NEW RAMSEY BOUNDS FROM CYCLIC GRAPHS OF PRIME ORDER

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ABSTRACT. We present new explicit lower bounds for some Ramsey numbers. All the graphs are cyclic, and are on a prime number of vertices. We give a partial probabilistic analysis which suggests that the cyclic Ramsey numbers grow exponentially. We show that the standard expectation arguments are insufficient to prove such a result. These arguments motivated our searching for Ramsey graphs of prime order.

1. INTRODUCTION

A red–blue coloring of the edges of the complete graph K_n (which we will regard as having vertex set $\{0, 1, 2, 3, \ldots, n-1\}$) is *cyclic* if it is invariant under the rotation $i \to i+1$ (mod n). For integers $k, l \geq 2$, define the *cyclic Ramsey number* $C(k,l)$ to be the least N so that for all $n \geq N$, every cyclic coloring of K_n contains either a red K_k or a blue K_l. Clearly $C(k,l) \leq R(k,l)$. We note, however, that not every $n < C(k,l)$ is such that there exists a cyclic coloring without a red K_k or a blue K_l.

Many authors have searched for lower bounds for Ramsey numbers amongst cyclic graphs, and most of the best known explicit lower bounds come either from cyclic graphs or from cyclic graphs together with a small number of additional vertices.

Our motivating question is: does $C(k,l)$ grow exponentially?

Our paper will have two parts: in the first part we give a partial analysis of random cyclic colorings. We show that standard expectation arguments cannot be used to answer the question above. The analysis suggests that colorings on a prime number of vertices may be slightly more likely to give extremal cyclic Ramsey graphs, motivating our search for the graphs given below. In the second part, we present some cyclic graphs which improve the previously best known bounds for $R(4,12)$, $R(4,15)$, $R(5,7)$ and $R(5,9)$.

2. THE STANDARD PROBABILISTIC ANALYSIS, AND WHY IT FAILS HERE

The standard probabilistic lower bounds for $R(k,l)$ are obtained as follows: let $0 < p < 1$, randomly 2-color the edges of K_n, red with probability p, and blue with probability $1-p$. Compute the expected number of red K_k's and blue K_l's: if this expectation satisfies

$$\sum_{|K|=k} \Pr(K \text{ is a red clique}) + \sum_{|L|=l} \Pr(L \text{ is a blue clique}) < 1$$

then there exists a coloring of K_n with no red K_k and no blue K_l.

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In the cyclic case, the existence of one monochromatic subgraph implies the existence of many, since the image of a monochromatic clique under the rotation $i \to i+1$ (mod n) is also a monochromatic clique. It is easy to see that in fact the existence of one monochromatic clique of order k implies the existence of at least $\frac{n}{(n,k)}$ distinct cliques, and in particular, if n is prime, at least n distinct cliques. Hence, if

$$\frac{(n,k)}{n}\sum_{|K|=k}\Pr(K \text{ is a red clique}) + \frac{(n,l)}{n}\sum_{|L|=l}\Pr(L \text{ is a blue clique}) < 1$$

where the expectations are now computed over all random *cyclic* colorings, then there exists a cyclic coloring of the edges of K_n without a red K_k or a blue K_l. This dependence on the greatest common divisors (n,k) and (n,l) suggests that we may be slightly more successful in finding graphs of prime order.

However, as we shall see, the computation of the expectation is not sufficient to obtain *any* bounds for $C(k,l)$: indeed, we shall see that the expression above grows at least as fast as $n/\sqrt{k}$ for large n.

We shall concentrate on the first part of the sum: fix k, n, and for now set $p = 1/2$. We wish to compute

$$\sum_{|K|=k}\Pr(K \text{ is a red clique}).$$

Define the *difference* of a pair of vertices i and j as $\min\{|i-j|, n-|i-j|\}$. Note that if a coloring is cyclic, then all edges with the same difference are the same color. The differences $D(K)$ of a set K of vertices are the differences between the pairs comprising $K \times K$.

If a set $K = \{x_1, x_2, \ldots, x_k\} \subseteq \{0, 1, 2, \ldots, n-1\}$ has exactly i distinct differences, i.e. $D(K) = i$, then the probability that K is a red clique in a random cyclic coloring is 2^{-i}. Define $N_{i,k,n}$ to be the number of k-subsets of $\{0, 1, 2, \ldots, n-1\}$ having exactly i distinct differences. Then the expected number of red k-cliques in a random cyclic coloring of K_n is

$$\sum_{i=\lfloor k/2\rfloor}^{\binom{k}{2}} N_{i,k,n}2^{-i}.$$

If $k \not| n$ then $N_{j,k,n} = 0$ for $j \le k-2$. Since the 2^{-i} part of the summand is largest in the range $i \le k-2$, this appears as another slight advantage for prime values of n.

Proposition 2.1. *For n prime and $k < \sqrt{n/2}$,*

$$\sum_{i=k-2}^{\binom{k}{2}} N_{i,k,n}2^{-i} = \Omega(n^2/\sqrt{k})$$

Proof: Clearly,

$$\sum_{i=k-2}^{\binom{k}{2}} N_{i,k,n}2^{-i} > 2^{-(2k-3)}\sum_{i=k-1}^{2k-3} N_{i,k,n}.$$

To bound the latter sum, consider the $\lfloor n/2\rfloor$ arithmetic progressions mod n of $2k-2$ terms beginning at 0 with common difference $d:\ 0 < d < n/2$. Each of these sets has $2k-3$ distinct differences. From each progression we may remove $k-2$ nonzero elements in $\binom{2k-3}{k-2}$ ways to form a collection of k-subsets. We claim these are all distinct. It is obvious that those from the same progression are distinct, so it suffices to show that no two progressions of $2k-2$ terms, starting at 0, can contain the same k-subset. To see this, we first show that arithmetic progressions of integers with initial term 0 can't intersect in too many elements: let

$$A = \{0, a, 2a, \ldots, ka\}$$

and

$$B = \{0, b, 2b, \ldots, kb\}$$

be two arithmetic progressions, with $a < b$, and $(a, b) = 1$ (otherwise just divide both a and b by their greatest common divisor);we will show that

$$|A \cap B| > \lfloor \frac{k}{b} \rfloor + 1.$$

Since $(a, b) = 1$, if an element x is in their intersection, it is of the form ja and lb, where $b|j$ and $a|l$. Thus the elements of A in the intersection are a subset of

$$0, ba, 2ba, 3ba, \ldots b\lfloor \frac{k}{b} \rfloor a.$$

Hence there are at most $\lfloor \frac{k}{b} \rfloor + 1$ of them.

We now consider arbitrary arithmetic progressions: by translating both progressions, we may assume that

$$A = \{0, a, 2a, 3a, \ldots, ka\}$$

and

$$B = \{c, c+b, c+2b, c+3b, \ldots, c+kb\}.$$

Now, if $c > 0$ we can replace A by $A \setminus \{0\} \cup \{(k+1)a\}$ without decreasing the size of the intersection. Iterating this process, we see that we can translate until the arithmetic progressions both start with 0, and we are in the case handled above.

We now show that two arithmetic progressions taken modulo n have the same property, provided that k is much less than n (clearly it fails to be true if k is close to n).

Since n is prime, by multiplying both arithmetic progressions by $a^{-1} \bmod p$, and by rotating, we may assume

$$A = \{0, 1, 2, 3, \ldots, k\}$$

and

$$B = \{c, c+b, c+2b, c+3b, \ldots, c+kb\}.$$

Now, if we knew that B didn't wrap around modulo n, then we would be able to appeal to the statement for arithmetic progressions of integers above: we shall show that there is a value $d \bmod n$ so that neither $dA \bmod n$ nor $dB \bmod n$ wrap around. Observe that since the progressions intersect in at least two elements then we have e, f, g, h so that $e = c + gb$ and $f = c + hb$, where each of e, f, g, h are at most k and we may assume $f > e$. Then

$$f - e = (h - g)b;$$

if $h < g$, then we will replace the arithmetic progression B by the reverse arithmetic progression (with common difference $n - b$ and initial term $c + kb$). Thus we are now in the situation where we have $0 \le e < f \le k$, $0 \le g < h \le k$, and

$$f - e = (h - g)b.$$

If we now let $d = h - g$, and consider the progressions $A' = dA$ and $B' = dB$ we see that

$$A' = \{0, d, 2d, \dots, dk\}$$

and

$$B' = \{cd, cd + bd, cd + 2bd, \dots, cd + kbd\}$$

(taken modulo n). Now, since $d \le k$ and $bd = f - e \le k \pmod{n}$, each of A' and B' has a small common difference. Indeed, the difference of A is $d \le k$ and the difference of B is $bd \le k$. Thus, provided that $k^2 < \frac{n}{2}$, A' doesn't wrap around (mod n), and B' wraps around at most once: moreover, if B' wraps around we can rotate both arithmetic progressions so that B' starts at 0 and neither progression wraps around, reducing us to the cases handled above. Thus we have shown that if the arithmetic progressions modulo n intersect in many elements then they are the same arithmetic progression.

Now since n is prime and $d < n/2$ each subset may be rotated $n - 1$ times to yield a total of

$$(\lfloor n/2 \rfloor) n \binom{2k-3}{k-2} \sim n^2 2^{2k-3}/\sqrt{k}$$

distinct k-subsets. Each of these subsets has at most $2k - 3$ distinct differences, since each is a subset of a progression having $2k - 3$ distinct differences. Therefore,

$$\sum_{i=k-1}^{2k-3} N_{i,k,n} = \Omega(n^2 2^{2k-3}/\sqrt{k})$$

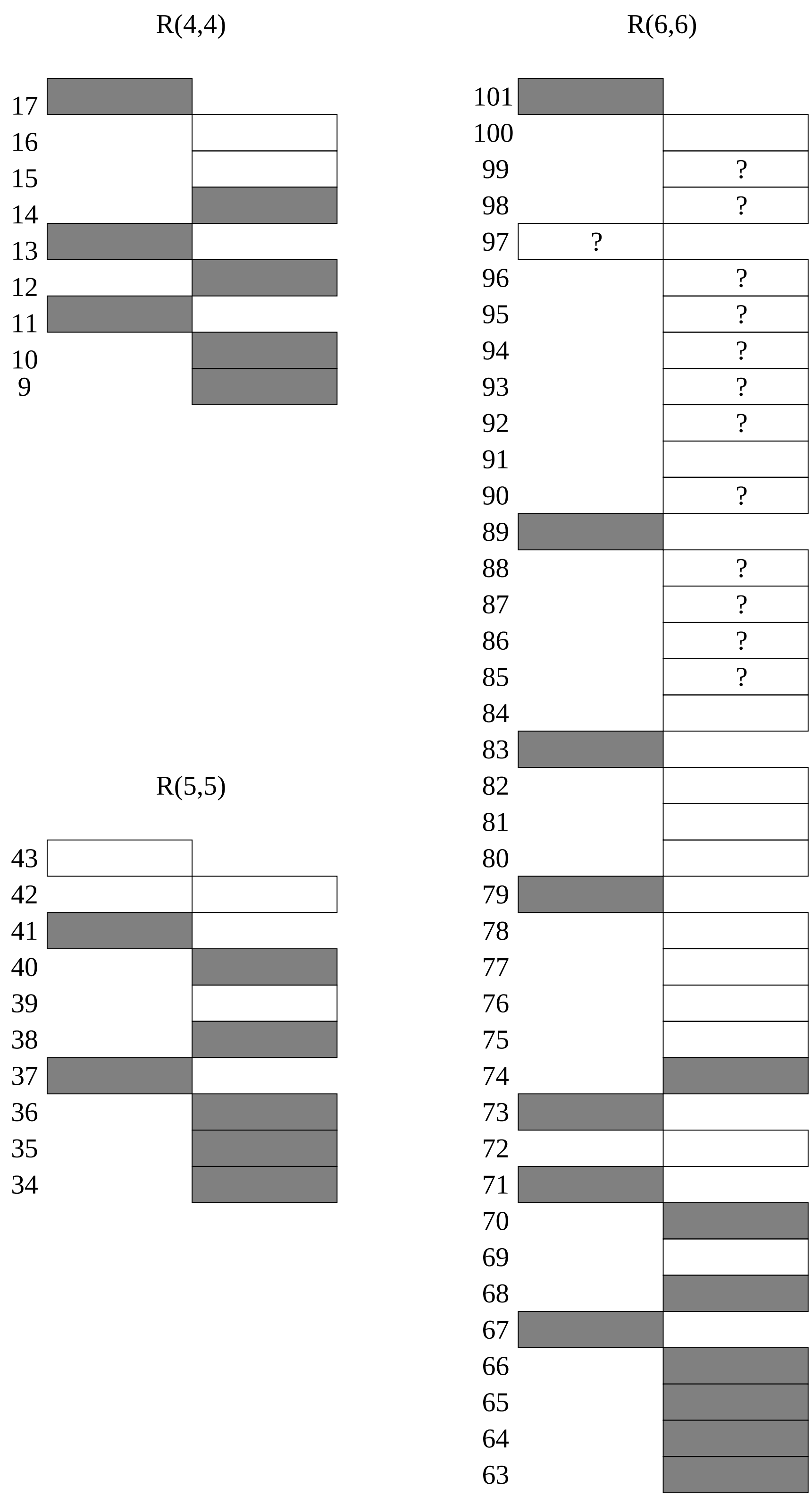
and the proposition follows.

From the proposition we see that the standard argument will not give bounds on $R(k, l)$ that are exponential in $\min\{k, l\}$.

An additional advantage of primes: a natural way to investigate bounds for $N_{i,k,n}$ is to "grow" a set K randomly, counting the number of new distinct differences when a vertex x is added to K. All $|K|$ differences will be distinct only if x does not satisfy any of a set of equations mod n derived from the vertices in K (e.g. x can not be the mean of two points in K). When n is prime these equations are solved over the field Z_n and have unique solutions. But when n is composite there can be multiple solutions, increasing the probability of duplicating a difference (e.g. both 3 and 0 are midpoints of 2 and 4, mod 6) .

3. New bounds on some classical Ramsey numbers

As we've discussed, primes seem to show some advantage at several points in the probabilistic analysis. We checked empirically for advantages of primes over composites with regard to bounds for $R(4, 4)$, $R(5, 5)$, and $R(6, 6)$. The results are given in in the figures below where shading denotes that a cyclic Ramsey graph is known to exist, and blank areas indicate that there are no cyclic Ramsey graphs. A question mark indicates cases where we do not know whether or not cyclic Ramsey graphs exist.


R(4,4)
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R(5,5)
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R(6,6)
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Primes show a slight advantage in the first two cases, and a dramatic advantage in the third case: the largest known ramsey graph of composite order has 74 vertices, while every prime number order through 101 yields a ramsey graph, with the possible exception of 97.

Having some empirical confirmation that prime order graphs are more apt to provide good Ramsey bounds, we successfully searched for graphs to improve best known lower bounds. The graphs were found by implicit enumeration of cyclic 2-colorings. The program was written in Pascal and run on Sun SPARCstations (2, 10, or 20). We emphasize that the algorithm is straightforward and the hardware unexceptional even by 1991 standards. The advantage that we had was knowing to look at graphs of prime order. We suspect that in the past, when a complete search revealed no cyclic Ramsey graphs of order n or $n+1$, researchers did not continue the search over larger orders. We hope that our computational results will encourage other researchers with better algorithms and hardware to look for further improvements, both by searching over larger orders, and by taking our graphs and modifying them.

We searched for cyclic graphs of order equal to the smallest prime greater than or equal to the best known bound. The required CPU times varied from 25 minutes (for $R(5,7)$) to 10 days (for $R(4,15)$).

$R(4,12) \geq 98$. This improves on the bound of 97 reported in [1]. In the 97 vertex graph, the following edge differences are present: $11, 19, 21, 22, 23, 29, 34, 35, 38, 39, 43, 44, 46, 47, 48$.

$R(4,15) \geq 128$. This improves on the bound of 123 reported in [1]. In the 127 vertex graph, the following edge differences are present: $14, 27, 28, 29, 38, 39, 41, 43, 44, 45, 47, 49, 51, 52, 58, 60, 62, 63$.

$R(7,5) \geq 80$. This improves on the bound of 76 reported in [1]. In the 79 vertex graph, the following edge differences are present: $1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 15, 16, 18, 19, 23, 27, 29, 30, 31, 32, 33, 35, 39$.There is no such cyclic graph on 83 vertices.

$R(5,9) \geq 114$. This appears to the be first bound reported [1]. In the 113 vertex graph, the following edge differences are present: $8, 9, 10, 11, 12, 13, 14, 15, 16, 20, 28, 32, 34, 35, 39, 42, 43, 44, 46, 48, 52, 54, 55$.

Primes do not always fare better than composites. Besides the trivial case of 3-vertex graphs for $R(3,3)$, the smallest example occurs for $R(4,5)$: there is no cyclic Ramsey graph on 23 vertices, but there is one on 24 vertices.

References

[1] Radziszowski, Stanislaw. Small Ramsey Numbers, *Electronic Journal of Combinatorics*, Dynamic Survey 1, 1994.

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