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FACETS OF THE WEAK ORDER POLYTOPE DERIVED FROM THE INDUCED PARTITION PROJECTION*

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Abstract. The weak order polytopes are studied in Gurgel and Wakabayashi [*Discrete Math.,* 175 (1997), pp. 163–172], Gurgel and Wakabayashi [*The Complete Pre-Order Polytope: Facets and Separation Problem,* manuscript, 1996], and Fiorini and Fishburn [*Weak order polytopes,* submitted]. We make use of their natural, affine projection onto the partition polytopes to determine several new families of facets for them. It turns out that not all facets of partition polytopes are lifted into facets of weak order polytopes. We settle the cases of all facet-defining inequalities established for partition polytopes by Grötschel and Wakabayashi [*Math. Programming,* 47 (1990), pp. 367–387]. Our method, although rather simple, allows us to establish general families of facets which contain two particular cases previously requiring long proofs.

Key words. weak order polytope, partition polytope, facet of convex polytope

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1. Introduction. In order to solve real-life problems which require finding an optimal linear ordering, Grötschel, Jünger, and Reinelt [8] have studied the facial structure of the "linear ordering polytope." They found several facets and used these results in the so-called branch-and-cut technique which combines branch-and-bound with cutting planes techniques. The same polytope had appeared before under the name "binary choice polytope" in the theory of probabilistic utility; see the references in Fishburn [7]. Recently, several papers were devoted to the geometry of this polytope and closely related ones; see Fiorini and Fishburn [6] or Fiorini [5] and their bibliographies.

For a general definition, let $\mathcal{F}$ be a family of reflexive relations on the set $\mathsf{n} = \{1, 2, \ldots, n\}$. Each element R of $\mathcal{F}$ is encoded by its characteristic vector x^R, which has a coordinate $x^R_{(i,j)}$ for each pair (i,j) of distinct elements in n; this coordinate equals 1 when iRj, and 0 otherwise. A subset of vertices of the unit cube in $\mathbb{R}^{n(n-1)}$ is thus associated to $\mathcal{F}$. The study of its convex hull, called the *$\mathcal{F}$-polytope*, includes, for instance, the determination of (many) facets, or, more ambitiously, of the full combinatorial structure. Several particular cases have been investigated. In the previous paragraph, $\mathcal{F}$ is the family of all linear orderings on n. (For recent references, see, e.g., [3] and [4].) When $\mathcal{F}$ is the set of all equivalence relations on n, the $\mathcal{F}$-polytope is also called the *partition polytope* $\mathrm{P}^n_{\mathrm{PA}}$, or the *clique-partitioning polytope*; see Grötschel and Wakabayashi [9]. Here we focus on the *weak order polytope* $\mathrm{P}^n_{\mathrm{WO}}$. A *weak order* on n is a reflexive, transitive, and complete relation on n. Such a weak order W is a ranking of the elements of n with ties allowed, typically

$$C_1 \prec C_2 \prec \cdots \prec C_k, \tag{1.1}$$

where $\{C_1, C_2, \ldots, C_k\}$ is the partition of n corresponding to the equivalence relation $W \cap W^{-1}$. The mapping $W \mapsto W \cap W^{-1}$ leads to the "induced partition projection"

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from the weak order polytope $\mathrm{P}^n_{\mathrm{WO}}$ onto the partition polytope $\mathrm{P}^n_{\mathrm{PA}}$, which we will exploit to produce new facets of weak order polytopes.

Adjacency of vertices in the weak order polytope $\mathrm{P}^n_{\mathrm{WO}}$ is studied by Gurgel and Wakabayashi [12], while facets are determined by Gurgel and Wakabayashi [11] (see also Gurgel [10]). The results of these two papers are extended by Fiorini and Fishburn [6], who also spell out some motivation for the study of $\mathrm{P}^n_{\mathrm{WO}}$. By "lifting" known facets from $\mathrm{P}^n_{\mathrm{PA}}$ (that is, taking in $\mathrm{P}^n_{\mathrm{WO}}$ preimages of facets of $\mathrm{P}^n_{\mathrm{PA}}$), we derive several new families of facets of $\mathrm{P}^n_{\mathrm{WO}}$. For instance, two isolated examples of facets are given by Gurgel and Wakabayashi [11], the 20-page proof for one of them being omitted (see, however, Gurgel [10]); they are the first two instances of a general family of facets which we establish easily along our approach.

It should be stressed that not any facet of $\mathrm{P}^n_{\mathrm{PA}}$ has a preimage which is a facet of $\mathrm{P}^n_{\mathrm{WO}}$. Thus a case by case analysis is required, whose outcome is reported here for all facet-defining inequalities established for $\mathrm{P}^n_{\mathrm{PA}}$ by Grötschel and Wakabayashi [9]. (Additional facets of $\mathrm{P}^n_{\mathrm{PA}}$ were recently provided in [1]; they are reserved for later work.) In summary, we derive facets of $\mathrm{P}^n_{\mathrm{WO}}$ from trivial, 2-chorded even wheel, and 2-chorded path facets of $\mathrm{P}^n_{\mathrm{PA}}$, and also from 2-partition facets, except for the "smallest" one. On the other hand, triangle and 2-chorded cycle facets are lifted into ridges; we describe the two facets of $\mathrm{P}^n_{\mathrm{WO}}$ containing such a ridge. Thus we show how a natural relation between the two polytopes can be useful in their study.

2. Permutation subspaces. Assume $n \geq 2$ throughout the paper. The weak order polytope $\mathrm{P}^n_{\mathrm{WO}}$ lies by definition in the real affine space $\mathbb{R}^{n(n-1)}$ and has one vertex x^W for each weak order W on $\mathsf{n} = \{1, 2, \ldots, n\}$. Its dimension equals $n(n-1)$. On the other hand, as equivalence relations are symmetric, it is simpler to see the partition polytope $\mathrm{P}^n_{\mathrm{PA}}$ in the real affine space $\mathbb{R}^{\binom{n}{2}}$ in which any point has one coordinate $y_{\{i,j\}}$ for each unordered pair $\{i, j\}$ in n. In fact, $\mathrm{P}^n_{\mathrm{PA}}$ is of dimension $\binom{n}{2}$. Its vertex corresponding to the equivalence relation R will be denoted as y^R.

The mapping $W \mapsto E = W \cap W^{-1}$, where W is a weak order on n and E is the equivalence relation of W, provides a mapping from $\operatorname{vert} \mathrm{P}^n_{\mathrm{WO}}$ (the set of all vertices of the weak order polytope) onto $\operatorname{vert} \mathrm{P}^n_{\mathrm{PA}}$: in the same notation, x^W is mapped onto y^E. The latter mapping extends to the affine projection (i.e., surjective, affine mapping)

$$\pi : \mathbb{R}^{n(n-1)} \to \mathbb{R}^{\binom{n}{2}} : x \mapsto y \quad \text{with} \quad y_{\{i,j\}} = x_{(i,j)} + x_{(j,i)} - 1, \tag{2.1}$$

called the *induced partition projection*. As $\pi(\mathrm{P}^n_{\mathrm{WO}}) = \mathrm{P}^n_{\mathrm{PA}}$, the linear mapping π induces an inclusion preserving correspondence from the face lattice of $\mathrm{P}^n_{\mathrm{PA}}$ to the face lattice of $\mathrm{P}^n_{\mathrm{WO}}$. If $f(y) \leq b$ is a linear inequality valid for $\mathrm{P}^n_{\mathrm{PA}}$ which defines the face F, then $\dot\pi^{-1}(F) = \mathrm{P}^n_{\mathrm{WO}} \cap \pi^{-1}(F)$ is the face of $\mathrm{P}^n_{\mathrm{WO}}$ defined by the inequality $(f \circ \pi)(x) \leq b$ (which is valid for $\mathrm{P}^n_{\mathrm{WO}}$). We say that the latter inequality is *lifted* from inequality $f(y) \leq b$ and also that the preimage $\dot\pi^{-1}(F)$ is *lifted* from F; here $\dot\pi$ denotes the restriction of π to $\mathrm{P}^n_{\mathrm{WO}}$.

In particular, when the face F consists of a single vertex y^E, its preimage $\dot\pi^{-1}(y^E)$ is a face of $\mathrm{P}^n_{\mathrm{WO}}$ which is affinely equivalent to the linear ordering polytope $\mathrm{P}^k_{\mathrm{LO}}$, with k the number of equivalent classes of E; indeed, vertices of $\dot\pi^{-1}(y^E)$ correspond to all weak orders which linearly order the k classes of E.

The *(affine) dimension* of a set S of points in $\mathbb{R}^{n(n-1)}$ is the dimension of the affine subspace $\operatorname{aff} S$ it spans and is denoted as $\dim S$. Now choose any point o in the affine space $\mathbb{R}^{n(n-1)}$ as an origin, thus transforming $\mathbb{R}^{n(n-1)}$ into a vector space $\mathbb{R}^{n(n-1)}_o$. (This point does not need to be $(0, 0, \ldots, 0)$.) The *rank* $\operatorname{rk} S$ of a set S of points in $\mathbb{R}_o^{n(n-1)}$ is the rank of the vector subspace it generates.

Given a vertex y^E of $\mathrm{P}^n_{\mathrm{PA}}$, the *permutation subspace* $\operatorname{permsub}(y^E)$ is the (unique) vector subspace of $\mathbb{R}_o^{n(n-1)}$ that forms a translate of $\operatorname{aff} \dot{\pi}^{-1}(y^E)$. Equivalently, $\operatorname{permsub}(y^E)$ consists of all linear combinations of vectors $\overrightarrow{op}$ with $p \in \dot{\pi}^{-1}(y^E)$, whose coefficients sum up to 0. Although $\dim \pi^{-1}(y^E) = \binom{n}{2}$, the face $\dot{\pi}^{-1}(y^E)$ can have a lower affine dimension; the latter equals $\operatorname{rk} \operatorname{permsub}(y^E)$. A basis for $\operatorname{permsub}(y^E)$ can be easily found, as we now indicate. The trick is to consider the difference between two vertices x^W, $x^{W'}$ of $\dot{\pi}^{-1}(y^E)$ with W and W', disagreeing only in the transposition of two consecutive classes. (The argument is classic for the linear ordering polytope.)

For two nonempty, disjoint subsets U and V of n, define in $\mathbb{R}_o^{n(n-1)}$ the *transposition vector* $\operatorname{transp}(U, V)$, located at o, by specifying as follows its components in the canonical base (also located at o):

$$(\operatorname{transp}(U,V))_{(i,j)} = \begin{cases} 1 & \text{if } i \in U \text{ and } j \in V, \\ -1 & \text{if } i \in V \text{ and } j \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The set of all transposition vectors $\operatorname{transp}(U, V)$, for U, V distinct classes of the equivalence relation E, span $\operatorname{permsub}(y^E)$ but is linearly dependent (since $\operatorname{transp}(U, V) = -\operatorname{transp}(V, U)$). A basis of $\operatorname{permsub}(y^E)$ is formed by selecting one of the two transposition vectors $\operatorname{transp}(U, V)$ and $\operatorname{transp}(V, U)$ for any unordered pair $\{U, V\}$ of classes of E. Such vectors $\operatorname{transp}(U, V)$, with U and V two classes of E, are also called *transposition vectors* of the vertex y^E. When $U = \{u\}$, $V = \{v\}$, we abbreviate the notation by writing $\operatorname{transp}(u, v)$.

PROPOSITION 2.1. *Let E and E' be two equivalence relations on n with $E \subseteq E'$. Then* $\operatorname{permsub}(y^E) \supseteq \operatorname{permsub}(y^{E'})$.

Proof. From the assumption, any two classes U', V' of E' are unions of classes of E, say $U' = \cup_{i=1}^k U_i$ and $V' = \cup_{j=1}^l V_j$. Then

$$\operatorname{transp}(U', V') = \sum_{i=1}^{k} \sum_{j=1}^{l} \operatorname{transp}(U_i, V_j), \tag{2.3}$$

and $\operatorname{transp}(U', V') \in \operatorname{permsub}(y^E)$. □

If F is a nonempty face of $\mathrm{P}^n_{\mathrm{PA}}$, its *permutation subspace* $\operatorname{permsub}(F)$ is the vector subspace spanned by the union of the permutation subspaces of the vertices of F. By Proposition 2.1, we can ignore here any vertex $y^{E'}$ of F for which there exists a vertex y^E of F with $E \subset E'$.

The next result is the main tool we will use to derive facets of $\mathrm{P}^n_{\mathrm{WO}}$.

PROPOSITION 2.2. *For any face F of* $\mathrm{P}^n_{\mathrm{PA}}$,

$$\dim F + \operatorname{rk} \operatorname{permsub}(F) \le \dim \dot{\pi}^{-1}(F) \le \dim F + \binom{n}{2}. \tag{2.4}$$

Proof. Choose a vertex o' of F as an origin in $\mathbb{R}^{\binom{n}{2}}$ and then choose a vertex o of $\dot{\pi}^{-1}(F)$ as an origin in $\mathbb{R}^{n(n-1)}$. As $\pi(o) = o'$, the affine mapping π becomes a linear mapping. Setting $f = \operatorname{rk} F$, we may select f linearly independent vertices $q_1, q_2, \ldots, q_f$ of F. Pick in $\dot{\pi}^{-1}(F)$ vertices $p_1, p_2, \ldots, p_f$ with $\pi(p_i) = q_i$ for $i = 1, 2, \ldots, f$. With $g = \operatorname{rk} \operatorname{permsub}(F)$, select next a basis $r_1, r_2, \ldots, r_g$ of $\operatorname{permsub}(F)$ which consists of transposition vectors of vertices of F. Thus any r_j is a difference between two vertices of $\dot\pi^{-1}(F)$. As is easily checked, the vectors p_1, p_2, …, p_f, r_1, r_2, …, r_g are linearly independent. Because with o they all belong to $\operatorname{aff} \dot\pi^{-1}(F)$, we have $f + g \leq \dim \dot\pi^{-1}(F)$, which is the first inequality to be proved. The second inequality follows at once from $\operatorname{rk} \pi^{-1}(o') = \binom{n}{2}$. ☐

Remark. Both inequalities in Proposition 2.2 become equalities at least when y^{id} belongs to F (where id is the equivalence relation with n classes) but not always (as will be seen later, e.g., in Proposition 4.1).

As we now show, Proposition 2.2 greatly helps understanding preimages by $\dot\pi$ of various facets of $\mathrm{P}^n_{\mathrm{PA}}$.

3. Lifting the trivial, triangle, and 2-partition inequalities. For i, j distinct elements in n, inequality $x_{\{ij\}} \geq 0$ defines a facet of $\mathrm{P}^n_{\mathrm{PA}}$; see Grötschel and Wakabayashi [9]. (Notice how we sometime abbreviate indices $\{i, j\}$ in $\{ij\}$.) The lifted inequality $x_{ij} + x_{ji} - 1 \geq 0$ defines a facet of $\mathrm{P}^n_{\mathrm{WO}}$ as shown by Gurgel and Wakabayashi [12]. (In index position, (i, j) is from now on abbreviated in ij or i, j.) On the other hand, if F is the facet of $\mathrm{P}^n_{\mathrm{PA}}$ defined by the *triangle inequality*

$$x_{\{ij\}} + x_{\{jk\}} - x_{\{ik\}} \leq 1, \tag{3.1}$$

where i, j, k are distinct elements in n, the preimage $\dot\pi^{-1}(F)$ defined by

$$x_{ij} + x_{ji} + x_{jk} + x_{kj} - x_{ik} - x_{ki} \leq 2 \tag{3.2}$$

turns out to be a ridge. (We skip the proof.) This ridge is the intersection of the two facets defined by the two following *transitivity inequalities* [11], [10]:

$$\begin{aligned} x_{ij} + x_{jk} - x_{ik} &\leq 1, \\ x_{kj} + x_{ji} - x_{ki} &\leq 1. \end{aligned}$$

Now consider two nonempty, disjoint subsets S, T of n with $|S| < |T|$. Grötschel and Wakabayashi [9] show the 2-*partition inequality*

$$\sum_{\substack{i\in S\\ j\in T}} x_{\{ij\}} - \sum_{\substack{i,j\in S\\ i<j}} x_{\{ij\}} - \sum_{\substack{i,j\in T\\ i<j}} x_{\{ij\}} \leq |S| \tag{3.3}$$

to be facet-defining for $\mathrm{P}^n_{\mathrm{PA}}$. Its lifted inequality in $\mathbb{R}^{n(n-1)}$, that we also call a 2-*partition inequality,*

$$\sum_{\substack{i\in S\\ j\in T}} (x_{ij} + x_{ji}) - \sum_{\substack{i,j\in S\\ i\neq j}} x_{ij} - \sum_{\substack{i,j\in T\\ i\neq j}} x_{ij} \leq |S| + |S|\,|T| - \binom{|S|}{2} - \binom{|T|}{2} \tag{3.4}$$

is valid for $\mathrm{P}^n_{\mathrm{WO}}$. When $S = \{j\}$ and $T = \{i, k\}$ this inequality coincides with inequality (3.2), so it is clearly not facet-defining.

THEOREM 3.1. *Let S and T be two disjoint subsets of n such that $0 < |S| < |T|$ and $(|S|, |T|) \neq (1, 2)$. Then inequality* (3.4) *defines a facet of* $\mathrm{P}^n_{\mathrm{WO}}$.

Proof. According to the trivial lifting lemma of Fiorini and Fishburn [6], any facet-defining inequality for $\mathrm{P}^n_{\mathrm{WO}}$ is also facet-defining for $\mathrm{P}^m_{\mathrm{WO}}$ when $m > n$. We may thus assume $S \cup T = \mathsf{n}$. Let F be the facet of $\mathrm{P}^n_{\mathrm{PA}}$ defined by inequality (3.3). Then, inequality (3.4) defines the preimage $\dot\pi^{-1}(F)$ in $\mathrm{P}^n_{\mathrm{WO}}$. We proceed by showing that all transposition vectors $\operatorname{transp}(i, j)$, for i, j distinct in n, belong to $\operatorname{permsub}(F)$; the thesis then follows from Proposition 2.2.

For every injective mapping $f : S \to T$, an equivalence relation E yields $y^E \in F$ when it has the classes $\{s, f(s)\}$ for all s in S and $\{t\}$ for all t in $T \setminus f(S)$.

When $|S| + 2 \leq |T|$, we deduce that all transposition vectors $\mathrm{transp}(j, k)$, for $j, k \in T$, and $j \neq k$, belong to $\mathrm{permsub}(F)$. Also, for $i \in S$ and $j, k \in T$ with $j \neq k$, we get $\mathrm{transp}(\{j\}, \{i, k\}) \in \mathrm{permsub}(F)$ which, after subtraction of $\mathrm{transp}(j, k)$, gives $\mathrm{transp}(j, i) \in \mathrm{permsub}(F)$. If S contains two distinct elements h, i, similar arguments give $\mathrm{transp}(h, i) \in \mathrm{permsub}(F)$. Thus $\mathrm{permsub}(F)$ contains all $\mathrm{transp}(u, v)$ for u, v distinct in $S \cup T = \mathsf{n}$.

When $|S| + 1 = |T|$, we have $|S| \geq 2$ by assumption. Let i, j be two distinct elements in S and k, l be two distinct elements in T. By considering appropriate partitions encoded as vertices of F, we find the following six linearly independent transposition vectors in $\mathrm{permsub}(F)$:

$$\begin{aligned}
&\mathrm{transp}(k, i) + \mathrm{transp}(k, l),\\
&\mathrm{transp}(k, j) + \mathrm{transp}(k, l),\\
&\mathrm{transp}(l, i) + \mathrm{transp}(l, k),\\
&\mathrm{transp}(l, j) + \mathrm{transp}(l, k),\\
&\mathrm{transp}(k, j) + \mathrm{transp}(k, l) + \mathrm{transp}(i, j) + \mathrm{transp}(i, l),\\
&\mathrm{transp}(i, j) + \mathrm{transp}(i, k) + \mathrm{transp}(l, j) + \mathrm{transp}(l, k).
\end{aligned}$$

All transposition vectors $\mathrm{transp}(u, v)$ with u, v distinct in $\{i, j, k, l\}$ are linear combinations of the six vectors in the above list. Thus again $\mathrm{permsub}(F)$ contains all possible transposition vectors. □

4. Lifting the 2-chorded cycle inequalities. Consider a cycle in n; to simplify notation, we relabel the elements of n in such a way that the cycle has vertices 1, 2, ..., k. We denote by $\oplus$ and $\ominus$ the addition and subtraction on $\mathsf{k} = \{1, 2, \ldots, k\}$ with results reduced modulo k to a value in k. The 2-*chorded cycle inequality*

$$\sum_{i=1}^{k} x_{\{i,i\oplus 1\}} - \sum_{i=1}^{k} x_{\{i,i\oplus 2\}} \leq \frac{k-1}{2} \tag{4.1}$$

is facet-defining for $\mathrm{P}^n_{\mathrm{PA}}$ when k is odd and at least 5 (see [9]). In what follows, we assume this double condition on k. Taking the preimage by the induced partition projection π, we derive the following inequality valid for $\mathrm{P}^n_{\mathrm{WO}}$:

$$\sum_{i=1}^{k} (x_{i,i\oplus 1} + x_{i\oplus 1,i}) - \sum_{i=1}^{k} (x_{i,i\oplus 2} + x_{i\oplus 2,i}) \leq \frac{k-1}{2}. \tag{4.2}$$

As will be shown in Proposition 4.1, this inequality defines a ridge of $\mathrm{P}^n_{\mathrm{WO}}$; the two facets containing the ridge are specified in Theorem 4.2. We use the notation

$$\begin{aligned}
\mathcal{D}_k &= \{(i, j) \in \mathsf{k} \times \mathsf{k} \mid i \ominus j \text{ is odd and } i \neq j\},\\
\mathcal{E}_k &= \{(i, j) \in \mathsf{k} \times \mathsf{k} \mid i \ominus j \text{ is even}\}.
\end{aligned}$$

PROPOSITION 4.1. *For k odd and at least 5, inequality* (4.2) *defines a ridge of* $\mathrm{P}^n_{\mathrm{WO}}$. *Any vertex of* $\mathrm{P}^n_{\mathrm{WO}}$ *satisfying inequality* (4.2) *with equality also satisfies the linear equation*

$$\sum_{ij \in \mathcal{E}_k} x_{ij} - \sum_{ij \in \mathcal{D}_k} x_{ij} = 0. \tag{4.3}$$

Proof. Let F denote the facet of $\mathrm{P}^n_{\mathrm{PA}}$ defined by inequality (4.1); then inequality (4.2) defines the face $G = \dot{\pi}^{-1}(F)$ of $\mathrm{P}^n_{\mathrm{WO}}$. We show that any vertex x^W of G also satisfies (4.3). Indeed, $\pi(x^W) \in F$, and a vertex y^E belongs to F iff the equivalence relation E, hereafter denoted as $\sim$, enjoys up to cyclic rotation of 1, 2, ..., k either the conditions

$$1 \not\sim 2 \sim 3 \not\sim 4 \sim 5 \not\sim 6 \sim 7 \not\sim \cdots \not\sim k-1 \sim k \not\sim 1 \tag{4.4}$$

or the conditions

$$1 \sim 2 \sim 3 \not\sim 4 \sim 5 \not\sim 6 \sim 7 \not\sim \cdots \not\sim k-1 \sim k \not\sim 1. \tag{4.5}$$

Either set of conditions implies that x^W also satisfies (4.3).

Now to prove that G is a ridge, it suffices by Proposition 2.2 to prove that permsub(F) plus the transposition vector transp$(1,2)$ generate all transposition vectors transp(i,j) with i,j distinct in n. Take a minimal equivalence relation E such that y^E satisfies (4.1) with equality. Then E admits the class $\{i\}$ for all $i \in \mathsf{n} \setminus \mathsf{k}$; moreover, we may assume, after relabeling if necessary, that E admits the class $\{1\}$. As $y^E \in F$, we get transp$(1,i)$ and transp(i,j) in permsub(F) for all i,j distinct in $\mathsf{n} \setminus \mathsf{k}$. Taking appropriate cyclic images of E, we see that the same holds if 1 is replaced with any element in k. Now using the minimal equivalence relation defined by (4.4) together with its cyclically rotated images, we can check that permsub(F) plus transp$(1,2)$ generate transp$(1,3)$, transp$(2,3)$, transp$(2,4)$, transp$(3,4)$, transp$(1,4)$, transp$(3,5)$, etc., thus all transp(i,j) with i,j distinct in k. ∎

THEOREM 4.2. *Let k be odd and at least 5. One of the two facets of $\mathrm{P}^n_{\mathrm{WO}}$ containing the ridge obtained in Proposition 4.1 is defined by the inequality*

$$\begin{aligned} &\frac{k+1}{2}\sum_{i=1}^{k}(x_{i,i\oplus 1} - x_{i,i\oplus 2}) + \frac{k-3}{2}\sum_{i=1}^{k}(x_{i\oplus 1,i} - x_{i\oplus 2,i}) \\ &\qquad + \sum_{j=3}^{(k-1)/2}\sum_{i=1}^{k}(-1)^{j+1}(x_{i,i\oplus j} - x_{i\oplus j,i}) \quad \le \quad \frac{(k-1)^2}{4}, \end{aligned} \tag{4.6}$$

and the other facet containing the ridge is defined by the similar inequality obtained by substituting x_{ij} with x_{ji} for all i,j in k with $i<j$.

Gurgel and Wakabayashi [11] present two facet-defining inequalities for $\mathrm{P}^n_{\mathrm{WO}}$ which are the two particular cases of Theorem 4.2 for $k=5$ and $k=7$; the 20-page proof for $k=7$ is omitted there but appears in Gurgel [10]. Our proof of Theorem 4.2 is rather short and uses the following known result on tournaments (see Bermond [2] or Laslier [13]).

LEMMA 4.3. *For k odd, the relation $\mathcal{D}_k$ is a tournament on k whose Slater index (or feedback arc-number) equals $\frac{k^2-1}{8}$. The same holds for the relation $\mathcal{E}_k$.*

Proof. As k is assumed to be odd, $i \ominus j$ is even iff $j \ominus i$ is odd; thus $\mathcal{D}_k$ is a tournament. Let L be any linear ordering on k obtained by reversing certain arcs of $\mathcal{D}_k$. For $i = 1, 2, \ldots, (k-1)/2$, the outdegree of the element of rank i in L equals $k-i$, while in $\mathcal{D}_k$ it equals $(k-1)/2$. Thus the number of reversed arcs is at least $\sum_{i=1}^{(k-1)/2}((k-i)-(k-1)/2) = (k^2-1)/8$. The following linear ordering T requires no more arc reversings than this number:

$$1 \prec 3 \prec 5 \prec \cdots \prec k-2 \prec k \prec 2 \prec 4 \prec 6 \prec \cdots \prec k-3 \prec k-1. \tag{4.7}$$

Finally, notice that $\mathcal{D}_k$ and $\mathcal{E}_k$ are dual relations. □

Proof of Theorem 4.2. Let G be the ridge defined by inequality (4.2), written more compactly as

$$(4.8)\quad \langle c, x\rangle \le \frac{k-1}{2} \quad \text{for} \quad \sum_{i=1}^{k}(x_{i,i\oplus 1}+x_{i\oplus 1,i}) - \sum_{i=1}^{k}(x_{i,i\oplus 2}+x_{i\oplus 2,i}) \le \frac{k-1}{2}.$$

We know that G also satisfies (4.3), which we summarize as

$$(4.9)\qquad \langle d, x\rangle = 0 \qquad \text{for} \qquad \sum_{ij\in\mathcal{E}_k} x_{ij} - \sum_{ij\in\mathcal{D}_k} x_{ij} = 0.$$

Then

$$(4.10)\qquad \langle 2(k-1)c + 4d, x\rangle \le (k-1)^2$$

is inequality (4.6) up to a factor 4. There remains to show that inequality (4.10) is valid for any vertex of $\mathrm{P}^n_{\mathrm{WO}}$ and that it becomes an equality at some vertex of $\mathrm{P}^n_{\mathrm{WO}}$ not in G.

Let W be a weak order on n. By a *class segment* we mean a subset S of k of the form $\{i,\, i\oplus 1,\, \ldots,\, i\oplus \ell\}$ such that all elements of S belong to a same class of W, and, moreover, S cannot be extended in k while keeping this double condition. The *length* of S is $\ell+1$. (Thus k itself may be a class segment, and its length is by convention k.) Now denote by s the number of class segments of length strictly greater than 1 and by r the number of class segments of odd length. It is not difficult to check $\langle c, x^W\rangle = s$.

We now prove

$$(4.11)\qquad 2(k-1)\langle c, x^W\rangle + 4\langle d, x^W\rangle \le (k-1)^2.$$

Only coordinates x^W_{ij} with $i, j \in \mathsf{k}$ appear in inequality (4.11); thus we may assume that any element in $\mathsf{n}\setminus\mathsf{k}$ is isolated in its equivalence class. Next, if a class segment $S = \{i,\, i\oplus 1,\, \ldots,\, i\oplus\ell\}$ has $\ell > 1$, we modify W by breaking only the class C containing S into two successive classes, namely $\{i, i\oplus 1\}$ and $C\setminus\{i, i\oplus 1\}$; this modification does not decrease $\langle c, x^W\rangle$ and leaves $\langle d, x^W\rangle$ unchanged (as easily verified). We may now assume that all class segments of W have length 1 or 2.

If a class C strictly contains a one-element segment $\{i^*\}$, let W^+ and W^- be the weak orders obtained from W by pushing i^* out of its equivalence class one position down and one position up, respectively; that is,

$$\begin{aligned} W^- &= W \setminus \{(j, i^*) \mid j \sim i^* \text{ and } j \ne i^*\},\\ W^+ &= W \setminus \{(i^*, j) \mid j \sim i^* \text{ and } j \ne i^*\}. \end{aligned}$$

(As before, $\sim$ denotes the equivalence relation $W\cap W^{-1}$.) Either $\langle d, x^{W^+}\rangle = \langle d, x^{W^-}\rangle = \langle d, x^W\rangle$ or $\langle d, x^{W^+}\rangle - \langle d, x^W\rangle = \langle d, x^W\rangle - \langle d, x^{W^-}\rangle \ne 0$; thus we may extract i^* from its equivalence class and get another weak order, also called W, without decreasing either $\langle d, x^W\rangle$ or $\langle c, x^W\rangle$. We may now further assume that any element i^* isolated in its segment is also isolated in its class.

In the evaluation of $\langle d, x^W\rangle$, all pairs of W touching one class segment of length 2 have a total contribution zero. Thus, we need look only at the linear ordering T induced by W on the set R of the r elements of k which are alone in their classes.

Notice that $\mathcal{D}_k$ induces on R a relation which is naturally isomorphic to $\mathcal{D}_r$. Thus using Lemma 4.3 we get

$$\begin{aligned}\langle d, x^W\rangle &= |T \cap \mathcal{E}_r| - |T \cap \mathcal{D}_r| \\ &= |T| - 2|T \cap \mathcal{D}_r| \\ &\leq \frac{r(r-1)}{2} - 2 \cdot \frac{r^2-1}{8} \\ &= \frac{(r-1)^2}{4}.\end{aligned}$$

Hence, as $s = (k-r)/2$,

$$\begin{aligned}2(k-1)\langle c, x^W\rangle + 4\langle d, x^W\rangle &\leq 2(k-1)\frac{k-r}{2} + 4\frac{(r-1)^2}{4} \\ &= \left(r - \frac{k+1}{2}\right)^2 + \frac{3(k-1)^2}{4}.\end{aligned}$$

As k is odd, we cannot have $r = 0$. For r varying in k, the last expression has maximum value $(k-1)^2$, attained for $r = 1$ or $r = k$.

We have thus established that inequality (4.10) defines a face of $\mathrm{P}^n_{\mathrm{WO}}$ which contains the ridge G. A vertex x^T in this face but not in G is obtained for T, the linear ordering specified in the proof of Lemma 4.3.

To get the other facet of $\mathrm{P}^n_{\mathrm{WO}}$ which contains the ridge G, we apply the linear permutation mapping (x_{ij}) onto (x_{ji}); this permutation stabilizes both $\mathrm{P}^n_{\mathrm{WO}}$ and G. ∎

5. Lifting the 2-chorded path and 2-chorded wheel inequalities. Again, by making use of our fundamental tool (Proposition 2.2), we infer additional facets of $\mathrm{P}^n_{\mathrm{WO}}$ from two families of facets of $\mathrm{P}^n_{\mathrm{PA}}$.

A 2-*chorded path inequality* for $\mathrm{P}^n_{\mathrm{PA}}$ is obtained as follows (under a specific choice of labels for the elements in n). Let $1, 2, \ldots, \ell-1$ be (a path) in n, and let ℓ be an additional element in n. Grötschel and Wakabayashi [9] prove that the inequality

$$\sum_{i=1}^{\ell-2} x_{\{i,i+1\}} - \sum_{i=1}^{\ell-3} x_{\{i,i+2\}} + \sum_{\substack{j=2\\ j\,\mathrm{even}}}^{\ell-2} x_{\{j\ell\}} - \sum_{\substack{j=1\\ j\,\mathrm{odd}}}^{\ell-1} x_{\{j\ell\}} \leq \frac{\ell-2}{2} \tag{5.1}$$

is facet-defining for $\mathrm{P}^n_{\mathrm{PA}}$ when ℓ is even and at least 4.

THEOREM 5.1. *The inequality lifted from inequality* (5.1),

$$\begin{aligned}&\sum_{i=1}^{\ell-2}(x_{i,i+1} + x_{i+1,i}) - \sum_{i=1}^{\ell-3}(x_{i,i+2} + x_{i+2,i}) \\ &\quad + \sum_{\substack{j=2\\ j\,even}}^{\ell-2}(x_{j\ell} + x_{\ell j}) - \sum_{\substack{j=1\\ j\,odd}}^{\ell-1}(x_{j\ell} + x_{\ell j}) \quad \leq \quad \frac{\ell-2}{2},\end{aligned} \tag{5.2}$$

defines a facet of $\mathrm{P}^n_{\mathrm{WO}}$ *for* ℓ *even,* $\ell \geq 4$.

Proof. Because of Proposition 2.2, it suffices to show that permsub(F), for F the facet of $\mathrm{P}^n_{\mathrm{PA}}$ defined by inequality (5.1), contains all transposition vectors transp(u, v) for u, v distinct in n. To this aim, we notice that vertex y^E belongs to F at least when E is an equivalence relation with the following equivalence classes, where i, $j \in \{1, 2, \ldots, \ell-1\}$ and $i < j$:

(i) for i and j odd,

$$\begin{array}{l}\{1,2\},\ \{3,4\},\ \ldots,\ \{i-2,i-1\},\\ \{i\},\\ \{i+1,\ell\},\\ \{i+2,i+3\},\ \{i+4,i+5\},\ \ldots,\ \{j-2,j-1\},\\ \{j\},\\ \{j+1,j+2\},\ \{j+3,j+4\},\ \ldots,\ \{\ell-2,\ell-1\},\\ \{\ell+1\},\ \{\ell+2\},\ \ldots,\ \{n\};\end{array} \tag{5.3}$$

(ii) for i even and j odd,

$$\begin{array}{l}\{1,2\},\ \{3,4\},\ \ldots,\ \{i-1,i\},\ \ldots,\ \{j-2,j-1\},\\ \{j\},\\ \{j+1,j+2\},\ \{j+3,j+4\},\ \ldots,\ \{\ell-2,\ell-1\},\\ \{\ell\},\ \{\ell+1\},\ \ldots,\ \{n\};\end{array} \tag{5.4}$$

(iii) for i odd and j even,

$$\begin{array}{l}\{1,2\},\ \{3,4\},\ \ldots,\ \{i-2,i-1\},\\ \{i\},\\ \{i+1,i+2\},\ \{i+3,i+4\},\ \ldots,\ \{j,j+1\},\ \ldots,\ \{\ell-2,\ell-1\},\\ \{\ell\},\ \{\ell+1\},\ \ldots,\ \{n\};\end{array} \tag{5.5}$$

(iv) for i and j even ,

$$\begin{array}{l}\{1,2\},\ \{3,4\},\ \ldots,\ \{i-1,i\},\ \ldots,\ \{j-1,j\},\ \ldots,\ \{\ell-3,\ell-2\},\\ \{\ell-1\},\ \{\ell\},\ \ldots,\ \{n\}.\end{array} \tag{5.6}$$

By (i), we get $\operatorname{transp}(i,j) \in \operatorname{permsub}(F)$ when i and j are odd. For i even and j odd, (ii) gives $\operatorname{transp}(\{i-1,i\},\{j\}) \in \operatorname{permsub}(F)$, and by subtracting $\operatorname{transp}(i-1,j)$ (which was just shown to belong to $\operatorname{permsub}(F)$), we get $\operatorname{transp}(i,j) \in \operatorname{permsub}(F)$. The cases ($i$ odd, j even) and (i, j even) follow in a similar way from (iii) and (iv). The other transposition vectors $\operatorname{transp}(u,v)$, for u, $v \in \mathsf{n}$ with $u \neq v$, also belong to $\operatorname{permsub}(F)$. □

With an adequate relabeling of the elements in n, a 2-*chorded even wheel inequality* for $\mathrm{P}^n_{\mathrm{PA}}$, with $1 \leq k \leq n-1$ and k even, is written as

$$\sum_{i=1}^{k} x_{\{i,i\oplus 1\}} - \sum_{i=1}^{k} x_{\{i,i\oplus 2\}} + \sum_{\substack{j=1\\ j\text{even}}}^{k} x_{\{jn\}} - \sum_{\substack{j=1\\ j\text{odd}}}^{k} x_{\{jn\}} \leq \frac{k}{2} \tag{5.7}$$

(where i and j are taken modulo k in $\{1, 2, \ldots, k\}$). According to [9], this inequality defines a facet of $\mathrm{P}^n_{\mathrm{PA}}$ when k is even and $k \geq 8$.

THEOREM 5.2. *The inequality lifted from inequality* (5.7) *gives a facet-defining inequality for* $\mathrm{P}^n_{\mathrm{WO}}$ *when* k *is even and* $k \geq 8$*, which reads*

$$\begin{aligned}\sum_{i=1}^{k}(x_{i,i\oplus 1} + x_{i\oplus 1,i}) - \sum_{i=1}^{k}(x_{i,i\oplus 2} + x_{i\oplus 2,i})\\ - \sum_{\substack{j=1\\ j\,even}}^{k}(x_{jn} + x_{nj}) - \sum_{\substack{j=1\\ j\,odd}}^{k}(x_{jn} + x_{nj}) \quad \leq \quad \frac{k}{2}.\end{aligned} \tag{5.8}$$

Proof. As the arguments are similar to those in the previous proof, we just list one example of a useful partition of n; the corresponding equivalence relation E provides a vertex y^E of $\mathrm{P}^n_{\mathrm{PA}}$ which belongs to the facet defined by inequality (5.7). For i, j odd in 1, 2, …, k with $j \notin \{i \ominus 2, i, i \oplus 2\}$ (the case $j \in \{i \ominus 2, i \oplus 2\}$ requires a slight modification), we list the classes of E:

$$
\begin{array}{l}
\{i\}, \\
\{i \oplus 2, i \oplus 3\},\ \{i \oplus 4, i \oplus 5\},\ \ldots,\ \{j \ominus 2, j \ominus 1\}, \\
\{j\}, \\
\{j \oplus 2, j \oplus 3\},\ \{j \oplus 4, j \oplus 5\},\ \ldots,\ \{k \ominus 1, k\}, \\
\{1, 2\},\ \{3, 4\},\ \ldots,\ \{i \ominus 2, i \ominus 1\}, \\
\{i \oplus 1, j \oplus 1, n\}, \\
\{k+1\},\ \{k+2\},\ \ldots,\ \{n-1\}.
\end{array}
\tag{5.9}
$$

This partition and similar ones obtained by cyclically rotating 1, 2, …, k help in showing that $\mathrm{permsub}(F)$ contains all transposition vectors $\mathrm{transp}(u, v)$, where F is the facet of $\mathrm{P}^n_{\mathrm{PA}}$ defined by inequality (5.7), and $1 \le u < v \le n$. ◻

Grötschel and Wakabayashi [9] present still one more facet-defining inequality for $\mathrm{P}^n_{\mathrm{PA}}$, an isolated example which is not "symmetric," and has some coefficients equal to 2. We leave it to the reader to verify that this inequality is lifted into an additional facet-defining inequality for $\mathrm{P}^n_{\mathrm{WO}}$.

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