# Improved bounds for the crossing numbers of $K_{m, n}$ and $K_{n}$ 

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#### Abstract

It has been long-conjectured that the crossing number $\operatorname{cr}\left(K_{m, n}\right)$ of the complete bipartite graph $K_{m, n}$ equals the Zarankiewicz Number $Z(m, n):=\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor$. Another long-standing conjecture states that the crossing number $\operatorname{cr}\left(K_{n}\right)$ of the complete graph $K_{n}$ equals $Z(n):=\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$. In this paper we show the following improved bounds on the asymptotic ratios of these crossing numbers and their conjectured values: (i) for each fixed $m \geq 9, \lim _{n \rightarrow \infty} \operatorname{cr}\left(K_{m, n}\right) / Z(m, n) \geq 0.83 m /(m-1)$; (ii) $\lim _{n \rightarrow \infty} \operatorname{cr}\left(K_{n, n}\right) / Z(n, n) \geq 0.83$; and (iii) $\lim _{n \rightarrow \infty} \operatorname{cr}\left(K_{n}\right) / Z(n) \geq 0.83$.

The previous best known lower bounds were $0.8 m /(m-1), 0.8$, and 0.8 , respectively. These improved bounds are obtained as a consequence of the new bound $\operatorname{cr}\left(K_{7, n}\right) \geq$ $2.1796 n^{2}-4.5 n$. To obtain this improved lower bound for $\operatorname{cr}\left(K_{7, n}\right)$, we use some elementary topological facts on drawings of $K_{2,7}$ to set up a quadratic program on 6! variables whose minimum $p$ satisfies $\operatorname{cr}\left(K_{7, n}\right) \geq(p / 2) n^{2}-4.5 n$, and then use state-of-the-art quadratic optimization techniques combined with a bit of invariant theory of permutation groups to show that $p \geq 4.3593$.


Keywords: crossing number, semidefinite programming, copositive cone, invariants and centralizer rings of permutation groups

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## 1 Introduction

In the earliest known instance of a crossing number question, Paul Turán raised the problem of calculating the crossing number of the complete bipartite graphs $K_{m, n}$. Turán's interesting account of the origin of this problem can be found in [23.

We recall that in a drawing of a graph in the plane, different vertices are drawn as different points, and each edge is drawn as a simple arc whose endpoints coincide with the drawings of the endvertices of the edge. Furthermore, the interior of the arc for an edge is disjoint from all the vertex points. We often make no distinction between a graph object, such as a vertex, edge, or cycle, and the subset of the plane that represents it in a drawing of the graph.

The crossing number $\operatorname{cr}(G)$ of a graph $G$ is the minimum number of pairwise intersections of edges (at a point other than a vertex) in a drawing of $G$ in the plane.

Exact crossing numbers of graphs are in general very difficult to compute. Long-standing conjectures involve the crossing numbers of interesting families of graphs, such as $K_{m, n}$ and $K_{n}$. On a positive note, it was recently proved by Glebskii and Salazar [9] that the crossing number of the Cartesian product $C_{m} \times C_{n}$ of the cycles of sizes $m$ and $n$ equals its long-conjectured value, namely $(m-2) n$, at least for $n \geq m(m+1)$. For a recent survey of crossing number results, see for instance Shahrokhi et al. [20].

Zarankiewicz published a paper [25], in which he claimed that $\operatorname{cr}\left(K_{m, n}\right)=Z(m, n)$ for all positive integers $m$, $n$, where

$$
\begin{equation*}
Z(m, n)=\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor . \tag{1}
\end{equation*}
$$

However, several years later Ringel and Kainen independently found a hiatus in Zarankiewicz's argument. A comprehensive account of the history of the problem, including a discussion of the gap in Zarankiewicz's argument, is given by Guy [10].

Figure $\rceil$ shows a drawing of $K_{4,5}$ with 8 crossings. As Zarankiewicz observed, such a drawing strategy can be naturally generalized to construct, for any positive integers $m, n$, drawings of $K_{m, n}$ with exactly $Z(m, n)$ crossings. This observation implies the following well-known upper bound for $\operatorname{cr}\left(K_{m, n}\right)$ :

$$
\operatorname{cr}\left(K_{m, n}\right) \leq Z(m, n) .
$$

No one has yet exhibited a drawing of any $K_{m, n}$ with fewer than $Z(m, n)$ crossings. In allusion to Zarankiewicz's failed attempt to prove that this is the crossing number of $K_{m, n}$, the following is commonly known as Zarankiewicz's Crossing-Number Conjecture:

$$
\operatorname{cr}\left(K_{m, n}\right) \stackrel{?}{=} Z(m, n), \quad \text { for all positive integers } m, n
$$



Figure 1: A drawing of $K_{4,5}$ with 8 crossings. A similar strategy can be used to construct drawings of $K_{m, n}$ with exactly $Z(m, n)$ crossings.

In 1973, Guy and Erdős wrote "[Al]most all questions that one can ask about crossing numbers remain unsolved" 6]. More than three decades later, despite some definite progress in our understanding of this elusive parameter, most of the fundamental and more important questions about crossing numbers remain open. Zarankiewicz's Conjecture has been verified by Kleitman 11 for $\min \{m, n\} \leq 6$ and by Woodall [24] for the special cases $7 \leq m \leq 8$, $7 \leq n \leq 10$.

Since the crossing number of $K_{m, n}$ is unknown for all other values of $m$ and $n$, it is natural to ask what are the best general lower bounds known for $\operatorname{cr}\left(K_{m, n}\right)$. A standard counting argument, together with the fact that $\operatorname{cr}\left(K_{5, n}\right)$ is as conjectured, yields the best general lower bound (2) known for $\operatorname{cr}\left(K_{m, n}\right)$. It goes as follows: suppose we know a lower bound $c_{r}$ on $\operatorname{cr}\left(K_{r, n}\right)$, for $2<r<m \leq n$. Each crossing in the embedding of $K_{m, n}$ lies in $\binom{m-2}{r-2}$ distinct $K_{r, n} \subset K_{m, n}$. As there are in total $\binom{m}{r}$ distinct $K_{r, n}$ 's, one obtains

$$
\begin{equation*}
\operatorname{cr}\left(K_{m, n}\right) \geq \frac{c_{r}\binom{m}{r}}{\binom{m-2}{r-2}}, \quad \text { for } r=5 \text { one derives } \quad \operatorname{cr}\left(K_{m, n}\right) \geq 0.8 Z(m, n) \tag{2}
\end{equation*}
$$

A small improvement on the 0.8 factor (roughly to something around 0.8001 ) was recently reported by Nahas [15].

Zarankiewicz's Conjecture for $K_{7, n}$ states that

$$
\operatorname{cr}\left(K_{7, n}\right) \stackrel{?}{=} 9\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor=\left\{\begin{array}{lr}
2.25 n^{2}-4.5 n+2.25, & n \text { odd, } \\
2.25 n^{2}-4.5 n, & n \text { even, }
\end{array} \quad n \geq 8\right.
$$

As we observed above, this has been verified only for $n=7,8,9$, and 10 . Using $\operatorname{cr}\left(K_{7,10}\right)=$ 180, a standard counting argument gives the best known lower bounds for $\operatorname{cr}\left(K_{7, n}\right)$ for $11 \leq n \leq 22$. However, for $n \geq 23$, the best known lower bounds for $\operatorname{cr}\left(K_{7, n}\right)$ are obtained
by the same counting argument, but using the known value of $\operatorname{cr}\left(K_{5, n}\right)$ instead of $\operatorname{cr}\left(K_{7,10}\right)$. Summarizing, previous to this paper, the best known lower bounds for $\operatorname{cr}(7, n)$ were:

$$
\operatorname{cr}\left(K_{7, n}\right) \geq\left\{\begin{array}{lr}
2 n(n-1), & 11 \leq n \leq 22,  \tag{3}\\
2.1 n^{2}-4.2 n+2.1, & \text { odd } n \geq 23, \\
2.1 n^{2}-4.2 n, & \text { even } n \geq 24,
\end{array}\right.
$$

In this paper we prove the following.
Theorem 1. For all integers n,

$$
\operatorname{cr}\left(K_{7, n}\right)>2.1796 n^{2}-4.5 n .
$$

An elementary calculation shows that this is an improvement, for all $n \geq 23$, on the bounds for $\operatorname{cr}\left(K_{7, n}\right)$ given in (3).

The strategy of the proof can be briefly outlined as follows. Let $(A, B)$ be the bipartition of the vertex set of $K_{7, n}$, where $|A|=7$ and $|B|=n \geq 2$. Let $b, b^{\prime}$ be vertices in $B$. In any drawing $\mathcal{D}$ of $K_{7, n}$, the number of crossings that involve an edge incident with $b$ and an edge incident with $b^{\prime}$ is bounded from below by a function of the cyclic rotation schemes of $b$ and $b^{\prime}$. This elementary topological observation on drawings of $K_{2,7}$ naturally yields a standard quadratic (minimization) program whose minimum $p$ satisfies $\operatorname{cr}\left(K_{7, n}\right) \geq(p / 2) n^{2}-4.5 n$ (see Lemma (2). We then use state-of-the-art quadratic programming techniques to show that $p \geq 4.3593$ (see Lemma 3), thus implying Theorem 1

The rest of this paper is organized as follows. In Section 2 we review some elementary topological observations about drawings of $K_{2, n}$, and use these facts to set up the quadratic program mentioned in the previous paragraph. The bound for $\operatorname{cr}\left(K_{7, n}\right)$ in terms of the minimum of this quadratic program is the content of Lemma 2 In Section 3 we prove Proposition 3 which gives a lower bound for the quadratic program. As we observe at the end of Section 3 Theorem [1 is an obvious consequence of Lemmas [2] and 3) In Section [4 we discuss consequences of Theorem the improved bound for $\mathrm{cr}\left(K_{7, n}\right)$ implies improved asymptotic bounds for the crossing numbers of $\operatorname{cr}\left(K_{m, n}\right)$ and $\operatorname{cr}\left(K_{n}\right)$.

## 2 Quadratic optimization problem yielding a lower bound for $\operatorname{cr}\left(K_{m, n}\right)$

Our goal in this section is to establish Lemma 20 a statement that gives a lower bound for $\operatorname{cr}\left(K_{m, n}\right)$, for $m \leq n$, (and thus for $\operatorname{cr}\left(K_{7, n}\right)$ ) in terms of the solution of a quadratic minimization problem on $(m-1)$ ! variables.

Let $n \geq m$ be fixed. Let $V$ denote the vertex set of $K_{m, n}$, and let $(A, B)$ denote the


Figure 2: Here $m=7$. Vertices $b_{i}$ and $b_{j}$ have cyclic orderings (0134526) and (0265341), respectively (we write $i$ for $a_{i}$ for the sake of brevity). It is easy to check that the minimum number of interchanges among adjacent elements in (0134526) required to obtain $(0265341)^{-1}$ (namely $\left.(0143562)\right)$ is 2 . Thus, $Q((0134526),(0134526))=2$. Therefore, there must be at least two crossings (as it is indeed the case in the drawing above) that involve edges incident with $b_{i}$ and $b_{j}$.
bipartition of $V$ such that each vertex of $A=\left\{a_{0}, a_{1}, \ldots, a_{m-1}\right\}$ is adjacent to each vertex of $B=\left\{b_{0}, b_{1}, \ldots, b_{n-1}\right\}$.

Consider a fixed drawing $\mathcal{D}$ of $K_{m, n}$. To each vertex $b_{i}$ we associate a cyclic ordering $\pi_{\mathcal{D}}\left(b_{i}\right)$ of the elements in $A$, defined by the (clockwise) cyclic order in which the edges incident with $b_{i}$ leave $b_{i}$ towards the vertices in $A$ (see Figure (2). Let $\Pi$ denote the set of all cyclic orderings of $\left\{a_{0}, a_{1}, \ldots, a_{m-1}\right\}$. Note that $|\Pi|=m!/ m=(m-1)!$.

Following Kleitman [11, let $\operatorname{cr}_{\mathcal{D}}\left(b_{i}, b_{j}\right)$ denote the number of crossings in $\mathcal{D}$ that involve an edge incident with $b_{i}$ and an edge incident with $b_{j}$. Further, let $\rho_{1}, \rho_{2} \in \Pi$ and $Q\left(\rho_{1}, \rho_{2}\right)$ be the minimum number of interchanges of adjacent elements of $\rho_{1}$ required to produce $\rho_{2}^{-1}$. Then, for all $b_{i}, b_{j}$ with $b_{i} \neq b_{j}$,

$$
\begin{equation*}
\operatorname{cr}_{\mathcal{D}}\left(b_{i}, b_{j}\right) \geq Q\left(\pi_{\mathcal{D}}\left(b_{i}\right), \pi_{\mathcal{D}}\left(b_{j}\right)\right) . \tag{4}
\end{equation*}
$$

This inequality is stated in [11] and proved in 24]. This observation alone yields a lower bound for $\operatorname{cr}\left(K_{m, n}\right)$, as follows. Fix any drawing $\mathcal{D}$ of $K_{m, n}$. For each $\rho \in \Pi$, let

$$
x_{\rho}:=\frac{1}{n}\left|\left\{b_{i} \in B \mid \pi_{\mathcal{D}}\left(b_{i}\right)=\rho\right\}\right|
$$

The matrix $Q$ can be viewed as the matrix of quadratic form $Q(\cdot, \cdot)$ on the space $\mathbb{R}^{|\Pi|}$. It
follows from (4) that

$$
\begin{aligned}
\operatorname{cr}(\mathcal{D}) & \geq \sum_{\substack{\rho, \rho^{\prime} \in \Pi \\
\rho \neq \rho^{\prime}}} Q\left(\rho, \rho^{\prime}\right)\left(x_{\rho} n\right)\left(x_{\rho^{\prime}} n\right)+\sum_{\rho \in \Pi} Q(\rho, \rho)\binom{x_{\rho} n}{2}= \\
& =\frac{n}{2}\left(n \sum_{\rho, \rho^{\prime} \in \Pi} Q\left(\rho, \rho^{\prime}\right) x_{\rho} x_{\rho^{\prime}}-\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\right)
\end{aligned}
$$

where we have used the (easily verifiable, see e.g. [24) fact that $Q(\rho, \rho)=\lfloor m / 2\rfloor\lfloor(m-1) / 2\rfloor$ for every $\rho \in \Pi$.

Since the drawing $\mathcal{D}$ was arbitrary, we have proved the following.
Lemma 2. Let $Q$ be the $(m-1)!\times(m-1)$ ! matrix of the form $Q(\cdot, \cdot)$, and let $e$ denote the all ones vector. Then, for every integer $n \geq m \geq 2$,

$$
\begin{aligned}
& c r\left(K_{m, n}\right) \geq \frac{n}{2}\left(n \min \left\{x^{T} Q x \mid x \in \mathbb{R}_{+}^{(m-1)!}, e^{T} x=1\right\}-\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\right), \\
& c r\left(K_{7, n}\right) \geq \frac{n}{2}\left(n \min \left\{x^{T} Q x \mid x \in \mathbb{R}_{+}^{6!}, e^{T} x=1\right\}-9\right)
\end{aligned}
$$

Remark For obvious reasons ( $Q$ is a $720 \times 720$-matrix) we do not include in this paper the matrix $Q$ in table form. As we mentioned above, $Q(\rho, \rho)=9$ for every $\rho \in \Pi$, and therefore all the diagonal entries of $Q$ are 9 . It is not difficult to show that $Q\left(\rho, \rho^{\prime}\right) \leq 8$ if $\rho \neq \rho^{\prime}$, so every non-diagonal entry of $Q$ is at most 8 . The calculation of the entries of $Q$, using the definition of $Q(\cdot, \cdot)$, and taking its symmetries into account (see Section 3.2 on this) takes only a few seconds of computer time.

## 3 Finding a lower bound for the optimization problem

Our aim in this section is to find a (reasonably good) lower bound for the quadratic programming problem with $m=7$ given in Lemma 2 in order to obtain a (reasonably good) lower bound for $\operatorname{cr}\left(K_{7, n}\right)$. The main result in this section is the following.

Proposition 3. Let $Q$ be the $6!\times 6!$ matrix of the quadratic form $Q(\cdot, \cdot)$. Then

$$
\min \left\{x^{T} Q x \mid x \in \mathbb{R}_{+}^{6!}, e^{T} x=1\right\} \geq 4.3593
$$

We devote this section to the proof of Proposition 3 It involves computer calculations; more details on this are given in Section 3.8

### 3.1 The standard quadratic programming problem

The problem we have formulated is known as standard quadratic optimization problem. The standard quadratic optimization problem (standard QP) is to find the global minimizers of a quadratic form over the standard simplex, i.e. we consider the global optimization problem

$$
\begin{equation*}
\underline{p}:=\min _{x \in \Delta} x^{T} Q x \tag{5}
\end{equation*}
$$

where $Q$ is an arbitrary symmetric $d \times d$ matrix, $e$ is the all ones vector, and $\Delta$ is the standard simplex in $\mathbb{R}^{d}$,

$$
\Delta=\left\{x \in \mathbb{R}_{+}^{d}: e^{T} x=1\right\}
$$

We will now reformulate the standard QP problem as a convex optimization problem in conic form. First, we will review the relevant convex cones as well as the duality theory of conic optimization. We define the following convex cones.

- The $d \times d$ symmetric matrices:

$$
\mathcal{S}_{d}=\left\{X \in \mathbb{R}^{d} \times \mathbb{R}^{d}, X=X^{T}\right\} ;
$$

- The $d \times d$ symmetric positive semidefinite matrices:

$$
\mathcal{S}_{d}^{+}=\left\{X \in \mathcal{S}_{d}, y^{T} X y \geq 0 \quad \forall y \in \mathbb{R}^{d}\right\} ;
$$

- The $d \times d$ symmetric copositive matrices:

$$
\mathcal{C}_{d}=\left\{X \in \mathcal{S}_{d}, y^{T} X y \geq 0 \quad \forall y \in \mathbb{R}^{d}, y \geq 0\right\} ;
$$

- The $d \times d$ symmetric completely positive matrices:

$$
\mathcal{C}_{d}^{*}=\left\{X=\sum_{i=1}^{k} y_{i} y_{i}^{T}, y_{i} \in \mathbb{R}^{d}, y_{i} \geq 0(i=1, \ldots, k)\right\} ;
$$

- The $d \times d$ symmetric nonnegative matrices:

$$
\mathcal{N}_{d}=\left\{X \in \mathcal{S}_{d}, X_{i j} \geq 0 \quad(i, j=1, \ldots, d)\right\} .
$$

Recall that the completely positive cone is the dual of the copositive cone, and that the nonnegative and semidefinite cones are self-dual for the inner product $\langle X, Y\rangle:=\operatorname{Tr}(X Y)$, where ' $\operatorname{Tr}$ ' denotes the trace operator.

For a given cone $\mathcal{K}_{d}$ and its dual cone $\mathcal{K}_{d}^{*}$ we define the primal and dual pair of conic linear programs:

$$
\begin{align*}
& p^{*}:=\inf _{X \in \mathcal{K}_{d}}\left\{\operatorname{Tr}(C X) \mid \operatorname{Tr}\left(A_{i} X\right)=b_{i}(i=1, \ldots, M)\right\}  \tag{P}\\
& d^{*}:=\sup _{y \in \mathbb{R}^{m}}\left\{b^{T} y \mid \sum_{i=1}^{M} y_{i} A_{i}+S=C, S \in \mathcal{K}_{d}^{*}\right\} . \tag{D}
\end{align*}
$$

If $\mathcal{K}_{d}=\mathcal{S}_{d}^{+}$we refer to semidefinite programming, if $\mathcal{K}_{d}=\mathcal{N}_{d}$ to linear programming, and if $\mathcal{K}_{d}=\mathcal{C}_{d}$ to copositive programming.

The well-known conic duality theorem (see e.g. Renegar [17]) gives the duality relations between $(P)$ and $(D)$.

Theorem 4 (Conic duality theorem). If there exists an interior feasible solution $X^{0} \in$ $\operatorname{int}\left(\mathcal{K}_{d}\right)$ of $(P)$, and a feasible solution of $(D)$ then $p^{*}=d^{*}$ and the supremum in $(D)$ is attained. Similarly, if there exist feasible $y^{0}, S^{0}$ for $(D)$ where $S^{0} \in \operatorname{int}\left(\mathcal{K}_{d}^{*}\right)$, and a feasible solution of $(P)$, then $p^{*}=d^{*}$ and the infimum in $(P)$ is attained.

Optimization over the cones $\mathcal{S}_{d}^{+}$and $\mathcal{N}_{d}$ can be done in polynomial time (to compute an $\epsilon$-optimal solution), but some NP-hard problems can be formulated as copositive programs, see e.g. De Klerk and Pasechnik [12].

### 3.1.1 Convex reformulation of the standard QP

We rewrite problem (5) in the following way:

$$
\underline{p}:=\min _{x \in \Delta} \operatorname{Tr}\left(Q x x^{T}\right) .
$$

Now we define the cone of matrices

$$
\mathcal{K}=\left\{X \in \mathcal{S}_{d}: X=x x^{T}, x \geq 0\right\} .
$$

Note that the requirement $x \in \Delta$ corresponds to $X \in \mathcal{K}$ with $\operatorname{Tr}\left(e e^{T} X\right)=1$.
We arrive at the following reformulation of problem (5):

$$
\begin{equation*}
\underline{p}=\min \left\{\operatorname{Tr}(Q X): \operatorname{Tr}\left(e e^{T} X\right)=1, X \in \mathcal{K}\right\} \tag{6}
\end{equation*}
$$

The last step is to replace the cone $\mathcal{K}$ by its convex hull, which is simply the cone of completely positive matrices, i.e.

$$
\operatorname{conv}(\mathcal{K})=\mathcal{C}_{d}^{*}=\left\{X=\sum_{i=1}^{k} y_{i} y_{i}^{T}, y_{i} \in \mathbb{R}^{n}, y_{i} \geq 0(i=1, \ldots, k)\right\}
$$

Replacing the feasible set by its convex hull does not change the optimal value of problem (6), since its objective function is linear. Thus we obtain the well-known convex reformulation:

$$
\begin{equation*}
\underline{p}=\min \left\{\operatorname{Tr}(Q X) \mid \operatorname{Tr}\left(e e^{T} X\right)=1, X \in \mathcal{C}_{d}^{*}\right\} . \tag{7}
\end{equation*}
$$

The dual problem takes the form:

$$
\begin{equation*}
\underline{p}=\max \left\{t \mid Q-t e e^{T} \in \mathcal{C}_{d}\right\}, \tag{8}
\end{equation*}
$$

where $\mathcal{C}_{d}$ is the cone of copositive matrices, as before. Note that both problems have the same optimal value, in view of the conic duality theorem.

### 3.2 Exploiting group symmetries

We can reduce the number of variables in the optimization problems in (78) considerably by exploiting the invariance properties of the quadratic function $x^{T} Q x$. This will also prove to be computationally necessary for the problems we intend to solve.

Consider the situation where the matrix $Q$ is invariant under the action of a group $G$ of order $k=|G|$ of permutation matrices $P \in G$, in the sense that

$$
Q=P^{T} Q P \quad \text { for all } \quad P \in G
$$

Then we have

$$
\begin{aligned}
\underline{p} & =\min \left\{\operatorname{Tr}(Q X) \mid \operatorname{Tr}\left(e e^{T} X\right)=1, X \in \mathcal{C}_{d}^{*}\right\} \\
& =\min \left\{\operatorname{Tr}\left(P^{T} Q P X\right) \mid \operatorname{Tr}\left(P e e^{T} P X\right)=1, X \in \mathcal{C}_{d}^{*}\right\} \text { for any } P \in G \\
& =\min \left\{\operatorname{Tr}\left(Q P^{T} X P\right) \mid \operatorname{Tr}\left(e e^{T} P^{T} X P\right)=1, X \in \mathcal{C}_{d}^{*}\right\} \text { for any } P \in G \\
& =\min \left\{\operatorname{Tr}\left(Q \frac{1}{k}\left[\sum_{P \in G} P^{T} X P\right]\right) \left\lvert\, \operatorname{Tr}\left(e e^{T}\left[\frac{1}{k} \sum_{P \in G} P^{T} X P\right]\right)=1\right., X \in \mathcal{C}_{d}^{*}\right\} .
\end{aligned}
$$

We can therefore restrict the optimization to the subset of the feasible set obtained by replacing each feasible $X$ by the group average $\frac{1}{k} \sum_{P \in G} P^{T} X P$, i.e. replacing $X$ by its image under what is known in invariant theory as the Reynolds operator. Note that if $X \in \mathcal{C}_{d}^{*}$, then so is its image under the group average.

In particular, we wish to compute a basis for the so-called fixed point subspace:

$$
\mathcal{A}:=\left\{Y \in \mathcal{S}_{d} \left\lvert\, Y=\frac{1}{k} \sum_{P \in G} P^{T} X P\right., X \in \mathcal{S}_{d}\right\} .
$$

Note that $Q$ and $e e^{T}$ are elements of $\mathcal{A}$ (set $X=Q$, respectively, $X=e e^{T}$ ). Hence $Q-t e e^{T} \in \mathcal{A}$ for any $t$, and

$$
\underline{p}=\max \left\{t \mid Q-t e e^{T} \in \mathcal{C}_{d}\right\}=\max \left\{t \mid Q-t e e^{T} \in \mathcal{C}_{d} \cap \mathcal{A}\right\} .
$$

The right-hand side here is the dual of the primal problem when it was restricted to $\mathcal{A}$ as above.

The next step is to compute a basis for the subspace $\mathcal{A}$.

### 3.3 Computing a basis for the fixed point subspace

We assume for simplicity that $G$ acts transitively as a permutation group on the standard basis vectors. (This holds in our setting. A more general, and computationally less efficient,
setting, can be found in Gatermann and Parrilo [8].) The theory here is well-known, and goes back to Burnside, Schur and Wielandt. See e.g. Cameron [5] for details. Although we need a basis of $\mathcal{A}$, the subspace of symmetric matrices fixed by $G$, it is more natural to compute the basis $\mathcal{X}$ of the subspace $\mathcal{B}$ of all fixed by $G$ matrices, and then pass on to $\mathcal{A}$.

The dimension of $\mathcal{B}$ equals the number $r$ of orbits of $G$ on the Cartesian square of the standard basis. The set of the latter orbits, also known as 2-orbits, naturally corresponds to certain set $\mathcal{X}$ of $d \times d$ zero-one matrices. Namely, for each $X \in \mathcal{X}$ one has $X_{i j}=1$ if and only if $X_{P(i), P(j)}=1$ for all $P \in G$ and all $1 \leq i \leq j \leq|\Pi|$. As $G$ is transitive on the standard basis vectors, the identity matrix $I$ belongs to $\mathcal{X}$. We also have $\sum_{X \in \mathcal{X}} X=e e^{T}$.
As $\mathcal{X}$ is closed under the matrix transposition, i.e. $X^{T} \in \mathcal{X}$ for any $X \in \mathcal{X}$,

$$
\mathcal{X}_{\mathcal{A}}=\left\{A_{1}, \ldots, A_{M}\right\}=\left\{X \mid X=X^{T} \in \mathcal{X}\right\} \cup\left\{X+X^{T} \mid X \in \mathcal{X}, X \neq X^{T}\right\}
$$

is a basis of $\mathcal{A}$. Each $A \in \mathcal{X}_{\mathcal{A}}$ is a symmetric zero-one matrix and $\sum_{A \in \mathcal{X}_{\mathcal{A}}} A=e e^{T}$. Moreover

$$
\left\{Y \in \mathcal{S}_{d} \mid Y=\sum_{i=1}^{M} y_{i} A_{i}\right\}=\mathcal{A} \equiv\left\{Y \in \mathcal{S}_{d} \left\lvert\, Y=\frac{1}{k} \sum_{P \in G} P^{T} X P\right., X \in \mathcal{S}_{d}\right\} .
$$

Since $Q \in \mathcal{A}$, we will write $Q=\sum_{i=1}^{M} b_{i} A_{i}$.
It is worth mentioning that algebraically the vector space $\mathcal{B}$ behaves very nicely: it is closed under multiplication. In other words, $\mathcal{B}$ is a matrix algebra of dimension $r$, also known as the centralizer ring of the permutation group $G$.

We proceed to describe $G$ and $\mathcal{B}$ in our case. For us $G$ is isomorphic to the direct product $\operatorname{Sym}(m) \times \operatorname{Sym}(2)$, of symmetric groups $\operatorname{Sym}(m)$ and $\operatorname{Sym}(2)$, where $\operatorname{Sym}(m)$ acts (as a permutation group) by conjugation on the $d=(m-1)$ ! elements of $\Pi$, and $\operatorname{Sym}(2)$ acts (as a permutation group) on $\Pi$ by switching $\pi \in \Pi$ with $\pi^{-1} \in \Pi$.

Computing $\mathcal{X}$ is an elementary combinatorial procedure, that can be found in one form or another in many computer algebra systems, so one does not have to program this again. First, the permutations that generate $\operatorname{Sym}(m) \times \operatorname{Sym}(2)$ in its action on $\Pi$ are computed. The action of $\operatorname{Sym}(2)$ is already known, and is described by the permutation $g_{0}$, say. In its usual action on $m$ symbols, $\operatorname{Sym}(m)$ is generated by $h_{1}=(0,1, \ldots, m-1)$ and $h_{2}=(0,1)$. These $h_{i}$ (for $i=1,2$ ) act on $\Pi$ by mapping each $\pi \in \Pi$ to $h_{i} \pi h_{i}^{-1}$. Denote by $g_{i}$ (for $i=1,2$ ) the permutations of $\Pi$ that realize these actions.

Next, one computes the orbits of the permutation group $\operatorname{Sym}(m) \times \operatorname{Sym}(2)=\left\langle g_{0}, g_{1}, g_{2}\right\rangle$ on the Cartesian square $\Pi \times \Pi$ of $\Pi$, by "spinning" $\left(\pi_{i}, \pi_{j}\right) \in \Pi \times \Pi$ : begin with $S_{i j}=\left\{\left(\pi_{i}, \pi_{j}\right)\right\}$ and apply the generators $g_{i}, 0 \leq i \leq 2$, in a loop until $S_{i j}$ stops growing. Then one sets $\Pi:=\Pi-S_{i j}$ and repeats until $\Pi$ is exhausted.

When $m=7$ one has $r=78$ and $M=56$. Note that here the algebra $\mathcal{B}$ is not commutative.
When $m=5$ one has $r=M=6$, and $\mathcal{B}$ commutative.

### 3.4 Reformulation of the optimization problem

We can now reformulate the dual problem by using the basis of $\mathcal{A}$ to obtain:

$$
\underline{p}=\max \left\{t \mid Q-t e e^{T} \in \mathcal{C}_{d} \cap \mathcal{A}\right\}=\max \left\{t \mid \sum_{i=1}^{M}\left(b_{i}-t\right) A_{i} \in \mathcal{C}_{d}\right\} .
$$

We will now proceed to derive a lower bound on $\underline{p}$ by solving the dual problem approximately.

### 3.5 Approximations of the copositive cone

The problem of determining whether a matrix is not copositive is NP-complete, as shown by Murty and Kabadi [14. We therefore wish to replace the copositive cone $\mathcal{C}_{d}$ by a conic subset, in such a way that the resulting optimization problem becomes tractable. We can represent the copositivity requirement for a $(d \times d)$ symmetric matrix $S$ as

$$
\begin{equation*}
P(x):=(x \circ x)^{T} S(x \circ x)=\sum_{i, j=1}^{d} S_{i j} x_{i}^{2} x_{j}^{2} \geq 0, \forall x \in \mathbb{R}^{d}, \tag{9}
\end{equation*}
$$

where ' $o$ ' indicates the componentwise (Hadamard) product. We therefore wish to know whether the polynomial $P(x)$ is nonnegative for all $x \in \mathbb{R}^{d}$. Although one apparently cannot answer this question in polynomial time in general, as it is an NP-hard problem, one can decide using semidefinite programming whether $P(x)$ can be written as a sum of squares.

Parrilo [16] showed that $P(x)$ in (9) allows a sum of squares decomposition if and only if $S \in \mathcal{S}_{d}^{+}+\mathcal{N}_{d}$, which is a well-known sufficient condition for copositivity. Set $\mathcal{K}_{d}^{0}$ to be the convex cone $\mathcal{K}_{d}^{0}=\mathcal{S}_{d}^{+}+\mathcal{N}_{d}$.

Higher order sufficient conditions can be derived by considering the polynomial:

$$
\begin{equation*}
P^{(\ell)}(x)=P(x)\left(\sum_{i=1}^{d} x_{i}^{2}\right)^{\ell}=\left(\sum_{i, j=1}^{d} S_{i j} x_{i}^{2} x_{j}^{2}\right)\left(\sum_{i=1}^{d} x_{i}^{2}\right)^{\ell} \tag{10}
\end{equation*}
$$

and asking whether $P^{(\ell)}(x)$ - which is a homogeneous polynomial of degree $2(\ell+2)$ - has a sum of squares decomposition, or whether it only has nonnegative coefficients.

For $\ell=1$, Parrilo [16] showed that a sum of squares decomposition exists if and only if ${ }^{1}$

[^1]the following system of linear matrix inequalities has a solution:
\[

$$
\begin{align*}
S-S^{(i)} & \in \mathcal{S}_{d}^{+}, \quad i=1, \ldots, d  \tag{11}\\
S_{i i}^{(i)} & =0, \quad i=1, \ldots, d  \tag{12}\\
S_{j j}^{(i)}+2 S_{i j}^{(j)} & =0, \quad i \neq j  \tag{13}\\
S_{j k}^{(i)}+S_{i k}^{(j)}+S_{i j}^{(k)} & \geq 0, \quad i<j<k \tag{14}
\end{align*}
$$
\]

where $S^{(i)}(i=1, \ldots, d)$ are symmetric matrices. Similar to the $\ell=0$ case, we define $\mathcal{K}_{d}^{1}$ as the (convex) cone of matrices $S$ for which the above system has a solution.
We will consider the lower bounds we get by replacing the copositive cone by either $\mathcal{K}_{d}^{0}$ or $\mathcal{K}_{d}^{1}$ :

$$
\begin{equation*}
\underline{p} \geq p_{\ell}:=\max \left\{t \mid Q-t e e^{T} \in \mathcal{K}_{d}^{\ell}\right\}, \quad \ell \in\{0,1\} \tag{15}
\end{equation*}
$$

### 3.6 Approximations (relaxations) of the copositive cone

We will now study the relaxation obtained by replacing the copositive cone by its proper subset $\mathcal{K}_{d}^{0}$. In other words, we study the relaxation:

$$
\begin{aligned}
\underline{p} & =\max \left\{t \mid \sum_{i=1}^{M}\left(b_{i}-t\right) A_{i} \in \mathcal{C}_{d}\right\} \\
& \geq p_{0}:=\max \left\{t \mid \sum_{i=1}^{M}\left(b_{i}-t\right) A_{i} \in \mathcal{K}_{d}^{0}=\mathcal{S}_{d}^{+}+\mathcal{N}_{d}\right\}
\end{aligned}
$$

We rewrite $\sum_{i=1}^{M}\left(b_{i}-t\right) A_{i} \in \mathcal{K}_{d}^{0}$ as

$$
\sum_{i=1}^{M}\left(b_{i}-t\right) A_{i}=\sum_{i=1}^{M} y_{i} A_{i}+\sum_{i=1}^{M} z_{i} A_{i}, \quad \text { where } \sum_{i=1}^{M} y_{i} A_{i} \in \mathcal{S}_{d}^{+} \text {and } \sum_{i=1}^{M} z_{i} A_{i} \in \mathcal{N}_{d}
$$

Note that, since the $A_{i}$ 's are $0-1$ matrices that sum to $e e^{T}$, it follows that $z_{i} \geq 0$. Moreover,

$$
b_{i}-t=y_{i}+z_{i} \quad \text { implies } \quad b_{i}-t-y_{i} \geq 0
$$

We obtain the relaxation:

$$
\begin{equation*}
p_{0}=\max \left\{t \mid b_{i}-t-y_{i} \geq 0(i=1, \ldots, M), \sum_{i=1}^{M} y_{i} A_{i} \in \mathcal{S}_{d}^{+}\right\} \tag{16}
\end{equation*}
$$

### 3.7 Block factorization

The next step in reducing the problem size is to perform a similarity transformation that simultaneously block-diagonalizes the matrices $A_{1}, \ldots, A_{M}$. In particular, we want to find an orthogonal matrix $V$ such that the matrices

$$
\tilde{A}_{i}:=V A_{i} V^{-1} \quad i=1, \ldots, M
$$

all have the same block diagonal structure, and the maximum block size is as small as possible. Note that the conjugation preserves spectra, and orthogonality of $V$ preserves symmetry.

This will further reduce the size of the relaxation (16) via

$$
\begin{aligned}
p_{0} & =\max \left\{t \mid b_{i}-t-y_{i} \geq 0(i=1, \ldots, M), \sum_{i=1}^{M} y_{i} A_{i} \in \mathcal{S}_{d}^{+}\right\} \\
& =\max \left\{t \mid b_{i}-t-y_{i} \geq 0(i=1, \ldots, M), \sum_{i=1}^{M} y_{i} V A_{i} V^{-1} \in \mathcal{S}_{d}^{+}\right\} \\
& =\max \left\{t \mid b_{i}-t-y_{i} \geq 0(i=1, \ldots, M), \sum_{i=1}^{M} y_{i} \tilde{A}_{i} \in \mathcal{S}_{d}^{+}\right\} .
\end{aligned}
$$

The necessity to restrict to orthogonal $V$ 's lies in the fact that there is currently no software (or algorithms) available that would be able to deal with non-symmetric $\tilde{A}_{i}$ 's.

Computing the finest possible block decomposition (this would mean finding explicitly the orthogonal bases for the irreducible submodules of the natural module of $G$ in its action by the matrices $P$ ) is computationally not easy, especially due to the orthogonality requirement on $V$. We restricted ourselves to decomposing into two blocks of equal size $\frac{d}{2} \times \frac{d}{2}$. Namely, each row corresponds to a cyclic permutation $g \in \Pi$, and the natural pairing $\left(g, g^{-1}\right)$ can be used to construct $V=\frac{\sqrt{2}}{2} V^{\prime}$, as follows:

- the first half of the rows of $V^{\prime}$ are characteristic vectors of the 2-subsets $\left\{g, g^{-1}\right\}$, $g \in \Pi$;
- the second half of the rows of $V^{\prime}$ consists of "twisted" rows from the first half: namely one of the two 1 s is replaced by -1 .

It is obvious that $V^{\prime} V^{\prime T}=2 I$ and thus $V$ is orthogonal.

Remark It is worth mentioning that in [19 Schrijver essentially dealt, in a different context, with a similar setup, except that in his case the elements of the basis $\mathcal{X}$ of $\mathcal{B}$ were
symmetric and (hence) the algebra $\mathcal{B}$ commutative. In such a situation the elements of $\mathcal{X}$ can be simultaneously diagonalized, and the corresponding optimization problem becomes a linear programming problem.

### 3.8 Computational results: proof of Theorem 3

The combinatorial/group theoretic part of the computations, namely of the $A_{i}$ 's, $V$, and $Q=\sum_{i} b_{i} A_{i}$ was performed using a computer algebra system GAP [7] version 4.3, and its shared package GRAPE by Soicher [21]. Semidefinite programs were solved using SeDuMi by Sturm [22], version 1.05 under Matlab 6.5. The biggest SDP took about 10 minutes of CPU time of a Pentium 4 with 1 GB of RAM.

In addition, the results were verified using Maple. Namely, for $t=p_{0}$ and $y$, the variables computed upon solving (16), we checked that the corresponding (matrix and scalar) inequalities in (16) hold. As $p_{0}$ is a lower bound on $\underline{p}$, we thus validated the computed value of $p_{0}$ independently of the SDP solver used.

For the test case of $K_{5, n}$ we solved the relaxed problem (15) with $\ell=1$ to obtain

$$
p_{1} \approx 1.9544, \quad \text { that is } \quad \operatorname{cr}\left(K_{5, n}\right) \geq \frac{1}{2}(1.9544) n^{2}=0.9772 n^{2}
$$

asymptotically. The correct asymptotic value is known to be $\operatorname{cr}\left(K_{5, n}\right)=n^{2}$, which shows the quality of the bound. The weaker bound for $\ell=0$ in (15) yields, still quite tight

$$
p_{1} \approx 1.94721, \quad \text { that is } \quad \operatorname{cr}\left(K_{5, n}\right) \geq \frac{1}{2}(1.94721) n^{2}=0.973605 n^{2}
$$

For the case $K_{7, n}$ we solved the relaxed problem (15) with $\ell=0$ to obtain

$$
p_{0} \approx 4.3593, \quad \text { that is } \quad \operatorname{cr}\left(K_{7, n}\right) \geq \frac{1}{2}(4.3593) n^{2}=2.1796 n^{2}
$$

asymptotically.

Proof of Theorem 1. For the sake of completeness, we close this section with the observation that Theorem 1] has been proved. It follows from Lemmas 2 and 3.

## 4 Improved bounds for the crossing numbers of $K_{m, n}$ and $K_{n}$

Perhaps the most appealing consequence of our improved bound for $\operatorname{cr}\left(K_{7, n}\right)$ is that it also allows us to give improved lower bounds for the crossing numbers of $K_{m, n}$ and $K_{n}$. The
quality of the new bounds is perhaps best appreciated in terms of the following asymptotic parameters:

$$
A(m):=\lim _{n \rightarrow \infty} \frac{\operatorname{cr}\left(K_{m, n}\right)}{Z(m, n)}, \quad B:=\lim _{n \rightarrow \infty} \frac{\operatorname{cr}\left(K_{n, n}\right)}{Z(n, n)}
$$

(see Richter and Thomassen [18]). These natural parameters give us a good idea of our current standing with respect to Zarankiewicz's Conjecture. It is not difficult to show that $A(m)$ (for every integer $m \geq 3$ ) and $B$ both exist [18.
Previous to the new bound we report in Theorem the best known lower bounds for $A(m)$ and $B$ were $A(m) \geq 0.8 \frac{m}{m-1}$ and (consequently) $B \geq 0.8$. Both bounds were obtained by using the known value of $\operatorname{cr}\left(K_{5, n}\right)$, and applying a standard counting argument.

By applying the same counting argument, but instead using the bound given by Theorem we improve these asymptotic quotients to $A(m)>0.83 \frac{m}{m-1}$ and $B>0.83$.

The improved lower bound for $B$ has an additional, important application. It has been long conjectured that $\operatorname{cr}\left(K_{n}\right)=Z(n)$, where

$$
Z(n)=\frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor,
$$

but this has been verified only for $n \leq 10$ (see for instance [6]). As we did with $K_{m, n}$, it is natural to inquire about the asymptotic parameter

$$
C:=\lim _{n \rightarrow \infty} \frac{\operatorname{cr}\left(K_{n}\right)}{Z(n)} .
$$

In [18] it is proved that $C$ exists, and, moreover, $C \geq B$. In view of this, our improved lower bound for $B$ yields $C>0.83$.

We summarize these results in the following statement.
Theorem 5. With $Z(m, n)$ and $Z(n)$ as above,

$$
\lim _{n \rightarrow \infty} \frac{c r\left(K_{m, n}\right)}{Z(m, n)} \geq 0.83 \frac{m}{m-1}, \quad \lim _{n \rightarrow \infty} \frac{c r\left(K_{n, n}\right)}{Z(n, n)} \geq 0.83, \quad \text { and } \lim _{n \rightarrow \infty} \frac{c r\left(K_{n}\right)}{Z(n)} \geq 0.83
$$

We close this section with a few words on some important recent developments involving the rectilinear crossing number of $K_{n}$.

The rectilinear crossing number $\overline{\operatorname{cr}}(G)$ of a graph $G$ is the minimum number of pairwise intersections of edges in a drawing of $G$ in the plane, with the additional restriction that all edges of $G$ must be drawn as straight segments.

It is known that $\overline{\operatorname{cr}}\left(K_{n}\right)$ and $\operatorname{cr}\left(K_{n}\right)$ may be different (for instance, $\overline{\operatorname{cr}}\left(K_{8}\right)=19$, whereas $\operatorname{cr}\left(K_{8}\right)=18$ ). While we have a (non-rectilinear) way of drawing $K_{n}$ that shows $\operatorname{cr}\left(K_{n}\right) \leq$
$Z(n)$ (equality is conjectured to hold, as we observed above), good upper bounds for $\overline{\mathrm{cr}}\left(K_{n}\right)$ are notoriously difficult to obtain. Currently, the best upper bound known is $\overline{\operatorname{cr}}\left(K_{n}\right) \leq$ $0.3807\binom{n}{4}$ (see Aichholzer et al. [2]).
For many years the best lower bounds known for $\overline{\mathrm{cr}}\left(K_{n}\right)$ were considerably smaller (around $0.32\binom{n}{4}$ ) than the best upper bounds available (currently around $0.380\binom{n}{4}$ ). However, remarkably better lower bounds have been recently proved independently by Ábrego and Fernández-Merchant [1] and by Lovász et al. [13], and refined by Balogh and Salazar [3]. In [1], the technique of allowable sequences is used to show that $\operatorname{cr}\left(K_{n}\right) \geq 0.375\binom{n}{4}$. Lovász et al. use similar methods to prove $\overline{\operatorname{cr}}\left(K_{n}\right)>0.37501\binom{n}{4}+O\left(n^{3}\right)$. Recently, Balogh and Salazar improved this to $\overline{\operatorname{cr}}\left(K_{n}\right)>0.37553\binom{n}{4}+O\left(n^{3}\right)$ 3]. The importance of establishing that $\overline{\operatorname{cr}}\left(K_{n}\right)$ is strictly greater than $0.375\binom{n}{4}+O\left(n^{3}\right)$ is that it effectively shows that the ordinary and the rectilinear crossing numbers of $K_{n}$ are different in the asymptotically relevant term, namely $n^{4}$.

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## References

[1] B.M. Ábrego and S. Fernández-Merchant, A lower bound for the rectilinear crossing number. Manuscript, 2003.
[2] O. Aichholzer, F. Aurenhammer, and H. Krasser, On the crossing number of complete graphs. in Proc. $18^{\text {th }}$ Ann. ACM Symp. Comp. Geom., Barcelona, Spain (2002), 19-24.
[3] J. Balogh and G. Salazar, On $k$-sets, convex quadrilaterals, and the rectilinear crossing number of $K_{n}$, Manuscript (2004).
[4] I.M. Bomze and E. De Klerk, Solving standard quadratic optimization problems via linear, semidefinite and copositive programming, J. Global Opt. 24(2002), 163-185.
[5] P.J. Cameron, Permutation groups, Cambridge University Press, 1999.
[6] P. Erdős and R.K. Guy, Crossing number problems, Amer. Math. Monthly 80 (1973), 52-58.
[7] The GAP Group, GAP - Groups, Algorithms, and Programming, Version 4.3; 2002, http://www.gap-system.org
[8] K. Gatermann and P.A. Parrilo, Symmetry groups, semidefinite programs, and sums of squares, J. Pure Appl. Algebra, to appear.
[9] L. Glebsky and G. Salazar, The crossing number of $C_{m} \times C_{n}$ is as conjectured for $n \geq m(m+1)$, J. Graph Theory, to appear.
[10] R.K. Guy, The decline and fall of Zarankiewicz's Theorem. Proof Techniques in Graph Theory (Proc. Second Ann Arbor Graph Theory Conf., Ann Arbor, Mich., 1968), 6369. Academic Press, New York.
[11] D.J. Kleitman, The crossing number of $K_{5, n}$. J. Combinatorial Theory 9 (1970), 315323.
[12] E. De Klerk, D.V. Pasechnik, Approximation of the stability number of a graph via copositive programming. SIAM J. Optim. 12(2002), 875-892.
[13] L. Lovász, K. Vesztergombi, U. Wagner, and E. Welzl, Convex Quadrilaterals and $k$-sets. Manuscript, 2003.
[14] K.G. Murty and S.N. Kabadi, Some NP-complete problems in quadratic and linear programming, Math. Programming 39(1987), 117-129.
[15] N. Nahas, On the crossing number of $K_{m, n}$. Electron. J. Combin. 10 (2003), Note 8.
[16] P.A. Parrilo, Structured Semidefinite Programs and Semi-algebraic Geometry Methods in Robustness and Optimization, PhD thesis, California Institute of Technology, CA, 2000.
[17] J. Renegar, A mathematical view of interior-point methods in convex optimization, SIAM, Philadelphia, PA, 2001.
[18] R.B. Richter and C. Thomassen, Relations between crossing numbers of complete and complete bipartite graphs. Amer. Math. Monthly 104 (1997), no. 2, 131-137.
[19] A. Schrijver, A comparison of the Delsarte and Lovász bounds. IEEE Trans. Inform. Theory, 25(1979), 425-429.
[20] F. Shahrokhi, O. Sýkora, L.A. Székely, I. Vrťo, Crossing numbers: bounds and applications. Intuitive geometry. Bolyai Soc. Math. Stud., 6, János Bolyai Math. Soc., Budapest (1997), 179-206.
[21] L.H. Soicher, GRAPE: a system for computing with graphs and groups, in Groups and Computation (L. Finkelstein and W.M. Kantor, eds), DIMACS Series in Discrete Mathematics and Theoretical Computer Science, 11(1991), 287-291, http://www.gap-system.org/Share/grape.html
[22] J.F. Sturm, Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones, Optim. Methods Softw., 11-12(1999), 625-653, http://fewcal.kub.nl/sturm/software/sedumi.html
[23] P. Turán, A note of welcome. J. Graph Theory 1 (1977), 7-9.
[24] D.R. Woodall, Cyclic-order graphs and Zarankiewicz's crossing-number conjecture. J. Graph Theory 17 (1993), no. 6, 657-671.
[25] K. Zarankiewicz, On a problem of P. Turán concerning graphs. Fund. Math. 41, (1954). 137-145.


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[^1]:    ${ }^{1}$ In fact, Parrilo [16] only proved the 'if'-part; the converse is proven in Bomze and De Klerk 4.

