in [15]), we deduce from (51) the existence of a continuously differentiable and symmetric function $\bar{P}(t)$, of a continuous and symmetric function Q(t) and of constants c_1 , c_2 and c_3 such that

$$\frac{dP(t)}{dt} + \bar{P}(t)\mathbf{A}(\mathbf{z}(t,\mathbf{z}_0)) + \mathbf{A}^{\mathrm{T}}(\mathbf{z}(t,\mathbf{z}_0))\bar{P}(t) = -Q(t)$$

with

$$0 < c_1 I \le \overline{P}(t) \le c_2 I$$
 and $Q(t) \ge c_3 I > 0$

for all $t \geq 0$.

Bearing in mind this result, we return now to the lower subsystem of (47) which can be more conveniently seen as a *time-varying nonlinear* system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{z}(t, \mathbf{z}_0))\mathbf{x} + \mathbf{h}(\mathbf{z}(t, \mathbf{z}_0), \mathbf{x}) + \nu(\mathbf{z}(t, \mathbf{z}_0)).$$
(52)

In particular note that, as far as the term $\mathbf{h}(\mathbf{z}(t, \mathbf{z}_0), \mathbf{x})$ is concerned, for any $\epsilon > 0$ there is $\delta_{\epsilon} > 0$ such that

$$|\mathbf{x}| \le \delta_{\epsilon} \qquad \Rightarrow \qquad |\mathbf{h}(\mathbf{z}(t, \mathbf{z}_0), \mathbf{x})| \le \epsilon |\mathbf{x}|$$

for all $t \ge 0$ and all $\mathbf{z}_0 \in \mathbf{Z}$. Moreover note that, by Assumption (v), there exist positive numbers M_z , a_z and d such that, for any \mathbf{z}_0 satisfying dist $(\mathbf{z}_0, \mathcal{A}_0) \le d$, the following bound holds

$$\operatorname{dist}(\mathbf{z}(t), \mathcal{A}_0) \le M_z e^{-a_z t} \operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0), \qquad (53)$$

for all $t \ge 0$. From this and from the definition of $\nu(\cdot)$ (and in particular from the fact that $\nu(\cdot)$ is differentiable and vanishes on \mathcal{A}_0) it follows that there is a constant $\gamma > 0$ such that

$$|\nu(\mathbf{z}(t, \mathbf{z}_0))| \le \gamma \operatorname{dist}(\mathbf{z}(t), \mathcal{A}_0) \le \gamma M_z e^{-a_z t} \operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0),$$

for all $t \ge 0$ and all \mathbf{z}_0 satisfying dist $(\mathbf{z}_0, \mathcal{A}_0) \le d$. Consider now the candidate Lyapunov function $W(\mathbf{x}, t) = \mathbf{x}^T \bar{P}(t)\mathbf{x}$, whose time derivative along the solution of (52) yields

$$\begin{aligned} \dot{W}(\mathbf{x},t) &= -\mathbf{x}^{\mathrm{T}}Q(t)\mathbf{x} + 2\mathbf{x}^{\mathrm{T}}\bar{P}(t)\mathbf{h}(\mathbf{z}(t,\mathbf{z}_{0}),\mathbf{x}) + 2\mathbf{x}^{\mathrm{T}}\bar{P}(t)\nu(\mathbf{z}(t,\mathbf{z}_{0})) \\ &\leq -c_{3}|\mathbf{x}|^{2} + 2c_{2}|\mathbf{x}||\mathbf{h}(\mathbf{z}(t,\mathbf{z}_{0}),\mathbf{x})| + 2c_{2}|\mathbf{x}||\nu(\mathbf{z}(t,\mathbf{z}_{0}))| \\ &\leq -c_{3}|\mathbf{x}|^{2} + 2c_{2}|\mathbf{x}||\mathbf{h}(\mathbf{z}(t,\mathbf{z}_{0}),\mathbf{x})| + 2c_{2}|\mathbf{x}|\gamma M_{z}e^{-a_{z}t}\operatorname{dist}(\mathbf{z}_{0},\mathcal{A}_{0}). \end{aligned}$$

Picking $\epsilon \leq \frac{c_3}{4c_2}$ and δ_{ϵ} accordingly, it follows that

$$\begin{aligned} |\mathbf{x}(t)| &\leq \delta_{\epsilon} \qquad \Rightarrow \qquad \dot{W}(\mathbf{x}(t), t) \leq -\frac{c_3}{2} |\mathbf{x}(t)|^2 + 2c_2 \gamma M_z |\mathbf{x}(t)| e^{-a_z t} \operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0) \\ \Rightarrow \qquad \dot{W}(\mathbf{x}(t), t) \leq -\frac{c_3}{4c_2} W(\mathbf{x}(t), t) + 4 \frac{(c_2 \gamma M_z)^2}{c_3} e^{-2a_z t} \left[\operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0)\right]^2. \end{aligned}$$

¿From this, standard arguments can be invoked to claim the existence of positive numbers $r_x < \delta_{\epsilon}$, $r_z < d$, A_x , A_y , λ_x , λ_z such that that if $|\mathbf{x}_0| < r_x$ and $\operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0) \leq r_z$ then the trajectory $\mathbf{x}(t)$ of (52) can be bounded as

$$|\mathbf{x}(t)| \le A_x e^{-\lambda_x t} |\mathbf{x}_0| + A_z e^{-\lambda_z t} \operatorname{dist}(\mathbf{z}_0, \mathcal{A}_0) \,.$$

This proves the Lemma. \triangleleft

5 Adaptive output regulation

We return now to the closed loop system obtained from the interconnection of (20), (4) and (21). As mentioned before, this system, viewed as a system with input v and output $e = e_1$ has relative degree 1. To put it in "normal form", we use, instead of (23), the change variables

$$\widetilde{\theta} = \widehat{\theta} - \theta(\rho) - \beta x$$

$$\eta = \xi - M[\widehat{\theta} - \theta(\rho)] - Kx.$$
(54)

This, after some simple algebra and some obvious rearrangement of terms, yields a system of the form

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w)$$

$$\dot{z} = f_0(\varrho, w, z) + f_1(\varrho, w, z, e)e$$

$$\dot{\eta} = A\eta + b\beta^{\mathrm{T}}\tilde{\theta} - K[q(\varrho, w, z, 0) + \eta_1] + \phi(\eta_1) + \Omega(\eta_1)\theta + \delta_1(\varrho, w, z, e, X, \eta_1)e$$

$$\dot{\tilde{\theta}} = -\beta[q(\varrho, w, z, 0) + \eta_1] + \delta_2(\varrho, w, z, e, X, \eta_1)e - \mathrm{dzv}_{\ell}(\tilde{\theta} + \theta(\varrho))$$

$$\dot{X} = FX + G\Omega(\eta_1) + \delta_3(\eta_1, e)e$$

$$\dot{e} = -[q(\varrho, w, z, 0) + \eta_1] + \vartheta(\varrho, w, z, e)e + v.$$
(55)

in which $\delta_1(\cdot), \delta_2(\cdot), \delta_3(\cdot)$ and $\vartheta(\cdot)$ are continuously differentiable functions of their arguments.

A more succinct form can be obtained setting \mathbf{w} as in section (3) and

$$\mathbf{x} = \operatorname{col}(\eta, \hat{\theta}, X_1, \dots X_q),$$

(where X_i denotes the *i*-th column of X) in which case, the system in question can be rewritten in the form³

$$\dot{\mathbf{w}} = \mathbf{s}(\mathbf{w})$$

$$\dot{z} = f_0(\mathbf{w}, z) + f_1(\mathbf{w}, z, e)e$$

$$\dot{\mathbf{x}} = \mathbf{g}_0(\mathbf{w}, z, \mathbf{x}) + \mathbf{g}_1(\mathbf{w}, z, \mathbf{x}, e)e$$

$$\dot{e} = \mathbf{h}(\mathbf{w}, z, \mathbf{x}) + \mathbf{k}(\mathbf{w}, z, \mathbf{x}, e)e + v.$$
(56)

In this notation, the set of equations

$$\dot{\mathbf{w}} = \mathbf{s}(\mathbf{w}) \dot{z} = f_0(\mathbf{w}, z)$$

$$\dot{\mathbf{x}} = \mathbf{g}_0(\mathbf{w}, z, \mathbf{x})$$

$$(57)$$

is a succinct version for the set of equations (30), whose asymptotic properties have been analyzed in the previous section. More precisely, under the hypotheses introduced earlier, the positive orbit of $\mathbf{Z} \times \Xi \times \Theta \times \mathbf{X}$ under the flow (57) is bounded and all trajectories uniformly converge to the compact invariant set $\omega(\mathbf{B})$ described in Proposition 1. Moreover, the function $\mathbf{h}(\mathbf{w}, z, \mathbf{x})$, which is a succinct version of the quantity $-[q(\varrho, w, z, 0) + \eta_1]$ in (55), vanishes on the set $\omega(\mathbf{B})$. With this in mind we are now in the position to formulate the final result of the paper which states that the controller (21) completed with

$$v = -ke \tag{58}$$

solves the problem of output regulation if k is chosen sufficiently large.

Proposition 3 Consider system (20) with exosystem (4). Let \mathbf{W}, Z, E be fixed compact sets of initial conditions, for which the assumptions (i)-(iv) indicated in section 3 are supposed to hold. Suppose, in addition, that assumption (v) introduced in section 4.3 holds. Consider the controller (21) completed with (58) and initial conditions in a fixed compact set \mathbf{K} . Then, there exists a number $k^* > 0$ such that if $k \ge k^*$ the positive orbit of $\mathbf{W} \times Z \times E \times \mathbf{K}$ in the closed loop system is bounded and $e(t) \to 0$ as $t \to \infty$.

Proof. The result directly follows from Proposition 4 of Appendix A. In particular it is easy to check that system (56) - (58) can be viewed as a system of the form (59), the role of x in (59) being played here by the one-dimensional variable e. The properties established for (57) and the fact that $\mathbf{h}(\mathbf{w},\mathbf{z},\mathbf{x})$ vanishes on $\omega(\mathbf{B})$ show that all the assumptions of Proposition 4 are satisfied. Thus, the desired result follows by taking a large value of k. \triangleleft

Remark. The previous Proposition indicates that the proposed controller (21) completed with (58) solves the problem of output regulation for the relative degree one system (20).

³With a minor abuse of notation we have replaced $f_0(\varrho, w, z)$ and $f_1(\varrho, w, z, e)$ by $f_0(\mathbf{w}, z)$ and, respectively, $f_1(\mathbf{w}, z, e)$.

Bearing in mind the discussion at the end of section 3, though, it follows that a controller of the form (21), completed with

$$v = -k(e_r + g^{r-1}a_0e_1 + g^{r-2}a_1e_2 + \ldots + ga_{r-2}e_{r-1}),$$

is able to solve the problem of output regulation for the original plant (2), if g is large enough. In this respect, it is worth stressing that the assumptions under which the proposed controller solves the problem need only to be checked on the original system (2) and not necessarily on the transformed, relative degree one, system (12). As a matter of fact, we have already shown, in Lemma 2, that assumptions (i) through (iv) on system (2) imply identical properties on system (12). For the sake of coherence, it remains to show that the fulfilment of assumption (v) on system (2) implies the fulfillment of the corresponding assumption on system (12). But this is a trivial matter, in view of the fact that the assumption in question is determined (once the matrices A, C, F, G and the map $\Omega(\cdot)$ have been fixed) only by the restriction of $\tau_1(\mathbf{z})$ to the invariant set \mathcal{A}_0 . As shown in the proof Lemma 2, the map $\tilde{\tau}(\tilde{\mathbf{z}})$ which makes assumption (iv) satisfied for (12) is such that $\tilde{\tau}(\tilde{\mathbf{z}}) = \tau(\mathbf{z})$ for $\tilde{\mathbf{z}} = (\mathbf{z}, 0)$ and $\mathbf{z} \in \mathcal{A}_0$, and therefore, if system (2) has the property (v), an identical property holds for the transformed system (12). \triangleleft

Appendix

A A small-gain property

Consider a system of the form

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w)$$

$$\dot{z} = f_0(\varrho, w, z) + \ell(\varrho, w, z, x)$$

$$\dot{x} = q_0(\varrho, w, z) + r(\varrho, w, z, x) + gAx$$
(59)

in which $(\varrho, w, z, x) \in \mathbb{R}^p \times \mathbb{R}^s \times \mathbb{R}^n \times \mathbb{R}^m$. Let the functions $f_0(\cdot), q_0(\cdot), g(\cdot), \ell(\cdot), s(\cdot)$ be continuously differentiable and, moreover, let $\ell(\varrho, w, z, 0) = 0$ and $r(\varrho, w, z, 0) = 0$ for all $(\varrho, w, z) \in \mathbb{R}^p \times \mathbb{R}^s \times \mathbb{R}^n$. A is a given Hurwitz matrix and g is a positive number. As in section 3, let $\mathcal{P} \subset \mathbb{R}^p$, $W \subset \mathbb{R}^s$, $Z \subset \mathbb{R}^n$ denote compact sets of initial conditions for ϱ, w , z, set $\mathbf{z} = \operatorname{col}(\varrho, w, z)$ and $\mathbf{Z} = \mathcal{P} \times W \times Z$. Suppose that the autonomous system

$$\dot{\varrho} = 0$$

$$\dot{w} = s(\varrho, w) \tag{60}$$

$$\dot{z} = f_0(\varrho, w, z),$$

with initial conditions in the compact set \mathbf{Z} , satisfies assumptions (i), (ii), (iii) of section 3 and, coherently with the abbreviated notation used throughout the paper, set $\mathcal{A}_0 = \omega(\mathbf{Z})$. The following lemma presents describes some relevant properties of (59), proven [4], which instrumental in proving the desired results. **Lemma 7** Consider system (59) under the assumptions specified above, with initial conditions in $\mathbf{Z} \times X$ with $X \subset \mathbb{R}^m$ a compact set. Then the following holds:

(a) there exist positive numbers d, M, a and γ such that if

$$\operatorname{dist}(\mathbf{z}(0), \mathcal{A}_0) \le d$$
 and $|x(t)| \le d$ for all $t \ge 0$

then

$$\operatorname{dist}(\mathbf{z}(t), \mathcal{A}_0) \le M e^{-at} \operatorname{dist}(\mathbf{z}(0), \mathcal{A}_0) + \gamma \max_{\tau \in [0, t]} |x(\tau)| \qquad \text{for all } t \ge 0.$$

(b) for all $\epsilon > 0$ there exist $g^* > 0$ and T > 0 such that for all $g \ge g^*$ the positive orbit of $\mathbf{Z} \times X$ under the flow of (59) is bounded and

$$\operatorname{dist}(\mathbf{z}(t), \mathcal{A}_0) \leq \epsilon, \qquad |x(t)| \leq \epsilon \qquad \text{for all } t \geq T.$$

The previous lemma provides the tools needed to study the asymptotic behavior of the system (59) under the additional hypothesis that the function $q_0(\varrho, w, z)$ vanishes on \mathcal{A}_0 (or, what is the same, that $\mathcal{A}_0 \times \{0\}$ is invariant for (59)). This is specified in the next proposition.

Proposition 4 Consider system (59) under the assumptions specified above and assume, in addition, that $q_0(\varrho, w, z) = 0$ for all $(\varrho, w, z) \in \mathcal{A}_0$. Then for any compact set X there exists $g_1^* > 0$ such that, for all $g \ge g_1^*$, the positive orbit of $\mathbf{Z} \times X$ under the flow of (59) is bounded and $\lim_{t\to\infty} x(t) = 0$.

Proof. The proof is an easy consequence of the results of Lemma 7 and of the small gain theorem. As a matter of fact, pick $\epsilon \leq d$ and set

$$\mathcal{S}_{\epsilon} = \{ (\mathbf{z}, x) \in \mathbb{R}^{p+s+n} \times \mathbb{R}^m : \operatorname{dist}(\mathbf{z}, \mathcal{A}_0) \le \epsilon, \quad |x| \le \epsilon \}$$

From property (b), it is seen that if $g \ge g^*$, any initial condition in $\mathbf{Z} \times X$ produces a trajectory of (59) which is bounded in forward time and satisfies $(\mathbf{z}(t), x(t)) \in S_{\epsilon}$ for all $t \ge T$. From property (a), it is seen that

$$\operatorname{dist}(\mathbf{z}(t-T), \mathcal{A}_0) \le M e^{-a(t-T)} \operatorname{dist}(\mathbf{z}(T), \mathcal{A}_0) + \gamma \max_{\tau \in [T, t-T]} |x(\tau)|$$

for all $t \ge T$. Note that the differentiable function $r(\varrho, w, z, x)$, which vanishes for x = 0, can be estimated as

$$|r(\varrho, w, z, x)| \le \alpha |x|$$

for all $(\mathbf{z}, x) \in S_{\varepsilon}$, while the differentiable function $q_0(\varrho, w, z)$ which vanishes on \mathcal{A}_0 , can be estimated as

$$|q_0(\varrho, w, z)| \le \beta \operatorname{dist}(\mathbf{z}, \mathcal{A}_0)$$

for all $\mathbf{z} \in S_{\varepsilon}$. Now, let P > 0 denote the solution of $PA + A^{\mathrm{T}}P = -I$ and by $\underline{\lambda}$ and $\overline{\lambda}$ respectively the smallest and largest eigenvalue of P. Standard arguments can be used to show that, for all $t \geq T$,

$$|x(t)| \le \sqrt{\frac{\overline{\lambda}}{\underline{\lambda}}} e^{-\frac{\lambda_g}{4\lambda}(t-T)} |x(T)| + \frac{4\beta}{\lambda_g} |P| \max_{\tau \in [T,t-T]} \operatorname{dist}(\mathbf{z},\mathcal{A}_0)$$

where $\lambda_g = g - 2|P|\alpha$. Hence the result follows by classical small gain arguments if $g_1^* \ge g^*$ is picked so that the small gain condition

$$\lambda_{g_1^\star} > 4\beta |P|\gamma$$

is fulfilled. This completes the proof of Proposition 4. \lhd

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