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Exponential Stability of an Abstract Non-dissipative Linear System

Kangsheng Liu * Zhuangyi Liu † Bopeng Rao ‡

Abstract

In this paper we consider an abstract linear system with perturbation of the form

$$\frac{dy}{dt} = Ay + \varepsilon By$$

on a Hilbert space $\mathcal{H}$, where A is skew-adjoint, B is bounded, and ε is a positive parameter. Motivated by a work of Freitas and Zuazua on the one-dimensional wave equation with indefinite viscous damping [4], we obtain a sufficient condition for exponential stability of the above system when B is not a dissipative operator. We also obtain a Hautus-type criterion for exact controllability of system (A, G), where G is a bounded linear operator from another Hilbert space to $\mathcal{H}$. Our result about the stability is then applied to establish the exponential stability of several elastic systems with indefinite viscous damping, as well as the exponential stabilization of the elastic systems with non-co-located observation and control.

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1 Introduction

We consider a linear evolution equation

$$\begin{cases} \dfrac{d}{dt}y(t) = \mathcal{A}_\varepsilon y(t) \equiv (A + \varepsilon B)y(t), \\ y(0) = y_0 \end{cases} \tag{1.1}$$

in a Hilbert space $\mathcal{H}$, where A is a densely defined, closed linear operator with domain $\mathcal{D}(\mathcal{A})$. We assume that

(H1) A is skew-adjoint ($A^* = -A$), and has a compact resolvent.

(H2) B is a bounded linear operator on $\mathcal{H}$ with $\|B\| = N_b$.

Under assumptions (H1) and (H2), we know that the operator $\mathcal{A}_\varepsilon$ generates a C_0 semigroup $S_\varepsilon(t)$ on $\mathcal{H}$ (see [9]). In this paper we study mainly exponential stability of the above system, i.e., that there exist $\mu > 0$ and $M \geq 1$ such that

$$\|S_\varepsilon(t)\| \leq Me^{-\mu t}, \quad \forall\, t \geq 0. \tag{1.2}$$

When $\varepsilon = 1$, this problem has been investigated extensively for both bounded and unbounded operator B. These works are based on the assumption of *dissipativeness* of B,

$$\mathrm{Re}\langle By, y\rangle \leq 0, \quad \forall\, y \in \mathcal{D}(A), \tag{1.3}$$

which implies that the energy of the system, $E(t) = \|S_\varepsilon(t)\|^2$, is a decreasing function of time. Clearly, this is not a necessary condition for $E(t)$ being bounded upper by a function which tends to zero exponentially. A natural question to ask is the following: Without the dissipativeness of B, can we still obtain (1.2) under some extra conditions? This problem is quite significant in the control theory for distributed parameter systems, because

(a) the optimality systems resulting from the regulators are non-dissipative;

(b) the closed-loop systems by feedback with non-co-located sensors and actuators are non-dissipative;

(c) the perturbations arising from undetermined parts of models are non-dissipative, in general.

Such a question was first raised in [2] for the one-dimensional wave equation

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - d(x)w_t(x,t), \quad 0 < x < 1, \ t > 0, \\ w(0,t) = w(1,t) = 0, \quad t > 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 < x < 1 \end{cases} \tag{1.4}$$

where d is a smooth function and changes sign on $(0,1)$. It was conjectured that (1.2) holds if

$$I_n \equiv \int_0^1 d(x) \sin^2 n\pi x \mathrm{d}x \geq C_0 > 0, \quad n = 1, 2, \cdots. \tag{1.5}$$

It turns out that (1.5) is not enough to ensure exponential stability. When $\|d\|_{L^\infty}$ becomes large enough, there will be eigenvalues of the system (1.2) with positive real part (see [3]). Thus, in order to have exponential stability, the damping coefficient must not only satisfy (1.5), but also has a small L^∞ norm. Later on, Freitas and Zuazua [4] considered the modified system of (1.4):

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - \varepsilon d(x)w_t(x,t), \quad 0 < x < 1, \ t > 0, \\ w(0,t) = w(1,t) = 0, \quad t > 0, \\ w(0,t) = w_0(x), \ w_t(x,0) = w_1(x), \quad 0 < x < 1. \end{cases} \tag{1.6}$$

They proved that if $d \in BV(0,1)$ and the condition (1.5) holds, then there exist positive constants ε_0, M, ω, depending only on the function d, such that for all $0 < \varepsilon < \varepsilon_0$

$$E(t) = \int_0^1 (|w_x|^2 + |w_t|^2) \mathrm{d}x \leq M e^{-\varepsilon \omega t} E(0), \quad \forall t > 0, \tag{1.7}$$

for every finite energy solution of (1.6). Their result was further extended in [1] to the equation

$$w_{tt} = w_{xx} - 2\varepsilon d(x) w_t - b(x) w \tag{1.8}$$

where $b \in L^1(0,1)$.

These works lead us to the current study in this paper. In stead of working on a particular PDE system, we would like to obtain general result along the line developed in [2]. Although the shooting method used in [4] and [1] is no longer applicable to our abstract problem, the analysis in these papers do provide us valuable information on how to impose additional conditions in order to guarantee (1.2). In the next section, we estimate the growth rate of the semigroup $S_\varepsilon(t)$, by a formula for the type of a C_0 semigroup on a Hilbert space due to Huang [7] and Prüss [10]. The exponential stability of the semigroup follows from a negative growth rate. It is well-known that the exponential stability of a linear system reversible in time is always connected with exact controllability of the corresponding system. Thanks to the fact that our sufficient condition for exponential stability of (1.1) is also necessary when $-B$ is symmetrical and nonnegative, in section 3 we obtain a Hautus-type criterion for exact controllability of system (A, G), where G is a bounded linear operator from another Hilbert space to $\mathcal{H}$. In section 4, we apply the result on exponential stability to several elastic systems (such as string, Euler-Bernoulli beam, Timoshenko beam and 2-dimensional Schrödinger equations) with indefinite viscous damping or non-dissipative perturbation arising from feedback by non-co-located observation and control.

2 Sufficient Condition for Exponential Stability

Under the condition (H1), there is an orthonormal base of $\mathcal{H}$ consisting of eigenvectors of A,

$$\{\phi_n \mid n = 1, 2, \cdots\}, \tag{2.1}$$

such that

$$\begin{cases} A\phi_n = i\beta_n\phi_n, & n = 1, 2, \cdots, \ \beta_n \in \mathbb{R}, \\ 0 \le |\beta_1| \le |\beta_2| \le \cdots \le |\beta_n| \le |\beta_{n+1}| \to \infty. \end{cases} \tag{2.2}$$

We have taken multiple eigenvalues into account. Every eigenvalue has a finite multiplicity.

For each $\gamma > 0$, set

$$\Sigma_\gamma = \{\psi = \sum_{n\in I_{\gamma,m}} a_n\phi_n \Big| \sum_{n\in I_{\gamma,m}} |a_n|^2 = 1, \ m \in \mathbb{N}, \ a_n \in \mathbb{C}\} \tag{2.3}$$

where

$$I_{\gamma,m} = \{n \in \mathbb{N} | \quad |\beta_n - \beta_m| < \gamma\}. \tag{2.4}$$

Let

$$C_\gamma = \inf_{\psi\in\Sigma_\gamma} \mathrm{Re}\langle -B\psi, \psi\rangle. \tag{2.5}$$

Note that $I_{\gamma,m} = I_{\gamma,l}$ if $\beta_m = \beta_l$ and $C_{\gamma_1} \ge C_{\gamma_2}$ for $0 < \gamma_1 < \gamma_2$. We further assume that

(H3) $\qquad C_\gamma > 0, \qquad$ for some $\quad \gamma > 0.$

Denote the type of semigroup $S_\varepsilon(t)$ by

$$\omega_0(\mathcal{A}_\varepsilon) = \lim_{t\to\infty} \frac{\ln \|S_\varepsilon(t)\|}{t} \tag{2.6}$$

and the spectral bound of $\mathcal{A}_\varepsilon$ by

$$\sigma_0(\mathcal{A}_\varepsilon) = \sup\{\mathrm{Re}\lambda |\ \lambda \in \sigma(\mathcal{A}_\varepsilon)\}. \tag{2.7}$$

We shall use a result in [7, 10], which states

$$\omega_0(\mathcal{A}_\varepsilon) = \inf\{s > \sigma_0(\mathcal{A}_\varepsilon) \mid \sup_{Re\lambda=s} \|(\lambda I - \mathcal{A}_\varepsilon)^{-1}\| < +\infty\}. \tag{2.8}$$

Theorem 2.1 *Under the assumptions (H1)-(H3), for every $C \in [0, C_\gamma)$ it holds that*

$$\omega_0(\mathcal{A}_\varepsilon) < -\varepsilon C \tag{2.9}$$

whenever

$$0 < \varepsilon < \frac{\gamma(\sqrt{N_b^2 + C_\gamma(C_\gamma - C)} - N_b)}{2C_\gamma\sqrt{N_b^2 - C^2}}. \tag{2.10}$$

In particular, $S_\varepsilon(t)$ is exponentially stable if

$$0 < \varepsilon < \frac{\gamma}{2N_bC_\gamma}\left(\sqrt{N_b^2 + C_\gamma^2} - N_b\right). \tag{2.11}$$

Proof. We will prove that for every $\sigma \geq -C$ there exists $\delta_\varepsilon > 0$ such that

$$\|(\varepsilon\sigma + i\tau)y - \mathcal{A}_\varepsilon y\| \geq \delta_\varepsilon\|y\| \qquad \forall\, \tau \in \mathbb{R},\ y \in \mathcal{D}(\mathcal{A}_\varepsilon). \tag{2.12}$$

Since for $y \in \mathcal{D}(A)$, $\sigma, \tau \in \mathbb{R}$,

$$\|(\varepsilon\sigma + i\tau)y - \mathcal{A}_\varepsilon y\|\|y\| \geq \mathrm{Re}\langle(\varepsilon\sigma + i\tau)y - \mathcal{A}_\varepsilon y\ ,\ y\rangle \geq \varepsilon(\sigma - N_b)\|y\|^2,$$

(2.12) holds for all $\sigma > N_b$. If (2.12) is false for some $\sigma \in [-C, N_b]$, then there exist a sequence of real numbers τ_p and a sequence of normalized vectors $y_p \in \mathcal{D}(\mathcal{A}_\varepsilon)$ such that

$$((\varepsilon\sigma + i\tau_p)I - \mathcal{A}_\varepsilon)y_p \equiv f_p \to 0 \quad \text{in} \quad \mathcal{H} \quad \text{as} \quad p \to +\infty. \tag{2.13}$$

From (2.13) we have

$$\sigma = \frac{1}{\varepsilon}\langle f_p - (i\tau_p I - A)y_p + \varepsilon By_p\ ,\ y_p\rangle = \lim_{p\to+\infty} \mathrm{Re}\langle By_p, y_p\rangle. \tag{2.14}$$

Moreover, (2.13)-(2.14) imply that

$$\begin{aligned}\|(i\tau_p I - A)y_p\|^2 &= \|f_p - \varepsilon(\sigma I - B)y_p\|^2 \\ &\leq \|f_p\|^2 + 2\varepsilon\|\sigma I - B\|\|f_p\| + \varepsilon^2\|(\sigma I - B)y_p\|^2 \\ &= o(1) + \varepsilon^2(\sigma^2 - 2\sigma \mathrm{Re}\langle By_p, y_p\rangle + \|By_p\|^2) \\ &\leq \varepsilon^2(N_b^2 - \sigma^2) + o(1).\end{aligned}$$

Thus, for any $\delta > 0$ there exists $N \in \mathbb{N}$ such that

$$\|(i\tau_p I - A)y_p\|^2 \leq \varepsilon^2(N_b^2 - \sigma^2 + \delta) \quad \forall\, p > N. \tag{2.15}$$

We expand y_p for $p > N$ in the eigenvectors of A:

$$y_p = \sum_{n=1}^{\infty} \langle y_p, \phi_n\rangle \phi_n. \tag{2.16}$$

Substituting (2.16) into (2.15) yields

$$\sum_{n=1}^{\infty} |\tau_p - \beta_n|^2 |\langle y_p, \phi_n\rangle|^2 \leq \varepsilon^2(N_b^2 - \sigma^2 + \delta). \tag{2.17}$$

Choose $m = m(p)$ such that

$$|\tau_p - \beta_m| = \min\{|\tau_p - \beta_n| \mid n \in \mathbb{N}\}. \tag{2.18}$$

Then we have

$$\frac{\gamma}{2} < |\tau_p - \beta_n| \qquad \forall n \notin I_{\gamma,m}. \tag{2.19}$$

In fact, if $|\tau_p - \beta_m| \geq \frac{\gamma}{2}$, (2.19) holds obviously. If $|\tau_p - \beta_m| < \frac{\gamma}{2}$, then (2.19) follows from $|\tau_p - \beta_n| \geq |\beta_n - \beta_m| - |\tau_p - \beta_m|$. Combination of (2.17) and (2.19) gives that

$$\frac{\gamma^2}{4} \sum_{n \notin I_{\gamma,m}} |\langle y_p, \phi_n\rangle|^2 \leq \varepsilon^2(N_b^2 - \sigma^2 + \delta). \tag{2.20}$$

Define

$$z_p = \sum_{n \in I_{\gamma,m}} \langle y_p, \phi_n\rangle \phi_n, \qquad p > N. \tag{2.21}$$

Then (2.20) implies that

$$\|y_p - z_p\| \leq \frac{2\varepsilon}{\gamma}\sqrt{N_b^2 - \sigma^2 + \delta}, \qquad 1 \geq \|z_p\|^2 \geq 1 - \frac{4\varepsilon^2}{\gamma^2}(N_b^2 - \sigma^2 + \delta). \tag{2.22}$$

Note that $-\sigma \leq C < C_\gamma \leq N_b$. Since the function

$$g(x) \equiv \frac{\sqrt{N_b^2 + C_\gamma(C_\gamma + x)} - N_b}{\sqrt{N_b^2 - x^2}} \tag{2.23}$$

is monotonically increasing on $(-C_\gamma, N_b)$ (see Supplement 1), the inequality (2.10) implies (2.11) and, therefore,

$$2N_b\varepsilon/\gamma < \frac{\sqrt{N_b^2 + C_\gamma^2} - N_b}{C_\gamma} = \frac{C_\gamma}{\sqrt{N_b^2 + C_\gamma^2} + N_b} < 1.$$

From (2.22) we know $z_p \neq 0$ if δ is small enough. Hence, We have $z_p/\|z_p\| \in \Sigma_\gamma$ and

$$-\mathrm{Re}\langle Bz_p, z_p\rangle \geq C_\gamma \|z_p\|^2 \geq C_\gamma\Big(1 - \frac{4\varepsilon^2}{\gamma^2}(N_b^2 - \sigma^2 + \delta)\Big). \tag{2.24}$$

It follows from (2.14), (2.22) and (2.24) that

$$\begin{aligned}
\sigma &= \lim_{p\to+\infty} \mathrm{Re}\langle By_p, y_p\rangle \\
&\leq \sup_{p>N}[\mathrm{Re}\langle Bz_p, z_p\rangle + \mathrm{Re}\langle B(y_p - z_p), y_p\rangle + \mathrm{Re}\langle Bz_p, y_p - z_p\rangle] \\
&\leq \sup_{p>N}[-C_\gamma\|z_p\|^2 + (1 + \|z_p\|)\|B\|\|y_p - z_p\|] \\
&\leq -C_\gamma\Big(1 - \frac{4\varepsilon^2}{\gamma^2}(N_b^2 - \sigma^2 + \delta)\Big) + \frac{4N_b\varepsilon}{\gamma}\sqrt{N_b^2 - \sigma^2 + \delta}.
\end{aligned} \tag{2.25}$$

We take $\delta \to 0$ in (2.25) to get

$$C_\gamma + \sigma \leq 4C_\gamma\left(\frac{\varepsilon}{\gamma}\sqrt{N_b^2 - \sigma^2}\right)^2 + 4N_b\left(\frac{\varepsilon}{\gamma}\sqrt{N_b^2 - \sigma^2}\right). \tag{2.26}$$

If $\sigma = N_b$, then (2.26) is an obvious contradiction. For $-C \leq \sigma < N_b$, (2.26) implies

$$\varepsilon \geq \frac{\gamma}{2C_\gamma}g(\sigma) \geq \frac{\gamma}{2C_\gamma}g(-C), \tag{2.27}$$

which contradicts (2.10).

Since $\mathcal{A}_\varepsilon$ also has compact resolvent, from (2.12) we deduce that the resolvent of $\mathcal{A}_\varepsilon$ is bounded on $\varepsilon\sigma + i\mathbb{R}$ for all $\sigma \geq -C$. By the resolvent equation, the resolvent is bounded on $\{\varepsilon\sigma + i\tau \,|\, \sigma \geq -C - \delta, \tau \in \mathbb{R}\}$ for some $\delta > 0$ small enough. The proof is complete from (2.8). □

It is easy to see that (H3) is satisfied with $\gamma = \gamma_0$ when the following conditions hold:

(H4) The spectrum of A satisfies the gap condition

$$\inf\{|\beta_j - \beta_k| : j,k = 1,2,\cdots,\beta_j \neq \beta_k\} \equiv \gamma_0 > 0. \tag{2.28}$$

(H5) For any normalized eigenvector ϕ of A,

$$-\mathrm{Re}\langle B\phi, \phi\rangle \geq C_0 > 0. \tag{2.29}$$

Corollary 2.1 *Assume that the conditions (H1), (H2),(H4) and (H5) hold. Then the semigroup $S_\varepsilon(t)$ is exponentially stable if*

$$0 < \varepsilon < \frac{\gamma_0}{2N_bC_0}\left(\sqrt{N_b^2 + C_0^2} - N_b\right). \tag{2.30}$$

Moreover, for every $C \in (0, C_0)$, it holds that

$$\omega_0(\mathcal{A}_\varepsilon) < -\varepsilon C \quad \forall\, 0 < \varepsilon < \frac{\gamma_0(\sqrt{N_b^2 + C_0(C_0 - C)} - N_b)}{2C_0\sqrt{N_b^2 - C^2}}. \tag{2.31}$$

Proof. This follows from Theorem 2.1 with $\gamma = \gamma_0$, $C_\gamma \geq C_0$ and the fact that the function

$$g_1(x) = \frac{\sqrt{N_b^2 + x(x-C)} - N_b}{x} = \left(\frac{N_b}{x-C} + \sqrt{\left(\frac{N_b}{x-C}\right)^2 + \frac{C}{x-C} + 1}\right)^{-1}$$

is monotonically increasing for $x > C \geq 0$. □

Remark 2.1: In the analysis above, we provided not only the sufficient conditions for the exponential stability of semigroup $S_\varepsilon(t)$, but also an explicit negative bound of the type $\omega_0(\mathcal{A}_\varepsilon)$ of semigroup $S_\varepsilon(t)$ with the perturbation parameter ε in an explicit range.

Remark 2.2: Chen *et al.* [2] discussed the exponential stability of (1.1) with $\varepsilon = 1$ and the dissipative operator B. In addition to the assumptions similar to (H1)-(H3), they needed the condition that $\mathrm{Re}\langle By_p, y_p\rangle \to 0$ implies $By_p \to 0$ for any sequence of normalized vectors y_p. This condition does not hold generally if B is nondissipative. On the other hand, If $-B$ is symmetrical and nonnegative, i. e., $-B = -B^* \geq 0$, then the assumption (H3) is necessary for the exponential stability of semigroup $S_\varepsilon(t)$ (see the proof of Theorem 3.1 below).

Remark 2.3: The spectral gap condition (H4) is very restrictive. Corollary 2.1 applies primarily to the one-dimensional problems. Actually, as we will see in Section 4, the condition (H4) can be absent even for some PDE system on the region of one spatial dimension.

Roughly speaking, (H5) means that the damping operator B is uniformly effective for all the normalized eigenvectors. When the spectral gap condition fails, (H3) means that the damping operator B is uniformly effective for all the normalized linear combinations of eigenvectors corresponding to the eigenvalues located in the $\gamma-$neighborhood of any eigenvalue.

3 Hautus-type Criterion for Exact Controllability

Let $\mathcal{H}$ and U be Hilbert spaces. Consider the control system (A, G)

$$y(u,t) = e^{tA}y_0 + \int_0^t e^{(t-s)A}Gu(s)ds \tag{3.1}$$

where A generates a C_0 semigroup e^{tA} on $\mathcal{H}$, $G \in \mathcal{L}(U;\mathcal{H})$, $y_0 \in \mathcal{H}$. When $\mathcal{H} = \mathbb{C}^n$ and $U = \mathbb{C}^m$ are finite dimensional, the famous Hautus lemma [6] says that the system (A, G) is controllable if and only if

$$\text{Rank}[\lambda I - A,\ G] = n \qquad \forall\, \lambda \in \sigma(A), \tag{3.2}$$

or, equivalently,

$$\|(\lambda I - A)^* y\| + \|G^* y\|_U > 0 \qquad \forall\, \lambda \in \sigma(A),\ \|y\| = 1. \tag{3.3}$$

When $A^* = -A$, (3.3) is equivalent to

$$\|G^*\phi\|_U > 0 \qquad \forall\, \phi \text{ being normalized eigenvectors of } A. \tag{3.4}$$

In this section, we will give a counterpart of the special Hautus criterion (3.4) for infinite-dimensional systems. We need the following Lemma given in K. Liu [8, Thm 2.3]. Concerning the definitions of exact controllability and exponential stabilizability of (A, G), we refer the reader to [8].

Lemma 3.1 *Let $A^* = -A$, $G \in \mathcal{L}(U;\mathcal{H})$. Then the following propositions are equivalent:*

(a) The system (A, G) is exactly controllable.

(b) The system (A, G) is exponentially stabilizable.

(c) *For every positive-definite self-adjoint $K \in \mathcal{L}(U)$ the operator $A - GKG^*$ generates an exponentially stable C_0 semigroup on $\mathcal{H}$.*

By Lemma 3.1 and the frequency domain condition for exponential stability [7, 10], K. Liu [8] gave a Hautus-type criterion for exact controllability of the second order conservative systems in Hilbert spaces. Also by Lemma 3.1, Zhou and Yamamoto [11] gave a counterpart of (3.3) for the conservative system (A, G), $A^* = -A$. Our result is the following:

Theorem 3.1 *Suppose that the assumption (H1) holds and $G \in \mathcal{L}(U; \mathcal{H})$. Then the following propositions are equivalent:*

(a) *The system (A, G) is exactly controllable.*

(b) *The assumption (H3) holds for $B = -GG^*$, that is,*

$$\lim_{\gamma \to 0^+} \inf_{\psi \in \Sigma_\gamma} \|G^* \psi\|_U > 0. \tag{3.5}$$

(c) *There exists $F \in \mathcal{L}(\mathcal{H}; U)$ such that the assumption (H3) holds for $B = GF$.*

Proof. The implication (b)$\Rightarrow$(c) is trivial, (c)$\Rightarrow$(a) follows readily from Theorem 2.1 and Lemma 3.1.

(a)$\Rightarrow$(b): By Lemma 3.1, the exact controllability of (A, G) is equivalent to the exponential stability of the semigroup $S_\varepsilon(t)$ with $B = -GG^*$, $\varepsilon > 0$. Thus, it suffices to prove that (H3) is necessary for the exponential stability of $S_\varepsilon(t)$ when $-B = -B^* \geq 0$. In this case, $C_\gamma \geq 0$ for all $\gamma > 0$. If for any $\gamma > 0$, $C_\gamma = 0$, then there exist β_m and a normalized vector of the form

$$\psi_\gamma = \sum_{n \in I_{\gamma,m}} a_n \phi_n \tag{3.6}$$

such that

$$\|B\psi_\gamma\| \leq \|(-B)^{\frac{1}{2}}\| \langle -B\psi_\gamma, \psi_\gamma \rangle^{\frac{1}{2}} < \gamma.$$

Thus,

$$\|(i\beta_m I - \mathcal{A}_\varepsilon)\psi_\gamma\| \leq \varepsilon\gamma + \|\sum_{n \in I_{\gamma,m}} (\beta_m - \beta_n) i a_n \phi_n\| \leq (1 + \varepsilon)\gamma. \tag{3.7}$$

This means that the resolvent of $\mathcal{A}_\varepsilon$ is unbounded on $i\mathbb{R}$ if it exists. Thus, $S_\varepsilon(t)$ is not exponentially stable. $\square$

Remark 3.1: If the spectral gap condition (H4) holds, then the condition (3.5) takes the form

$$\|G^*\phi\|_U \geq \delta > 0 \qquad \forall\, \phi \text{ being normalized eigenvectors of } A. \tag{3.8}$$

This is just a counterpart of the finite-dimensional case (3.4).

4 Applications

In this section, we apply our result about exponential stability to the wave, beam, and 2-dimensional Shrödinger equations with indefinite viscous damping or non-dissipative perturbation arising from feedback by non-co-located observation and control.

Example 1: The 1-d wave equation with indefinite viscous damping

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) - \varepsilon d(x) w_t(x,t), & 0 < x < 1, \ t > 0, \\ w(0,t) = w_x(1,t) = 0, \quad t > 0, \\ w(0,t) = w_0(x), \ \ w_t(x,0) = w_1(x), & 0 < x < 1, \end{cases} \tag{4.1}$$

where $d \in L^\infty(0,1)$ is real-valued. The underlying Hilbert space is

$$\mathcal{H} = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \in H^1(0,1) \times L^2(0,L) \Big| \ w(0) = 0 \right\},$$

with the inner product

$$\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \rangle = \int_0^1 [w_1' \bar{w}_2' + v_1 \bar{v}_2] dx.$$

Define

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \Big| \ w \in H^2(0,1), v \in H^1(0,1), w(0) = v(0) = w'(1) = 0 \right\},$$

$$A = \begin{bmatrix} 0 & I \\ \partial_x^2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -d(x) \end{bmatrix}, \quad y = \begin{bmatrix} w \\ w_t \end{bmatrix}.$$

Then, the system (4.1) can be rewritten as (1.1). A has a complete orthonormal set of eigenfunctions

$$\phi_n^{\pm} = \frac{1}{(n+\frac{1}{2})\pi}\begin{bmatrix} \sin(n+\frac{1}{2})\pi x \\ \pm \boldsymbol{i}(n+\frac{1}{2})\pi \sin(n+\frac{1}{2})\pi x \end{bmatrix}$$

with eigenvalues

$$\boldsymbol{i}\beta_n^{\pm} = \pm \boldsymbol{i}(n+\frac{1}{2})\pi, \qquad n = 0, 1, 2, \cdots.$$

It is easy to see that (H1), (H2) and (H4) are satisfied with $\gamma_0 = \pi$, and

$$-\langle B\phi_n^{\pm}, \phi_n^{\pm}\rangle = \int_0^1 d(x)\sin^2(n+\frac{1}{2})\pi x dx. \tag{4.2}$$

Thus, (H5) holds if and only if

$$\inf_{n\geq 0}\int_0^1 d(x)[1-\cos(2n+1)\pi x]dx > 0. \tag{4.3}$$

For example, we take

$$d(x) = 1 + \alpha\cos 2k\pi x, \quad \alpha \in \mathbb{R} \tag{4.4}$$

with k being any positive integer. Then,

$$\gamma_0 = \pi, \quad N_b = 1 + |\alpha|, \quad -\langle B\phi_n^{\pm}, \phi_n^{\pm}\rangle = \frac{1}{2} \quad \forall\, n \geq 0.$$

Therefore, $S_\varepsilon(t)$ is exponentially stable if

$$0 < \varepsilon < \frac{\pi}{2(1+|\alpha|)}\Big(\sqrt{4(1+|\alpha|)^2+1} - 2(1+|\alpha|)\Big). \tag{4.5}$$

Moreover, for every $C \in (0, \frac{1}{2})$, it holds that

$$\omega_0(\mathcal{A}_\varepsilon) < -\varepsilon C \qquad \forall\ 0 < \varepsilon < \frac{\pi(\sqrt{4(1+|\alpha|)^2+1-2C} - 2(1+|\alpha|))}{2\sqrt{(1+|\alpha|)^2 - C^2}}. \tag{4.6}$$

We can also choose $d(x)$ with local support, such as

$$d(x) = \begin{cases} \sin 4\pi x, & \frac{1}{4} \leq x \leq \frac{1}{2}, \\ 2\sin 4\pi x, & \frac{1}{2} \leq x \leq \frac{3}{4}, \\ 0, & \text{otherwise}. \end{cases} \tag{4.7}$$

For this case,

$$-\langle B\phi_n^{\pm}, \phi_n^{\pm}\rangle \geq \frac{3}{20\pi}, \quad \forall\, n \geq 0.$$

Example 2: Exponential stabilization of the Euler-Bernoulli beam equation with non-co-located observation and control.

Consider the following system with distributed control and locally distributed observation

$$\begin{cases} w_{tt}(x,t) + w_{xxxx}(x,t) = f(x,t), & 0 < x < 1,\ t > 0, \\ w(0,t) = w(1,t) = w_{xx}(0,t) = w_{xx}(1,t) = 0, & t > 0, \\ w(0,t) = w_0(x),\ \ w_t(x,0) = w_1(x), & 0 < x < 1,\ t > 0, \\ h(x,t) = w_t(x,t), & x \in (\frac{1}{2}, 1),\ t > 0. \end{cases} \tag{4.8}$$

Our purpose is to find a bounded linear operator K from $L^2(\frac{1}{2}, 1)$ to $L^2(0,1)$ so that the feedback control law

$$f(\cdot,t) = -\varepsilon K h(\cdot,t), \qquad t > 0 \tag{4.9}$$

stablizes the system (4.8) for some $\varepsilon > 0$. The simplest form of operator K would be

$$[Kh](x) = \begin{cases} 2c(x)h(1-x), & 0 < x < \frac{1}{2}, \\ 2d(x)h(x), & \frac{1}{2} < x < 1, \end{cases} \tag{4.10}$$

where $c \in L^\infty(0, \frac{1}{2})$, $d \in L^\infty(\frac{1}{2}, 1)$ are real-valued. The underlying Hilbert space is

$$\mathcal{H} = [H^2(0,1) \cap H_0^1(0,1)] \times L^2(0,1)$$

with the inner product

$$\langle \begin{bmatrix} w_1 \\ v_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ v_2 \end{bmatrix} \rangle = \int_0^1 [w_1'' \bar{w}_2'' + v_1 \bar{v}_2] dx.$$

Define

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} w \\ v \end{bmatrix} \middle| \ w \in H^4(0,1), v \in H^2(0,1), w, v \in H_0^1(0,1), w''(0) = w''(1) = 0 \right\},$$

$$A = \begin{bmatrix} 0 & I \\ -\partial_x^4 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & -KI_0 \end{bmatrix}, \quad y = \begin{bmatrix} w \\ w_t \end{bmatrix}$$

where I_0 is the embedding from $L^2(0,1)$ to $L^2(\frac{1}{2}, 1)$. Then, the closed-loop system (4.8)-(4.9) can be rewritten as (1.1). A has a complete orthonormal set of eigenfunctions

$$\phi_n^\pm = \frac{1}{n^2\pi^2} \begin{bmatrix} \sin n\pi x \\ \pm i n^2 \pi^2 \sin n\pi x \end{bmatrix}$$

with eigenvalues

$$i\beta_n^{\pm} = \pm i n^2\pi^2, \qquad n = 1, 2, \cdots.$$

It is easy to see that (H1), (H2) and (H4) are satisfied with $\gamma_0 = 3\pi^2$. For K defined in (4.10), we have

$$\begin{aligned} -\langle B\phi_n^{\pm}, \phi_n^{\pm}\rangle &= 2\int_0^{\frac{1}{2}} c(x)\sin n\pi(1-x)\sin n\pi x \mathrm{d}x + 2\int_{\frac{1}{2}}^1 d(x)\sin^2 n\pi x \mathrm{d}x \\ &= \int_0^{\frac{1}{2}} [d(1-x) + (-1)^{n+1}c(x)](1-\cos 2n\pi x)\mathrm{d}x. \end{aligned} \tag{4.11}$$

Thus, (H5) holds if and only if

$$\inf_{n\geq 1}\int_0^{\frac{1}{2}} [d(1-x) + (-1)^{n+1}c(x)](1-\cos 2n\pi x)\mathrm{d}x > 0. \tag{4.12}$$

For example, we take

$$c(x) = \cos 4\pi x, \quad d(x) = \frac{1}{2} + \cos 4\pi x. \tag{4.13}$$

Then,

$$N_b = 3, \quad -\langle B\phi_n^{\pm}, \phi_n^{\pm}\rangle = \frac{1}{4} \quad \forall\, n \geq 1.$$

Therefore, $S_\varepsilon(t)$ is exponentially stable if

$$0 < \varepsilon < \frac{\pi^2}{2}\Big(\sqrt{145} - 12\Big). \tag{4.14}$$

Moreover, for every $C \in (0, \frac{1}{4})$, it holds that

$$\omega_0(\mathcal{A}_\varepsilon) < -\varepsilon C \qquad \forall\ 0 < \varepsilon < \frac{3\pi^2\Big(\sqrt{145 - 4C} - 12\Big)}{2\sqrt{9 - C^2}}. \tag{4.15}$$

Remark 4.1: Let $d \equiv 0$ in (4.10), then the observation and control are completely non-co-located. In this case, by (4.11) we know that K can be uniformly effective only for finite many eigenmodes. The problem of whether there exists $K \in \mathcal{L}(L^2(\frac{1}{2}, 1), L^2(0,1))$ with $\mathrm{supp}Kh \subset (0, \frac{1}{2})$ for any $h \in L^2(\frac{1}{2}, 1)$ such that (H5) holds remains open.

Example 3. Two-dimensional Schrödinger equation with distributed control and locally distributed observation:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) = i\Delta y(x,t) + f(x,t), \ \ x \in \Omega = (0,a)\times(0,b), \ t > 0 \\ y|_{\partial\Omega} = 0, \ t > 0, \quad y(x,0) = y_0(x), \ x \in \Omega \\ h(x,t) = y(x,t), \qquad x \in \Omega_1 = (\frac{a}{2}, a)\times(0,b), \ t > 0. \end{cases} \tag{4.16}$$

Let

$$\mathcal{H} = L^2(\Omega)$$

with the standard L^2 (complex) inner product. Define

$$A = i\Delta, \quad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega). \tag{4.17}$$

Then, the operator A is skew-adjoint, and has eigenvalues

$$\lambda_{k,l} = i\beta_{k,l} = i\left(\frac{k^2}{a^2} + \frac{l^2}{b^2}\right)\pi^2, \quad k, l \in \mathbb{N},$$

and the corresponding normalized eigenfunctions

$$\phi_{k,l}(x) = \frac{2}{\sqrt{ab}} \sin\frac{k\pi x_1}{a} \sin\frac{l\pi x_2}{b}, \quad k, l \in \mathbb{N}.$$

Set the feedback control law

$$f(\cdot, t) = -\varepsilon K h(\cdot, t), \qquad K \in \mathcal{L}(L^2(\Omega_1), L^2(\Omega), \tag{4.18}$$

$$[Kh](x) = \begin{cases} c(x_1)h(a - x_1, x_2), & 0 < x_1 < \frac{a}{2},\ 0 < x_2 < b, \\ d(x_1)h(x), & \frac{a}{2} < x_1 < a,\ 0 < x_2 < b, \end{cases} \tag{4.19}$$

where $c \in L^\infty(0, \frac{a}{2})$, $d \in L^\infty(\frac{a}{2}, a)$ are real-valued. Then, the closed-loop system (4.16)-(4.18) can be rewritten as (1.1) with $B = -KI_1$, where I_1 is the embedding from $L^2(\Omega)$ to $L^2(\Omega_1)$. When a/b is a rational number, the gap condition (H4) is true, but there are multiple eigenvalues. While a/b is an irrational number, the gap condition (H4) is false (see [2]). So, we have to check the condition (H3). We recount the eigenvalues and the corresponding normalized eigenfunctions:

$$i\beta_n = i\beta_{k_n,l_n}, \quad \beta_n \le \beta_{n+1}, \qquad \phi_n = \phi_{k_n,l_n}, \quad n \in \mathbb{N}.$$

Choose $\gamma = (\frac{\pi}{a})^2$, then

$$I_{\gamma,m} = \{n \in \mathbb{N} \mid |\frac{k_n^2 - k_m^2}{a^2} + \frac{l_n^2 - l_m^2}{b^2}| < \frac{1}{a^2}\}.$$

We note that for $p, q \in I_{\gamma,m}$, $p = q$ if and only if $l_p = l_q$, for any $\psi \in \Sigma_\gamma$,

$$\psi = \sum_{n \in I_{\gamma,m}} a_n \phi_n = \sum_{n \in I_{\gamma,m}} \frac{2a_n}{\sqrt{ab}} \sin\frac{k_n \pi x_1}{a} \sin\frac{l_n \pi x_2}{b}, \quad \sum_{n \in I_{\gamma,m}} |a_n|^2 = 1. \tag{4.20}$$

Using the orthogonality of $\{\sin(l\pi x_2/b)\}_{l=1}^{\infty}$ in $L^2(0,b)$, we have

$$\begin{aligned}
-\langle B\psi, \psi\rangle &= \int_0^{\frac{a}{2}} c(x_1) \int_0^b \sum_{p,q\in I_{\gamma,m}} a_p \bar{a}_q \phi_p(a-x_1, x_2)\phi_q(x_1,x_2)\mathrm{d}x_2\mathrm{d}x_1 \\
&\quad + \int_{\frac{a}{2}}^a d(x_1)\int_0^b \left| \sum_{n\in I_{\gamma,m}} \frac{2a_n}{\sqrt{ab}} \sin\frac{k_n\pi x_1}{a} \sin\frac{l_n\pi x_2}{b}\right|^2 \mathrm{d}x_2\mathrm{d}x_1 \\
&= \frac{2}{a}\sum_{n\in I_{\gamma,m}} |a_n|^2 \int_0^{\frac{a}{2}} [d(a-x_1) + (-1)^{1+k_n}c(x_1)] \sin^2\frac{k_n\pi x_1}{a}\mathrm{d}x_1. \qquad (4.21)
\end{aligned}$$

Thus, (H3) holds if and only if

$$\inf_{k\in\mathbb{N}} \int_0^{\frac{a}{2}} [d(a-x_1) + (-1)^{1+k}c(x_1)] \sin^2\frac{k\pi x_1}{a}\mathrm{d}x_1 > 0. \qquad (4.22)$$

The sufficiency follows from (4.21). If (4.22) is false, then there exists a sequence ξ_n of positive integers such that

$$\lim_{n\to+\infty} \int_0^{\frac{a}{2}} [d(a-x_1) + (-1)^{1+\xi_n}c(x_1)] \sin^2\frac{\xi_n\pi x_1}{a}\mathrm{d}x_1 = \alpha \leq 0.$$

Thus, for the sequence $\phi_{\xi_n,1}$ of normalized eigenfunctions of A we have

$$-\langle B\phi_{\xi_n,1}\ ,\ \phi_{\xi_n,1}\rangle = \frac{2}{a}\int_0^{\frac{a}{2}} [d(a-x_1) + (-1)^{1+\xi_n}c(x_1)] \sin^2\frac{\xi_n\pi x_1}{a}\mathrm{d}x_1 \to \frac{2\alpha}{a} \leq 0,$$

which implies that (H3) is false. For example, we take

$$c(x_1) = \cos\frac{4\pi x_1}{a}, \quad d(x_1) = \frac{1}{2} + \cos\frac{4\pi x_1}{a}. \qquad (4.23)$$

Then,

$$N_b = \frac{3}{2}, \qquad \gamma = (\frac{\pi}{a})^2, \qquad C_\gamma = \frac{1}{4}.$$

Therefore, $S_\varepsilon(t)$ is exponentially stable if

$$0 < \varepsilon < \frac{\pi^2}{3a^2}\big(\sqrt{37} - 6\big). \qquad (4.24)$$

Moreover, for every $C \in (0, \frac{1}{4})$, it holds that

$$\omega_0(\mathcal{A}_\varepsilon) < -\varepsilon C \qquad \forall\ 0 < \varepsilon < \frac{\pi^2\big(\sqrt{37-4C} - 6\big)}{a^2\sqrt{9-4C^2}}. \qquad (4.25)$$

Example 4. Timoshenko beam equation with indefinite viscous damping

$$\begin{cases} u_{tt} = pu_{xx} - p\phi_x - \varepsilon d_1(x)u_t, & 0 < x < \pi,\ t > 0, \\ \phi_{tt} = q\phi_{xx} + pu_x - p\phi - \varepsilon d_2(x)\phi_t, & 0 < x < \pi,\ t > 0, \\ u(0,t) = u(\pi,t) = \phi_x(0,t) = \phi_x(\pi,t) = 0, \quad t > 0, \\ u(x,0) = u_0,\ u_t(x,0) = u_1,\ \phi(x,0) = \phi_0,\ \phi_t(x,0) = \phi_1,\ 0 < x < \pi, \end{cases} \tag{4.26}$$

where $p,\ q > 0$ are constants and $d_1,\ d_2 \in L^\infty(0,\pi)$ are real-valued.

The underlying Hilbert space is

$$\mathcal{H} = H_0^1(0,\pi) \times L^2(0,\pi) \times H^1(0,\pi) \times L^2(0,\pi)$$

with the inner-product induced by the quadratic form of energy,

$$\|[u,v,\phi,\psi]^T\|^2 = \int_0^\pi \left(p|u_x - \phi|^2 + q|\phi_x|^2 + |v|^2 + |\psi|^2\right) dx.$$

Define in $\mathcal{H}$

$$\mathcal{D}(A) = \{[u,v,\phi,\psi]^T \mid u, v, \in H_0^1(0,\pi),\ u, \phi \in H^2(0,\pi),\ \psi \in H^1(0,\pi)\},$$

$$A = \begin{bmatrix} 0 & I & 0 & 0 \\ p\partial_x^2 & 0 & -p\partial_x & 0 \\ 0 & 0 & I & 0 \\ p\partial_x & 0 & q\partial_x^2 - pI & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -d_1(x) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -d_2(x) \end{bmatrix}.$$

Then, the system (4.26) can be rewritten as (1.1) by setting $y = [u, u_t, \phi, \phi_t]^T$. It is easy to verify that (H1) and (H2) are satisfied. To compute the eigenvalues of A, we solve the eigenequation

$$A[u,v,\phi,\psi]^T = \lambda[u,v,\phi,\psi]^T, \quad [u,v,\phi,\psi]^T \in \mathcal{D}(A).$$

Eliminating the unknowns v, ϕ, ψ, we obtain

$$\begin{cases} pqu_{xxxx} - (p+q)\lambda^2 u_{xx} + \lambda^2(\lambda^2 + p)u = 0, \\ u(0) = u(\pi) = u_{xx}(0) = u_{xx}(\pi) = 0. \end{cases}$$

A straight forward calculation leads to

$$\begin{cases} \lambda_{n,1}^2 = -\frac{1}{2}[(p+q)n^2 + p] + \frac{1}{2}\sqrt{[(p-q)n^2 + p]^2 + 4pqn^2}, \\ \lambda_{n,2}^2 = -\frac{1}{2}[(p+q)n^2 + p] - \frac{1}{2}\sqrt{[(p-q)n^2 + p]^2 + 4pqn^2}, \end{cases} \tag{4.27}$$

$$\lambda_{n,i}^{\pm} = \pm \boldsymbol{i}\sqrt{-\lambda_{n,i}^2}, \quad \text{(counting multiple eigenvalues)}$$

and the corresponding normalized eigenvectors

$$Z_{n,i}^{\pm} = \frac{1}{R_{n,i}}[\sin nx,\ \lambda_{n,i}^{\pm}\sin nx,\ S_{n,i}\cos nx,\ \lambda_{n,i}^{\pm}S_{n,i}\cos nx]^T, \qquad i = 1,\ 2, \quad n \in \mathbb{N}$$

where

$$\begin{aligned} S_{n,i} &= \frac{1}{n}\left(n^2 + \frac{1}{p}\lambda_{n,i}^2\right) \quad (\neq 0) \\ &= \frac{1}{2pn}\left([(p-q)n^2 - p] + (-1)^{1+i}\sqrt{[(p-q)n^2+p]^2 + 4pqn^2}\right), \end{aligned} \tag{4.28}$$

$$R_{n,i} = \sqrt{\frac{\pi}{2}}\left[p(n - S_{n,i})^2 + qn^2S_{n,i}^2 - \lambda_{n,i}^2 - \lambda_{n,i}^2S_{n,i}^2|\right]^{\frac{1}{2}}. \tag{4.29}$$

These sequences have asymptotic expansions:

$$\text{for } p = q, \qquad \begin{cases} \lambda_{n,j}^{\pm} = \pm\boldsymbol{i}\sqrt{p}(n + \frac{(-1)^j}{2} + O(\frac{1}{n})), & S_{n,j}^2 = 1 + O(\frac{1}{n}), \\ R_{n,j}^2 = 2\pi pn^2 + O(n), & j = 1,\ 2, \end{cases} \tag{4.30}$$

$$\text{for } p \neq q, \qquad \begin{cases} (i,j) = (1,2) \text{ for } p > q, & (i,j) = (2,1) \text{ for } p < q, \\ \lambda_{n,i}^2 = -qn^2 + O(1), & \lambda_{n,j}^2 = -pn^2 + O(1), \\ S_{n,i}^2 = (1 - \frac{q}{p})^2n^2 + O(1), & S_{n,j}^2 = O(\frac{1}{n^2}), \\ R_{n,i}^2 = \pi q(1-\frac{q}{p})^2n^4 + O(n^2), & R_{n,j}^2 = \pi pn^2 + O(1). \end{cases} \tag{4.31}$$

In order to verify condition (H3), we need the following observations of the eigenvalues:

1. Each of the two branches of eigenvalues are distinct within itself. This can be verified by showing that $-\lambda_{n,i}^2, i = 1,2$ are strictly monotonically increasing functions of n^2, see Supplement 2. Moreover, $\inf_{n\geq 1}|\lambda_{n+1,i}^{\pm} - \lambda_{n,i}^{\pm}| = \gamma_i > 0,\ i = 1,2$, by the asymptotic expansions (4.30) and (4.31).

2. $\inf_{n\geq 1}|\lambda_{n,1}^{\pm} - \lambda_{n,2}^{\pm}| = \gamma_3 > 0$. This follows from the fact that $\lambda_{n,1}^2 - \lambda_{n,2}^2 > 0$ for all $n \geq 1$ and the asymptotic expansions (4.30) and (4.31).

3. Multiplicity of each eigenvalue is less than two. For any $1 \leq m < n$, there exists $r = p/q$ such that $\lambda_{n,1} = \lambda_{m,2}$, i. e., double eigenvalues occur. This can be verified by showing that the function $f(r) = (\lambda_{n,1}^2 - \lambda_{m,2}^2)/q$ changes sign on $(0, \infty)$ (see Supplement 3).

4. When $p \neq q$, the gap condition (H4) does never hold. In fact, if $\sqrt{p/q} = k/l,\ k,l \in \mathbb{N}$ is a rational number we have $\lambda_{kj,1}^2 - \lambda_{lj,2}^2 = O(1)$ for $p > q$ and $\lambda_{lj,1}^2 - \lambda_{kj,2}^2 = O(1)$ for $p < q$ as $j \to \infty$. See Supplement 4 for the case when $\sqrt{p/q}$ is an irrational number.

Choose $\gamma = \min\{\gamma_1, \gamma_2, \gamma_3, 2|\lambda_{1,1}^+|\}$. From the above observations, we know that

$$\Sigma_\gamma = \{c_1 Z_{n,1}^\pm + c_2 Z_{m,2}^\pm \ \big|\ |c_1|^2 + |c_2|^2 = 1,\ |\lambda_{n,1}^\pm - \lambda_{m,2}^\pm| < \gamma,\ m,n \in \mathbb{N},\ c_1, c_2 \in \mathbb{C}\}. \quad (4.32)$$

For any $c_1 Z_{n,1}^\pm + c_2 Z_{m,2}^\pm \in \Sigma_\gamma$, by $\gamma \le \gamma_3$ we have $n \neq m$, and

$$\begin{aligned}\Gamma(m,n,c_1,c_2) &\equiv -\langle B(c_1 Z_{n,1}^\pm + c_2 Z_{m,2}^\pm)\ ,\ c_1 Z_{n,1}^\pm + c_2 Z_{m,2}^\pm\rangle \\ &= \int_0^\pi d_1(x)\left|\frac{c_1\lambda_{n,1}^+}{R_{n,1}}\sin nx + \frac{c_2\lambda_{m,2}^+}{R_{m,2}}\sin mx\right|^2 dx \\ &\quad + \int_0^\pi d_2(x)\left|\frac{c_1\lambda_{n,1}^+ S_{n,1}}{R_{n,1}}\cos nx + \frac{c_2\lambda_{m,2}^+ S_{m,2}}{R_{m,2}}\cos mx\right|^2 dx \end{aligned} \quad (4.33)$$

Thus, (H3) holds if and only if

$$\inf\{\Gamma(m,n,c_1,c_2) \mid m \neq n, |c_1|^2 + |c_2|^2 = 1, m,n \in \mathbb{N}, c_1, c_2 \in \mathbb{C}\} > 0. \quad (4.34)$$

For example, we take

$$d_i(x) = \alpha_i(1 + \beta_i \cos k_i x), \quad \alpha_i \ge 0,\ \alpha_1^2 + \alpha_2^2 \neq 0,\ |\beta_i| < 2,\ k_i \in \mathbb{N},\ i = 1,\ 2. \quad (4.35)$$

Using

$$\left|\int_0^\pi \cos k_2 x \cos nx \cos mx dx\right| \le \frac{\pi}{4},$$

$$\left|\int_0^\pi \cos k_1 x \sin nx \sin mx dx\right| \begin{cases} = 0, & k_1 = 2n \text{ or } 2m, \\ \le \frac{\pi}{4}, & \text{otherwise}, \end{cases}$$

we deduce that

$$\begin{aligned}\Gamma(m,n,c_1,c_2) &\ge \frac{\pi}{2}\alpha_1(1 - \frac{|\beta_1|}{2})\left(\frac{-\lambda_{n,1}^2}{R_{n,1}^2}|c_1|^2 + \frac{-\lambda_{m,2}^2}{R_{m,2}^2}|c_2|^2\right) \\ &\quad + \frac{\pi}{2}\alpha_2(1 - \frac{|\beta_2|}{2})\left(\frac{-\lambda_{n,1}^2 S_{n,1}^2}{R_{n,1}^2}|c_1|^2 + \frac{-\lambda_{m,2}^2 S_{m,2}^2}{R_{m,2}^2}|c_2|^2\right) \\ &\ge \frac{\pi}{2}\min(\eta_1, \eta_2)\end{aligned}$$

where we have put

$$\eta_1 = \inf_{n \ge 1}\left(\alpha_1(1 - \frac{|\beta_1|}{2})\frac{-\lambda_{n,1}^2}{R_{n,1}^2} + \alpha_2(1 - \frac{|\beta_2|}{2})\frac{-\lambda_{n,1}^2 S_{n,1}^2}{R_{n,1}^2}\right),$$

$$\eta_2 = \inf_{m\geq 1}\left(\alpha_1(1-\frac{|\beta_1|}{2})\frac{-\lambda_{m,2}^2}{R_{m,2}^2} + \alpha_2(1-\frac{|\beta_2|}{2})\frac{-\lambda_{m,2}^2 S_{m,2}^2}{R_{m,2}^2}\right).$$

Now, using the asymptotic expansions (4.30) and (4.31) we can easily conclude the following:

1. When $p = q$, (H3) holds even if either α_1 or α_2 is zero. Therefore, by Theorem 2.1, $S_\varepsilon(t)$ is exponentially stable if ε is small enough. This means that when the two wave speeds are the same, only one displacement or rotation angle damping is sufficient for the exponential energy decay in the Timoshenko beam.

2. When $p \neq q$, (H3) holds if both α_1 and α_2 are positive. Therefore, $S_\varepsilon(t)$ is exponentially stable if ε is small enough. On the other hand, for $d_1(x) \equiv 0$ or $d_2(x)) \equiv 0$, it is easy to see that $S_\varepsilon(t)$ is not exponentially stable for any $\varepsilon > 0$. This means that when the two wave speeds are different, the displacement and rotation angle dampings are necessary for the exponential energy decay in the Timoshenko beam.

5 Technical Supplements

Supplement 1. *Let $0 < C_\gamma \leq N_b$,*

$$g(x) = \frac{\sqrt{N_b^2 + C_\gamma(C_\gamma + x)} - N_b}{\sqrt{N_b^2 - x^2}}.$$

Then, g is monotonically increasing function on $(-C_\gamma, N_b)$.

Proof. Write the function g in the following form

$$g(x) = \frac{C_\gamma}{\sqrt{N_b - x}}\sqrt{1 - \frac{N_b - C_\gamma}{N_b + x}}\left(\sqrt{\frac{N_b^2}{C_\gamma + x} + C_\gamma} + \frac{N_b}{\sqrt{C_\gamma + x}}\right)^{-1}.$$

Then the monotony of g follows from the fact that all the factors are positive monotonically increasing functions on $(-C_\gamma, N_b)$. □

Supplement 2 $-\lambda_{n,i}^2, i = 1, 2$ *are monotonic increasing functions of n^2.*

Proof. It is obvious that the conclusion holds for $-\lambda_{n,2}^2$. Let

$$\begin{aligned} f(x) &= (p+q)x + p - \sqrt{[(p-q)x+p]^2 + 4pqx} \\ &= (p+q)x + p - \sqrt{[(p+q)x+p]^2 - 4pqx^2}. \end{aligned}$$

Then, $2f(n^2) = -\lambda_{n,2}^2$. Since

$$\begin{aligned} f'(x) &= (p+q) - \frac{[(p+q)x+p](p+q) - 4pqx}{\sqrt{[(p+q)x+p]^2 - 4pqx^2}} \\ &= (p+q)\left[1 - \frac{(p+q)x + p - 4pq(p+q)^{-1}x}{\sqrt{[(p+q)x+p]^2 - 4pqx^2}}\right], \end{aligned}$$

and the fraction in the bracket is strictly less than one, we know that $f'(x) > 0$ for all $x > 0$.
□

Supplement 3 *For any* $1 \le m < n$, *there exist* $p, q > 0$ *such that* $\lambda_{n,1}^2 = \lambda_{m,2}^2$.

Proof. Let $r = p/q$, and

$$\begin{aligned} f(r) &= \frac{1}{q}(\lambda_{n,1}^2 - \lambda_{m,2}^2) \\ &= (r+1)(m^2 - n^2) + \sqrt{[(r-1)n^2 + r]^2 + 4rn^2} + \sqrt{[(r-1)m^2 + r]^2 + 4rm^2}, \end{aligned}$$

which is continuous for $r \in (0, +\infty)$. Moreover, we have

$$\lim_{r \to 0^+} f(r) = m^2 - n^2 < 0, \qquad \lim_{r \to +\infty} f(r) = +\infty. \tag{5.1}$$

This proves that for certain ratio of p and q, there exis double eigenvalues. □

Supplement 4 *Let* $\sqrt{p/q}$ *be an irrational number, then there exist sequences of integers* $k_j \to +\infty$, $l_j \to +\infty$, *such that* $qk_j^2 - pl_j^2 = O(1)$.

Proof. By a result in number theory [5, p.140], for any $j \ge 1$, there exist a rational number $\frac{k_j}{l_j}$ with $l_j > j$ such that

$$\left|\sqrt{\frac{p}{q}} - \frac{k_j}{l_j}\right| \le \frac{1}{l_j^2}, \quad \text{i. e.,} \quad \left|l_j\sqrt{\frac{p}{q}} - k_j\right| \le \frac{1}{l_j}.$$

Thus,

$$\begin{aligned} |qk_j^2 - pl_j^2| &= q(l_j\sqrt{\frac{p}{q}} + k_j)\left|l_j\sqrt{\frac{p}{q}} - k_j\right| \\ &\le ql_j(2\sqrt{\frac{p}{q}} + 1) \cdot \frac{1}{l_j} = 2\sqrt{pq} + q. \end{aligned}$$

□

References

[1] A. Benaddi and B. Rao, Energy Decay Rate of Wave Equations with Indefinite Damping, *J. Diff. Eqns.*, 161(2000), pp. 337-357.

[2] G. Chen, S.A. Fulling, F.J. Narcowich and S. Sun, Exponential decay of energy of evolution equation with locally distributed damping, *SIAM J. Appl. Math.*, 51(1991), pp. 266-301.

[3] P. Freitas, On Some Eigenvalue Problems Related to the Wave Equation with Indefinite Damping, *J. Diff. Eqns.*, 127(1996), pp. 320-335.

[4] P. Freitas and E. Zuazua, Stability Results for the Wave Equation with Indefinite Damping, *J. Diff. Eqns.*, 132(1996), pp. 338-352.

[5] G. H. Hardy and E. M. Wright, *An Introduction to The Theory of Numbers*, Fifth Ed., Clarendon Press, Oxford, 1979.

[6] M.L.J. Hautus, Controllability and observability conditions for linear autonomous systems, *Nederl. Akad. Wetenschappen, Proc. Ser. A*, 72(1969), pp. 443-448.

[7] F.L. Huang, Characteristic condition for exponential stability of linear dynamical systems in Hilbert spaces, *Ann. of Diff. Eqs*, 1(1985), pp. 43-56.

[8] K. Liu, Locally distributed control and damping for the conservative systems, *SIAM J. Cont. Optim.*, 35(1997)5, pp. 1574-1590.

[9] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, New York, 1983.

[10] J. Prüss, On the spectrum of C_0-semigroups, *Trans. Amer. Math. Soc.* 284(1984), pp. 847-857.

[11] Q. Zhou and M. Yamamoto, Hautus condition on the exact controllability of conservative systems, *Internat. J. Cont.*, 67(1997), pp. 371-379.

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