

Algorithms for perfectly contractile graphs

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Abstract. We consider the class \mathcal{A} of graphs that contain no odd hole, no antihole of length at least 5, and no “prism” (a graph consisting of two disjoint triangles with three disjoint paths between them) and the class \mathcal{A}' of graphs that contain no odd hole, no antihole of length at least 5, and no odd prism (prism whose three paths are odd). These two classes were introduced by Everett and Reed and are relevant to the study of perfect graphs. We give polynomial-time recognition algorithms for these two classes. We proved previously that every graph $G \in \mathcal{A}$ is “perfectly contractile”, as conjectured by Everett and Reed [see the chapter “Even pairs” in the book *Perfect Graphs*, J.L. Ramírez-Alfonsín and B.A. Reed, eds., Wiley Interscience, 2001]. The analogous conjecture concerning graphs in \mathcal{A}' is still open.

1 Introduction

A graph G is *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$, where $\chi(G')$ is the chromatic number of G' and $\omega(G')$ is the maximum clique size in G' . Berge [1, 2, 3] introduced perfect graphs and conjectured that *a graph is perfect if and only if it does not contain as an induced subgraph an odd hole or an odd antihole* (the Strong Perfect Graph Conjecture), where a *hole* is a chordless cycle with at least four vertices and an *antihole* is the complement of a hole. We follow the tradition of calling *Berge graph* any graph that contains no odd hole and no odd antihole. The Strong Perfect Graph Conjecture was the object of much research (see the book [15]), until it was finally proved by Chudnovsky, Robertson, Seymour and Thomas [7]: *Every Berge graph is perfect*. Moreover, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković [6, 9, 8] gave polynomial-time algorithms to decide if a graph is Berge.

Despite those breakthroughs, some conjectures about Berge graphs remain open. An *even pair* in a graph G is a pair of non-adjacent vertices such that every chordless path between them has even length (number of edges). Given two vertices x, y in a graph G , the operation of *contracting* them means removing x and y and adding one vertex with edges to every vertex of $G \setminus \{x, y\}$ that is

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adjacent in G to at least one of x, y ; we denote by G/xy the graph that results from this operation. Fonlupt and Uhry [11] proved that *if G is a perfect graph and $\{x, y\}$ is an even pair in G , then the graph G/xy is perfect and has the same chromatic number as G* . In particular, given a $\chi(G/xy)$ -coloring c of the vertices of G/xy , one can easily obtain a $\chi(G)$ -coloring of the vertices of G as follows: keep the color for every vertex different from x, y ; assign to x and y the color assigned by c to the contracted vertex. This idea could be the basis for a conceptually simple coloring algorithm for Berge graphs: as long as the graph has an even pair, contract any such pair; when there is no even pair find a coloring c of the contracted graph and, applying the procedure above repeatedly, derive from c a coloring of the original graph. The polynomial-time algorithm for recognizing Berge graphs mentioned at the end of the preceding paragraph can be used to detect an even pair in a Berge graph G ; indeed, two non-adjacent vertices a, b form an even pair in G if and only if the graph obtained by adding a vertex adjacent only to a and b is Berge. The problem of deciding if a graph contains an even pair is NP-hard in general graphs [5]. Given a Berge graph G , one can try to color its vertices by keeping contracting even pairs until none can be found. Then some questions arise: what are the Berge graphs with no even pair? What are, on the contrary, the graphs for which a sequence of even-pair contractions leads to graphs that are easy to color?

As a first step towards getting a better grasp on these questions, Bertschi [4] proposed the following definitions. A graph G is *even-contractile* if either G is a clique or there exists a sequence G_0, \dots, G_k of graphs such that $G = G_0$, for $i = 0, \dots, k-1$ the graph G_i has an even pair $\{x_i, y_i\}$ such that $G_{i+1} = G_i/x_iy_i$, and G_k is a clique. A graph G is *perfectly contractile* if every induced subgraph of G is even-contractile. Perfectly contractile graphs include many classical families of perfect graphs, such as Meyniel graphs, weakly chordal graphs, perfectly orderable graphs, see [10]. Everett and Reed proposed a conjecture aiming at a characterization of perfectly contractile graphs. To understand it, one more definition is needed: say that a graph is a *prism* if it consists of two vertex-disjoint triangles (cliques of size 3) $\{a_1, a_2, a_3\}$, $\{b_1, b_2, b_3\}$, with three vertex-disjoint paths P_1, P_2, P_3 between them, such that for $i = 1, 2, 3$ path P_i is from a_i to b_i , and with no other edge than those in the two triangles and in the three paths. We may also say that the three paths P_1, P_2, P_3 *form* the prism. Say that a prism is *odd* (or *even*) if all three paths have odd length (respectively all have even length). See Figure 1.

Define two classes $\mathcal{A}, \mathcal{A}'$ of graphs as follows:

- \mathcal{A} is the class of graphs that do not contain odd holes, antiholes of length at least 5, or prisms.
- \mathcal{A}' is the class of graphs that do not contain odd holes, antiholes of length at least 5, or odd prisms.

Clearly $\mathcal{A} \subset \mathcal{A}'$.

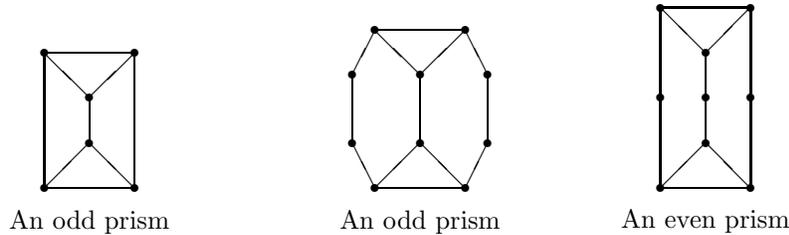


Figure 1: Some prisms

Conjecture 1.1 (Everett and Reed [10, 16]) *A graph is perfectly contractile if and only if it is in class \mathcal{A}' .*

The if part of this conjecture remains open. The only if part is not hard to establish, but it requires some careful checking; this was done formally in [13]. A weaker form of this conjecture was also proposed by Everett and Reed; that statement is now a theorem:

Theorem 1.2 (Maffray and Trotignon [14]) *If G is a graph in class \mathcal{A} and G is not a clique, then G has an even pair whose contraction yields a graph in \mathcal{A} (and so G is perfectly contractile).*

The preceding conjecture and theorem suggest that it may be interesting to recognize the classes \mathcal{A} and \mathcal{A}' in polynomial time; this is the aim of this manuscript.

In order to decide if a graph is in class \mathcal{A} , it would suffice to decide separately if it is Berge, if it has an antihole of length at least 5, and if it contains a prism. The first question, deciding if a graph is Berge, is now settled [6, 8, 9]. In Section 2 we will find it convenient for our purpose to give a summary of the polynomial time algorithm from [6, 8] that solves this problem. The second question is not hard: to decide if a graph G contains a hole of length at least 5, it suffices to test, for every chordless path $a-b-c$, whether a and c are in the same connected component of the subgraph of G obtained by removing the vertices of $N(a) \cap N(c)$ and those of $N(b) \setminus \{a, c\}$. This takes time $O(|V(G)|^5)$. To decide if a graph contains an antihole of length at least 5, we need only apply this algorithm on its complementary graph. However, the third question, to decide if a graph contains a prism, turns out to be NP-complete; this is established in Section 8 below. Likewise, we will see that it is NP-complete to decide if a graph contains an odd prism. Thus we cannot solve the recognition problem for class \mathcal{A} (or for class \mathcal{A}') in the fashion that is suggested at the beginning of this paragraph. Instead, we will adapt the Berge graph recognition algorithm to our purpose. This is done in Sections 3–7.

2 Recognizing Berge graphs

We give here a brief outline of the Berge graph recognition algorithm which follows from [6] and [8]. Given a graph G and a hole C in G , say that a vertex $x \in V(G) \setminus V(C)$ is a *major neighbour* of C if the set $N(x) \cap V(C)$ is not included in a 3-vertex subpath of C . Say that set $X \subseteq V(G)$ is a *cleaner* for the hole C if X contains all the major neighbours of C and $X \cap V(C)$ is included in a 3-vertex subpath of C . The algorithm is based on the results summarized in the following theorem.

Theorem 2.1 ([6, 8])

1. *There exist five types of configurations (graphs), types T_1, \dots, T_5 , such that, for $i = 1, \dots, 5$, we have: (a) if a graph G contains a configuration of type T_i then G is not a Berge graph, and (b) there exists a polynomial time algorithm A_i that decides if a graph contains a configuration of type T_i .*

2. *There is a polynomial-time algorithm which, given a graph G that does not contain a configuration of type T_i ($i = 1, \dots, 5$), returns a family \mathcal{F} of $|V(G)|^5$ subsets of $V(G)$ such that for any shortest odd hole C of G , some member of \mathcal{F} is a cleaner for C .*

3. *There is a polynomial-time algorithm which, given a graph G that does not contain a configuration of type T_i ($i = 1, \dots, 5$) and the family \mathcal{F} produced by step 2, decides if G contains an odd hole (and if it does, returns a shortest odd hole of G).*

We will not give the definition of all five types of configurations, but we recall from [6, 8] that, for $i = 1, \dots, 5$, the complexity of algorithm A_i is respectively $O(|V(G)|^5)$, $O(|V(G)|^6)$, $O(|V(G)|^6)$, $O(|V(G)|^6)$, $O(|V(G)|^9)$. We need to dwell on the configuration of type T_5 , which is called a *pyramid* in [8]. A pyramid is a graph that consists in three pairwise adjacent vertices b_1, b_2, b_3 (called the triangle vertices of the pyramid), a fourth vertex a (called the apex of the pyramid), and three chordless paths P_1, P_2, P_3 such that:

- For $i = 1, 2, 3$, path P_i is between a and b_i ;
- For $1 \leq i < j \leq 3$, $V(P_i) \cap V(P_j) = \{a\}$ and $b_i b_j$ is the only edge between $V(P_i) \setminus \{a\}$ and $V(P_j) \setminus \{a\}$;
- a is adjacent to at most one of b_1, b_2, b_3 .

We may say that the three paths P_1, P_2, P_3 form a pyramid. It is easy to see that a pyramid contains an odd hole (since two of the paths P_1, P_2, P_3 have the same parity, the union of their vertex sets induce an odd hole); so Berge graphs do not contain pyramids.

The pyramid-testing algorithm from [8] is the slowest algorithm in Step 1 of the Berge graph recognition algorithm. The algorithm of Step 2 has complexity

$O(|V(G)|^6)$ [6], and the algorithm of Step 3 has complexity $O(|V(G)|^9)$ [8]. Testing if a graph G is Berge can be done by running the algorithms described in the previous theorem on G and on its complementary graph \overline{G} . Thus the total complexity is $O(|V(G)|^9)$.

3 Recognizing pyramids and prisms

We present a polynomial-time algorithm that decides if a graph contains a pyramid or a prism. This algorithm has the same flavor as the pyramid-testing algorithm from [8]. We describe this algorithm now.

If a graph contains a pyramid or a prism, it contains a pyramid or a prism that is *smallest* in the sense that there is no pyramid or prism induced by strictly fewer vertices. Smallest pyramids or prisms have properties that make them easier to handle. These properties are expressed in the next two lemmas.

Lemma 3.1 *Let G be a graph. Let K be a smallest pyramid or prism in G . Suppose that K is a pyramid, formed by paths P_1, P_2, P_3 , with triangle $\{b_1, b_2, b_3\}$ and apex a . Let R_1 be a shortest path from b_1 to a whose interior vertices are not adjacent to b_2 or b_3 . Then the subgraph induced by $V(R_1) \cup V(P_2) \cup V(P_3)$ is a smallest pyramid or prism in G .*

Proof. Note that $|V(R_1)| \leq |V(P_1)|$ since P_1 is a path from b_1 to a whose interior vertices are not adjacent to b_2 or b_3 . Let P be the path induced by $(V(P_2) \setminus \{b_2\}) \cup (V(P_3) \setminus \{b_3\})$. If no vertex of $R_1 \setminus \{a\}$ has any neighbour in $P \setminus \{a\}$, then R_1, P_2, P_3 form a pyramid in G , and its number of vertices is not larger than $|V(K)|$, so the lemma holds. So we may assume that some vertex c of $R_1 \setminus \{a\}$ has a neighbour in $P \setminus \{a\}$, and we choose c closest to b_1 along R_1 . Recall that c is not adjacent to b_2 or b_3 , by the definition of R_1 . For $j = 2, 3$, let b'_j be the neighbour of b_j along P_j (so b'_2, b'_3 are the ends of P) and let c_j be the neighbour of c closest to b'_j along P .

Suppose $c_2 = c_3$. We have $c_3 \neq a$ since c has a neighbour along $P \setminus \{a\}$. Then the three chordless paths $c_2-c-R_1-b_1$, c_2-P-b_2 , c_2-P-b_3 form a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c_2 ; this pyramid is strictly smaller than K , because it is included in $(V(R_1) \setminus \{a\}) \cup V(P_2) \cup V(P_3)$, a contradiction. So $c_2 \neq c_3$. If c_2, c_3 are not adjacent, then the three chordless paths $c-R_1-b_1$, $c-c_2-P-b_2$, $c-c_3-P-b_3$ form a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c ; again this pyramid has strictly fewer vertices than K , a contradiction. So c_2, c_3 are adjacent. Then the three chordless paths $c-R_1-b_1$, c_2-P-b_2 and c_3-P-b_3 form a prism K' , with triangles $\{b_1, b_2, b_3\}$ and $\{c, c_2, c_3\}$. If $a \notin \{c_2, c_3\}$ then K' is smaller than K , a contradiction. So $a \in \{c_2, c_3\}$ and the prism K' has the same size as K , so the lemma holds. \square

Lemma 3.2 *Let G be a graph. Let K be a smallest pyramid or prism in G . Suppose that K is a prism, formed by paths P_1, P_2, P_3 , with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$, so that, for $i = 1, 2, 3$, path P_i is from a_i to b_i . Then:*

- *If R_1 is any shortest path from a_1 to b_1 whose interior vertices are not adjacent to b_2 or b_3 , then R_1, P_2, P_3 form a prism of size $|V(K)|$ in G , with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$.*
- *If R_2 is any shortest path from a_1 to b_2 whose interior vertices are not adjacent to b_1 or b_3 , then either the three paths $P_1, R_2 \setminus a_1, P_3$ form a smallest prism in G , or the three paths $P_1, R_2, P_3 + a_1$ form a pyramid of size $|V(K)|$ in G , with triangle $\{b_1, b_2, b_3\}$ and apex a_1 .*

Proof. Let us prove the first item of the lemma. Note that $|V(R_1)| \leq |V(P_1)|$ since P_1 is a path from a_1 to b_1 whose interior vertices are not adjacent to b_2 or b_3 . Let P be the path induced by $(V(P_2) \setminus \{b_2\}) \cup (V(P_3) \setminus \{b_3\})$. If no interior vertex of R_1 is adjacent to any vertex of $V(P)$, then the three paths R_1, P_2, P_3 form a prism in G whose size is not larger than the size of K , so it must be a smallest prism and the lemma holds. So we may assume that there is an interior vertex c of R_1 that has a neighbour in $V(P)$ and we choose c closest to b_1 along R_1 . For $j = 2, 3$, let b'_j be the neighbour of b_j along P_j (so b'_2, b'_3 are the ends of P) and let c_j be the neighbour of c closest to b'_j along P .

Suppose $c_2 = c_3$. Then the three paths $c_2-c-R_1-b_1$, c_2-P-b_2 , c_2-P-b_3 form a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c_2 ; this pyramid is strictly smaller than K (since $|V(R_1 \setminus \{a\})| < |V(P_1)|$), a contradiction. So $c_2 \neq c_3$. If c_2, c_3 are adjacent, then the three paths $c-R_1-b_1$, c_2-P-b_2 , c_3-P-b_3 form a prism, with triangles $\{b_1, b_2, b_3\}$ and $\{c, c_2, c_3\}$, that is strictly smaller than K , a contradiction. So c_2, c_3 are not adjacent. But then the three paths $c-R_1-b_1$, $c-c_2-P-b_2$, $c-c_3-P-b_3$ form a pyramid with triangle $\{b_1, b_2, b_3\}$, apex c , and this pyramid is strictly smaller than K , a contradiction. So the first item is proved.

Now we prove the second item of the lemma. Note that $|V(R_2)| \leq |V(P_2)| + 1$ since $P_2 + a_1$ is a path from a_1 to b_2 whose interior vertices are not adjacent to b_2 or b_3 . Let P be the path induced by $(V(P_1) \setminus \{b_1\}) \cup (V(P_3) \setminus \{b_3\})$. If no interior vertex of R_2 has any neighbour in $V(P \setminus a_1)$ then $P_1, R_2, P_3 + a_1$ form a pyramid, which is not larger than K ; so it is a smallest pyramid and the theorem holds. Now assume that some interior vertex of R_2 has a neighbour in $V(P)$, and choose the vertex c that has this property and is closest to b_2 . For $i = 1, 3$, let b'_i be the neighbour of b_i along P_i (so b'_1, b'_3 are the ends of P) and let c_i be the neighbour of c along P that is closest to b'_i .

Suppose $c_1 = c_3$. Then $c_1 \neq a_1$ since c has a neighbour in $V(P \setminus a_1)$. Then the three paths $c_1-c-R_2-b_2$, c_1-P-b_1 , c_1-P-b_3 form a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c_1 . This pyramid is strictly smaller than K , a contradiction. So $c_1 \neq c_3$. If c_1, c_3 are not adjacent, then the three paths $c-R_2-b_2$, $c-c_1-P-b_1$, $c-c_3-P-b_3$ form a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c ; this pyramid has

size strictly smaller than K , a contradiction. So c_1, c_3 are adjacent. Then the three paths $c-R_2-b_2$, c_1-P-b_1 , c_3-P-b_3 form a prism K' , with triangles $\{b_1, b_2, b_3\}$ and $\{c, c_1, c_3\}$. If $a_1 \notin \{c_1, c_3\}$ then this prism is strictly smaller than K , a contradiction. So $a_1 \in \{c_1, c_3\}$ and K' has the same size as K , and the lemma holds. This completes the proof of the lemma. \square

On the basis of the preceding lemmas we can present an algorithm for testing if a graph contains a pyramid or a prism.

Algorithm 1 (DETECTION OF A PYRAMID OR PRISM)

Input: A graph G .

Output: An induced pyramid or prism of G , if G contains any; else the negative answer “ G contains no pyramid and no prism.”

Method: For every quadruple a, b_1, b_2, b_3 of vertices of G such that b_1, b_2, b_3 are pairwise adjacent and a is adjacent to at most one of them, do: Compute a shortest path P_1 from a to b_1 whose interior vertices are not adjacent to b_2, b_3 , if any. Compute paths P_2 and P_3 similarly. If the three paths P_1, P_2, P_3 exist, and if $V(P_1) \cup V(P_2) \cup V(P_3)$ induces a pyramid or a prism, then return this subgraph of G , and stop.

If no quadruple has produced a pyramid or a prism, return the negative answer.

Complexity: $O(|V(G)|^6)$.

Proof of correctness. If G contains no pyramid and no prism then clearly the algorithm will return the negative answer. Conversely, suppose that G contains a pyramid or a prism. Let K be a smallest pyramid or prism. Let b_1, b_2, b_3 be the vertices of a triangle of K , and let a be such that if K is a pyramid then a is its apex and if K is a prism then a is a vertex of the other triangle of K . When our algorithm considers the quadruple a, b_1, b_2, b_3 , it will find paths P_1, P_2, P_3 since some paths in K do have the required properties. Then, three applications of lemmas 3.1 and 3.2 imply that P_1, P_2, P_3 do form a pyramid or a prism of G . So the algorithm will detect this subgraph.

Complexity analysis: Testing all quadruples take time $O(|V(G)|^4)$. For each quadruple, finding the three paths takes time $O(|V(G)|^2)$ and checking that the corresponding subgraph is a pyramid or prism takes time $O(|V(G)|)$. Thus the overall complexity is $O(|V(G)|^6)$. \square

We now show how the results of the preceding algorithm can be performed a little bit faster with a simple trick.

Lemma 3.3 *Let H be a connected graph and let V_1, V_2, V_3 be non-empty subsets of $V(H)$. Then H has an induced subgraph F such that either:*

1. F is a chordless path such that, up to a permutation of V_1, V_2, V_3 , one end of F is in V_1 , the other is in V_3 , some vertex of F is in V_2 and no interior vertex of F is in $V_1 \cup V_3$;
2. F consists of three chordless paths F_1, F_2, F_3 of length at least 1 such that: for $i = 1, 2, 3$, F_i is from f to v_i and $v_i \in V_i$; for $1 \leq i < j \leq 3$, $V(F_i) \cap V(F_j) = \{f\}$ and there is no edge between $F_i \setminus f$ and $F_j \setminus f$; and $F \setminus \{v_1, v_2, v_3\}$ contains no vertex of $V_1 \cup V_2 \cup V_3$;
3. F consists of three vertex-disjoint chordless paths F_1, F_2, F_3 (possibly of length 0) such that: for $i = 1, 2, 3$, F_i is from w_i to v_i and $v_i \in V_i$; vertices w_1, w_2, w_3 are pairwise adjacent; for $1 \leq i < j \leq 3$ there is no edge between F_i and F_j other than $w_i w_j$; and $F \setminus \{v_1, v_2, v_3\}$ contains no vertex of $V_1 \cup V_2 \cup V_3$.

Proof. Let P be a shortest path in H such that P has one end in V_1 and the other in V_3 ; let $v_1 \in V_1, v_3 \in V_3$ be the ends of P . Thus no interior vertex of P is in $V_1 \cup V_3$. If P contains a vertex of V_2 then we have outcome 1 of the lemma with $F = P$. Therefore let us assume that P contains no vertex of V_2 . Let Q be a shortest path such that one end v_2 of Q is in V_2 and the other end v of Q has a neighbour on P . Let w, x be the neighbours of v on P that are closest to v_1 and v_3 respectively. Note that $Q \setminus v_2$ contains no vertex of V_2 by the definition of Q . If Q contains vertices of both V_1, V_3 then some subpath F of Q contains vertices of each of V_1, V_2, V_3 and is minimal with this property, and so F satisfies outcome 1 of the lemma. If Q contains vertices of V_1 and not of V_3 , then v_3 - P - x - v - Q - v_2 is a path F that satisfies outcome 1. A similar outcome happens if Q contains vertices of V_3 and not of V_1 . So we may assume that Q contains no vertex of $V_1 \cup V_3$.

Suppose $w = x$. If $x \in \{v_1, v_3\}$, we have outcome 1 with $F = P + Q$. If $x \notin \{v_1, v_3\}$, the three paths x - P - v_1 , x - v - Q - v_2 , x - P - v_3 form a subgraph F that satisfies outcome 2. Now suppose that w, x are different and not adjacent. If $v = v_2$, then v_1 - P - w - v_2 - x - P - v_3 is a path F that satisfies outcome 1. If $v \neq v_2$, the three paths v - w - P - v_1 , v - Q - v_2 , v - x - P - v_3 form a subgraph F that satisfies outcome 2. Finally, suppose that w, x are different and adjacent. Then the three paths w - P - v_1 , v - Q - v_2 , x - P - v_3 form a subgraph F that satisfies the properties of outcome 3. This completes the proof of the lemma. \square

Now we can give an algorithm:

Algorithm 2 (DETECTION OF A PYRAMID OR PRISM)

Input: A graph G .

Output: The positive answer “ G contains a pyramid or a prism” if it does; else the negative answer “ G contains no pyramid and no prism.”

Method: For every triple b_1, b_2, b_3 of vertices of G such that b_1, b_2, b_3 are pairwise adjacent, do:

Step 1. Compute the set X_1 of those vertices of $V(G)$ that are adjacent to b_1 and not adjacent to b_2 or b_3 , and the similar sets X_2, X_3 , and compute the set X of those vertices of $V(G)$ that are not adjacent to any of b_1, b_2, b_3 . If some vertex of any X_i has a neighbour in each of the other two X_j 's, return the positive answer and stop. Else:

Step 2. Compute the connected components of X in G .

Step 3. For each component H of X , and for $i = 1, 2, 3$, if some vertex of H has a neighbour in X_i then mark H with label i . If any component H of X gets the three labels 1, 2, 3, return the positive answer and stop.

If no triple yields the positive answer, return the negative answer.

Complexity: $O(|V(G)|^5)$.

Proof of correctness. Suppose that G contains a pyramid or a prism K . Let b_1, b_2, b_3 be the vertices of a triangle of K , and for $i = 1, 2, 3$ let c_i be the neighbour of b_i in $K \setminus \{b_1, b_2, b_3\}$. The algorithm will place the three vertices c_1, c_2, c_3 in the sets X_1, X_2, X_3 respectively, one vertex in each set. If K has only six vertices, the algorithm will find that one of the c_i 's is adjacent to the other two, so it will return the positive answer at the end of Step 1. If K has at least seven vertices, then the algorithm will place the vertices of $K' = K \setminus \{b_1, b_2, b_3, c_1, c_2, c_3\}$ in X ; at Step 2 these vertices will all be in one component of X since K' is connected, and at Step 3 this component will get the three labels 1, 2, 3 since K' contains a neighbour of c_i for each $i = 1, 2, 3$, so the algorithm will return the positive answer.

Conversely, suppose that the algorithm returns the positive answer when it is examining a triple $\{b_1, b_2, b_3\}$ that induces a triangle of G . If this is at the end of Step 1, this means that, up to a permutation of $\{1, 2, 3\}$, the algorithm has found a vertex $c_1 \in X_1$ that has a neighbour $c_2 \in X_2$ and a neighbour $c_3 \in X_3$. Then the six vertices $b_1, b_2, b_3, c_1, c_2, c_3$ induce a pyramid if c_2, c_3 are not adjacent or a prism if c_2, c_3 are adjacent; so the positive answer is correct. Now suppose that the positive answer is returned at the end of step 3. This means that some component H of X gets the three labels 1, 2, 3. So, for each $i = 1, 2, 3$, the set V_i of vertices of H that have a neighbour in X_i is not empty. We can apply Lemma 3.3 to H , with the same notation, and we consider the subgraph F of H described in the lemma, which leads to the following three cases. In each case we will see that G contains a prism or a pyramid.

Outcome 1 of Lemma 3.3: F is a chordless path such that, up to a permutation of V_1, V_2, V_3 , one end of F is a vertex $v_1 \in V_1$, the other is a vertex $v_3 \in V_3$, no interior vertex of F is in $V_1 \cup V_3$, and F has a vertex of V_2 . There exists a neighbour c_1 of v_1 in X_1 , a neighbour c_3 of v_3 in X_3 , and a vertex c_2 of X_2 that has a neighbour in F . Note that there is at most one edge among c_1, c_2, c_3 , for otherwise we would have stopped at Step 1. Let x, y be the neighbours of c_2 along F that are closest respectively to v_1 and v_3 . If c_1, c_2 are adjacent and

$y \neq v_1$ then $c_2-c_1-b_1, c_2-b_2, c_2-y-F-v_3-c_3-b_3$ form a pyramid, while if c_1, c_2 are adjacent and $y = v_1$ then $c_1-b_1, c_2-b_2, v_1-F-v_3-c_3-b_3$ form a prism. So suppose c_2 is not adjacent to c_1 and likewise not to c_3 . If c_1, c_3 are adjacent and $v_1 = v_3$ then $c_1-b_1, v_1-c_2-b_2, c_3-b_3$ form a prism. If c_1, c_3 are adjacent and $v_1 \neq v_3$ then either $x \neq v_3$ or $y \neq v_1$, so let us assume up to symmetry that $x \neq v_3$; then $c_1-b_1, c_1-v_1-F-x-c_2-b_2, c_1-c_3-b_3$ form a pyramid. So suppose c_1, c_3 are not adjacent. If $x = y$ then $x-F-v_1-c_1-b_1, x-c_2-b_2, x-F-v_3-c_3-b_3$ form a pyramid. If x, y are different and not adjacent, then $c_2-x-F-v_1-c_1-b_1, c_2-b_2, c_2-y-F-v_3-c_3-b_3$ form a pyramid. If x, y are different and adjacent, then $x-F-v_1-c_1-b_1, c_2-b_2, y-F-v_3-c_3-b_3$ form a prism.

Outcome 2) of Lemma 3.3, with the same notation. For $i = 1, 2, 3$, there exists a neighbour c_i of v_i in X_i . Since the vertices v_1, v_2, v_3 are pairwise different, for each $i = 1, 2, 3$, vertex c_i has no other neighbour in F than v_i . If c_1, c_2 are adjacent, then $c_1-b_1, c_1-c_2-b_2, c_1-v_1-F_1-f-F_3-v_3-c_3-b_3$ form a pyramid. So suppose, by symmetry, that c_1, c_2, c_3 are pairwise not adjacent. Then for $i = 1, 2, 3$ the paths $f-F_i-v_i-c_i-b_i$ form a pyramid.

Outcome 3) of Lemma 3.3, with the same notation. For $i = 1, 2, 3$, there exists a neighbour c_i of v_i in X_i . Since the vertices v_1, v_2, v_3 are pairwise different, for each $i = 1, 2, 3$ vertex c_i has no other neighbour in F than v_i . If c_1, c_2 are adjacent, then $c_1-b_1, c_1-c_2-b_2, c_1-v_1-F_1-w_1-w_3-F_3-v_3-c_3-b_3$ form a pyramid. So suppose, by symmetry, that c_1, c_2, c_3 are pairwise non adjacent. Then for $i = 1, 2, 3$, the paths $w_i-F_i-v_i-c_i-b_i$ form a prism. So in either case G contains a pyramid or a prism, and the proof of correctness is complete.

Complexity analysis: Finding all triples takes time $O(|V(G)|^3)$. For each triple, computing the sets X_1, X_2, X_3, X takes time $O(|V(G)|)$. Finding the components of X takes time $O(|V(G)|^2)$. Marking the components can be done as follows: for each edge uv of G , if u is in a component H of X and v is in some X_i then mark H with label i ; so this takes time $O(|V(G)|^2)$. Thus the overall complexity is $O(|V(G)|^5)$. \square

We observe that the above two algorithms are faster than the algorithm from [8] for finding a pyramid.

4 Recognition of graphs in class \mathcal{A}

We can now present the algorithm for recognizing graphs in the class \mathcal{A} .

Algorithm 3 (RECOGNITION OF GRAPHS IN CLASS \mathcal{A})

Input: A graph G .

Output: The positive answer “ G is in class \mathcal{A} ” if it is; else the negative answer “ G is not in class \mathcal{A} ”.

Method:

Step 1. Test whether G contains no antihole of length at least 5 as explained at the end of the introduction.

Step 2. Test whether G has no pyramid or prism using Algorithm 2 above.

Step 3. Test whether G is Berge using the algorithm from the preceding section.

Complexity: $O(|V(G)|^9)$.

The correctness of the algorithm is immediate from the correctness of the algorithms it refers to and from the fact that Berge graphs contain no pyramid. The complexity is dominated by the last step of the Berge recognition algorithm, which is $O(|V(G)|^9)$. Note that the other step of complexity $O(|V(G)|^9)$ in the Berge recognition algorithm (deciding if the input graph contains a pyramid) can be replaced by Step 2. Additionally, we can remark that it is not necessary to test for the existence of configurations of types T1, ..., T4 when we call the Berge recognition algorithm, because—this is not very hard to prove—any such configuration contains an antihole of length at least 5, so it is already excluded by Step 2. But this does not bring the overall complexity down from $O(|V(G)|^9)$.

The algorithm for recognizing graphs in class \mathcal{A} can also be used to color graphs in class \mathcal{A} . Recall that Theorem 1.2 states that: *If a graph G is in class \mathcal{A} and is not a clique, it admits a pair of vertices whose contraction yields a graph in class \mathcal{A} .* Therefore we could enumerate all pairs of non-adjacent vertices of G and test whether their contraction produces a graph in class \mathcal{A} ; Theorem 1.2 insures that at least one pair will work. We can then iterate this procedure until the contractions turn the graph into a clique. Since each vertex of the clique is the result of contracting a stable set of G , a coloring of this clique corresponds to an optimal coloring of G . In terms of complexity, we may need to check $O(|V(G)|^2)$ pairs at each contraction step, and there may be $O(|V(G)|)$ steps. So we end up with complexity $O(|V(G)|^{12})$. This is not as good as the direct method from [14], which has complexity $O(|V(G)|^6)$.

5 Even prisms

In this section we show how to decide in polynomial-time if a graph that contains no odd hole contains an even prism. Let K be an even prism, formed by paths P_1, P_2, P_3 , with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ so that for $1 \leq i \leq 3$ path P_i is from a_i to b_i . Let m_i be the middle vertex of path P_i . We say that the 9-tuple $(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3)$ is the *frame* of K . When we talk about a prism, the word *small* refers to its number of vertices.

Lemma 5.1 *Let G be a graph that contains no odd hole and contains an even prism, and let K be a smallest even prism in G . Let K be formed by paths*

P_1, P_2, P_3 and have frame $(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3)$, with $a_i, m_i, b_i \in V(P_i)$ ($1 \leq i \leq 3$). Let R be any path of G whose ends are a_1, m_1 , whose interior vertices are not adjacent to a_2, a_3, b_2 or b_3 , and which is shortest with these properties. Then $a_1-R-m_1-P_1-b_1$ is a chordless path R_1 and R_1, P_2, P_3 form a smallest even prism in G .

Proof. Let k be the length (number of edges) of path P_1 ; so k is even. Note that $|E(R)| \leq k/2$ since the path $a_1-P_1-m_1$ satisfies the properties required for R . Call Q the chordless path induced by $V(P_2) \cup V(P_3) \setminus \{a_2, a_3\}$ and call a'_2, a'_3 the ends of Q so that for $j = 2, 3$ vertex a'_j is adjacent to a_j .

Suppose that no interior vertex of R has any neighbour in Q . Let R_1 be a shortest path from a_1 to b_1 contained in $a_1-R-m_1-P_1-b_1$. So $|E(R_1)| \leq k$ and R_1, P_2, P_3 form a prism K' with $|V(K')| \leq |V(K)|$. Since G contains no odd hole, R_1 has even length (else $V(R_1) \cup V(P_2)$ would induce an odd hole), so K' is an even prism. Thus K' is a smallest even prism, and we have equality in the above inequalities; in particular R_1 is equal to $a_1-R-m_1-P_1-b_1$ and the theorem holds.

We may now assume that some vertex c of R has a neighbour in Q , and we choose c closest to m_1 along R . Let S be a chordless path from c to b_1 contained in $c-R-m_1-P_1-b_1$. We have $|E(S)| < k$ since $|E(R)| \leq k/2$ and $c \neq a_1$. By the choice of c no vertex of $S \setminus b_1$ has a neighbour in P_2 or P_3 . Let x, y be the neighbours of c along Q that are closest respectively to a'_2 and to a'_3 . If $x = y$ then $V(S) \cup V(P_2) \cup V(P_3)$ induces a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex x , so G contains an odd hole, a contradiction. Thus $x \neq y$. If x, y are not adjacent then $V(S) \cup V(P_2) \cup V(P_3)$ contains a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c , a contradiction. So x, y are different and adjacent and, up to symmetry, we may assume that they lie in the interior of P_2 . Now $V(S) \cup V(P_2) \cup V(P_3)$ induces a prism K' , with triangles $\{b_1, b_2, b_3\}$ and $\{c, x, y\}$, and $|V(K')| < |V(K)|$ since $|E(S)| < k$. Thus K' is an odd prism, which means that $y-P_2-b_2$ is an odd path, and so a_2-P_2-x is an even path. Let R' be a chordless path from c to a_1 contained in $c-R-m_1-P_1-a_1$. We have $|E(R')| < k$ since $|E(R)| \leq k/2$. By the choice of c no vertex of $R' \setminus a_1$ has a neighbour in P_2 or P_3 . Then R' has even length for otherwise $V(R') \cup V(a_2-P_2-x)$ induces an odd hole. Now $V(R') \cup V(P_2) \cup V(P_3)$ induces a prism K'' with triangles $\{a_1, a_2, a_3\}$ and $\{c, x, y\}$, and K'' is an even prism, and we have $|V(K'')| < |V(K)|$ since $|E(R')| < k$. This is a contradiction, which completes the proof. \square

Now we can give an algorithm:

Algorithm 4 (DETECTION OF AN EVEN PRISM IN A GRAPH THAT CONTAINS NO ODD HOLE)

Input: A graph G that contains no odd hole.

Output: An induced even prism of G if G contains any; else the negative answer “ G does not contain an even prism.”

Method: For every 9-tuple $(a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3)$ of vertices of G such that $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ induce triangles, do: For $i = 1, 2, 3$, compute the set F_i of those vertices that are not adjacent to $a_{i+1}, a_{i+2}, b_{i+1}, b_{i+2}$ (with indices modulo 3); look for a shortest path R_i from a_i to m_i whose interior vertices are in F_i , and look for a shortest path S_i from m_i to b_i whose interior vertices are in F_i . If the six paths $R_1, R_2, R_3, S_1, S_2, S_3$ exist and their vertices induce an even prism, then return this prism and stop.

If no 9-tuple yields an even prism, return the negative answer.

Complexity: $O(|V(G)|^{11})$.

Proof of correctness. If the algorithm returns an even prism then clearly G contains this prism. So suppose conversely that G contains an even prism. Let K be a smallest even prism, and let vertices $a_1, a_2, a_3, b_1, b_2, b_3, m_1, m_2, m_3$ be the frame of K . When the algorithm considers this 9-tuple, it will find paths $R_1, R_2, R_3, S_1, S_2, S_3$ since some paths in K do have the required properties. Then, six applications of Lemma 5.1 imply that the vertices of these six paths induce an even prism of G . So the algorithm will detect this subgraph.

Complexity analysis: Testing all 9-tuples take time $O(|V(G)|^9)$. For each 9-tuple, finding the six paths takes time $O(|V(G)|^2)$ and checking that the corresponding subgraph is an even prism takes time $O(|V(G)|)$. Thus the overall complexity is $O(|V(G)|^{11})$. \square

6 Line-graphs of subdivisions of K_4

The *line-graph* of a graph R is the graph whose vertices are the edges of R and where two vertices are adjacent if the corresponding edges of R have a common endvertex. *Subdividing* an edge xy in a graph means replacing it by a path of length at least two. A *subdivision* of a graph R is any graph obtained by repeatedly subdividing edges. Berge graphs that do not contain the line-graph of a bipartite subdivision of K_4 play an important role in the proof of the Strong Perfect Graph Theorem [7]. Thus recognizing them may be of interest on its own. Moreover, solving this question is also useful for later use in the recognition of graphs in the class \mathcal{A}' (see Section 7). Again it turns out that decide if a graph contains the line-graph of a subdivision of K_4 is NP-complete in general, see Section 8.

We will first deal with subdivisions of K_4 that are not necessarily bipartite, but are not too trivial in the following sense: say that a subdivision of K_4 is *proper* if at least one edge of the K_4 is subdivided. It is easy to see that the line-graph of a subdivision of K_4 is proper if and only if it has a vertex that lies in only one triangle. If F is the line-graph of a proper subdivision R of K_4 , let us denote by a, b, c, d the four vertices of K_4 , i.e., the vertices of degree 3 in R . Then the three edges incident to each vertex $x = a, b, c, d$ form a triangle in F ,

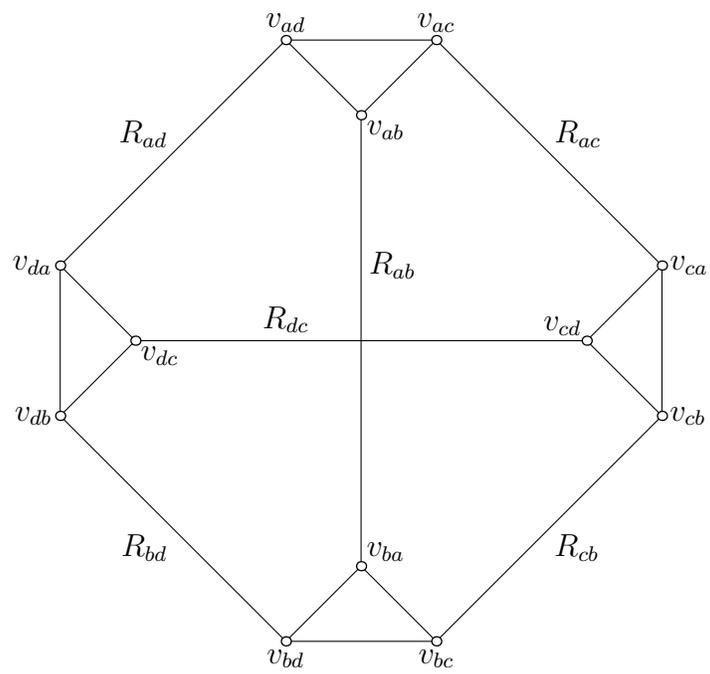


Figure 2: Line-graph of a subdivision of K_4

which will be labelled T_x and called a *basic triangle* of F . (F may have as many as two more, non-basic, triangles.) In F there are six paths, each path being between vertices x, y of distinct triangles of F (and so this path can be labelled R_{xy} accordingly). Note that $R_{xy} = R_{yx}$, and the six distinct paths are vertex disjoint. Some of these paths may have length 0. In the basic triangle T_x , we denote by v_{xy} the vertex that is the end of the path R_{xy} . Thus F has paths $R_{ab}, R_{ac}, R_{ad}, R_{bc}, R_{bd}, R_{cd}$, and the vertices of the basic triangles of F are $v_{ab}, v_{ac}, v_{ad}, v_{ba}, v_{bc}, v_{bd}, v_{ca}, v_{cb}, v_{cd}, v_{da}, v_{db}$ and v_{dc} . The subgraph F has no other edge than those in the four basic triangles and those in the six paths.

For each of the six paths R_{xy} of F , we call m_{xy} one vertex that is roughly in the middle of R_{xy} , so that if α denotes the length of $v_{xy}-R_{xy}-m_{xy}$ and β denotes the length of $m_{xy}-R_{xy}-v_{yx}$, then $\alpha - \beta \in \{-1, 0, 1\}$. Paths R_{xy} are called the *rungs* of F ; vertices v_{xy} are called the *corners* of F ; and the 18-tuple $(v_{ab}, v_{ac}, \dots, v_{cd}, m_{ab}, \dots, m_{cd})$ is called a *frame* of F .

Lemma 6.1 *Let G be a graph that contains no pyramid. Let F be an induced subgraph of G that is the line-graph of a proper subdivision of K_4 and F has smallest size with this property, and let $(v_{ab}, v_{ac}, \dots, v_{cd}, m_{ab}, \dots, m_{cd})$ be a frame of F . Let P be a path from v_{ab} to m_{ab} such that the interior vertices of P are non adjacent to every corner of F other than v_{ab} and P is a shortest path with these properties. Then $(V(F) \setminus V(R_{ab})) \cup V(P)$ induces the line-graph of a proper subdivision of K_4 of smallest size.*

Proof. Put $F' = F \setminus R_{ab}$. If v_{ab}, m_{ab} are equal or adjacent, then $P = v_{ab}-R_{ab}-m_{ab}$ and the conclusion is immediate. So we may assume that v_{ab}, m_{ab} are distinct and not adjacent, which also implies $m_{ab} \neq v_{ba}$.

Claim 6.2 *If the interior vertices of P have no neighbour in F' then the lemma holds.*

Proof. Let u be the vertex of $v_{ab}-P-m_{ab}$ that has neighbours in $m_{ab}-R_{ab}-v_{ba}$ and is closest to v_{ab} . Let u' be the neighbour of u in $m_{ab}-R_{ab}-v_{ba}$ closest to v_{ba} . Then $v_{ab}-P-u-u'-R_{ab}-v_{ba}$ is a chordless path R , and $V(F') \cup V(R)$ induce the line-graph of a proper subdivision of K_4 . So this subgraph has size at least the size of F , which is possible only if $u = m_{ab}$, and this case $V(F') \cup V(R)$ induce the line-graph of a proper subdivision of K_4 of smallest size, so the lemma holds. \square

Now we may assume that there exists a vertex $c_1 \in V(P)$ that has neighbours in F' , and choose c_1 closest to v_{ab} along P . Also there exists a vertex $d_1 \in V(P)$ that has neighbours in F' and is chosen closest to m_{ab} along P . Let us show that this leads to a contradiction. One may look at Figure 3.

Claim 6.3

1. The set $N(c_1) \cap V(F')$ consists of an edge of F' .
2. The set $N(d_1) \cap F'$ consists of an edge of F' .

Proof. Call H the hole induced by $V(R_{ac}) \cup V(R_{bc}) \cup V(R_{bd}) \cup V(R_{ad})$.

First suppose that c_1 has no neighbour on H . So c_1 has neighbours in the interior of R_{cd} . Let c_2, c_3 be the neighbours of c_1 respectively closest to v_{cd} and to v_{dc} along R_{cd} . If $c_2 = c_3$, the three paths $c_2-c_1-P-v_{ab}$, $c_2-R_{cd}-v_{cd}-v_{ca}-R_{ca}-v_{ac}$, $c_2-R_{cd}-v_{dc}-v_{da}-R_{ad}-v_{ad}$ form a pyramid with triangle $\{v_{ab}, v_{ac}, v_{ad}\}$ and apex c_2 , a contradiction. If c_2, c_3 are distinct and not adjacent, the three paths c_1-P-v_{ab} , $c_1-c_2-R_{cd}-v_{cd}-v_{ca}-R_{ca}-v_{ac}$, $c_1-c_3-R_{cd}-v_{dc}-v_{da}-R_{ad}-v_{ad}$ form a pyramid with triangle $\{v_{ab}, v_{ac}, v_{ad}\}$, and apex c_1 , a contradiction. If c_2, c_3 are adjacent, we have item 1 of the claim.

Now suppose that c_1 has neighbours on H . Define two chordless subpaths of H : $H_{ac} = H \setminus v_{ad}$ and $H_{ad} = H \setminus v_{ac}$. Let c_2 be the neighbour of c_1 on H_{ac} closest to v_{ac} , and let c_3 be the neighbour of c_1 on H_{ad} closest to v_{ad} . If $c_2 = c_3$ then $V(H) \cup V(c_1-P-v_{ab})$ induces a pyramid with triangle $\{v_{ab}, v_{ac}, v_{ad}\}$ and apex c_2 , a contradiction. So $c_2 \neq c_3$. If c_2, c_3 are not adjacent then the three paths c_1-P-v_{ab} , $c_1-c_2-H_{ac}-v_{ac}$, $c_1-c_3-H_{ad}-v_{ad}$ form a pyramid with triangle $\{v_{ab}, v_{ac}, v_{ad}\}$ and apex c_1 , a contradiction. So c_2, c_3 are adjacent and are the only neighbours of c_1 on H . Up to a symmetry, and by the definition of R , we may assume that c_2, c_3 are in the interior of R_{ac} or R_{bc} . If c_1 has no neighbour on R_{cd} then conclusion 1 holds. So suppose that c_1 has a neighbour c_4 on R_{cd} and c_4 is closest to v_{dc} . Then the three paths c_1-P-v_{ab} , $c_1-c_4-R_{cd}-v_{dc}-v_{da}-R_{da}-v_{ad}$, $c_1-c_2-H_{ac}-v_{ac}$ form a pyramid with triangle $\{v_{ab}, v_{ac}, v_{ad}\}$ and apex c_1 , a contradiction. This complete the proof of item 1.

The proof of item 2 is similar, with the following adjustment: whenever path c_1-P-v_{ab} was used for item 1, we can use for item 2 a chordless path from d_1 to v_{ba} contained in $d_1-P-m_{ab}-R_{ab}-v_{ba}$. This completes the proof of the claim. \square

Claim 6.4 *If J is the line-graph of a subdivision of K_4 with $V(J) \subseteq V(F') \cup V(P)$ and c_1 is a corner of J , then J is the line-graph of a proper subdivision of K_4 .*

Proof. This claim follows immediately from the fact that c_1 belongs to exactly one triangle of J . \square

In view of Claim 6.3, let c_2, c_3 be the two neighbours of c_1 in F' and d_2, d_3 be the two neighbours of d_1 in F' , with $c_2c_3, d_2d_3 \in E(G)$.

Claim 6.5 *We may assume that c_2, c_3 lie in R_{ac} and d_2, d_3 in R_{cb} or R_{bd} .*

Proof. Recall from the definition of P that c_2, c_3, d_2, d_3 cannot be corners of F . If c_2c_3 is an edge of R_{cd} , then $V(v_{ab}-P-c_1) \cup V(R_{ac}) \cup V(R_{ad}) \cup V(R_{cd})$ induces the line-graph of a subdivision of K_4 , which is proper by Claim 6.4 and is strictly smaller than F , a contradiction. If c_2c_3 is an edge of R_{bc} , then $V(v_{ab}-P-c_1) \cup V(F')$ induces the line-graph of a subdivision of K_4 , which is proper by Claim 6.4 and is strictly smaller than F , a contradiction. So c_2c_3 is an edge of

R_{ac} or R_{ad} . Similarly we may assume that d_2d_3 is an edge of R_{bc} or R_{bd} . Then by symmetry the claim holds. \square

We may assume that $v_{ac}, c_2, c_3, v_{ca}, d_2, d_3, v_{ad}$ appear in this order along H .

Claim 6.6 *Vertices c_1, d_1 are distinct and not adjacent.*

Proof. By Claims 6.3 and 6.5, we know that c_1, d_1 are distinct. If they are adjacent, the set $V(H') \cup \{c_1, d_1\}$ induces the line-graph of a subdivision of K_4 , which is proper by Claim 6.4 and is strictly smaller than F , a contradiction. \square

Let e_1 be the vertex of c_1-P-v_{ab} that has a neighbour e_2 in the interior of $m_{ab}-R_{ab}-v_{ab}$ and is closest to c_1 . Let e_4 be the vertex of d_1-P-m_{ab} that has neighbour e_3 in the interior of $m_{ab}-R_{ab}-v_{ab}$, and is closest to d_1 . Given e_1, e_4 , take e_2, e_3 as close to each other as possible along R_{ab} .

Claim 6.7 $e_1 \neq v_{ab}$.

Proof. For suppose $e_1 = v_{ab}$. Then the three paths $v_{ab}-P-c_1$, $v_{ab}-v_{ac}-R_{ac}-c_2$, $v_{ab}-R_{ab}-e_3-e_4-P-d_1-d_2-H_{ac}-c_3$ form a pyramid with triangle $\{c_1, c_2, c_3\}$ and apex v_{ab} , a contradiction. \square

At this point we have obtained that $c_1-P-e_1-e_2-R_{ab}-e_3-e_4-P-d_1$ is a chordless path R whose interior vertices have no neighbour in F' . Moreover the subgraph F_R induced by $V(F') \cup V(R)$ is the line graph of a subdivision of K_4 , and it is proper by Claim 6.4.

Claim 6.8 $|V(F_R)| < |V(F)|$.

Proof. We need only show that the total length of the rungs of F_R is strictly smaller than the total length of the rungs of F . Let α be the length of $v_{ab}-R_{ab}-m_{ab}$, let β be the length of $v_{ba}-R_{ab}-m_{ab}$, and let δ be the number of those edges of F' that belong to the rungs of F .

The total length l of the rungs of F is equal to $\alpha + \beta + \delta = 2\alpha - \varepsilon + \delta$, with $\varepsilon = \alpha - \beta \in \{-1, 0, 1\}$ by the definition of m_{ab} .

The total length l_R of the rungs of F_R is at most $\delta + 2\alpha - 3$, and it is equal to this value only in the following case: $e_4 = m_{ab}$, there is only one vertex of R_{ab} between c_1 and d_1 , $e_1v_{ab} \in E(G)$, $e_2v_{ab} \in E(G)$, and the paths P and $v_{ab}-R_{ab}-m_{ab}$ have the same length. Indeed in this case the length of the rung of F_R whose ends are c_1, d_1 is equal to $2\alpha - 3$.

Thus in either case we have $l_R < l$ and the claim holds. \square

Now the preceding claim leads to a contradiction, which proves the lemma. \square

Lemma 6.1 is the basis of an algorithm for deciding if a graph contains a pyramid or the line-graph of a proper subdivision of K_4 .

Algorithm 5 (DETECTION OF A LINE-GRAPH OF A PROPER SUBDIVISION OF K_4 IN A GRAPH THAT CONTAINS NO PYRAMID)

Input: A graph G that contains no pyramid.

Output: An induced subgraph of G that is the line-graph of a proper subdivision of K_4 (if G contains any); else the negative answer “ G does not contain the line-graph of a proper subdivision of K_4 ”.

Method: For every 18-tuple of vertices $(v_{ab}, v_{ac}, \dots, v_{cd}, m_{ab}, \dots, m_{cd})$, do the following:

For each $i, j \in \{a, b, c, d\}$ with $i \neq j$, find a shortest path S_{ij} from v_{ij} to m_{ij} ;

If the subgraph induced by the union of the twelve paths S_{ij} ($i, j \in \{a, b, c, d\}, i \neq j$) is the line-graph of a proper subdivision of K_4 , return this subgraph and stop.

If no 18-tuple has produced such a subgraph, return the negative answer.

Complexity: $O(|V(G)|^{20})$.

Proof of correctness. When the algorithm returns the line-graph of a proper subdivision of K_4 , clearly this answer is correct.

Conversely, suppose that G contains the line-graph of a proper subdivision of K_4 . Then G has an induced subgraph F that is the line-graph of a proper subdivision of K_4 and has minimal size.

At some step the algorithm will consider an 18-tuple $(v_{ab}, v_{ac}, \dots, v_{cd}, m_{ab}, \dots, m_{cd})$ which is a frame of F . The algorithm will find the paths S_{ij} since the corresponding paths of F do have the required properties. With twelve applications of Lemma 6.1, it follows that the subgraph formed by these twelve paths is the line-graph of a proper subdivision of K_4 and is actually a smallest such subgraph. So the algorithm will detect this subgraph.

Complexity analysis: There are $O(|V(G)|^{18})$ frames to test. For each such subset, finding the shortest paths S_{ij} takes time $O(|V(G)|^2)$, and checking that the subgraph they form is the line-graph of a proper subdivision of K_4 takes time $O(|V(G)|)$. Thus the algorithm finishes in time $O(|V(G)|^{20})$. \square

Let us now focus on finding line-graphs of *bipartite* subdivisions of K_4 .

Lemma 6.9 *Let R be a subdivision of K_4 and F be the line-graph of R . Then either $R = K_4$, or F contains an odd hole, or R is a bipartite subdivision of K_4 .*

Proof. Suppose $R \neq K_4$. Call a, b, c, d the four vertices of the K_4 of which R is a subdivision (i.e., the vertices of degree 3 in R), and for $i, j \in \{a, b, c, d\}$ with $i \neq j$, call C_{ij} the subdivision of edge ij . Suppose that F contains no odd hole and R is not bipartite. Then R contains an odd cycle Z . This cycle must be a

triangle, for otherwise $L(R)$ contains an odd hole, a contradiction. So we may assume up to symmetry that a, b, c induce a triangle. Since $R \neq K_4$, we may assume that C_{ad} has length at least 2. But then one of $E(C_{ad}) \cup \{ad\} \cup E(C_{cd})$ or $E(C_{ad}) \cup \{ab\} \cup \{bc\} \cup C_{cd}$ is the edge set of an odd cycle of R , of length at least 5, so $L(R)$ contains an odd hole, a contradiction. \square

Now we can devise an algorithm that decides if a graph with no odd hole contains the line-graph of a bipartite subdivision of K_4 . This algorithm is simply Algorithm 5 applied to graphs that contain no odd hole, by the preceding lemma.

7 Recognition of graphs in class \mathcal{A}'

To decide if a graph is in class \mathcal{A}' , it suffices to decide separately if it is Berge, if it has an antihole of length at least 5, and if it contains an odd prism. But again it turns out that this third question—deciding if a graph contains an odd prism—is NP-complete (see Section 8). However, we can decide in polynomial time if a graph with no odd hole contains an odd prism. For this purpose the next lemmas will be useful.

Lemma 7.1 *Let F be the line-graph of a bipartite subdivision of K_4 . Then F contains an odd prism.*

Proof. Let R be a bipartite subdivision of K_4 such that F is the line-graph of R , and let a, b, c, d be the four vertices of degree 3 in R . We may suppose without loss of generality that a, b lie on the same side of the bipartition of R . Thus edge ab is subdivided to a path R_{ab} of even length, with the usual notation. Now it is easy to see that $F \setminus V(R_{cd})$ is an odd prism. \square

Before we present an algorithm for recognizing graphs in class \mathcal{A}' , we can remark that the technique which worked well for detecting even prisms tends to fail for odd prisms. The graph featured in Figure 4 illustrates this problem. This graph G is the line-graph of a bipartite graph, so it is a Berge graph. For any two grey triangles, there exists one (and only one) odd prism that contain these two triangles. Moreover, the paths P_1, P_2, P_3 form an odd prism of G of minimal size. Yet, replacing P_1 (or the path $a_1-P_1-m_1$) by a shortest path with the same ends does not produce an odd prism. Thus an algorithm that would be similar to the even prism testing algorithm presented above may work incorrectly. We note however that in this example the graph G contains the line-graph of a proper subdivision of K_4 (the subgraph obtained by forgetting the black vertices). The next lemma shows that this remark holds in general.

Lemma 7.2 *Let G be a graph that contains no odd hole and no line-graph of a proper subdivision of K_4 . Let H be a prism in G , with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Let P be any chordless path from a_1 to b_1 whose interior vertices*

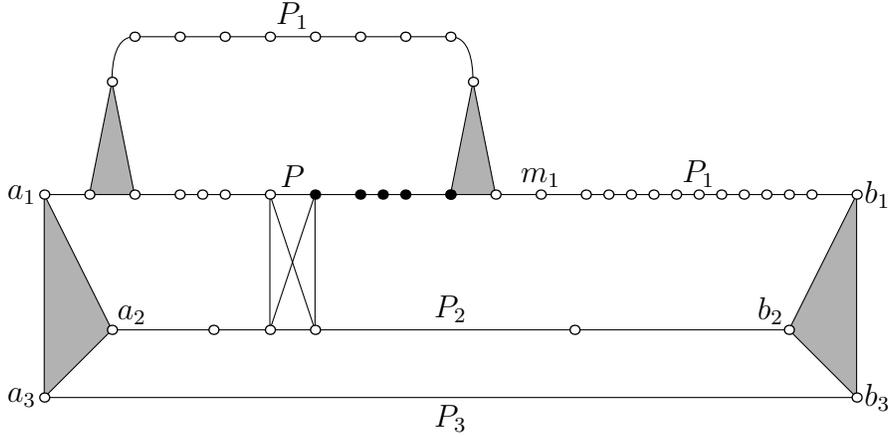


Figure 4: A graph with six odd prisms

are not adjacent to a_2, a_3, b_2, b_3 . Then the three paths P, P_2, P_3 form a prism of G of the same parity as H .

Proof. If the interior vertices of P have no neighbour on $P_2 \cup P_3$ then the lemma holds. So suppose that some interior vertex c_1 of P has neighbours on $P_2 \cup P_3$, and choose c_1 closest to a_1 along P . Define paths $H_2 = P_2 + P_3 \setminus \{b_3\}$ and $H_3 = P_2 + P_3 \setminus \{b_3\}$. For $i = 2, 3$, let c_i be the neighbour of c_1 closest to b_i along H_i .

If $c_2 = c_3$, then the three paths $c_2-c_1-P-a_1, c_2-P_2-a_2, c_2-P_2-b_2-b_3-P_3-a_3$ form a pyramid with triangle $\{a_1, a_2, a_3\}$ and apex c_2 , a contradiction. So $c_2 \neq c_3$. If c_2, c_3 are not adjacent, then the three paths $c_1-P-a_1, c_1-c_2-P_2-a_2, c_1-c_3-P_2-b_2-b_3-P_3-a_3$ form a pyramid with triangle $\{a_1, a_2, a_3\}$ and apex c_1 , a contradiction. So c_2, c_3 are adjacent. Up to symmetry, c_2c_3 is an edge of P_2 . If c_1, b_1 are adjacent, then the three paths $c_1-b_1, c_1-c_3-P_2-b_2, c_1-P-a_1-a_3-P_3-b_3$ form a pyramid with triangle $\{b_1, b_2, b_3\}$ and apex c_1 . So we may assume that c_1, b_1 are not adjacent. Let a'_1 be the neighbour of a_1 in P_1 . Let d_1 be the vertex of a'_1-P-c_1 that has neighbours in P_1 and is closest to c_1 . Let d_2, d_3 be the neighbours of d_1 along P_1 that are closest to a_1 and b_1 respectively.

If $d_2 = d_3$, then the three paths $d_2-d_1-P-c_1, d_2-P_1-a_1-a_2-P_2-c_2, d_2-P_1-b_1-b_2-P_2-c_3$ form a pyramid with triangle $\{c_1, c_2, c_3\}$ and apex d_2 , a contradiction. So $d_2 \neq d_3$. If d_2, d_3 are not adjacent, then the three paths $d_1-d_2-P_1-a_1, d_1-d_3-P_1-b_1-b_3-P_3-a_3, d_1-P-c_1-c_2-P_2-a_2$ form a pyramid with triangle $\{a_1, a_2, a_3\}$ and apex d_1 , a contradiction. So d_2, d_3 are adjacent. Then the four triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}, \{c_1, c_2, c_3\}, \{d_1, d_2, d_3\}$ and the six paths $P_3, a_2-P_2-c_2, a_1-P_1-d_2, b_2-P_2-c_3, b_1-P_1-d_3, c_1-P-d_1$ form the line-graph of a subdivision of K_4 , and it is not the line-graph of K_4 since $a_3 \neq b_3$; so G contains the line-graph of a proper

subdivision of K_4 , a contradiction. \square

Now we can present an algorithm that decides if a graph with no odd hole contains an odd prism.

Algorithm 6 (DETECTION OF AN ODD PRISM IN A GRAPH THAT CONTAINS NO ODD HOLE)

Input: A graph G that contains no odd hole.

Output: An odd prism induced in G , if G contains any, else the negative answer “ G contains no odd prism”.

Method: Using Algorithm 5, test whether G contains the line-graph of a proper subdivision of K_4 . If G contains such a subgraph F , for each of the six rungs R of F , test if $F \setminus V(R)$ is an odd prism, and if it is, return this odd prism. If Algorithm 5 answers that G does not contain the line-graph of a proper subdivision of K_4 , then for every 6-tuple $(a_1, a_2, a_3, b_1, b_2, b_3)$ do:

For $i = 1, 2, 3$ compute a shortest path P_i from a_i to b_i whose interior vertices are not adjacent to a_{i+1} , a_{i+2} , b_{i+1} and b_{i+2} (subscripts are understood modulo 3). If paths P_1, P_2, P_3 exist and form an odd prism, return the answer no and stop.

If no 6-tuple has produced a prism, return the answer yes.

Complexity: $O(|V(G)|^{20})$.

Proof of correctness. If G contains the line-graph of proper subdivision of K_4 , this will be detected by Algorithm 5. If G contains no odd hole and no odd prism, then Lemma 7.1 ensures that G cannot contain the line-graph of a proper subdivision of K_4 . So the algorithm will return the correct answer.

Now suppose that G does not contain the line graph of a proper subdivision of K_4 and G contains an odd prism, with triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Then in some step the algorithm will consider these six vertices, and it will find paths P_i since the corresponding paths of the prism have the required properties. By three applications of Lemma 7.2, we obtain that P_1, P_2, P_3 form an odd prism, and so the algorithm will detect it.

Complexity analysis: The complexity is clearly determined by its costliest step, which is Algorithm 4. \square

Now deciding if a graph is in class \mathcal{A}' can be done as follows: test if G contains an antihole of length at least 5 as explained earlier; test if G is Berge using the algorithm from Section 2; then use Algorithm 6 to test if G contains no odd prism. The complexity is the same as that of Algorithm 6.

We note that if Conjecture 1.1 is true then the algorithm for recognizing graphs in class \mathcal{A}' can be used to color optimally the vertices of any graph $G \in \mathcal{A}'$ (even if a proof of Conjecture 1.1 is not algorithmic); this can be done similarly

to the remark made at the end of Section 4, as follows. Enumerate all pairs of non-adjacent vertices of G and test whether their contraction produces a graph in class \mathcal{A} ; the assumed validity of Conjecture 1.1 insures that at least one pair will work. Then iterate this procedure until the contractions turn the graph into a clique. In terms of complexity, since we may need to check $O(|V(G)|^2)$ pairs at each contraction step, and there may be $O(|V(G)|)$ steps, we end up with total complexity $O(|V(G)|^{23})$; thus it is desirable to find a proof of Conjecture 1.1 that produces an algorithm with lower complexity.

8 NP-complete problems

In this section we show that the following problems are NP-complete:

- Decide if a graph contains a prism.
- Decide if a graph contains an even prism.
- Decide if a graph contains an odd prism.
- Decide if a graph contains the line-graph of a proper subdivision of K_4 .
- Decide if a graph contains the line-graph of a bipartite subdivision of K_4 .

We have seen in the preceding sections that all these problems are polynomial when the input is restricted to the class of graphs that contain no odd hole.

The above NP-completeness results can all be derived from the following theorem. Let us call problem Π the decision problem whose input is a triangle-free graph G and two non-adjacent vertices a, b of G of degree 2 and whose question is: “Does G have a hole that contains both a, b ?” Bienstock [5] mentions that this problem is NP-complete in general (i.e., not restricted to triangle-free graphs). We adapt his proof here for triangle-free graphs.

Theorem 8.1 *Problem Π is NP-complete.*

Proof. Let us give a polynomial reduction from the problem 3-SATISFIABILITY of Boolean functions to problem Π . Recall that a Boolean function with n variables is a mapping f from $\{0, 1\}^n$ to $\{0, 1\}$. A Boolean vector $\xi \in \{0, 1\}^n$ is a *truth assignment* for f if $f(\xi) = 1$. For any Boolean variable x on $\{0, 1\}$, we write $\bar{x} := 1 - x$, and each of x, \bar{x} is called a *literal*. An instance of 3-SATISFIABILITY is a Boolean function f given as a product of clauses, each clause being the Boolean sum \vee of three literals; the question is whether f admits a truth assignment. The NP-completeness of 3-SATISFIABILITY is a fundamental result in complexity theory, see [12].

Let f be an instance of 3-SATISFIABILITY, consisting of m clauses C_1, \dots, C_m on n variables x_1, \dots, x_n . Let us build a graph G_f with two specialized vertices a, b , such that there will be a hole containing both a, b in G if and only if there exists a truth assignment for f .

For each variable x_i ($i = 1, \dots, n$), make a graph $G(x_i)$ with eight vertices $a_i, b_i, t_i, f_i, a'_i, b'_i, t'_i, f'_i$, and ten edges $a_i t_i, a_i f_i, b_i t_i, b_i f_i$ (so that $\{a_i, b_i, t_i, f_i\}$ induces a hole), $a'_i t'_i, a'_i f'_i, b'_i t'_i, b'_i f'_i$ (so that $\{a'_i, b'_i, t'_i, f'_i\}$ induces a hole) and $t_i f'_i, t'_i f_i$. See Figure 5.

For each clause C_j ($j = 1, \dots, m$), with $C_j = u_j^1 \vee u_j^2 \vee u_j^3$, where each u_j^p ($p = 1, 2, 3$) is a literal from $\{x_1, \dots, x_n, \bar{x}_1, \dots, \bar{x}_n\}$, make a graph $G(C_j)$ with five vertices $c_j, d_j, v_j^1, v_j^2, v_j^3$ and six edges so that each of c_j, d_j is adjacent to each of v_j^1, v_j^2, v_j^3 . See Figure 6. For $p = 1, 2, 3$, if $u_j^p = x_i$ then add two edges $u_j^p f_i, u_j^p f'_i$, while if $u_j^p = \bar{x}_i$ then add two edges $u_j^p t_i, u_j^p t'_i$.

The graph G_f is obtained from the disjoint union of the $G(x_i)$'s and the $G(C_j)$'s as follows. For $i = 1, \dots, n-1$, add edges $b_i a_{i+1}$ and $b'_i a'_{i+1}$. Add an edge $b'_n c_1$. For $j = 1, \dots, m-1$, add an edge $d_j c_{j+1}$. Introduce the two specialized vertices a, b and add edges aa_1, aa'_1 and bd_m, bb_n . See Figure 7. Clearly the size of G_f is polynomial (actually linear) in the size $n + m$ of f . Moreover, it is easy to see that G_f contains no triangle, and that a, b are non-adjacent and both have degree 2.

Suppose that f admits a truth assignment $\xi \in \{0, 1\}^n$. We build a hole in G by selecting vertices as follows. Select a, b . For $i = 1, \dots, n$, select a_i, b_i, a'_i, b'_i ; moreover, if $\xi_i = 1$ select t_i, t'_i , while if $\xi_i = 0$ select f_i, f'_i . For $j = 1, \dots, m$, since ξ is a truth assignment for f , at least one of the three literals of C_j is equal to 1, say $u_j^p = 1$ for some $p \in \{1, 2, 3\}$. Then select c_j, d_j and v_j^p . Now it is a routine matter to check that the selected vertices induce a cycle Z that contains a, b , and that Z is chordless, so it is a hole. The main point is that there is no chord in Z between some subgraph $G(C_j)$ and some subgraph $G(x_i)$, for that would be either an edge $t_i v_j^p$ (or $t'_i v_j^p$) with $u_j^p = x_i$ and $\xi_i = 1$, or, symmetrically, an edge $f_i v_j^p$ (or $f'_i v_j^p$) with $u_j^p = \bar{x}_i$ and $\xi_i = 0$, in either case a contradiction to the way the vertices of Z were selected.

Conversely, suppose that G_f admits a hole Z that contains a, b . Clearly Z contains a_1, a'_1 since these are the only neighbours of a in G_f .

Claim 8.2 *For $i = 1, \dots, n$, Z contains exactly six vertices of G_i : four of them are a_i, a'_i, b_i, b'_i , and the other two are either t_i, t'_i or f_i, f'_i .*

Proof. First we prove the claim for $i = 1$. Since a, a_1 are in Z and a_1 has only three neighbours a, t_1, f_1 , exactly one of t_1, f_1 is in Z . Likewise exactly one of t'_1, f'_1 is in Z . If t_1, f'_1 are in Z then the vertices a, a_1, a'_1, t_1, f'_1 are all in Z and they induce a hole that does not contain b , a contradiction. Likewise we do not have both t'_1, f_1 in Z . Therefore, up to symmetry we may assume that t_1, t'_1 are in Z and f_1, f'_1 are not. If a vertex u_j^p of some $G(C_j)$ ($1 \leq j \leq m, 1 \leq p \leq 3$)

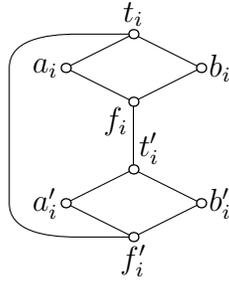


Figure 5: Graph $G(x_i)$

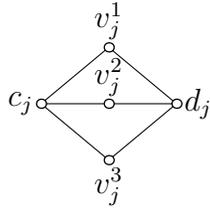


Figure 6: Graph $G(C_j)$

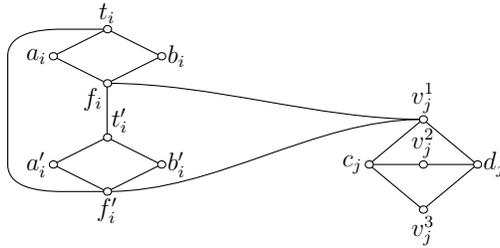


Figure 7: The two edges added to G_f in the case $u_j^p = x_i$

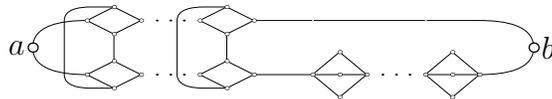


Figure 8: Graph G_f

is in Z and is adjacent to t_1 then, since this u_j^p is also adjacent to t'_1 , we see that the vertices $a, a_1, a'_1, t_1, t'_1, u_j^p$ are all in Z and induce a hole that does not contain b , a contradiction. Thus the neighbour of t_1 in $Z \setminus a_1$ is not in any $G(C_j)$ ($1 \leq j \leq m$), so that neighbour is b_1 . Likewise b'_1 is in Z . So the claim holds for $i = 1$. Since b_1 is in Z and exactly one of t_1, f_1 is in Z , and b_1 has degree 3 in G_f , we obtain that a_2 is in Z , and similarly b_2 is in Z . Now the proof of the claim for $i = 2$ is essentially the same as for $i = 1$, and by induction the claim holds up to $i = n$. \square

Claim 8.3 *For $j = 1, \dots, m$, Z contains c_j, d_j and exactly one of v_j^1, v_j^2, v_j^3 .*

Proof. First we prove this claim for $j = 1$. By Claim 8.2, b'_n is in Z and exactly one of t'_n, f'_n is in Z , so (since b'_n has degree 3 in G_f) c_1 is in Z . Consequently exactly one of u_1^1, u_1^2, u_1^3 is in Z , say u_1^1 . The neighbour of u_1^1 in $Z \setminus c_1$ cannot be a vertex of some $G(x_i)$ ($1 \leq i \leq n$), for that would be either t_i (or f_i) and thus, by Claim 8.2, t'_i (or f'_i) would be a third neighbour of u_1^1 in Z , a contradiction. Thus the other neighbour of u_1^1 in Z is d_1 , and the claim holds for $j = 1$. Since d_1 has degree 4 in G_f and exactly one of v_1^1, v_1^2, v_1^3 is in Z , it follows that its fourth neighbour c_2 is in Z . Now the proof of the claim for $j = 2$ is the same as for $j = 1$, and by induction the claim holds up to $j = m$. \square

We can now make a Boolean vector ξ as follows. For $i = 1, \dots, n$, if Z contains t_i, t'_i set $\xi_i = 1$; if Z contains f_i, f'_i set $\xi_i = 0$. By Claim 8.2 this is consistent. Consider any clause C_j ($1 \leq j \leq m$). By Claim 8.3 and up to symmetry we may assume that v_j^1 is in Z . If $u_j^1 = x_i$ for some $i \in \{1, \dots, n\}$, then the construction of G_f implies that f_i, f'_i are not in Z , so t_i, t'_i are in Z , so $\xi_i = 1$, so clause C_j is satisfied by x_i . If $u_j^1 = \bar{x}_i$ for some $i \in \{1, \dots, n\}$, then the construction of G_f implies that t_i, t'_i are not in Z , so f_i, f'_i are in Z , so $\xi_i = 0$, so clause C_j is satisfied by \bar{x}_i . Thus ξ is a truth assignment for f . This completes the proof of the theorem. \square

Now we can prove the main result of this section.

Theorem 8.4 *The following problems are NP-complete:*

1. *Decide if a graph contains a prism.*
2. *Decide if a graph contains an odd prism.*
3. *Decide if a graph contains an even prism.*
4. *Decide if a graph contains the line-graph of a proper subdivision of K_4 .*
5. *Decide if a graph contains the line-graph of a bipartite subdivision of K_4 .*

Proof. For each of these five problems we show a reduction from problem Π to this problem. So let (G, a, b) be any instance of problem Π , where G is a

triangle-free graph and a, b are non-adjacent vertices of G of degree 2. Let us call a', a'' the two neighbours of a and b', b'' the two neighbours of b in G .

Reduction to Problem 1: Starting from G , build a graph G' as follows (see Figure 9): replace vertex a by five vertices a_1, a_2, a_3, a_4, a_5 with five edges $a_1a_2, a_1a_3, a_2a_3, a_2a_4, a_3a_5$, and put edges a_4a' and a_5a'' . Do the same with b , with five vertices named b_1, \dots, b_5 instead of a_1, \dots, a_5 and with the analogous edges. Add an edge a_1b_1 . Since G has no triangle, G' has exactly two triangles $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$. Moreover we see that G' contains a prism if and only if G contains a hole that contains a and b . So every instance of Π can be reduced polynomially to an instance of Problem 1, which proves that Problem 1 is NP-complete.

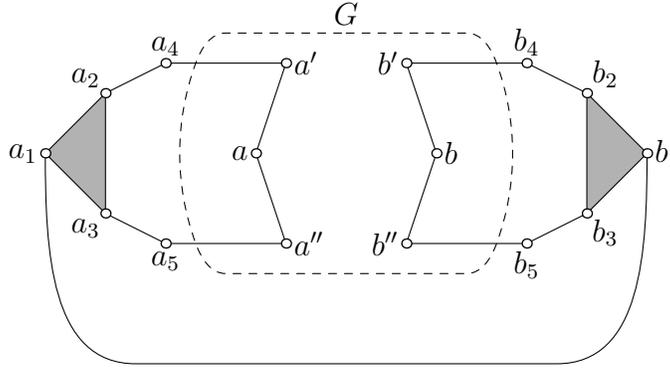


Figure 9: Problem 1: G and G'

Reduction to Problem 2: Starting from G , build the same graph G' as above. Then build eight graphs $G_{i,j,k}$ ($i, j, k \in \{0, 1\}$) as follows: if $i = 1$, subdivide the edge a_2a_4 into a path of length 2; else do not subdivide it. Likewise, subdivide the edge a_3a_5 if and only if $j = 1$; and subdivide the edge a_1b_1 if and only if $k = 1$. Now G contains a hole that contains a and b if and only if at least one of the eight graphs $G_{i,j,k}$ contains an odd prism. So every instance of Π can be reduced polynomially to eight instances of Problem 2.

Reduction to Problem 3: Starting from G , build the eight graphs $G_{i,j,k}$ as above. Then G contains a hole that contains a and b if and only if at least one of the eight graphs $G_{i,j,k}$ contains an even prism. So every instance of Π can be reduced polynomially to eight instances of Problem 3.

Reduction to Problem 4: Starting from G , build a graph G'' as follows (see Figure 10): remove vertices a and b and add twelve vertices $v_{ab}, v_{ac}, v_{ad}, v_{ba}, v_{bc}, v_{bd}, v_{ca}, v_{cb}, v_{cd}, v_{da}, v_{db}, v_{dc}$. Add edges such that each of $\{v_{ab}, v_{ac}, v_{ad}\}$, $\{v_{ba}, v_{bc}, v_{bd}\}$, $\{v_{ca}, v_{cb}, v_{cd}\}$ and $\{v_{da}, v_{db}, v_{dc}\}$ is a triangle. Add edges $v_{ab}v_{ba}, v_{dc}v_{cd}, v_{bd}v_{db}, v_{bc}v_{cb}, v_{ad}v_{da}, v_{ac}v_{ca}, v_{da}v_{db}, v_{ca}v_{cb}$. The graph G'' contains exactly four triangles, and G contains a hole through a and b if and only if G'' contains

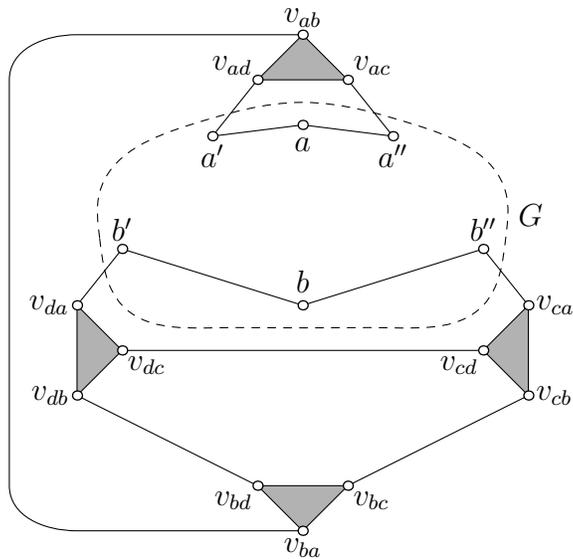


Figure 10: Problem 4: G and G''

the line-graph of a proper subdivision of K_4 . So every instance of Π can be reduced polynomially to an instance of Problem 4.

Reduction to Problem 5: Starting from G'' , make four graphs $G''_{i,j}$ ($i, j \in \{0, 1\}$) as follows: if $i = 1$ subdivide the edge $v_{ad}a'$ into a path of length 2, else do not subdivide it. Subdivide likewise the edge $v_{ac}a''$ if and only if $j = 1$. Now G contains a hole through a and b if and only if one of the four graphs $G''_{i,j}$ contains the line-graph of a bipartite subdivision of K_4 . So every instance of Π can be reduced polynomially to four instances of Problem 5. This completes the proof of the theorem. \square

9 Conclusion

We summarize the complexity results mentioned in this paper in the following table, whose columns correspond to the class of graphs taken as instances and whose lines correspond to the subgraph that we look for. The symbol n refers to the number of vertices of the input graph; 0 means trivial, NPC means NP-complete, and a question mark means unsolved.

	General graphs	Graphs with no pyramid	Graphs with no odd hole
Pyramid or prism	n^5	n^5	n^5
Pyramid	n^9 [8]	0	0
Prism	NPC	n^5	n^5
LGPSK ₄	NPC	n^{20}	n^{20}
LGBSK ₄	NPC	?	n^{20}
Odd prism	NPC	?	n^{20}
Even prism	NPC	?	n^{11}

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