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SMOOTHNESS OF MULTIPLE REFINABLE FUNCTIONS AND MULTIPLE WAVELETS*

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Abstract. We consider the smoothness of solutions of a system of refinement equations written in the form

$$\phi = \sum\nolimits_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\cdot-\alpha),$$

where the vector of functions $\phi = (\phi_1, \ldots, \phi_r)^T$ is in $(L_p(\mathbb{R}))^r$ and a is a finitely supported sequence of $r \times r$ matrices called the refinement mask. We use the generalized Lipschitz space $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$, $\nu > 0$, to measure smoothness of a given function.

Our method is to relate the optimal smoothness, $\nu_p(\phi)$, to the p-norm joint spectral radius of the block matrices A_ε, $\varepsilon = 0, 1$, given by $A_\varepsilon = (a(\varepsilon + 2\alpha - \beta))_{\alpha,\beta}$, when restricted to a certain finite dimensional common invariant subspace V. Denoting the p-norm joint spectral radius by $\rho_p(A_0|_V, A_1|_V)$, we show that $\nu_p(\phi) \geq 1/p - \log_2 \rho_p(A_0|_V, A_1|_V)$ with equality when the shifts of $\phi_1, \ldots, \phi_r$ are stable and the invariant subspace is generated by certain vectors induced by difference operators of sufficiently high order. This allows an effective use of matrix theory. Also the computational implementation of our method is simple.

When $p = 2$, the optimal smoothness is also given in terms of the spectral radius of the transition matrix associated with the refinement mask.

To illustrate the theory, we give a detailed analysis of two examples where the optimal smoothness can be given explicitly. We also apply our methods to the smoothness analysis of multiple wavelets. These examples clearly demonstrate the applicability and practical power of our approach.

Key words. refinement equations, multiple refinable functions, multiple wavelets, vector subdivision schemes, joint spectral radii, transition operators

AMS subject classifications. Primary, 39B12, 41A25, 42C15, 65F15

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1. Introduction.

The purpose of this paper is to investigate the smoothness properties of multiple refinable functions and multiple wavelets.

Suppose $\phi_1, \ldots, \phi_r$ are compactly supported distributions on $\mathbb{R}$. Denote by ϕ the vector $(\phi_1, \ldots, \phi_r)^T$, the transpose of $(\phi_1, \ldots, \phi_r)$. We say that ϕ is *refinable* if it satisfies the following *refinement equation*:

$$\phi = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\cdot-\alpha), \tag{1.1}$$

where each $a(\alpha)$ is an $r \times r$ matrix of complex numbers and $a(\alpha) = 0$ except for finitely many α. We view a as a sequence from $\mathbb{Z}$ to $\mathbb{C}^{r\times r}$ and call it the *refinement mask*.

In our previous paper [20], we gave a characterization for the accuracy of a vector of multiple refinable functions in terms of the corresponding mask. In another paper [21], we characterized the L_p-convergence ($1 \leq p \leq \infty$) of a subdivision scheme in

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terms of the p-norm joint spectral radius of two matrices derived from the mask. (See [17] for the definition of the p-norm joint spectral radius of a finite collection of matrices.) In this paper, we will take the same approach as we did in [21] to a study of the smoothness properties of the solutions of the refinement equation (1.1).

Taking the Fourier transform of both sides of (1.1), we obtain

$$\hat{\phi}(\xi) = H(\xi/2)\hat{\phi}(\xi/2), \qquad \xi \in \mathbb{R}, \tag{1.2}$$

where

$$H(\xi) := \sum_{\alpha\in\mathbb{Z}} a(\alpha)e^{-i\alpha\xi}/2, \qquad \xi \in \mathbb{R}.$$

Evidently, H is 2π-periodic. Let

$$M := H(0) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2.$$

If ϕ is a solution of (1.1), then it follows from (1.2) that $\hat{\phi}(0) = M\hat{\phi}(0)$. In other words, either $\hat{\phi}(0) = 0$, or $\hat{\phi}(0)$ is an eigenvector of M corresponding to the eigenvalue 1.

Before proceeding, we introduce some notation. For $1 \le p \le \infty$, let $(L_p(\mathbb{R}))^r$ denote the linear space of all vectors $f = (f_1, \ldots, f_r)^T$ such that $f_1, \ldots, f_r \in L_p(\mathbb{R})$. The norm on $(L_p(\mathbb{R}))^r$ is defined by

$$\|f\|_p := \left(\sum_{j=1}^r \|f_j\|_p^p\right)^{1/p}, \qquad f = (f_1, \ldots, f_r)^T \in (L_p(\mathbb{R}))^r.$$

By $(C(\mathbb{R}))^r$ we denote the linear space of all $r \times 1$ vectors of continuous functions.

The shifts of functions $\phi_1, \ldots, \phi_r \in L_p(\mathbb{R})$ are said to be *stable* if there exist two positive constants C_1 and C_2 such that, for arbitrary $b_1, \ldots, b_r \in \ell_p(\mathbb{Z})$,

$$C_1 \sum_{j=1}^r \|b_j\|_p \le \left\|\sum_{j=1}^r \sum_{\alpha\in\mathbb{Z}} b_j(\alpha)\phi_j(\cdot - \alpha)\right\|_p \le C_2 \sum_{j=1}^r \|b_j\|_p.$$

It was proved by Jia and Micchelli [19] that the shifts of the functions $\phi_1, \ldots, \phi_r$ are stable if and only if, for any $\xi \in \mathbb{R}$, the sequences $(\hat{\phi}_j(\xi + 2\pi\beta))_{\beta\in\mathbb{Z}}$ $(j = 1, \ldots, r)$ are linearly independent.

Let $\ell_0(\mathbb{Z})$ denote the linear space of all finitely supported sequences on $\mathbb{Z}$. Similarly, we denote by $\ell_0(\mathbb{Z} \to \mathbb{C}^r)$ (resp., $\ell_0(\mathbb{Z} \to \mathbb{C}^{r\times r})$) the linear space of all finitely supported sequences of $r \times 1$ vectors (resp., $r \times r$ matrices). We identify $\ell_0(\mathbb{Z} \to \mathbb{C}^r)$ with $(\ell_0(\mathbb{Z}))^r$ and identify $\ell_0(\mathbb{Z} \to \mathbb{C}^{r\times r})$ with $(\ell_0(\mathbb{Z}))^{r\times r}$.

For $\beta \in \mathbb{Z}$, we use δ_β to denote the sequence given by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \in \mathbb{Z} \setminus \{\beta\}. \end{cases}$$

In particular, we write δ for δ_0.

We denote by ∇ the difference operator on $\ell_0(\mathbb{Z})$:

$$\nabla v := v - v(\cdot - 1), \qquad v \in \ell_0(\mathbb{Z}).$$

The domain of the difference operator ∇ can be naturally extended to include $(\ell_0(\mathbb{Z}))^r$ and $(\ell_0(\mathbb{Z}))^{r\times r}$.

Let a be an element of $(\ell_0(\mathbb{Z}))^{r\times r}$. For $\varepsilon = 0, 1$, let A_ε be the linear operator on $(\ell_0(\mathbb{Z}))^r$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon + 2\alpha - \beta)v(\beta), \quad \alpha \in \mathbb{Z}, \quad v \in (\ell_0(\mathbb{Z}))^r. \tag{1.3}$$

Our first concern is the existence and uniqueness of the refinement equation (1.1) with the mask a. Under the condition that $\lim_{n\to\infty} M^n$ exists, Heil and Colella [13] established existence and uniqueness of distributional solutions of (1.1). In this case, convergence of the subdivision scheme was studied by Cohen, Dyn, and Levin in [4]. Without assuming the condition that $\lim_{n\to\infty} M^n$ exists, the existence and uniqueness of distributional solutions of (1.1) were studied by several authors, including Cohen, Daubechies, and Plonka [3], Jiang and Shen [23], and Zhou [34].

In section 2, we will investigate the existence of L_p-solutions of the refinement equation (1.1) with the mask a. For $j = 1, \ldots, r$, we use e_j to denote the jth column of the $r \times r$ identity matrix. Let A_0 and A_1 be the linear operators given by (1.3) and V the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla\delta)$, $j = 1, \ldots, r$, in $(\ell_0(\mathbb{Z}))^r$. We will prove that, for an eigenvector y of the matrix $M := \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ corresponding to the eigenvalue 1, there exists a compactly supported solution $\phi \in (L_p(\mathbb{R}))^r$ ($\phi \in (C(\mathbb{R}))^r$ in the case $p = \infty$) of the refinement equation (1.1) subject to $\hat{\phi}(0) = y$, provided

$$\rho_p(A_0|_V, A_1|_V) < 2^{1/p},$$

where $\rho_p(A_0|_V, A_1|_V)$ denotes the p-norm joint spectral radius of $A_0|_V$ and $A_1|_V$. This condition is necessary if, in addition, the shifts of $\phi_1, \ldots, \phi_r$ are stable.

We use the generalized Lipschitz space to measure smoothness of a given function. Let us recall from [8] the definition of the generalized Lipschitz space. For $y \in \mathbb{R}$, the difference operator ∇_y is defined by

$$\nabla_y f = f - f(\cdot - y),$$

where f is a function from $\mathbb{R}$ to $\mathbb{C}$. The *modulus of continuity* of a function f in $L_p(\mathbb{R})$ ($1 \le p \le \infty$) is defined by

$$\omega(f, h)_p := \sup_{|y|\le h} \left\|\nabla_y f\right\|_p, \qquad h \ge 0.$$

Let k be a positive integer. The kth *modulus of smoothness* of $f \in L_p(\mathbb{R})$ is defined by

$$\omega_k(f, h)_p := \sup_{|y|\le h} \left\|\nabla_y^k f\right\|_p, \qquad h \ge 0.$$

For $\nu > 0$, let k be an integer greater than ν. The *generalized Lipschitz space* $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$ consists of those functions $f \in L_p(\mathbb{R})$ for which

$$\omega_k(f, h)_p \le Ch^\nu \qquad \forall\, h > 0,$$

where C is a positive constant independent of h.

By $(\mathrm{Lip}^*(\nu, L_p(\mathbb{R})))^r$ we denote the linear space of all vectors $f = (f_1, \ldots, f_r)^T$ such that $f_1, \ldots, f_r \in \mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$. The optimal smoothness of a vector $f \in (L_p(\mathbb{R}))^r$ in the L_p-norm is described by its *critical exponent* $\nu_p(f)$ defined by

$$\nu_p(f) := \sup\Big\{ \nu : f \in \big(\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))\big)^r \Big\}.$$

In section 3, we will establish our main result on characterization of the smoothness of multiple refinable functions. Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is a compactly supported solution of the refinement equation (1.1) with the mask a. Let k be a positive integer and V the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla^k \delta)$, $j = 1, \ldots, r$. If ϕ lies in $(L_p(\mathbb{R}))^r$ for $1 \le p < \infty$ (ϕ lies in $(C(\mathbb{R}))^r$ for $p = \infty$), then

$$\nu_p(\phi) \ge 1/p - \log_2 \rho_p(A_0|_V, A_1|_V). \tag{1.4}$$

If, in addition, the shifts of $\phi_1, \ldots, \phi_r$ are stable, and if $k > 1/p - \log_2 \rho_p(A_0|_V, A_1|_V)$, then equality holds in (1.4). When $p = 2$, the critical exponent is also given in terms of the spectral radius of the transition operator associated with the refinement mask a.

Regularity of multiple refinable functions was studied by Cohen, Daubechies, and Plonka in [3] and by Micchelli and Sauer in [25]. Both approaches are based on the factorization technique introduced by Plonka [26]. Our approach is different from theirs and does not rely on factorization. Thus, our methods can be applied to multiple refinable functions and multiple wavelets of several variables. For smoothness analysis of a single multivariate refinable function, the reader is referred to [18] and [27]. Even in the univariate case our methods have advantages over the factorization technique. Indeed, our methods use the joint spectral radius of finite matrices. This allows a more effective use of matrix theory to reduce the size of the matrices by a restriction to a certain common invariant subspace. Thus, the computational implementation of our method becomes much simpler. In fact, in the multiple case, the factorization would usually enlarge the support of the mask making the order of the matrices larger, hence computationally more complex. For a discussion of the size of the support of vector scaling functions, see So and Wang [30].

To illustrate the general theory, we shall give detailed analysis of smoothness for two examples in section 4. One example is taken from [10], the other from [21]. In particular, for the example of Donovan et al. [10], our method gives explicitly the exact smoothness in all p-norms. In comparison, Cohen, Daubechies, and Plonka [3] partially recovered the result of [10] for the regularity in the L_∞-norm, while Micchelli and Sauer [25] gave a crude estimate for the regularity in the L_1-norm for a special case.

In section 5, applying our study to multiple wavelets, we construct a family of orthogonal double wavelets which includes the one of Chui and Lian [2]. We give a complete smoothness analysis in L_2 for this family, and in all L_p for the example of Chui and Lian (who did not discuss smoothness).

The examples in sections 4 and 5 clearly demonstrate the applicability and practical power of our approach.

2. Existence of L_p-solutions. In order to solve the refinement equation (1.1), we introduce the linear operator Q_a on $(L_p(\mathbb{R}))^r$ $(1 \le p \le \infty)$ as follows:

$$Q_a f := \sum_{\alpha \in \mathbb{Z}} a(\alpha) f(2 \cdot - \alpha), \qquad f \in (L_p(\mathbb{R}))^r. \tag{2.1}$$

If ϕ is a fixed point of Q_a, i.e., $Q_a\phi = \phi$, then ϕ is a solution of the refinement equation (1.1).

Let Q_a be the linear operator given in (2.1). For an initial vector $f \in (L_p(\mathbb{R}))^r$, we have

$$Q_a^n f = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha) f(2^n \cdot -\alpha), \qquad n = 1, 2, \ldots,$$

where each a_n is independent of the choice of f. In particular, $a_1 = a$. Consequently, for $n > 1$ we have

$$\begin{aligned} Q_a^n f = Q_a^{n-1}(Q_a f) &= \sum_{\beta\in\mathbb{Z}} a_{n-1}(\beta)(Q_a f)(2^{n-1} \cdot -\beta) \\ &= \sum_{\beta\in\mathbb{Z}} \sum_{\alpha\in\mathbb{Z}} a_{n-1}(\beta) a(\alpha) f(2^n \cdot -2\beta - \alpha) \\ &= \sum_{\alpha\in\mathbb{Z}} \left[\sum_{\beta\in\mathbb{Z}} a_{n-1}(\beta) a(\alpha - 2\beta) \right] f(2^n \cdot -\alpha). \end{aligned}$$

This establishes the following iteration relation for a_n $(n = 1, 2, \ldots)$:

$$a_1 = a \quad \text{and} \quad a_n(\alpha) = \sum_{\beta\in\mathbb{Z}} a_{n-1}(\beta) a(\alpha - 2\beta), \quad \alpha \in \mathbb{Z}. \tag{2.2}$$

For $\varepsilon \in \mathbb{Z}$, we denote by $A_\varepsilon = (A_\varepsilon(\alpha, \beta))_{\alpha,\beta\in\mathbb{Z}}$ the bi-infinite block matrix given by

$$A_\varepsilon(\alpha, \beta) := a(\varepsilon + 2\alpha - \beta), \qquad \alpha, \beta \in \mathbb{Z}. \tag{2.3}$$

For $a \in (\ell_0(\mathbb{Z}))^{r\times r}$ and $n = 1, 2, \ldots$, let $a_n \in (\ell_0(\mathbb{Z}))^{r\times r}$ be given by the iteration relation (2.2). If $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \ldots, \varepsilon_n, \gamma \in \mathbb{Z}$, then

$$a_n(\alpha - \beta) = A_{\varepsilon_n} \cdots A_{\varepsilon_1}(\gamma, \beta) \qquad \forall\, \beta \in \mathbb{Z}. \tag{2.4}$$

This can be proved easily by induction on n. For $n = 1$ and $\alpha = \varepsilon_1 + 2\gamma$, where $\varepsilon_1, \gamma \in \mathbb{Z}$, we have

$$a_1(\alpha - \beta) = a(\varepsilon_1 + 2\gamma - \beta) = A_{\varepsilon_1}(\gamma, \beta).$$

Suppose $n > 1$ and (2.4) has been verified for $n - 1$. For $\alpha = \varepsilon_1 + 2\alpha_1$, where $\alpha_1, \varepsilon_1 \in \mathbb{Z}$, by the iteration relation (2.2) we have

$$a_n(\alpha - \beta) = \sum_{\eta\in\mathbb{Z}} a_{n-1}(\eta) a(\alpha - \beta - 2\eta) = \sum_{\eta\in\mathbb{Z}} a_{n-1}(\alpha_1 - \eta) a(\varepsilon_1 + 2\eta - \beta). \tag{2.5}$$

Suppose $\alpha_1 = \varepsilon_2 + \cdots + 2^{n-2}\varepsilon_n + 2^{n-1}\gamma$, where $\varepsilon_2, \ldots, \varepsilon_n, \gamma \in \mathbb{Z}$. Then by the induction hypothesis we have

$$a_{n-1}(\alpha_1 - \eta) = A_{\varepsilon_n} \cdots A_{\varepsilon_2}(\gamma, \eta).$$

This in connection with (2.5) gives

$$a_n(\alpha - \beta) = \sum_{\eta\in\mathbb{Z}} A_{\varepsilon_n} \cdots A_{\varepsilon_2}(\gamma, \eta) A_{\varepsilon_1}(\eta, \beta) = A_{\varepsilon_n} \cdots A_{\varepsilon_2} A_{\varepsilon_1}(\gamma, \beta),$$

thereby completing the induction procedure.

The relation (2.4) motivates us to consider the joint spectral radius of a finite multiset of linear operators. The uniform joint spectral radius was introduced by Rota and Strang [28]. The use of the joint spectral radius to obtain regularity results was initiated by Daubechies and Lagarias [6, 7] for the scalar case. Colella and Heil [5] used the joint spectral radius to characterize continuous solutions of scalar refinement equations.

The p-norm joint spectral radius was introduced by Jia in [17]. Let us recall from [17] the definition of the p-norm joint spectral radius. Let V be a *finite-dimensional* vector space equipped with a vector norm $\|\cdot\|$. For a linear operator A on V, define

$$\|A\| := \max_{\|v\|=1} \big\{\|Av\|\big\}.$$

Let $\mathcal{A}$ be a finite multiset of linear operators on V. For a positive integer n we denote by $\mathcal{A}^n$ the nth Cartesian power of $\mathcal{A}$:

$$\mathcal{A}^n = \big\{(A_1, \dots, A_n) : A_1, \dots, A_n \in \mathcal{A}\big\}.$$

For $1 \le p < \infty$, let

$$\|\mathcal{A}^n\|_p := \left(\sum_{(A_1,\dots,A_n)\in\mathcal{A}^n} \|A_1 \cdots A_n\|^p \right)^{1/p},$$

and, for $p = \infty$, define

$$\|\mathcal{A}^n\|_\infty := \max\big\{\|A_1 \cdots A_n\| : (A_1, \dots, A_n) \in \mathcal{A}^n\big\}.$$

For $1 \le p \le \infty$, the *p-norm joint spectral radius* of $\mathcal{A}$ is defined to be

$$\rho_p(\mathcal{A}) := \lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n}.$$

It is easily seen that this limit indeed exists, and

$$\lim_{n\to\infty} \|\mathcal{A}^n\|_p^{1/n} = \inf_{n\ge1} \|\mathcal{A}^n\|_p^{1/n}.$$

Clearly, $\rho_p(\mathcal{A})$ is independent of the choice of the vector norm on V.

If $\mathcal{A}$ consists of a single linear operator A, then $\rho_p(\mathcal{A}) = \rho(A)$, where $\rho(A)$ denotes the spectral radius of A, which is independent of p. It is easily seen that $\rho(A) \le \rho_\infty(\mathcal{A})$ for any element A in $\mathcal{A}$.

Now let $\mathcal{A}$ be a finite multiset of linear operators on a normed vector space V, which is not necessarily finite dimensional. A subspace W of V is said to be invariant under $\mathcal{A}$, or *$\mathcal{A}$-invariant*, if it is invariant under every operator A in $\mathcal{A}$. For a vector $w \in V$, we define

$$\|\mathcal{A}^n w\|_p := \begin{cases} \left(\sum_{(A_1,\dots,A_n)\in\mathcal{A}^n} \|A_1 \cdots A_n w\|^p\right)^{1/p} & \text{for } 1 \le p < \infty, \\ \max\big\{\|A_1 \cdots A_n w\| : (A_1, \dots, A_n) \in \mathcal{A}^n\big\} & \text{for } p = \infty. \end{cases} \tag{2.6}$$

If the minimal $\mathcal{A}$-invariant subspace W generated by w is finite dimensional, then we have

$$\lim_{n\to\infty} \big\|\mathcal{A}^n w\big\|_p^{1/n} = \rho_p(\mathcal{A}|_W), \qquad 1 \le p \le \infty.$$

See [12, Lemma 2.4] for a proof of this result.

If $V = (\ell_0(\mathbb{Z}))^r$, it is often convenient to choose the ℓ_p-norm as the underlying vector norm in (2.6). We denote by $\ell_p(\mathbb{Z} \to \mathbb{C}^r)$ the linear space of all sequences $u : \mathbb{Z} \to \mathbb{C}^r$ such that $u(\alpha) = (u_1(\alpha), \ldots, u_r(\alpha))^T$ for some $u_1, \ldots, u_r \in \ell_p(\mathbb{Z})$ and all $\alpha \in \mathbb{Z}$. Obviously, $u \mapsto (u_1, \ldots, u_r)^T$ is a canonical isomorphism between $\ell_p(\mathbb{Z} \to \mathbb{C}^r)$ and $(\ell_p(\mathbb{Z}))^r$. Thus, we may identify $\ell_p(\mathbb{Z} \to \mathbb{C}^r)$ with $(\ell_p(\mathbb{Z}))^r$. The norm of $u = (u_1, \ldots, u_r)^T$ is given by

$$\|u\|_p := \Big(\sum\nolimits_{j=1}^r \|u_j\|_p^p\Big)^{1/p}.$$

Equipped with this norm, $(\ell_p(\mathbb{Z}))^r$ becomes a Banach space.

We denote by $\ell_p(\mathbb{Z} \to \mathbb{C}^{r\times r})$ the linear space of all sequences $b : \mathbb{Z} \to \mathbb{C}^{r\times r}$ such that $b(\alpha) = (b_{jk}(\alpha))_{1\le j,k\le r}$ for some $b_{jk} \in \ell_p(\mathbb{Z})$ $(j, k = 1, \ldots, r)$ and all $\alpha \in \mathbb{Z}$. We also identify $\ell_p(\mathbb{Z} \to \mathbb{C}^{r\times r})$ with $(\ell_p(\mathbb{Z}))^{r\times r}$. The norm of $b = (b_{jk})_{1\le j,k\le r}$ is defined by

$$\|b\|_p := \left(\sum_{j=1}^r \sum_{k=1}^r \|b_{jk}\|_p^p\right)^{1/p}.$$

Let a be an element of $(\ell_0(\mathbb{Z}))^{r\times r}$. The bi-infinite block matrices A_ε $(\varepsilon \in \mathbb{Z})$ defined in (2.3) may be viewed as the linear operators on $(\ell_0(\mathbb{Z}))^r$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon + 2\alpha - \beta)v(\beta), \quad \alpha \in \mathbb{Z}, \quad v \in (\ell_0(\mathbb{Z}))^r. \tag{2.7}$$

Suppose $y \in \mathbb{C}^r$ and $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \ldots, \varepsilon_n, \gamma \in \mathbb{Z}$. Then it follows from (2.4) that

$$a_n(\alpha - \beta)y = A_{\varepsilon_n} \cdots A_{\varepsilon_1}(y\delta_\beta)(\gamma) \qquad \forall\, \beta \in \mathbb{Z}. \tag{2.8}$$

For a bounded subset K of $\mathbb{R}$, we use $\ell(K)$ to denote the linear space of all sequences on $\mathbb{Z}$ supported in $K \cap \mathbb{Z}$. Suppose a is supported on $[0, N]$, where N is a positive integer. Then, for $j \le 0$ and $k \ge N - 1$, $(\ell([j, k]))^r$ is invariant under both A_0 and A_1. Consequently, the minimal common invariant subspace of A_0 and A_1 generated by a finite subset of $(\ell_0(\mathbb{Z}))^r$ is finite dimensional.

THEOREM 2.1. *For an element $a \in (\ell_0(\mathbb{Z}))^{r\times r}$, let y be an eigenvector of the matrix $M := \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ corresponding to the eigenvalue 1, and let V be the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla\delta)$, $j = 1, \ldots, r$, in $(\ell_0(\mathbb{Z}))^r$. If*

$$\rho_p(A_0|_V, A_1|_V) < 2^{1/p}, \tag{2.9}$$

then there exists a compactly supported solution $\phi \in (L_p(\mathbb{R}))^r$ ($\phi \in (C(\mathbb{R}))^r$ in the case $p = \infty$) of the refinement equation (1.1) with the mask a subject to $\hat{\phi}(0) = y$. Conversely, if $\phi = (\phi_1, \ldots, \phi_r)^T \in (L_p(\mathbb{R}))^r$ ($\phi \in (C(\mathbb{R}))^r$ in the case $p = \infty$) is a compactly supported solution of (1.1) such that the shifts of $\phi_1, \ldots, \phi_r$ are stable, then (2.9) holds true.

Proof. The proof follows the lines of [21, Theorem 5.3]. Let $f := yg$, where g is the hat function supported on $[0, 2]$ satisfying $g(x) = x$ for $0 \le x \le 1$ and $g(x) = 2-x$ for $1 < x \le 2$. Since $\hat{g}(0) = 1$, we have $\hat{f}(0) = y$.

Consider $f_n := Q_a^n f$, $n = 1, 2, \ldots$. We have

$$f_n = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha) f(2^n \cdot - \alpha),$$

where the sequences a_n $(n = 1, 2, \ldots)$ are given by the iteration relation (2.2). Since $g = g(2\,\cdot)/2 + g(2\cdot - 1) + g(2\cdot - 2)/2$, we have

$$f_n = \sum_{\alpha\in\mathbb{Z}} \frac{1}{2}\big[a_n(\alpha)y + a_n(\alpha-1)y\big] g(2^{n+1}\cdot - 2\alpha) + \sum_{\alpha\in\mathbb{Z}} \big[a_n(\alpha)y\big] g(2^{n+1}\cdot - 2\alpha - 1).$$

Moreover,

$$f_{n+1} = \sum_{\alpha\in\mathbb{Z}} \big[a_{n+1}(2\alpha)y\big] g(2^{n+1}\cdot - 2\alpha) + \sum_{\alpha\in\mathbb{Z}} \big[a_{n+1}(2\alpha+1)y\big] g(2^{n+1}\cdot - 2\alpha - 1).$$

Subtracting the first equation from the second, we obtain

$$f_{n+1} - f_n = \sum_{\alpha\in\mathbb{Z}} \big[b_n(\alpha)y\big] g(2^{n+1}\cdot - 2\alpha) + \sum_{\alpha\in\mathbb{Z}} \big[c_n(\alpha)y\big] g(2^{n+1}\cdot - 2\alpha - 1),$$

where

$$b_n(\alpha) := a_{n+1}(2\alpha) - \frac{1}{2}a_n(\alpha) - \frac{1}{2}a_n(\alpha-1) \quad \text{and} \quad c_n(\alpha) := a_{n+1}(2\alpha+1) - a_n(\alpha), \quad \alpha \in \mathbb{Z}.$$

It follows that

$$\|f_{n+1} - f_n\|_p \le 2^{1-(n+1)/p}\big(\|b_n y\|_p + \|c_n y\|_p\big). \tag{2.10}$$

Let us estimate $\|b_n y\|_p$ and $\|c_n y\|_p$. Suppose $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\gamma \in \mathbb{Z}$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$. Then $2\alpha = 0 + 2\varepsilon_1 + 2^2\varepsilon_2 + \cdots + 2^n\varepsilon_n + 2^{n+1}\gamma$, and an application of (2.8) gives

$$\begin{aligned} b_n(\alpha)y &= a_{n+1}(2\alpha)y - \frac{1}{2}a_n(\alpha)y - \frac{1}{2}a_n(\alpha-1)y \\ &= A_{\varepsilon_n}\cdots A_{\varepsilon_1}A_0(y\delta)(\gamma) - \frac{1}{2}A_{\varepsilon_n}\cdots A_{\varepsilon_1}(y\delta + y\delta_1)(\gamma) \\ &= A_{\varepsilon_n}\cdots A_{\varepsilon_1}u(\gamma), \end{aligned}$$

where $u := A_0(y\delta) - (y\delta + y\delta_1)/2$. Similarly, we have

$$c_n(\alpha)y = A_{\varepsilon_n}\cdots A_{\varepsilon_1}v(\gamma),$$

where $v := A_1(y\delta) - y\delta$.

Let $\mathcal{A} := \{A_0, A_1\}$. The norm in (2.6) is chosen to be the ℓ_p-norm. The discussion above tells us that

$$\|b_n y\|_p = \|\mathcal{A}^n u\|_p \quad \text{and} \quad \|c_n y\|_p = \|\mathcal{A}^n v\|_p. \tag{2.11}$$

Write ρ for $\rho_p(A_0|_V, A_1|_V)$. In order to prove that the sequence $(f_n)_{n=1,2,\ldots}$ converges in the L_p-norm, it suffices to show that

$$\lim_{n\to\infty} \|\mathcal{A}^n u\|_p^{1/n} \le \rho \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{A}^n v\|_p^{1/n} \le \rho. \tag{2.12}$$

Indeed, since $\rho < 2^{1/p}$, we may pick a number σ such that $2^{-1/p}\rho < \sigma < 1$. Hence, there exists a constant C independent of n such that

$$2^{-n/p}\|\mathcal{A}^n u\|_p \le C\sigma^n \quad \text{and} \quad 2^{-n/p}\|\mathcal{A}^n v\|_p \le C\sigma^n.$$

This together with (2.10) and (2.11) yields

$$\|f_{n+1} - f_n\|_p \le 2^{1-(n+1)/p}\big(\|b_n y\|_p + \|c_n y\|_p\big) \le 4C\sigma^n \qquad \forall\, n \in \mathbb{N}.$$

Since $\sigma < 1$, this shows that $(f_n)_{n=1,2,\ldots}$ converges to some $\phi \in (L_p(\mathbb{R}))^r$ in the L_p-norm. In the case $p = \infty$, since each f_n is continuous, the limit ϕ is also continuous. Furthermore, since y is an eigenvector of the matrix M corresponding to the eigenvalue 1, we have

$$\hat{f}_n(0) = M^n \hat{f}(0) = M^n y = y.$$

Taking the limit as $n \to \infty$ in the above equation, we obtain $\hat{\phi}(0) = y$.

Let us verify (2.12). For this purpose, we set $v_j := A_j(y\delta) - y\delta$ for $j = 0, 1$. Then $v = v_1$ and $u = v_0 + (y\nabla\delta)/2$. But

$$\lim_{n\to\infty} \|\mathcal{A}^n(y\nabla\delta)\|_p^{1/n} \le \rho.$$

Thus, it suffices to show that

$$\lim_{n\to\infty} \|\mathcal{A}^n(v_0 + v_1)\|_p^{1/n} \le \rho \quad \text{and} \quad \lim_{n\to\infty} \|\mathcal{A}^n(v_0 - v_1)\|_p^{1/n} \le \rho. \tag{2.13}$$

To verify the first inequality in (2.13), we observe that

$$v_0 + v_1 = A_0(y\delta) - y\delta + A_1(y\delta) - y\delta = \sum_{\alpha\in\mathbb{Z}} \big[a(2\alpha) + a(2\alpha+1)\big] y\delta_\alpha - 2y\delta.$$

But $\sum_{\alpha\in\mathbb{Z}}[a(2\alpha) + a(2\alpha+1)]y = 2My = 2y$. Hence it follows that

$$v_0 + v_1 = \sum\nolimits_{\alpha\in\mathbb{Z}} \big[a(2\alpha) + a(2\alpha+1)\big] y(\delta_\alpha - \delta).$$

Note that only finitely many terms in the above sum do not vanish, while $\delta_\alpha - \delta$ can be written as $-\sum_{\beta=0}^{\alpha-1} \nabla\delta_\beta$. Therefore, $v_0 + v_1$ can be written as a finite linear combination of $e_j(\nabla\delta_\beta)$, $j = 1, \ldots, r$, $\beta \in \mathbb{Z}$. We claim that

$$\|\mathcal{A}^n(e_j\nabla\delta_\beta)\|_p = \|\mathcal{A}^n(e_j\nabla\delta)\|_p \qquad \forall\, \beta \in \mathbb{Z}.$$

Indeed, for $w \in \mathbb{C}^r$ and $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ and $\gamma \in \mathbb{Z}$, by (2.8) we have

$$A_{\varepsilon_n} \cdots A_{\varepsilon_1}\big[w(\delta_\beta - \delta_{\beta+1})\big](\gamma) = a_n(\alpha - \beta)w - a_n(\alpha - \beta - 1)w.$$

Note that $\|a_n(\cdot - \beta)w - a_n(\cdot - \beta - 1)w\|_p = \|a_n w - a_n(\cdot - 1)w\|_p$. Consequently,

$$\|\mathcal{A}^n(w\nabla\delta_\beta)\|_p = \|\nabla a_n w\|_p = \|\mathcal{A}^n(w\nabla\delta)\|_p \qquad \forall\, \beta \in \mathbb{Z}. \tag{2.14}$$

This verifies our claim, and thereby establishes the first inequality in (2.13).

As to the second inequality in (2.13), we observe that

$$A_0(y\delta) = \sum_{\alpha\in\mathbb{Z}} a(2\alpha)y\delta_\alpha = A_1(y\delta_1).$$

It follows that

$$v_0 - v_1 = A_0(y\delta) - A_1(y\delta) = A_1(y\delta_1) - A_1(y\delta) = -A_1(y\nabla\delta).$$

Hence, for $n = 1, 2, \ldots$, we have

$$\|\mathcal{A}^n(v_0 - v_1)\|_p \le \|\mathcal{A}^{n+1}(y\nabla\delta)\|_p.$$

This verifies the second inequality in (2.13). The proof for the sufficiency part of the theorem is complete.

It remains to prove the necessity part of the theorem. Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is a solution of (1.1), where $\phi_1, \ldots, \phi_r$ are compactly supported functions in $L_p(\mathbb{R})$ ($\phi_1, \ldots, \phi_r$ are continuous in the case $p = \infty$). Iterating the refinement equation (1.1) n times, we obtain

$$\phi = \sum_{\alpha\in\mathbb{Z}} a_n(\alpha)\phi(2^n \cdot -\alpha),$$

where the sequences a_n ($n = 1, 2, \ldots$) are given by (2.2). It follows that

$$\phi - \phi(\cdot - 1/2^n) = \sum_{\alpha\in\mathbb{Z}} \nabla a_n(\alpha)\phi(2^n \cdot -\alpha).$$

If the shifts of $\phi_1, \ldots, \phi_r$ are stable, then there exists a constant $C_1 > 0$ such that

$$2^{-n/p}\|\nabla a_n\|_p \le C_1\|\phi - \phi(\cdot - 1/2^n)\|_p \qquad \forall\, n \in \mathbb{N}.$$

Consequently, there exists a constant $C > 0$ such that

$$2^{-n/p}\|\nabla a_n e_j\|_p \le C\|\phi - \phi(\cdot - 1/2^n)\|_p \qquad \forall\, j = 1, \ldots, r;\ n \in \mathbb{N}. \tag{2.15}$$

But (2.14) tells us that

$$\lim_{n\to\infty} \|\nabla a_n e_j\|_p^{1/n} = \lim_{n\to\infty} \|\mathcal{A}^n(e_j\nabla\delta)\|_p^{1/n}.$$

Note that $\rho = \max_{1\le j\le r}\{\lim_{n\to\infty} \|\mathcal{A}^n(e_j\nabla\delta)\|_p^{1/n}\}$. Since $\lim_{n\to\infty} \|\phi - \phi(\cdot - 1/2^n)\|_p = 0$, (2.15) holds true only if $2^{-1/p}\rho < 1$. This shows $\rho < 2^{1/p}$, as desired. □

3. Characterization of smoothness. In this section, we give a characterization for the smoothness of solutions of the refinement equation (1.1) in terms of the corresponding refinement mask. Our work is based on the following results from approximation theory: For a function f in $L_p(\mathbb{R})$ (f is continuous in the case $p = \infty$), f lies in $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}))$ ($\nu > 0$) if and only if, for some integer $k > \nu$, there exists a constant $C > 0$ such that

$$\|\nabla_{2^{-n}}^k f\|_p \le C2^{-n\nu} \qquad \forall\, n \in \mathbb{N}.$$

For these results we refer the reader to the work of Boman [1] and Ditzian [9].

Suppose $\phi = (\phi_1, \dots, \phi_r)^T$ is a solution of the refinement equation (1.1). Iterating (1.1) n times, we obtain

$$\phi = \sum_{\alpha \in \mathbb{Z}} a_n(\alpha)\phi(2^n \cdot - \alpha), \tag{3.1}$$

where a_n $(n = 1, 2, \dots)$ are given by (2.2). Applying the difference operator $\nabla_{2^{-n}}$ to both sides of (3.1), we obtain

$$\nabla_{2^{-n}}\phi = \sum_{\alpha \in \mathbb{Z}} a_n(\alpha)\big[\phi(2^n \cdot - \alpha) - \phi(2^n \cdot - \alpha - 1)\big] = \sum_{\alpha \in \mathbb{Z}} \nabla a_n(\alpha)\phi(2^n \cdot - \alpha).$$

For $k = 1, 2, \dots$, an induction argument tells us that

$$\nabla^k_{2^{-n}}\phi = \sum_{\alpha \in \mathbb{Z}} \nabla^k a_n(\alpha)\phi(2^n \cdot - \alpha). \tag{3.2}$$

Suppose $\phi_1, \dots, \phi_r$ are compactly supported functions in $L_p(\mathbb{R})$. It follows from (3.2) that

$$2^{n/p}\big\|\nabla^k_{2^{-n}}\phi\big\|_p \le C_1\big\|\nabla^k a_n\big\|_p \qquad \forall\, n \in \mathbb{N}, \tag{3.3}$$

where C_1 is a constant independent of n. If, in addition, the shifts of $\phi_1, \dots, \phi_r$ are stable, then there exists a constant $C_2 > 0$ such that

$$2^{n/p}\big\|\nabla^k_{2^{-n}}\phi\big\|_p \ge C_2\big\|\nabla^k a_n\big\|_p \qquad \forall\, n \in \mathbb{N}, \tag{3.4}$$

but, for $0 < \nu < k$, ϕ lies in $(\mathrm{Lip}^*(\nu, L_p(\mathbb{R})))^r$ if and only if there exists a constant $C > 0$ such that

$$\|\nabla^k_{2^{-n}}\phi\|_p \le C2^{-n\nu} \qquad \forall\, n \in \mathbb{N}.$$

Thus, we have established the following result.

LEMMA 3.1. *Suppose $\phi = (\phi_1, \dots, \phi_r)^T \in (L_p(\mathbb{R}))^r$ ($\phi \in (C(\mathbb{R}))^r$ in the case $p = \infty$) is a compactly supported solution of the refinement equation* (1.1) *with mask a. For $n = 1, 2, \dots$, let a_n be given by the iteration relation* (2.2). *Let $k > \nu > 0$, where k is an integer. If there exists a constant $C > 0$ such that*

$$\|\nabla^k a_n\|_p \le C2^{-n(\nu - 1/p)} \qquad \forall\, n \in \mathbb{N}, \tag{3.5}$$

then ϕ belongs to $(Lip^*(\nu, L_p(\mathbb{R})))^r$. Conversely, if ϕ lies in $(Lip^*(\nu, L_p(\mathbb{R})))^r$, then (3.5) *holds true, provided the shifts of $\phi_1, \dots, \phi_r$ are stable.*

For two elements b and c in $\ell_0(\mathbb{Z})$, the *discrete convolution* of b and c, denoted by $b{*}c$, is defined by

$$b{*}c(\alpha) = \sum_{\beta \in \mathbb{Z}} b(\alpha - \beta)c(\beta), \qquad \alpha \in \mathbb{Z}.$$

Evidently, $b{*}\delta = b$ for any $b \in \ell_0(\mathbb{Z})$. If $b \in (\ell_0(\mathbb{Z}))^{r\times r}$ and $c \in (\ell_0(\mathbb{Z}))^r$, then $b{*}c$ is defined in a similar way.

LEMMA 3.2. *Let a be an element of $(\ell_0(\mathbb{Z}))^{r\times r}$, and let a_n $(n = 1, 2, \dots)$ be given by the iteration relation* (2.2). *For $\varepsilon = 0, 1$, let A_ε be the linear operator on $(\ell_0(\mathbb{Z}))^r$ given by* (2.7). *Then, for each integer $k \ge 0$,*

$$\lim_{n\to\infty}\big\|\nabla^k a_n\big\|_p^{1/n} = \rho_p(A_0|_V, A_1|_V),$$

where V is the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla^k\delta)$, $j = 1, \ldots, r$.

Proof. Write $\mathcal{A}$ for $\{A_0, A_1\}$. For an element $w \in (\ell_0(\mathbb{Z}))^r$, the quantity $\|\mathcal{A}^n w\|_p$ is defined as in (2.6) with the ℓ_p-norm being the underlying vector norm on $(\ell_0(\mathbb{Z}))^r$.

Let v be an element in $(\ell_0(\mathbb{Z}))^r$. We observe that

$$(\nabla^k a_n)*v = a_n*(\nabla^k v).$$

Suppose $\alpha = \varepsilon_1 + 2\varepsilon_2 + \cdots + 2^{n-1}\varepsilon_n + 2^n\gamma$, where $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$ and $\gamma \in \mathbb{Z}$. Then by (2.4) we have

$$\begin{aligned}\big(\nabla^k a_n*v\big)(\alpha) &= \sum_{\beta\in\mathbb{Z}} a_n(\alpha-\beta)\nabla^k v(\beta)\\ &= \sum_{\beta\in\mathbb{Z}} A_{\varepsilon_n}\cdots A_{\varepsilon_1}(\gamma,\beta)\nabla^k v(\beta) = A_{\varepsilon_n}\cdots A_{\varepsilon_1}(\nabla^k v)(\gamma).\end{aligned}$$

It follows that

$$\|\nabla^k a_n*v\|_p = \|\mathcal{A}^n(\nabla^k v)\|_p. \tag{3.6}$$

Choosing $v = e_j\delta$ in (3.6), we obtain

$$\|(\nabla^k a_n)e_j\|_p = \|(\nabla^k a_n)*(e_j\delta)\|_p = \|\mathcal{A}^n(e_j\nabla^k\delta)\|_p.$$

This shows that

$$\lim_{n\to\infty}\|(\nabla^k a_n)e_j\|_p^{1/n} = \lim_{n\to\infty}\|\mathcal{A}^n(e_j\nabla^k\delta)\|_p^{1/n} = \rho_p(A_0|_{V_j}, A_1|_{V_j}),$$

where V_j is the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla^k\delta)$. But

$$\|\nabla^k a_n\|_p = \left(\sum_{j=1}^r \|(\nabla^k a_n)e_j\|_p^p\right)^{1/p}.$$

Therefore, we arrive at the conclusion that

$$\lim_{n\to\infty}\big\|\nabla^k a_n\big\|_p^{1/n} = \max\big\{\rho_p(A_0|_{V_j}, A_1|_{V_j}) : j = 1, \ldots, r\big\} = \rho_p(A_0|_V, A_1|_V),$$

where V is the sum of $V_1, \ldots, V_r$. ∎

We are in a position to prove the main result of this paper.

THEOREM 3.3. *Let a be an element of $(\ell_0(\mathbb{Z}))^{r\times r}$. For $\varepsilon = 0, 1$, let A_ε be the linear operator on $(\ell_0(\mathbb{Z}))^r$ given by (2.7). Let k be a positive integer and V the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla^k\delta)$, $j = 1, \ldots, r$. Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ is compactly supported and lies in $(L_p(\mathbb{R}))^r$ for $1 \le p < \infty$ (f lies in $(C(\mathbb{R}))^r$ for $p = \infty$). If ϕ is a solution of the refinement equation (1.1) with the mask a, then*

$$\nu_p(\phi) \ge 1/p - \log_2 \rho_p(A_0|_V, A_1|_V). \tag{3.7}$$

In addition, if the shifts of $\phi_1, \ldots, \phi_r$ are stable and if $k > 1/p - \log_2 \rho_p(A_0|_V, A_1|_V)$, then equality holds in (3.7).

Proof. Write ρ for $\rho_p(A_0|_V, A_1|_V)$. By Lemma 3.2, we have $\lim_{n\to\infty} \|\nabla^k a_n\|_p^{1/n} = \rho$. Thus, for $\varepsilon > 0$, there exists a constant $C > 0$ such that, $\forall n \in \mathbb{N}$,

$$\|\nabla^k a_n\|_p \le C(\rho+\varepsilon)^n = C2^{n\log_2(\rho+\varepsilon)} = C2^{-n(\nu-1/p)},$$

where $\nu := 1/p - \log_2(\rho+\varepsilon)$. By Lemma 3.1, ϕ belongs to $(\mathrm{Lip}^*(\nu, L_p(\mathbb{R})))^r$. This shows that

$$\nu_p(\phi) \ge 1/p - \log_2(\rho+\varepsilon),$$

but $\varepsilon > 0$ can be arbitrarily small; hence, we obtain

$$\nu_p(\phi) \ge 1/p - \log_2 \rho.$$

Now suppose $k > 1/p - \log_2 \rho$ and the shifts of $\phi_1, \dots, \phi_r$ are stable. We wish to show $\nu_p(\phi) \le 1/p - \log_2 \rho$. If this is not true, then there exists μ such that $1/p - \log_2 \rho < \mu < k$ and $\phi \in (\mathrm{Lip}^*(\mu, L_p(\mathbb{R})))^r$. By Lemma 3.1, there exists a constant $C > 0$ such that

$$\|\nabla^k a_n\|_p \le C2^{-n(\mu-1/p)} \qquad \forall\, n \in \mathbb{N}.$$

By Lemma 3.2 we get

$$\rho = \lim_{n\to\infty} \|\nabla^k a_n\|_p^{1/n} \le 2^{-\mu+1/p}.$$

It follows that

$$\mu \le 1/p - \log_2 \rho,$$

which contradicts the assumption $\mu > 1/p - \log_2 \rho$. Therefore, we obtain the desired result $\nu_p(\phi) \le 1/p - \log_2 \rho$. $\square$

The case $p = 2$ is of particular interest. In this case, the smoothness is usually measured by using Sobolev spaces. For $\nu \ge 0$ we denote by $W_2^\nu(\mathbb{R})$ the Sobolev space of all functions $f \in L_2(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \big|\hat{f}(\xi)\big|^2 \big(1+|\xi|^\nu\big)^2\, d\xi < \infty.$$

It is well known that, for $\nu > \varepsilon > 0$, the inclusion relations

$$W_2^\nu(\mathbb{R}) \subseteq \mathrm{Lip}^*(\nu, L_2(\mathbb{R})) \subseteq W_2^{\nu-\varepsilon}(\mathbb{R})$$

hold true. Therefore, for a vector $f = (f_1, \dots, f_r)^T \in (L_2(\mathbb{R}))^r$, we have

$$\nu_2(f) = \sup\big\{\nu : f \in (W_2^\nu(\mathbb{R}))^r\big\}.$$

In [29] Shen obtained lower bounds for the L_2-smoothness of refinable vectors.

When $p = 2$, the joint spectral radius in (3.7) can be computed by finding the spectral radius of a certain finite matrix associated to the mask a (see [11] and [21]).

Let us review some related results from [21]. For an element $a \in (\ell_0(\mathbb{Z}))^{r\times r}$, define the *transition operator* F_a to be the linear mapping from $(\ell_0(\mathbb{Z}))^{r\times r}$ to $(\ell_0(\mathbb{Z}))^{r\times r}$ given by

$$F_a w(\alpha) := \sum_{\beta,\gamma\in\mathbb{Z}} a(2\alpha-\beta)w(\beta+\gamma)a(\gamma)^*/2, \qquad \alpha \in \mathbb{Z},\ w \in (\ell_0(\mathbb{Z}))^{r\times r}, \tag{3.8}$$

where $a(\gamma)^*$ denotes the complex conjugate transpose of $a(\gamma)$. For $n = 1, 2, \ldots$, let a_n be the sequences given by the iteration relation (2.2). It was proved [21, Lemma 7.3] that, for any $v \in (\ell_0(\mathbb{Z}))^r$,

$$\lim_{n\to\infty} \|a_n * v\|_2^{1/n} = \sqrt{2\rho(F_a|_W)}\,, \tag{3.9}$$

where W is the minimal invariant subspace of F_a generated by the element $w \in (\ell_0(\mathbb{Z}))^{r\times r}$ given by

$$w(\beta) := \sum_{\gamma\in\mathbb{Z}} v(\beta+\gamma)v(\gamma)^*, \qquad \beta \in \mathbb{Z}.$$

Let Δ denote the difference operator on $\ell_0(\mathbb{Z})$ given by

$$\Delta v := -v(\cdot - 1) + 2v - v(\cdot + 1), \qquad v \in \ell_0(\mathbb{Z}).$$

In particular, $\Delta\delta := -\delta_{-1} + 2\delta - \delta_1$. Suppose V is the minimal common invariant subspace of A_0 and A_1 generated by $e_j(\nabla^k\delta)$, $j = 1, \ldots, r$. Then Lemma 3.2 and (3.9) tell us that

$$\rho_2(A_0|_V, A_1|_V) = \sqrt{2\rho(F_a|_W)},$$

where W is the minimal invariant subspace of F_a generated by $e_j e_j^T(\Delta^k\delta)$, $j = 1, \ldots, r$.

Thus, for the case $p = 2$, Theorem 3.3 can be strengthened as follows.

THEOREM 3.4. *Suppose $\phi = (\phi_1, \ldots, \phi_r)^T \in (L_2(\mathbb{R}))^r$ is a compactly supported solution of the refinement equation* (1.1) *with mask a. Let F_a be the transition operator given in* (3.8)*. Then, for any positive integer k,*

$$\nu_2(\phi) \ge -\log_2 \sqrt{\rho(F_a|_W)},$$

where W is the minimal invariant subspace of F_a generated by $e_j e_j^T(\Delta^k\delta)$, $j = 1, \ldots, r$. Moreover, $\nu_2(\phi) = -\log_2 \sqrt{\rho(F_a|_W)}$, provided $k > -\log_2 \sqrt{\rho(F_a|_W)}$ and the shifts of $\phi_1, \ldots, \phi_r$ are stable.

In order to apply Theorems 3.3 and 3.4 to smoothness analysis, one must check the stability of refinable functions in terms of the refinement mask. For the scalar case ($r = 1$), Jia and Wang [22] gave a characterization for the stability and linear independence of the shifts of a refinable function in terms of the refinement mask. Their results were extended by Zhou [33] to the case where the scaling factor is an arbitrary integer greater than 1. For the vector case ($r > 1$), stability of the shifts of multiple refinable functions was discussed by Hervé [15], Hogan [16], and Wang [32]. Assuming the vector of refinable functions lies in $(L_2(\mathbb{R}))^r$, Shen [29] gave a characterization for L_2-stability.

4. Examples. In this section, we give two examples to illustrate the general theory.

Let A be a linear operator on a linear space V with $\{v_1, \ldots, v_s\}$ as its basis. Suppose $Av_j = \sum_{k=1}^s a_{jk}v_k$ for $1 \le j \le s$. Then the matrix $(a_{jk})_{1\le j,k\le s}$ is said to be the matrix representation of A.

The definition of joint spectral radius given in section 2 also applies to a finite multiset of square matrices of the same size. Indeed, an $s \times s$ matrix can be viewed as a linear operator on $\mathbb{C}^s$. Obviously, the p-norm joint spectral radius of a finite multiset

of linear operators is the same as that of the multiset of the matrices representing those linear operators.

Now suppose $\mathcal{A} = \{A_0, A_1\}$, where $A_0 = (\lambda)$ and $A_1 = (\mu)$ are two 1×1 matrices. For $\varepsilon_1, \ldots, \varepsilon_n \in \{0, 1\}$, we have

$$A_{\varepsilon_1} \cdots A_{\varepsilon_n} = \prod_{j=1}^{n} (\lambda^{1-\varepsilon_j} \mu^{\varepsilon_j}).$$

Hence for $1 \le p < \infty$,

$$\|\mathcal{A}^n\|_p = \big(|\lambda|^p + |\mu|^p\big)^{n/p},$$

while

$$\|\mathcal{A}^n\|_\infty = \big(\max\{|\lambda|, |\mu|\}\big)^n.$$

Therefore we obtain

$$\rho_p(A_0, A_1) = \big(|\lambda|^p + |\mu|^p\big)^{1/p}, \qquad 1 \le p \le \infty, \tag{4.1}$$

where the right-hand side of (4.1) is interpreted as $\max\{|\lambda|, |\mu|\}$ for the case $p = \infty$.

Suppose $\mathcal{A} = \{A_1, \ldots, A_m\}$ and each A_j is a block triangular matrix:

$$A_j = \begin{pmatrix} E_j & 0 \\ G_j & F_j \end{pmatrix}, \qquad j = 1, \ldots, m,$$

where $E_1, \ldots, E_m$ are square matrices of the same size, and so are $F_1, \ldots, F_m$. It was proved [21, Lemma 4.2] that

$$\rho_p(A_1, \ldots, A_m) = \max\{\rho_p(E_1, \ldots, E_m), \rho_p(F_1, \ldots, F_m)\}, \qquad 1 \le p \le \infty. \tag{4.2}$$

Let A_0 and A_1 be two triangular matrices of the same type:

$$A_0 = \begin{bmatrix} \lambda_{11} & & & \\ \lambda_{12} & \lambda_{22} & & \\ \vdots & \vdots & \ddots & \\ \lambda_{s1} & \lambda_{s2} & \cdots & \lambda_{ss} \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} \mu_{11} & & & \\ \mu_{12} & \mu_{22} & & \\ \vdots & \vdots & \ddots & \\ \mu_{s1} & \mu_{s2} & \cdots & \mu_{ss} \end{bmatrix}.$$

Then (4.1) and (4.2) tell us that

$$\rho_p(A_0, A_1) = \max_{1 \le j \le s} \big(|\lambda_{jj}|^p + |\mu_{jj}|^p\big)^{1/p}, \qquad 1 \le p \le \infty. \tag{4.3}$$

Let us analyze the following example considered by Donovann et al. [10]. Suppose a is a sequence on $\mathbb{Z}$ supported on $[0, 3]$ and

$$a(0) = \begin{bmatrix} h_1 & 1 \\ h_2 & h_3 \end{bmatrix}, \qquad a(1) = \begin{bmatrix} h_1 & 0 \\ h_4 & 1 \end{bmatrix},$$

$$a(2) = \begin{bmatrix} 0 & 0 \\ h_4 & h_3 \end{bmatrix}, \qquad a(3) = \begin{bmatrix} 0 & 0 \\ h_2 & 0 \end{bmatrix},$$

where

$$h_1 = -\frac{s^2-4s-3}{2(s+2)}, \qquad h_2 = -\frac{3(s^2-1)(s^2-3s-1)}{4(s+2)^2},$$

$$h_3 = \frac{3s^2+s-1}{2(s+2)}, \qquad h_4 = -\frac{3(s^2-1)(s^2-s+3)}{4(s+2)^2}.$$

The matrix $M := \sum_{\alpha=0}^{3} a(\alpha)/2$ has two eigenvalues, 1 and s. We assume that $|s| < 1$. The eigenvectors of M corresponding to the eigenvalue 1 are cy, where $c \neq 0$ and

$$y = \begin{bmatrix} 1 \\ (s-1)^2/(s+2) \end{bmatrix}.$$

EXAMPLE 4.1. *Let $\phi = (\phi_1, \phi_2)^T$ be the solution of the refinement equation with the mask a such that $\hat{\phi}(0) = y$. Then*

$$\nu_p(\phi) = \begin{cases} 1 + 1/p & \text{if } |s| < 2^{-1-1/p}, \\ -\log_2 |s| & \text{if } 2^{-1-1/p} \le |s| < 1. \end{cases}$$

Proof. First, we prove that the solution ϕ is continuous, provided $|s| < 1$. For this purpose, let A_ε ($\varepsilon = 0, 1$) be the linear operators on $(\ell_0(\mathbb{Z}))^2$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta \in \mathbb{Z}} a(\varepsilon + 2\alpha - \beta) v(\beta), \qquad \alpha \in \mathbb{Z},\ v \in (\ell_0(\mathbb{Z}))^2.$$

Since a is supported on $[0, 3]$, the linear space $(\ell([0,3]))^2$ is invariant under both A_0 and A_1. Choose

$$\{e_1\delta_0, e_2\delta_0, e_1\delta_1, e_2\delta_1, e_1\delta_2, e_2\delta_2, e_1\delta_3, e_2\delta_3\}$$

as a basis for $(\ell([0,3]))^2$. With respect to this basis, the matrix representations of A_0 and A_1 are $(a(2\beta-\alpha)^T)_{0\le\alpha,\beta\le 3}$ and $(a(1+2\beta-\alpha)^T)_{0\le\alpha,\beta\le 3}$, respectively. We have

$$(a(2\beta-\alpha)^T)_{0\le\alpha,\beta\le 3} = \begin{bmatrix} h_1 & h_2 & 0 & h_4 & & & & \\ 1 & h_3 & 0 & h_3 & & & & \\ & & h_1 & h_4 & 0 & h_2 & & \\ & & 0 & 1 & 0 & 0 & & \\ & & h_1 & h_2 & 0 & h_4 & & \\ & & 1 & h_3 & 0 & h_3 & & \\ & & & & h_1 & h_4 & 0 & h_2 \\ & & & & 0 & 1 & 0 & 0 \end{bmatrix}$$

and

$$(a(1+2\beta-\alpha))^T_{0\le\alpha,\beta\le 3} = \begin{bmatrix} h_1 & h_4 & 0 & h_2 & & & & \\ 0 & 1 & 0 & 0 & & & & \\ h_1 & h_2 & 0 & h_4 & & & & \\ 1 & h_3 & 0 & h_3 & & & & \\ & & h_1 & h_4 & 0 & h_2 & & \\ & & 0 & 1 & 0 & 0 & & \\ & & h_1 & h_2 & 0 & h_4 & 0 & 0 \\ & & 1 & h_3 & 0 & h_3 & 0 & 0 \end{bmatrix}.$$

We observe that s is a common eigenvalue of both A_0 and A_1. Corresponding to this eigenvalue, A_0 and A_1 have a common eigenvector v_1 given by

$$v_1 := \begin{bmatrix} 1 \\ -\mu \end{bmatrix} \delta + \begin{bmatrix} 0 \\ -\mu \end{bmatrix} \delta_1,$$

where

$$\mu := \frac{3(1-s^2)}{2(s+2)}.$$

This motivates us to choose

$$v_2 := \begin{bmatrix} 1 \\ -\mu \end{bmatrix} \delta_1 + \begin{bmatrix} 0 \\ -\mu \end{bmatrix} \delta_2.$$

It is easily verified that $A_0 v_2 = s v_2$ and $A_1 v_2 = s v_1$. From Theorem 2.1, we need to add the generators $e_j \nabla\delta$, $j = 1, 2$, which motivates us to set $v_3 := e_2 \nabla\delta$ and $v_4 := e_2 \nabla\delta_1$, i.e.,

$$v_3 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_1 \qquad \text{and} \qquad v_4 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \delta_2.$$

Denote by V the linear span of v_1, v_2, v_3, and v_4. Then $e_1 \nabla\delta = v_1 - v_2 + \mu(v_3 + v_4) \in V$ and $e_2 \nabla\delta = v_3 \in V$. Using the matrix representations of A_0 and A_1 we find that

$$A_0 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} s & & & \\ 0 & s & & \\ 1 & 0 & 0.5 & \\ 0 & -1 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

and

$$A_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} s & & & \\ s & 0 & & \\ -1 & 0 & 0.5 & \\ 1 & 0 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Thus, V is invariant under both A_0 and A_1. Applying (4.3) to the two 4×4 matrices above, we obtain

$$\rho_\infty(A_0|_V, A_1|_V) = \max\{1/2, |s|\}.$$

Thus, $\rho_\infty(A_0|_V, A_1|_V) < 1$ for $|s| < 1$. Therefore, by Theorem 2.1, the solution ϕ is continuous, provided $|s| < 1$.

We claim that the shifts of ϕ_1 and ϕ_2 are stable. For this purpose it suffices to show that the shifts of ϕ_1 and ϕ_2 are linearly independent (see [19]), that is,

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)\phi_1(\cdot - \alpha) + \sum_{\alpha\in\mathbb{Z}} c(\alpha)\phi_2(\cdot - \alpha) = 0 \quad \Longrightarrow \quad b(\alpha) = c(\alpha) = 0 \quad \forall\, \alpha \in \mathbb{Z}. \tag{4.4}$$

In order to verify (4.4), we first compute $\phi(\alpha)$ for $\alpha \in \mathbb{Z}$. Since ϕ is supported on $[0, 3]$, we have $\phi(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \{1, 2\}$. The vector $\phi = (\phi_1, \phi_2)^T$ satisfies the refinement equation

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2x - \alpha) \qquad \forall\, x \in \mathbb{R}. \tag{4.5}$$

In particular,

$$\phi(\beta) = \sum_{\alpha=1}^{2} a(2\beta-\alpha)\phi(\alpha) \qquad \text{for } \beta = 1, 2.$$

Solving the above system of linear equations, we get

$$\phi(1) = (0,t)^T \quad \text{and} \quad \phi(2) = 0,$$

where t is a nonzero constant. Moreover, it follows from (4.5) that

$$\phi(\beta+1/2) = \sum_{\alpha=1}^{2} a(2\beta+1-\alpha)\phi(\alpha) \qquad \forall\, \beta \in \mathbb{Z}.$$

Consequently, $\phi(1/2) = t(1,h_3)^T$, $\phi(3/2) = t(0,h_3)^T$, and $\phi(\beta+1/2) = 0$ for $\beta \in \mathbb{Z}\setminus\{0,1\}$. Suppose

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)\phi_1(x-\alpha) + \sum_{\alpha\in\mathbb{Z}} c(\alpha)\phi_2(x-\alpha) = 0 \qquad \forall x \in \mathbb{R}. \tag{4.6}$$

Choosing $x = \beta$ for $\beta \in \mathbb{Z}$ in (4.6), we obtain $c(\beta-1) = 0$. This is true for all $\beta \in \mathbb{Z}$. Hence (4.6) implies

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)\phi_1(x-\alpha) = 0 \qquad \forall x \in \mathbb{R}.$$

For $\beta \in \mathbb{Z}$, setting $x = \beta + 1/2$ in the above equation gives $b(\beta) = 0$. Thus, (4.4) has been verified.

We are in a position to determine the smoothness of ϕ. For the subspace W, we retain the first two generators, v_1, v_2, but replace the others by $e_j(\nabla^2\delta)$, $j = 1, 2$, as required by Theorem 3.3:

$$w_1 := \begin{bmatrix} 1 \\ -\mu \end{bmatrix}\delta + \begin{bmatrix} 0 \\ -\mu \end{bmatrix}\delta_1, \qquad w_2 := \begin{bmatrix} 1 \\ -\mu \end{bmatrix}\delta_1 + \begin{bmatrix} 0 \\ -\mu \end{bmatrix}\delta_2,$$
$$w_3 := e_2(\nabla^2\delta), \qquad \text{and} \qquad w_4 := e_1(\nabla^2\delta).$$

Using the matrix representations of A_0 and A_1, we find

$$A_0 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} s & & & \\ 0 & s & & \\ 1 & 1 & 0.5 & \\ h_1 & -h_1 & \mu/2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$$

and

$$A_1 \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} s & & & \\ s & 0 & & \\ -2 & 0 & 0 & \\ -h_1 & h_1 & \mu/2 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

Let W be the linear span of w_1, w_2, w_3, and w_4. Then W is the minimal common invariant subspace of A_0 and A_1 generated by $e_1(\nabla^2\delta)$ and $e_2(\nabla^2\delta)$. Applying (4.3) to the two 4×4 matrices above, we obtain

$$\rho_p(A_0|_W, A_1|_W) = \max\{2^{1/p}|s|, 1/2\}.$$

If $|s| < 2^{-1-1/p}$, then $\rho_p(A_0|_W, A_1|_W) = 1/2$; hence $\nu_p(\phi) = 1 + 1/p$ for $1 < p \le \infty$, by Theorem 3.3. When $p = 1$, we have $\nu_1(\phi) \ge 2$. But $\nu_1(\phi) > 2$ is impossible. Indeed, $\nu_1(\phi) > 2$ would imply $\nu_p(\phi) > 2$ for some $p > 1$, by the embedding theorem. But we know that $\nu_p(\phi) < 2$ for $1 < p \le \infty$. Therefore, $\nu_1(\phi) = 2$ for $|s| < 1/4$. If $2^{-1-1/p} \le |s| < 1$, then $\rho_p(A_0|_W, A_1|_W) = 2^{1/p}|s|$. Theorem 3.3 tells us that $\nu_p(\phi) = -\log_2 |s|$ for $1 \le p \le \infty$. Thus, for $1 \le p \le \infty$, we have found the optimal L_p-smoothness of ϕ explicitly. ∎

Using fractal interpolation, Donovan et al. [10] showed that $\phi \in \operatorname{Lip} 1$ for $|s| < 1/2$ and $\phi \in \operatorname{Lip} \nu$ for $1/2 < |s| < 1$, where $\nu = -\log_2 |s|$. In [3], Cohen, Daubechies, and Plonka established the continuity of ϕ for $|s| < 1/2$. The case $p = 1$ was considered by Micchelli and Sauer [25], who obtained $\nu_1(\phi) > 1.1087$ for $s = -0.2$. In comparison with their result, our method gives $\nu_1(\phi) = 2$ for $|s| < 1/4$.

Our second example is taken from [20, 21]. Let a be the element in $(\ell_0(\mathbb{Z}))^{2\times 2}$ supported in $[0, 2]$ given by

$$a(0) = \begin{bmatrix} \frac{1}{2} & \frac{s}{2} \\ t & \lambda \end{bmatrix}, \qquad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad \text{and} \quad a(2) = \begin{bmatrix} \frac{1}{2} & -\frac{s}{2} \\ -t & \lambda \end{bmatrix}. \tag{4.7}$$

The matrix $M := \sum_{\alpha=0}^{2} a(\alpha)/2$ has two eigenvalues: 1 and $\lambda + \mu/2$. We assume that $|2\lambda + \mu| < 2$. Then there exists a unique distributional solution $\phi = (\phi_1, \phi_2)^T$ of the refinement equation with the mask a subject to $\hat{\phi}(0) = (1, 0)^T$. The distribution ϕ_1 is symmetric about 1, and ϕ_2 is antisymmetric about 1. It was proved [20, Example 4.3] that the shifts of ϕ_1 and ϕ_2 reproduce all quadratic polynomials if and only if

$$t \ne 0, \quad \mu = 1/2, \quad \text{and} \quad \lambda = 1/4 + 2st. \tag{4.8}$$

In this case, the condition $|2\lambda + \mu| < 2$ reduces to $-3/4 < st < 1/4$.

Example 4.2. *Let a be the mask given in* (4.7) *with $-3/4 < st < 1/4$. Let $\phi = (\phi_1, \phi_2)^T$ be the solution of the refinement equation with the mask a such that $\hat{\phi}(0) = (1, 0)^T$. Suppose the conditions in* (4.8) *are satisfied. Then, for $s \ne 0$ we have*

$$\nu_p(\phi) = \begin{cases} 2 + \frac{1}{p} & \text{if } |st + 1/4| \le 2^{-3-1/p}, \\ -\log_2 |\frac{1}{2} + 2st| & \text{if } 2^{-3-1/p} < |st + 1/4| < 1/2. \end{cases}$$

In the case $s = 0$, $\mu = 1/2$, and $\lambda = 1/4$, we have $\nu_p(\phi) = 1 + 1/p$.

Proof. First, we investigate the case $s = 0$. Under the conditions in (4.8), the refinement equation

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}} a(\alpha)\phi(2x - \alpha) \qquad \forall\, x \in \mathbb{R} \tag{4.9}$$

can be solved explicitly (see [20, Example 4.3]). The solution $\phi = (\phi_1, \phi_2)^T$ of (4.9) subject to $\hat{\phi}(0) = (1, 0)^T$ is given by

$$\phi_1(x) = \begin{cases} x & \text{for } 0 \le x < 1, \\ 2 - x & \text{for } 1 \le x \le 2, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 4tx(1 - x) & \text{for } 0 \le x < 1, \\ -4t(2 - x)(x - 1) & \text{for } 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, $\nu_p(\phi) = 1+1/p$ for $1 \le p \le \infty$. In this case, the shifts of ϕ_2 are linearly *dependent*.

Second, we consider the case $s \neq 0$. Under the conditions in (4.8), the solution $\phi = (\phi_1, \phi_2)^T$ is continuous, provided $-3/4 < st < 1/4$ (see [21, Example 6.3]). In this case, we claim that the shifts of ϕ_1 and ϕ_2 are linearly independent. To justify our claim, we find $\phi(\alpha)$ for $\alpha \in \mathbb{Z}$. Since ϕ is supported on $[0, 2]$, we have $\phi(\alpha) = 0$ for $\alpha \in \mathbb{Z} \setminus \{1\}$. From [20, Example 3.2] we see that $\sum_{\alpha\in\mathbb{Z}} \phi_1(\alpha) = 1$. Hence $\phi_1(1) = 1$. Moreover, it follows from (4.9) that $\phi(1) = a(1)\phi(1)$, which implies $\phi_2(1) = 0$. Next, we find $\phi(\beta + 1/2)$ for $\beta \in \mathbb{Z}$. Using the refinement equation (4.9), we obtain

$$\phi(\beta + 1/2) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\beta + 1 - \alpha) = a(2\beta)[1, 0]^T.$$

Therefore, $\phi(1/2) = [1/2, t]^T$, $\phi(3/2) = [1/2, -t]^T$, and $\phi(\beta+1/2) = 0\ \forall\beta \in \mathbb{Z}\setminus\{0, 1\}$. Furthermore, we can use (4.9) to find $\phi(\gamma + 1/4)\ \forall\gamma \in \mathbb{Z}$. As a result, we obtain $\phi(\gamma + 1/4) = 0\ \forall\gamma \in \mathbb{Z} \setminus \{0, 1\}$,

$$\phi(1/4) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(1/2 - \alpha) = a(0)\phi(1/2),$$

and

$$\phi(5/4) = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(5/2 - \alpha) = a(1)\phi(3/2) + a(2)\phi(1/2).$$

Suppose

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)\phi_1(x - \alpha) + \sum_{\alpha\in\mathbb{Z}} c(\alpha)\phi_2(x - \alpha) = 0 \qquad \forall x \in \mathbb{R}. \tag{4.10}$$

Choosing $x = \beta$ for $\beta \in \mathbb{Z}$ in the above equation, we obtain $b(\beta - 1) = 0$. This is true for all $\beta \in \mathbb{Z}$. Hence (4.10) implies

$$\sum_{\alpha\in\mathbb{Z}} c(\alpha)\phi_2(x - \alpha) = 0 \qquad \forall x \in \mathbb{R}. \tag{4.11}$$

For $\beta \in \mathbb{Z}$, setting $x = \beta + 1/2$ in the above equation gives

$$t\,[c(\beta) - c(\beta - 1)] = 0.$$

Since $t \neq 0$, we have $c(\beta) = c(\beta - 1)\ \forall\beta \in \mathbb{Z}$. Setting $x = 5/4$ in (4.11), we get

$$c(0)[\phi_2(1/4) + \phi_2(5/4)] = 0, \quad \text{i.e.}, \quad c(0)(2\lambda - \mu)t = 0.$$

But $(2\lambda - \mu)t = 4st^2 \neq 0$; hence $c(0) = 0$. So $c(\beta) = c(0) = 0$ for all $\beta \in \mathbb{Z}$. This justifies our claim that the shifts of ϕ_1 and ϕ_2 are linearly independent.

We are in a position to determine the smoothness of ϕ. Let A_ε $(\varepsilon = 0, 1)$ be the linear operators on $(\ell_0(\mathbb{Z}))^2$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon + 2\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z},\ v \in (\ell_0(\mathbb{Z}))^2.$$

We observe that $1/2 + 2st$ is a common eigenvalue of both A_0 and A_1. Corresponding to this eigenvalue, A_0 and A_1 have a common eigenvector v_1 given by

$$v_1 := \begin{bmatrix} 1 \\ 4t \end{bmatrix} \delta + \begin{bmatrix} -1 \\ 4t \end{bmatrix} \delta_1.$$

This motivates us to choose

$$v_2 := \begin{bmatrix} 1 \\ 4t \end{bmatrix} \delta_1 + \begin{bmatrix} -1 \\ 4t \end{bmatrix} \delta_2.$$

Then we have $A_0 v_2 = (1/2 + 2st)v_2$ and $A_1 v_2 = (1/2 + 2st)v_1$. We also must add $e_j(\nabla^3\delta)$, $j = 1, 2$, by Theorem 3.3 and from the action of A_0, A_1 on those vectors, we are led to our choice of v_3, v_4, v_5:

$$v_3 := e_2(\nabla^2\delta), \qquad v_4 := e_1(\nabla^3\delta), \qquad \text{and} \qquad v_5 := e_2(\nabla^3\delta).$$

By computation we find that

$$A_0 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1/2+2st & & & & \\ 0 & 1/2+2st & & & \\ s/2 & s/2 & 1/4 & & \\ 1/2 & -1/2 & -t & 0 & \\ s/2 & 3s/2 & 1/4 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}$$

and

$$A_1 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1/2+2st & & & & \\ 1/2+2st & 0 & & & \\ -s & 0 & 0 & & \\ -1/2 & 1/2 & -t & 0 & \\ -3s/2 & -s/2 & -1/4 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}.$$

Let V be the linear span of v_j, $j = 1, \ldots, 5$. Then V is the minimal common invariant subspace of A_0 and A_1 generated by $e_1(\nabla^3\delta)$ and $e_2(\nabla^3\delta)$. Applying (4.3) to the two 5×5 matrices above, we obtain

$$\rho_p(A_0|_V, A_1|_V) = \max\{2^{1/p}|1/2 + 2st|, 1/4\}.$$

If $|1/2 + 2st| < 2^{-2-1/p}$, then $\rho_p(A_0|_V, A_1|_V) = 1/4$; hence $\nu_p(\phi) = 2 + 1/p$ for $1 < p \le \infty$, by Theorem 3.3. When $p = 1$, we have $\nu_1(\phi) \ge 3$. But $\nu_1(\phi) > 3$ is impossible. Indeed, $\nu_1(\phi) > 3$ would imply $\nu_p(\phi) > 3$ for some $p > 1$, by the embedding theorem. But we know that $\nu_p(\phi) < 3$ for $1 < p \le \infty$. Therefore, $\nu_1(\phi) = 3$ for $|1/2 + 2st| < 2^{-2-1/p}$. If $2^{-2-1/p} \le |1/2 + 2st| < 1$, then $\rho_p(A_0|_V, A_1|_V) = 2^{1/p}|1/2 + 2st|$. Theorem 3.3 tells us that $\nu_p(\phi) = -\log_2 |1/2 + 2st|$ for $1 \le p \le \infty$. Thus, for $1 \le p \le \infty$, we have found the optimal L_p-smoothness of ϕ explicitly. ∎

The special case $s = 3/2$, $t = -1/8$, $\lambda = -1/8$, and $\mu = 1/2$ was discussed by Heil, Strang, and Strela [14]. In this case, ϕ can be solved explicitly as follows:

$$\phi_1(x) = \begin{cases} x^2(-2x+3) & \text{for } 0 \le x \le 1, \\ (2-x)^2(2x-1) & \text{for } 1 < x \le 2, \\ 0 & \text{for } x \in \mathbb{R} \setminus [0,2] \end{cases}$$

and

$$\phi_2(x) = \begin{cases} x^2(x-1) & \text{for } 0 \le x \le 1, \\ (2-x)^2(x-1) & \text{for } 1 < x \le 2, \\ 0 & \text{for } x \in \mathbb{R} \setminus [0,2]. \end{cases}$$

It is evident that $\nu_p(\phi) = 2 + 1/p$.

5. Multiple wavelets. In this section we apply the general theory to smoothness analysis of orthogonal multiple wavelets.

In Example 4.1, if $\phi = (\phi_1, \phi_2)^T$ is the solution of the refinement equation corresponding to the parameter $s = -0.2$, then the shifts of ϕ_1 and ϕ_2 are orthogonal. It was shown in the last section that the optimal smoothness of ϕ is $\nu_p(\phi) = 1 + 1/p$. This example of continuous symmetric orthogonal double refinable functions was first constructed by Donovan et al. [10] by means of fractal interpolation. On the basis of their work, Strang and Strela constructed symmetric orthogonal double wavelets in [31].

In this section we shall use refinement equations to study multiple wavelets. Let a be an element in $(\ell_0(\mathbb{Z}))^{r\times r}$ such that the matrix $M = \sum_{\alpha\in\mathbb{Z}} a(\alpha)/2$ has the following form:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda \end{bmatrix} \qquad \text{and} \qquad \lim_{n\to\infty} \Lambda^n = 0.$$

There exists a unique solution ϕ of the refinement equation

$$\phi = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\cdot - \alpha)$$

such that $\hat{\phi}(0) = (1, 0, \ldots, 0)^T$. This solution is called the *normalized solution*.

The following theorem summarizes the general theory on orthogonal multiple wavelets (see, e.g., [21]). Some different forms of this result were obtained by Long, Chen, and Yuan [24] and Shen [29].

THEOREM 5.1. *Let $\phi = (\phi_1, \ldots, \phi_r)^T$ be the normalized solution of the refinement equation with mask a. Then $\{\phi_j(\cdot - \alpha) : j = 1, \ldots, r;\ \alpha \in \mathbb{Z}\}$ forms an orthonormal system in $L_2(\mathbb{R})$ if and only if*

a. *$\sum_{\alpha\in\mathbb{Z}} a(\alpha)a(\alpha + 2\gamma)^* = 2\delta_{\gamma,0} I_r$ $\forall\gamma \in \mathbb{Z}$, where I_r denotes the $r \times r$ identity matrix, and*

b. *$\rho(F_a|_W) < 1$ where F_a is the linear operator on $(\ell_0(\mathbb{Z}))^{r\times r}$ given by (3.8) and W is the minimal invariant subspace of F_a generated by $e_1 e_1^T(\Delta\delta)$, $e_2 e_2^T\delta, \ldots, e_r e_r^T\delta$.*

Furthermore, if ψ is given by $\psi = (\psi_1, \ldots, \psi_r)^T = \sum_{\alpha\in\mathbb{Z}} b(\alpha)\phi(2\cdot - \alpha)$, where b is a sequence in $(\ell_0(\mathbb{Z}))^{s\times s}$ satisfying

$$\sum_{\alpha\in\mathbb{Z}} a(\alpha)b(\alpha + 2\gamma)^* = 0 \qquad \forall\gamma \in \mathbb{Z}$$

and

$$\sum_{\alpha\in\mathbb{Z}} b(\alpha)b(\alpha + 2\gamma)^* = 2\delta_{\gamma,0} I_r \qquad \forall\gamma \in \mathbb{Z},$$

then $\{\sqrt{2}^k \psi_j(2^k\cdot - \alpha) : j = 1, \ldots, r;\ k, \alpha \in \mathbb{Z}\}$ forms an orthonormal basis for $L_2(\mathbb{R})$. In other words, $\psi_1, \ldots, \psi_r$ are orthogonal multiple wavelets.

In [21] we constructed a class of continuous orthogonal double wavelets with symmetry. In our construction the mask a is supported on $[0, 2]$ and

$$a(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ t & t \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2 - 4t^2} \end{bmatrix}, \quad \text{and} \quad a(2) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -t & t \end{bmatrix},$$

where the parameter t is in the range $-1/\sqrt{2} \le t < -1/2$. Let $\phi = (\phi_1, \phi_2)^T$ be the normalized solution of the refinement equation with mask a. Then ϕ_1 is symmetric

about 1, ϕ_2 is antisymmetric about 1, and $\{\phi_j(\cdot-\alpha) : j=1,2,\ \alpha\in\mathbb{Z}\}$ forms an orthonormal system in $L_2(\mathbb{R})$. Moreover, for the coefficients

$$b(0) := \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{\mu}{2} & \frac{\mu}{2} \end{bmatrix}, \quad b(1) := \begin{bmatrix} 1 & 0 \\ 0 & -2t \end{bmatrix}, \quad \text{and} \quad b(2) := \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{\mu}{2} & \frac{\mu}{2} \end{bmatrix},$$

where $\mu=\sqrt{2-4t^2}$, the vector

$$\psi = (\psi_1,\psi_2)^T := \sum_{\alpha\in\mathbb{Z}} b(\alpha)\phi(2\cdot-\alpha)$$

gives orthogonal double wavelets ψ_1 and ψ_2 which are continuous. Furthermore, ψ_1 is symmetric about 1, and ψ_2 is antisymmetric about 1.

The special case $t=-\sqrt{7}/4$ was studied by Chui and Lian [2]. The following example gives a detailed analysis for the smoothness of the corresponding double-refinable functions.

EXAMPLE 5.2. *Let $\phi=(\phi_1,\phi_2)^T$ be the normalized solution of the refinement equation with mask a, where a is supported on $[0,2]$ and*

$$a(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ t & t \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad a(2) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -t & t \end{bmatrix}$$

with $t=-\sqrt{7}/4$. Then

$$\nu_p(\phi) = \frac{1}{p} - \frac{1}{p}\log_2\left[\left(\frac{\sqrt{7}}{4}\right)^p + \left(\frac{\sqrt{7}-2}{4}\right)^p\right], \qquad 1\le p\le\infty. \tag{5.1}$$

Proof. Denote by A_ε $(\varepsilon=0,1)$ the linear operators on $(\ell_0(\mathbb{Z}))^2$ given by

$$A_\varepsilon v(\alpha) = \sum_{\beta\in\mathbb{Z}} a(\varepsilon+2\alpha-\beta)v(\beta), \qquad \alpha\in\mathbb{Z},\ v\in(\ell_0(\mathbb{Z}))^2.$$

Set

$$v_1 := \begin{bmatrix} 1 \\ 2t \end{bmatrix}\delta + \begin{bmatrix} -1 \\ 2t-1 \end{bmatrix}\delta_1, \qquad v_2 := \begin{bmatrix} 1 \\ 2t-1 \end{bmatrix}\delta + \begin{bmatrix} -1 \\ 2t \end{bmatrix}\delta_1,$$
$$v_3 := \begin{bmatrix} 1 \\ 2t \end{bmatrix}\delta_1 + \begin{bmatrix} -1 \\ 2t-1 \end{bmatrix}\delta_2, \qquad \text{and} \qquad v_4 := \begin{bmatrix} 1 \\ 2t-1 \end{bmatrix}\delta_1 + \begin{bmatrix} -1 \\ 2t \end{bmatrix}\delta_2.$$

By computation we obtain

$$A_0\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} t+\frac{1}{2} & 0 & & \\ t & 0 & & \\ & & 0 & t \\ & & 0 & t+\frac{1}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

and

$$A_1\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 & t & & \\ 0 & t+\frac{1}{2} & & \\ t+\frac{1}{2} & 0 & 0 & 0 \\ t & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

Let V be the linear span of v_1 and v_2, and let W be the linear span of v_j, $j = 1, \ldots, 4$. Then V and W are invariant under both A_0 and A_1. It is easily seen that W is the minimal common invariant subspace of A_0 and A_1 generated by $e_1(\nabla^2\delta)$ and $e_2(\nabla^2\delta)$. By the remark made at the beginning of section 4 we see that

$$\rho_p(A_0|_W, A_1|_W) = \max\bigl\{\rho_p(A_0|_V, A_1|_V), |t + 1/2|\bigr\}.$$

It remains to compute $\rho_p(A_0|_V, A_1|_V)$. For this purpose, we choose the norm on V as follows:

$$\|\xi_1 v_1 + \xi_2 v_2\| := \max\{|\xi_1|, |\xi_2|\} \qquad \text{for } \xi_1, \xi_2 \in \mathbb{C}.$$

In particular, $\|v_1\| = \|v_2\| = 1$. Suppose $1 \le p < \infty$. Then

$$\|\mathcal{A}^n v_1\|_p^p = \sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{0,1\}} \|A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|^p.$$

Write s for $t + 1/2$. We claim that

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{0,1\}} \|A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|^p = (|s|^p + |t|^p)^n. \tag{5.2}$$

This will be proved by induction on n. For $n = 1$, we have $A_0 v_1 = s v_1$ and $A_1 v_1 = t v_2$. Hence

$$\|A_0 v_1\|^p + \|A_1 v_1\|^p = |s|^p + |t|^p.$$

This verifies (5.2) for $n = 1$. Suppose (5.2) is valid for n. We wish to establish it for $n + 1$. Recall that $A_0 v_2 = t v_1$ and $A_1 v_2 = s v_2$. Thus, either $A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1 = \xi v_1$ for some $\xi \in \mathbb{C}$, or $A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1 = \eta v_2$ for some $\eta \in \mathbb{C}$. In the former case, we have

$$A_0 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1 = s\xi v_1 \qquad \text{and} \qquad A_1 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1 = t\xi v_2.$$

It follows that

$$\|A_0 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|^p + \|A_1 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|^p = (|s|^p + |t|^p)|\xi|^p.$$

But $|\xi| = \|A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|$. Therefore, by the induction hypothesis, we obtain

$$\sum_{\varepsilon_1,\ldots,\varepsilon_n \in \{0,1\}} \bigl[\|A_0 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|^p + \|A_1 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1\|^p\bigr] = (|s|^p + |t|^p)^{n+1}. \tag{5.3}$$

In the latter case, we have

$$A_0 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1 = t\eta v_1 \qquad \text{and} \qquad A_1 A_{\varepsilon_n} \cdots A_{\varepsilon_1} v_1 = s\eta v_2.$$

Hence (5.3) is also valid. This completes the induction procedure.

Finally, we derive from (5.2) that

$$\rho_p(A_0|_V, A_1|_V) = \lim_{n\to\infty} \|\mathcal{A}^n v_1\|_p^{1/n} = \bigl(|s|^p + |t|^p\bigr)^{1/p}, \qquad 1 \le p < \infty.$$

For the case $p = \infty$, a similar argument gives

$$\rho_\infty(A_0|_V, A_1|_V) = \max\{|s|, |t|\}.$$

Note that $t = -\sqrt{7}/4$ and $s = t + 1/2 = (2-\sqrt{7})/4$. By Theorem 3.3, we obtain the desired result (5.1). In particular,

$$\nu_\infty(\phi) = -\log_2 \frac{\sqrt{7}}{4} \approx 0.59632,$$

$$\nu_2(\phi) = \frac{1}{2} - \frac{1}{2}\log_2 \frac{9-2\sqrt{7}}{8} \approx 1.05458,$$

and

$$\nu_1(\phi) = 1 - \log_2 \frac{\sqrt{7}-1}{2} \approx 1.28125. \qquad \square$$

We now return to the general mask and discuss the L_2-smoothness of the normalized solution ϕ.

EXAMPLE 5.3. *Let a be the mask*

$$a(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ t & t \end{bmatrix}, \quad a(1) = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad \text{and} \quad a(2) = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -t & t \end{bmatrix},$$

where $\mu := \sqrt{2-4t^2}$ and the parameter t is in the range $-1/\sqrt{2} \le t < -1/2$. Then

$$\nu_2(\phi) = -\log_2 \sqrt{\rho(B(t))}, \qquad t \ne -\sqrt{7}/4,$$

where $B(t)$ is the matrix

$$B(t) := \begin{bmatrix} 1/4 & t^2 & 0 & 0 \\ 1/4 & t\mu - t^2 & 1 & 0 \\ \frac{2t+2-\mu-4t\mu}{8} & -\frac{2t^2\mu - t\mu^2 + 4t^2 - 8t^3\mu}{4} & 1 & \frac{4t^2-1-8t^2\mu+2\mu}{8} \\ 1/2 & -\mu t & 0 & \frac{\mu+2t}{4} \end{bmatrix}.$$

Proof. To apply Theorem 3.4 with $k = 1$, we need to determine the minimal invariant subspace W of F_a generated by $e_1 e_1^T(\Delta\delta) =: w_1$ and $e_2 e_2^T(\Delta\delta) =: w_2$. From a computation, we find that W can be described as $W := \operatorname{span}\{w_1, w_2, w_3, w_4\}$ using the additional matrices

$$w_3 := \begin{bmatrix} 0 & (t/2-\mu/4)(\delta_{-1}-\delta_1) \\ -(t/2-\mu/4)(\delta_{-1}-\delta_1) & (2-4t\mu)\Delta\delta \end{bmatrix}$$

and

$$w_4 := \begin{bmatrix} 0 & \delta_{-1}-\delta_1 \\ \delta_1 - \delta_{-1} & 0 \end{bmatrix}.$$

Furthermore, $F_a|_W$ has the matrix representation given by

$$F_a \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = B(t) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}.$$

One of the eigenvalues of $B(t)$ is zero. The absolute value of the dominant eigenvalue has minimum $1/4$ precisely at $t = -\sqrt{7}/4$. Hence when $t \ne -\sqrt{7}/4$, $-\log_2 \sqrt{\rho(F_a|_W)} < 1$ and $\nu_2(\phi) = -\log_2 \sqrt{\rho(B(t))}$ by Theorem 3.4. $\square$

When $t = -\sqrt{7}/4$, the matrix B takes the form

$$B\left(-\frac{\sqrt{7}}{4}\right) = \begin{bmatrix} 1/4 & 7/16 & 0 & 0 \\ 1/4 & -\frac{2\sqrt{7}+7}{16} & 1 & 0 \\ 3/16 & -\frac{35+8\sqrt{7}}{64} & 1 & 0 \\ 1/2 & \frac{\sqrt{7}}{8} & 0 & \frac{1-\sqrt{7}}{8} \end{bmatrix}.$$

It is evident that in this case, the minimal invariant subspace W of F_a generated by $e_1 e_1^T(\Delta\delta) = w_1$ and $e_2 e_2^T(\Delta\delta) = w_2$ is span $\{w_1, w_2, w_3\}$. Moreover, the eigenvalues of the matrix representation of F_a on W are 0, $(9-2\sqrt{7})/16$, and $1/4$. From Example 5.2, we know that $\nu_2(\phi) = -\log_2 \sqrt{(9-2\sqrt{7})/16}$ for $t = -\sqrt{7}/4$; hence, this case provides an example to show that the condition $k > -\log_2 \sqrt{\rho(F_a|_W)}$ is necessary for the equality in Theorem 3.4.

We have seen above that the choice $t = -\sqrt{7}/4$ of the parameter t yields the best smoothness in $L_2(\mathbb{R})$ among all the orthonormal wavelets generated by the masks in Example 5.3. That same choice of the parameter also is the only one for which the resulting vector of functions $\phi = (\phi_1, \phi_2)^T$ achieves accuracy 2. However, if we measure the smoothness in L_∞, then the choice $t = -\sqrt{7}/4$ no longer provides optimal smoothness. Recall from [21] that the operators A_0 and A_1, when restricted to the subspace V generated by

$$\big\{2e_2\delta, \quad 2e_2\delta_1, \quad -e_1\nabla\delta + e_2\delta + e_2\delta_1\big\},$$

have the matrix representation

$$A_0|_V := \begin{bmatrix} \frac{1}{2}+t & \frac{1}{2}+t & -1 \\ 0 & \sqrt{2-4t^2} & 0 \\ 0 & \frac{2t+\sqrt{2-4t^2}}{2} & 0 \end{bmatrix} \quad \text{and} \quad A_1|_V := \begin{bmatrix} \sqrt{2-4t^2} & 0 & 0 \\ \frac{1}{2}+t & \frac{1}{2}+t & -1 \\ \frac{2t+\sqrt{2-4t^2}}{2} & 0 & 0 \end{bmatrix}.$$

Clearly, V contains the subspace generated by $e_1\nabla\delta$, $e_2\nabla\delta$.

A lower bound for the joint spectral radius $\rho_\infty(A_0|_V, A_1|_V)$ is given by the maximal spectral radius of the square root of the $\rho_\infty(A_{\varepsilon_1}|_V A_{\varepsilon_2}|_V)$, $\varepsilon_1, \varepsilon_2 \in \{0,1\}$. Now

$$f_1(t)^2 := \rho_\infty(A_0|_V^2) = \rho_\infty(A_1|_V^2) = \max\big(|t+1/2|, \mu\big)^2,$$

and

$$\begin{aligned} f_2(t)^2 &:= \rho_\infty(A_0|_V A_1|_V) = \rho_\infty(A_1|_V A_0|_V) \\ &= \max\big(0, |\eta \pm \sqrt{\eta^2 - 64t^2(\mu-1)^2}|/8\big), \end{aligned}$$

where $\eta = 8\mu t + 1 + 4t^2 - 4t$. In the interval $-1/\sqrt{2} \le t < -1/2$, the function f_1 is increasing, the function f_2 is decreasing, and they are equal in the point $t \approx -0.64268764$ at which the minimum value, $.5897545\ldots$, of $g := \max(f_1, f_2)$ is achieved. Perhaps surprisingly, the lower bound g for the joint spectral radius is exact for the point $t = -\sqrt{7}/4$. This suggests that the value $t \approx -0.64268764$ should give rise to ϕ which is smoother when measured in the L_∞ norm. This is indeed the case as a numerical computation shows that

$$\|\mathcal{A}^n\|^{\frac{1}{n}} < .6064 < \sqrt{7}/4 \quad \text{for} \quad t = -0.64268764 \quad \text{and} \quad n = 28.$$

Thus,

$$\nu_\infty(\phi) \ge -\log_2(.6064) = 0.721658\ldots \quad \text{for} \quad t = -0.64268764.$$

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