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NONLINEAR LAGRANGIAN FOR MULTIOBJECTIVE OPTIMIZATION AND APPLICATIONS TO DUALITY AND EXACT PENALIZATION*

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Abstract. Duality and penalty methods are popular in optimization. The study on duality and penalty methods for nonconvex multiobjective optimization problems is very limited. In this paper, we introduce vector-valued nonlinear Lagrangian and penalty functions and formulate nonlinear Lagrangian dual problems and nonlinear penalty problems for multiobjective constrained optimization problems. We establish strong duality and exact penalization results. The strong duality is an inclusion between the set of infimum points of the original multiobjective constrained optimization problem and that of the nonlinear Lagrangian dual problem. Exact penalization is established via a generalized calmness-type condition.

Key words. multiobjective optimization, nonlinear Lagrangian function, duality, exact penalization, stability

AMS subject classifications. 90C29, 90C46

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1. Introduction and preliminaries.

It is well known that the traditional Lagrange function plays an important role in both theory and methodology for single objective and multiobjective convex optimization problems, such as optimality condition, duality theory, saddle point theory, sensitivity analysis, and solution method [2, 19]. However, it becomes less effective for nonconvex optimization problems. For example, there may be a nonzero duality gap between the single objective nonconvex constrained optimization problem and its Lagrange dual problem. Thus the Lagrange method may fail for nonconvex optimization problems. Moreover, it is worth noting that a zero duality gap can be achieved for a single objective nonconvex optimization problem using an augmented Lagrangian function; see [14]. A more general scheme of the conjugate framework was established for convex and nonconvex cases in [12, 1], respectively. On the other hand, exact penalty functions and their applications in the study of optimality conditions were provided for single objective constrained optimization problems in, e.g., [4, 14, 15] under calmness conditions. See [3] for an excellent review.

Recently, a class of nonlinear Lagrangian functions was introduced and applied to establish a zero duality gap for single objective constrained continuous optimization problems without any convexity requirement [5, 17]. The terminology "nonlinear" refers to the nonlinearity of the objective function of the transformed problems with respect to the objective function of the original constrained optimization problem. The exact penalization result for nonconvex inequality constrained single objective optimization was obtained under a generalized calmness condition in [18]. It is worth

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noting that the early study on the nonlinear Lagrangian can be found in the work [23]. Moreover, a pth power transformation was introduced in [9] to guarantee a zero duality gap for an optimization problem, which is not necessarily convex.

In this paper, we introduce a class of nonlinear Lagrangian functions and nonlinear Lagrangian dual problems for (nonconvex) multiobjective optimization problems. In particular, we obtain a strong duality result between a constrained multiobjective optimization problem and its nonlinear Lagrangian dual problem without any convexity requirement. Several types of exact penalization for nonlinear penalty multiobjective optimization problems are investigated. We study conditions which guarantee

(i) there is a finite penalty parameter vector such that every infimum point of the original constrained multiobjective optimization problem is an infimum point of the nonlinear penalty multiobjective optimization problem (global exact penalization); and

(ii) for each infimum point of the original constrained multiobjective optimization problem, there is a finite penalty parameter vector such that this point is also an infimum point of the nonlinear penalty multiobjective optimization problem (local exact penalization).

The motivation of our study is that there is only limited study on duality and penalty methods for nonconvex multiobjective optimization problems. Yet these approaches are popular solution methods in single objective optimization. For convex multiobjective optimization problems, systematic study of Lagrangian duality and conjugate duality was given in [19, 10] and the references cited therein. To the best of our knowledge, investigation on the conventional penalty function method for constrained multiobjective optimization problems was only given in [16, 20]. We will establish strong duality for multiobjective optimization problems without any convexity requirement. The condition used is the lower semicontinuity of the functions involved, which is much weaker than the continuity assumption in [17]. Moreover, the conditions for exact penalization are a generalization of the ones for single objective optimization in [3, 4, 15, 18]. It is worth noting that nonlinear Lagrangian dual problems studied in this paper provide new models for convex composite optimization problems studied in [6, 7, 21].

Let R^l be an l-dimensional Euclidean space, $C = R^l_+$, and $\text{int}C$ be the interior of C. Define the following orderings: for any $z^1, z^2 \in R^l$,

$$z^1 \leq_C z^2 \Longleftrightarrow z^2 - z^1 \in C,\ z^1 \not\leq_C z^2 \Longleftrightarrow z^2 - z^1 \notin C,$$
$$z^1 \leq_{C\backslash\{0\}} z^2 \Longleftrightarrow z^2 - z^1 \in C\backslash\{0\},\ z^1 \not\leq_{C\backslash\{0\}} z^2 \Longleftrightarrow z^2 - z^1 \notin C\backslash\{0\},$$
$$z^1 \leq_{\text{int}C} z^2 \Longleftrightarrow z^2 - z^1 \in \text{int}C,\ z^1 \not\leq_{\text{int}C} z^2 \Longleftrightarrow z^2 - z^1 \notin \text{int}C.$$

Let $e = (1, \dots, 1) \in \text{int}C$, and $e_i = (0, 0, \dots, 1, 0, \dots, 0)$ (the ith component is 1 and the other components are 0's), $i = 1, \dots, l$.

Consider the following multiobjective constrained optimization problem:

$$\begin{array}{lll} \text{(MOP)} & \inf\limits_{x\in X} & f(x) \\ \text{such that (s.t.)} & & g_j(x) \leq 0, \quad j = 1, \dots, m, \end{array}$$

where $X \subseteq R^n$ is a nonempty closed set, $f = (f_1, \dots, f_l) : X \to R^l$ is a vector-valued function such that each of its component function f_i is lower semicontinuous (l.s.c.), and $g_j : X \to R^1$ is l.s.c. for any $j \in \{1, \dots, m\}$.

By X_0 we denote the set of feasible solutions of (MOP). That is, $X_0 = \{x \in X : g_j(x) \leq 0, j = 1, \dots, m\}$. It is clear that X_0 is closed.

We say that $x^* \in X_0$ is *an efficient solution* to (MOP) if there exists no $x \in X_0$ such that $f(x) \leq_{C\setminus\{0\}} f(x^*)$. The corresponding function value $f(x^*)$ is called an *efficient point* of (MOP). We denote by $E(0)$ the set of the efficient solutions of (MOP).

The point $x^* \in X_0$ is called *a weakly efficient solution* to (MOP) if there exists no $x \in X_0$ such that $f(x) \leq_{\text{int}C} f(x^*)$. The corresponding point $f(x^*)$ is called a *weakly efficient point* of (MOP). The set of weakly efficient solutions of (MOP) is denoted by $WE(0)$.

The point $x^* \in X_0$ is said to be *a locally weak efficient solution* to (MOP) if there exists $\delta > 0$ such that $f(x) \not\leq_{\text{int}C} f(x^*)$ for any $x \in X_0$ with $\|x - x^*\| \leq \delta$. The set of all locally weak efficient solutions of (MOP) is denoted by $LWE(0)$.

We denote by $V(0)$ the set of *infimum points* of (MOP), i.e., $V(0) = \inf_{x\in X_0} f(x)$. Namely, $z \in V(0)$ if and only if (i) $f(x) \not\leq_{C\setminus\{0\}} z \ \forall x \in X_0$ and (ii) $\exists x_k \in X_0$ such that $f(x_k) \to z$ as $k \to \infty$.

Clearly, if x_0 is an efficient solution to (MOP), then $f(x_0) \in V(0)$.

Without loss of generality, we assume throughout this paper that $\min_{1\leq i\leq l} \inf_{x\in X} f_i(x) \geq 0$. If this assumption does not hold, then consider the following optimization problem:

$$\begin{array}{lll} (\text{MOP}') & \inf\limits_{x\in X} & (\exp(f_1(x)) + 1, \ldots, \exp(f_l(x)) + 1) \\ & \text{s.t.} & g_j(x) \leq 0, \quad j = 1, \ldots, m. \end{array}$$

It is clear that the sets of efficient solutions and weakly efficient solutions of (MOP) are the same as that of (MOP$'$), respectively.

Throughout this paper, for simplicity, we shall use the notation $\|u\|_\gamma$ to denote the formula $[\sum_{j=1}^m |u_j|^\gamma]^{1/\gamma}$, where $u = (u_1, \ldots, u_m) \in R^m, \gamma \in (0, +\infty)$.

Let $y^1 = (y_1^1, \ldots, y_m^1), y^2 = (y_1^2, \ldots, y_m^2) \in R^m$, define the notation of componentwise product for y^1 and y^2:

$$y^1 * y^2 \equiv (y_1^1 y_1^2, \ldots, y_m^1 y_m^2).$$

Let Z_1 be a subset of a metric space Z, and $z \in Z$. Denote by $d(z, Z_1)$ the distance from the point z to the set Z_1.

The outline of the paper is as follows. In section 2, strong duality for (MOP) and its nonlinear Lagrangian dual problem (DMOP) (see next section) is established. In section 3, conditions are given which are necessary and sufficient for the existence of a global (local) exact penalty parameter. In section 4, we consider saddle points of the nonlinear Lagrangian.

2. Nonlinear Lagrangian functions and duality. Let $A \subseteq R^l \times R^m$. A vector-valued function $p : A \to R^l$ is called *increasing* on the set A if for any $(z^i, y^i) \in A (i = 1, 2)$ with $(z^1, y^1) - (z^2, y^2) \in C \times R_+^m$ we have $p(z^1, y^1) \geq_C p(z^2, y^2)$.

Let p be an increasing vector-valued function defined either on the domain $C \times R^m$ or on the domain $C \times R_+^m$ such that each of its component functions p_i is l.s.c. and p enjoys the following two properties:

(A) There exist positive real numbers $a_j (j = 1, \ldots, m)$ such that for any $z \in C, y = (y_1, \ldots, y_m)$ with (z, y) belonging to the domain of p, $p(z, y) \geq_C z$ and $p(z, y) \geq_C (\max_{1\leq j\leq m}\{a_j y_j\})e$.

(B) $\forall z \in C, p(z, 0, \ldots, 0) = z$.

Remark 2.1. This reduces to the function p of [17] when $l = 1$ and p is continuous.

It is easy to prove the following elementary proposition.

PROPOSITION 2.1. *Let $p(z, y) = p'(p'(z, y), y)$, where p' is an increasing function with properties* (A) *and* (B). *Then p is also an increasing function having properties* (A) *and* (B).

Example 2.1. Let $z = (z_1, \ldots, z_l), y = (y_1, \ldots, y_m)$, and $(z, y) \in C \times R^m$. Some examples of the increasing function p defined on $C \times R^m$ having properties (A) and (B) are as follows:

$p_\infty(z, y) = \sum_{i=1}^{l} \max\{z_i, y_1, \ldots, y_m\} e_i$;

$p_\gamma(z, y) = \sum_{i=1}^{l} \left(z_i^\gamma + \sum_{j=1}^{m} {y_j^+}^\gamma\right)^{1/\gamma} e_i, 0 < \gamma < \infty$, where $y_j^+ = \max\{y_j, 0\}, j = 1, \ldots, m$;

$p(z, y) = z + (\sum_{j=1}^{m} b_j y_j^+) e$, where $b_j > 0, j = 1, \ldots, m$.

Example 2.2. The restrictions of p_∞, p_γ, p (considered in Example 2.1) to $C \times R_+^m$ are increasing functions defined on $C \times R_+^m$ having properties (A) and (B).

In the rest of this section, p is assumed to be an increasing function defined on $C \times R^m$ with properties (A) and (B), and this section concludes with a remark for the case when p is defined on $C \times R_+^m$.

Let

$$F(x, d) = (f(x), d * g(x)),$$

where $d = (d_1, \ldots, d_m) \in R_+^m$ and $g(x) = (g_1(x), \ldots, g_m(x))$.

The *nonlinear Lagrangian* function corresponding to p for (MOP) is defined as

$$L(x, d) = p(F(x, d)). \tag{1}$$

DEFINITION 2.2. *The following problem,*

$$\text{(DMOP)} \sup_{d \in R_+^m} q(d),$$

where $q(d) = \inf_{x \in X} L(x, d)\ \forall d \in R_+^m$, is called the nonlinear Lagrangian dual problem to (MOP) *corresponding to p. Here by $z \in \sup_{d \in R_+^m} q(d)$ we mean that*

(i) $(z - q(d)) \bigcap (-C \backslash \{0\}) = \emptyset\ \forall d \in R_+^m$;

(ii) *$\exists d^j \in R_+^m$ and $z^j \in q(d^j)$ such that $z^j \to z$ as $j \to +\infty$.*

z is called a supremum point of (DMOP).

Remark 2.2. If p is convex, e.g., all the p's except p_γ in the case of $\gamma \in (0, 1)$ in Example 2.1, the problem of computing $q(d)$,

$$\inf_{x \in X} p(F(x, d)),$$

is a type of convex composite multiobjective optimization problem studied in [7].

It is elementary to prove the following results.

LEMMA 2.3. *Let p be an increasing function with properties* (A) *and* (B). *Then $p(F(x, d)) = f(x)\ \forall x \in X_0,\ d \in R_+^m$.*

PROPOSITION 2.4 (weak duality). $\forall x \in X_0, d \in R_+^m$, $(q(d) - f(x)) \bigcap (C \backslash \{0\}) = \emptyset$.

COROLLARY 2.5. *If $x^* \in X_0$ satisfies $f(x^*) \in \sup_{d \in R_+^m} q(d)$, then $x^* \in WE(0)$.*

COROLLARY 2.6. $[\sup_{d \in R_+^m} q(d) - V(0)] \bigcap \text{int}\, C = \emptyset$.

DEFINITION 2.7 (see [19]). *Let $X \subset R^n$ be a set and $f : X \to R^l$ be a vector-valued function. The set $f(X)$ is said to be externally stable if for any $x \in X$ there exists an efficient solution $x^* \in X$ of f on X such that $f(x^*) \leq_C f(x)$.*

DEFINITION 2.8. *Let $X \subset R^n$ be a set and $f : X \to R^l$ be a vector-valued function. The set $f(X)$ is said to be* inf*-externally stable if for any $x \in X$ there exists an infimum point z^* of $f(X)$ such that $z^* \leq_C f(x)$.*

Remark 2.3. The definition of external stability is given in [19], while the definition of inf-external stability is a weaker concept, which will be used later in this paper.

The following lemma on external stability can be derived from [19, Corollary 3.2.1].

LEMMA 2.9. *Let $X \subset R^n$ be a compact subset. Let $f : X \to R^l$ be a vector-valued function such that each of its component functions is l.s.c. Then $f(X)$ is externally stable.*

It is routine to prove the next lemma.

LEMMA 2.10. *Let $s : C \times R^m \to R^1$ be an increasing l.s.c. function. Let $f : X \to C$ be a vector-valued function such that each component function f_i is l.s.c. Let $g_j : X \to R^1$ $(j = 1, \dots, m)$ be l.s.c. Then $s(f(x), g(x))$ is l.s.c. on X.*

Let $\xi(z) = \max_{1 \leq i \leq l}\{z_i\}\ \forall z = (z_1, \dots, z_l)$.

Clearly, ξ is an increasing, continuous, subadditive, positively homogeneous, and convex function.

DEFINITION 2.11. *Let $X \subset R^n$ be an unbounded set. A vector-valued function $f : X \to R^l$ is said to be coercive on X if*

$$\lim_{\|x\| \to +\infty, x \in X} \xi(f(x)) \to +\infty,$$

where $\|.\|$ is a norm of R^n.

The following result establishes a proper relation between (MOP) and (DMOP).

THEOREM 2.12 (strong duality). *Assume that X is closed, $f(x) \geq_C 0\ \forall x \in X$, and f is coercive on X if X is unbounded. Then*

$$V(0) \subseteq \sup_{d \in R^m_+} q(d).$$

Proof. Let $z^* \in V(0)$. Then $\exists x^1_k \in X_0$ such that $f(x^1_k) \to z^*$ as $k \to +\infty$.

It follows that $\xi(f(x^1_k)) \to \xi(z^*)$ as $k \to +\infty$. Therefore, $\{x^1_k\}$ is a bounded sequence by the coercivity of f on X. Since X_0 is closed, there exists a subsequence $\{x^1_{k_j}\}$ such that $x^1_{k_j} \to x^*$ for some $x^* \in X_0$. Note that $f_i(x^*) \leq \liminf_{j \to +\infty} f_i(x^1_{k_j}) = (z^*)_i, i = 1, \dots, l$, where $(z^*)_i$ denotes the ith component of z^*. We have $f(x^*) \leq_C z^*$. This with $z^* \in V(0)$ implies that $f(x^*) = z^*$. Hence $x^* \in E(0)$. Since f is coercive on X, we deduce that $\exists N > 0$ such that

$$\xi(f(x)) \geq \xi(f(x^*)) + 1 \forall x \in X_1 = \{x \in X : \|x\| > N\}. \tag{2}$$

We claim that

$$f(x) \not\leq_{C \setminus \{0\}} f(x^*) \quad \forall x \in X_1. \tag{3}$$

Otherwise, $\xi(f(x)) \leq \xi(f(x^*))$, contradicting (2).

Let $d = ke, k = 1, 2, \dots$. Since $X_2 = \{x \in X : \|x\| \leq N\}$ is a nonempty compact set and $x^* \in X_2$, by Lemmas 2.9 and 2.10, we obtain a sequence $\{x^2_k\} \subseteq X_2$ such that each x^2_k is an efficient solution to the problem: $\min_{x \in X_2} p(f(x), kg(x))$ and

$$p(f(x^2_k), kg(x^2_k)) \leq_C p(f(x^*), kg(x^*)) = f(x^*). \tag{4}$$

We show that this fact combined with (3) yields that $p(F(x_k^2, d)) \in q(k, \dots, k) = \inf_{x \in X} p(F(x, d))$.

(i) It is obvious that if $x \in X_2$, $p(F(x_k^2, d)) \not\geq_{C\setminus\{0\}} p(F(x, d))$.

(ii) Suppose that $\exists\ \overline{x} \in X_1$ such that

$$p(F(x_k^2, d)) \geq_{C\setminus\{0\}} p(F(\overline{x}, d)). \tag{5}$$

Note that

$$p(F(x_k^2, d)) \leq_C f(x^*)$$

and

$$f(x^*) \not\geq_{C\setminus\{0\}} f(\overline{x}).$$

Then

$$p(F(x_k^2, d)) \not\geq_{C\setminus\{0\}} f(\overline{x}). \tag{6}$$

By (5) and (6),

$$p(F(\overline{x}, d)) \not\geq_{C\setminus\{0\}} f(\overline{x}),$$

a contradiction with the property (A).

It follows from $\{x_k^2\} \subset X_2$ that there exists a subsequence $\{x_{k_j}^2\}$ such that $x_{k_j}^2 \to x_0 \in X_2$.

Let us show that $x_0 \in X_0$. If not, $d(x_0, X_0) \geq \delta_0$ for some $\delta_0 > 0$. It follows that $d(x_{k_j}^2, X_0) \geq \delta_0/2$ when j is sufficiently large.

Let $X_3 = \{x \in X_2 : d(x, X_0) \geq \delta_0/2\}$ and $\bar{g}(x) = \max_{1 \leq j \leq m} g_j(x)$. Since $\bar{g}(x) > 0\ \forall x \in X_3$, X_3 is compact, and $\bar{g}$ is l.s.c, we deduce that $\min_{x \in X_3} \bar{g}(x) = m_0 > 0$.

By property (A) of the function p, there exist positive numbers $a_i (i = 1, \dots, m)$ such that

$$p(f(x_{k_j}^2), k_j g(x_{k_j}^2)) \geq_C \left(m_0 k_j \min_{1 \leq i \leq m} a_i\right) e$$

when j is sufficiently large, which contradicts (4). So $x_0 \in X_0$.

Applying property (A) and (4), we have

$$f(x_{k_j}^2) \leq_C p(f(x_{k_j}^2), k_j g(x_{k_j}^2)) \leq_C f(x^*).$$

Thus,

$$f_i(x_{k_j}^2) \leq p_i(f(x_{k_j}^2), k_j g(x_{k_j}^2)) \leq f_i(x^*), \quad i = 1, \dots, l. \tag{7}$$

Applying the lower limit to (7) by letting $j \to \infty$, we conclude that $f_i(x_0) \leq f_i(x^*), i = 1, \dots, l$, which implies that

$$f(x_0) = f(x^*) \tag{8}$$

since $x^* \in E(0)$.

Equation (8) combined with (7) as well as $x_{k_j}^2 \to x_0$ yields that

$$p(f(x_{k_j}^2), k_j g(x_{k_j}^2)) \to f(x^*) \text{ as } j \to +\infty.$$

Finally, it follows directly from Proposition 2.4 that

$$(q(d) - f(x^*)) \bigcap (C \backslash \{0\}) = \emptyset \quad \forall d \in R^m_+.$$

The proof is complete. $\square$

Remark 2.4. 1. When $l = 1$, this theorem improves Theorem 3.1 in [17] by relaxing the assumption of continuity of f and g_j as well as p to lower semicontinuity and dropping the assumption that X_0 is compact.

2. It is evident from the proof of Theorem 2.12 that to solve (MOP) we can solve a series of unconstrained multiobjective programming problems to approach the efficient points of (MOP).

3. The condition that f is coercive on X is important to guarantee the validity of Theorem 2.12. Otherwise, it may fail even if X_0 is compact. Example 2.3 shows this case.

Example 2.3. Let $l = 1, X = [0, +\infty), f(x) = 1/(x + 1)\ \forall x \in X,\ g_1(x) = x - 1$ if $0 \leq x \leq 1; g_1(x) = 1/\sqrt{x} - 1/x$ if $1 < x < +\infty$, $p(y_1, y_2) = \max\{y_1, y_2\}\ \forall y_1, y_2 \in R^1$.

Consider the problem

$$\inf_{x \in X} f(x) \text{ s.t. } g_1(x) \leq 0.$$

It is easy to see that $X_0 = [0, 1]$ (which is compact) and $V(0) = \{1/2\}$.

$p(f(x), dg_1(x)) = \max\{f(x), dg_1(x)\} = \max\{1/(x + 1), d(1/\sqrt{x} - 1/x)\}\ \forall x \in X \backslash X_0, d \geq 0$.

Clearly, $q(d) = 0\ \forall d \geq 0$. It follows that $\sup_{d \geq 0} q(d) = \{0\}$. Hence $V(0) \subseteq \sup_{d \geq 0} q(d)$ does not hold.

Despite Example 2.3, in actually designing an algorithm based on Theorem 2.12, if X_0 is compact, we can replace $f(x)$ with $f(x) + l(x)e$, where $l : X \to R^1_+$ is an l.s.c. function which satisfies the following condition: there exists a compact set X' such that $X_0 \subseteq X' \subseteq X$ with $l(x) = 0$ if $x \in X'$ and $l(x) \to +\infty$ if $x \in X$ and $\|x\| \to +\infty$. A simple example of such an l is $l(x) = d(x, X_0)\ \forall x \in X$. Thus Theorem 2.12 can be applied to the objective function $f(x) + l(x)e$, which has the same set of (weakly) efficient solutions and the same set of (weakly) efficient points as $f(x)$ on X_0.

Finally, we observe the following two points:

(i) for $z^* \in V(0)$ there may not exist $d^* \in R^m_+$ such that $z^* \in q(d^*)$ even if all the conditions in Theorem 2.12 hold;

(ii) for the conventional Lagrangian, Theorem 2.12 does not, in general, hold.

Counterexamples are given for these two cases in Examples 2.4 and 2.5, respectively.

Example 2.4. Let $l = 1, X = [1/2, +\infty)$, and $f(x) = 1/x$ if $x \in [1/2, 1]$; $f(x) = 2 - x$ if $x \in [1, 2]$; $f(x) = x - 2$ if $x \in (2, +\infty)$. Let $g_1(x) = x - 1$.

Consider the problem

$$\inf_{x \in X} f(x) \text{ s.t. } g_1(x) \leq 0.$$

Let $L(x, d) = \max\{f(x), dg_1(x)\},\ d \geq 0,\ x \in X$. Then it is not difficult to derive the following fact: $q(d) = d/(1 + d)\ \forall d \geq 0$. Clearly, $q(d) < 1 = \inf_{x \in X_0} f(x)\ \forall d > 0$.

Example 2.5. Let $l = 1,\ X = [0, +\infty),\ f(x) = x,\ g(x) = x - x^2$. Consider the problem

$$\begin{gathered}\inf_{x \in X} f(x) \\ \text{s.t. } g_1(x) \leq 0.\end{gathered}$$

It is clear that all the conditions of Theorem 2.12 hold. Let us look at the conventional Lagrangian for this problem: $l(x,\lambda) = f(x)+\lambda g_1(x) = x+\lambda(x-x^2)\ \forall x \in X,\ \lambda \geq 0$. It is easy to check that $\inf_{x\in X} l(x,\lambda) = -\infty\ \forall\lambda > 0$ and $\inf_{x\in X} l(x,0) = 0$. Thus, $\sup_{\lambda\geq 0}\inf_{x\in X} l(x,\lambda) = 0$. However, the optimal value of the original constrained problem is 1.

Based on some conditions on the constraint functions, we also have the following result.

THEOREM 2.13. *Let $\bar{g}(x) = \max_{1\leq j\leq m} g_j(x)$. Assume that there exist $N > 0$ and $m_1 > 0$ such that*

$$\bar{g}(x) \geq m_1 \quad \forall x \in X \text{ with } \|x\| > N. \tag{9}$$

Then $V(0) \subseteq \sup_{d\in R^m_+} q(d)$.

Proof. It follows from (9) that X_0 is a nonempty compact set. For any $z^* = f(x^*) \in V(0)$, by Proposition 2.4 we have that

$$(q(d) - f(x^*))\bigcap(C\backslash\{0\}) = \emptyset \quad \forall d \in R^m_+.$$

Furthermore, whenever $x \in X$ with $\|x\| > N$,

$$p(f(x), kg(x)) \geq_C \left(km_1 \min_{1\leq i\leq m}\{a_i\}\right) e \geq_{\mathrm{int}C} f(x^*) + e$$

when k is sufficiently large. Consequently, when k is sufficiently large, the set

$$\{x \in X : p(f(x), kg(x)) \leq_C f(x^*)\}(\subseteq \{x \in X : \|x\| \leq N\})$$

is a nonempty compact set. Therefore, when k is sufficiently large, $\exists x_k \in X$ with $\|x_k\| \leq N$ such that x_k is an efficient solution to the problem

$$\min_{x\in X}\ p(f(x), kg(x))$$

with

$$f(x_k) \leq_C p(f(x_k), kg(x_k)) \leq_C f(x^*). \tag{10}$$

Since $\|x_k\| \leq N$ for k sufficiently large, it follows that there exists a subsequence $\{x_{k_j}\}$ converging to $x' \in X$. We can show as in the proof of Theorem 2.12 that $x' \in X_0$. This fact combined with (10) yields that $f(x') \leq_C f(x^*)$. Therefore, $f(x') = f(x^*)$ since $x^* \in E(0)$. Hence, $p(f(x_{k_j}), k_j g(x_{k_j})) \to f(x^*)$. So $f(x^*) \in \sup_{d\in R^m_+} q(d)$ and the proof is complete. $\square$

The following proposition further clarifies the relation between (MOP) and (DMOP).

PROPOSITION 2.14. *Let $d^k \in R^m_+\ \forall k$ and $d^k \to +\infty$ as $k \to \infty$ (i.e., $d^k_i \to +\infty\ \forall i$ as $k \to +\infty$). Suppose that each x^k is a weakly efficient solution to* $\inf_{x\in X} L(x, d^k)$. *Then any limiting point of $\{x^k\}$ is a weakly efficient solution to* (MOP).

Proof. Without loss of generality, suppose that $x^k \to x^*$. We can show by contradiction that $x^* \in X_0$. In fact, if $d(x^*, X_0) \geq \delta_0$ for some $\delta_0 > 0$, then $d(x^k, X_0) \geq \delta_0/2$ when k is sufficiently large. Since $x^k \to x^*$, we deduce that $\|x^k - x^*\| \leq 1$ when k is sufficiently large.

Let $X_4 = \{x \in X : d(x, X_0) \geq \delta_0/2, \|x - x^*\| \leq 1\}$. Then $x^k \in X_4$ when k is sufficiently large. Let $\bar{g}(x) = \max_{1\leq i\leq m} g_i(x)$. Then $\bar{g}(x^k) \geq \min_{x\in X_3} \bar{g}(x) =$

$m_1 > 0$ when k is sufficiently large. So

$$\begin{aligned} p(f(x^k), d^k * g(x^k)) &\geq_C \bar{g}(x^k) \left(\min_{1\leq i\leq m} a_i \min_{1\leq i\leq m} d_i^k \right) e \\ &\geq_C \left(m_1 \min_{1\leq i\leq m} a_i \min_{1\leq i\leq m} d_i^k \right) e \\ &\geq_{\mathrm{int}C} f(x_0) \end{aligned} \tag{11}$$

for any fixed $x_0 \in X_0$ and k large enough. Moreover, by Lemma 2.3,

$$f(x_0) = p(f(x_0), d^k * g(x_0)). \tag{12}$$

The combination of (11) and (12) contradicts the fact that x^k is a weakly efficient solution to $\min_{x\in X} p(f(x), d^k * g(x))$. Therefore, $x^* \in X_0$.

Now we show that $x^* \in \mathrm{W}(0)$. Otherwise, $\exists x'' \in X_0$ such that $f(x'') \leq_{\mathrm{int}C} f(x^*)$. Therefore,

$$f(x'') \leq_{\mathrm{int}C} f(x^k) \tag{13}$$

when k is sufficiently large since each component function of f is l.s.c.

Note that

$$f(x'') = p(f(x''), d^k * g(x''))$$

and

$$p(f(x_k), d^k * g(x^k)) \geq_C f(x^k);$$

it follows from (13) that

$$p(f(x_0), d^k * g(x_0)) \leq_{\mathrm{int}C} p(f(x^k), d^k * g(x^k))$$

when k is sufficiently large. Namely, x^k is not a weakly efficient solution to

$$\min_{x\in X} \; p(f(x), d^k * g(x))$$

when k is sufficiently large, which cannot be true. The proof is complete. ∎

Remark 2.5. All the results in this section also hold for the case when p is defined on the domain $C \times R^m_+$, $F_+(x, d) = (f(x), d * g^+(x))$, $g^+(x) = (g_1^+(x), \ldots, g_m^+(x))$, and

$$L(x, d) = p(F_+(x, d)). \tag{14}$$

3. Exact penalization. Consider the following nonlinear penalty function:

$$L_\gamma(x, d) = p_\gamma(f(x), d * g^+(x)) = \sum_{i=1}^{l} \left[f_i^\gamma(x) + \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x) \right]^{1/\gamma} e_i,$$

where $0 < \gamma < +\infty$.

Let $u = (u_1, \ldots, u_m) \in R^m$. We associate (MOP) with a perturbed problem:

$$(\mathrm{MOP}_u) \quad \inf_{x\in X} \; f(x) \quad \text{s.t.} \quad g_j(x) \leq u_j, \quad j = 1, \ldots, m,$$

where X, f, g_j are defined as in (MOP).

Let

$$X(u) = \{x \in X : g_j(x) \le u_j, j = 1, \dots, m\}.$$

We will denote by $E(u), W(u)$, and $V(u)$ the sets of efficient solutions, efficient points, and infimum points of (MOP_u), respectively.

We need the following lemma.

LEMMA 3.1. *For any $x_0 \in X(u)$, there exists $z^* \in V(u)$ such that $z^* \le_C f(x_0)$.*

Proof. Let $Z = f(X(u))$, $Z_1 = \{z \in \text{cl}(Z) : z \le_C f(x_0)\}$. Clearly, Z_1 is nonempty and closed and $z \ge_C 0$ $\forall z \in Z$. Since $\le_C$ is a partial order in Z_1, by the well-known Hausdorff maximality principle (see, e.g., [11]), there exists a totally ordered subset Z_2 of Z_1, which is maximal with respect to the set inclusion. Let $z_i^* = \inf\{z_i : (z_1, \dots, z_i, \dots, z_l) \in Z_2\}, i = 1, \dots, l$, and $z^* = (z_1^*, \dots, z_l^*)$. It is obvious that $0 \le_C z^* \le_C f(x_0)$. Furthermore, by the definition of z^* and the fact that Z_2 is totally ordered, we deduce that $z^* \in \text{cl}(Z_2) \subset Z_1$. We assert that $z^* \in Z_2$. Otherwise, as $Z_2 \bigcup \{z^*\}$ is also a totally ordered subset of Z_1 and $Z_2 \subset Z_2 \bigcup \{z^*\}$, this contradicts the maximality of Z_2 with respect to the set inclusion. Finally, we show that $z^* \in V(u)$. We need to prove only that $z \not\le_{C\setminus\{0\}} z^*$ $\forall z \in \text{cl}(Z)$. Let $z \in \text{cl}(Z)$. If $z \not\le_C f(x_0)$, it can be shown by contradiction that $z \not\le_{C\setminus\{0\}} z^*$. If $z \le_C f(x_0)$ and

$$z \le_{C\setminus\{0\}} z^*, \tag{15}$$

then, by the maximality of Z_2, we have $z \in Z_2$, and thus $z^* \le_C z$ by the definition of z^*. This contradicts (15). The proof is complete. $\square$

DEFINITION 3.2. *We say that* (MOP) *is γ-rank uniformly weakly stable if there exist $\delta > 0$ and $M > 0$ such that*

$$\left[\frac{V(u) - V(0)}{\|u\|_\gamma^\gamma} + Me\right] \bigcap (-\text{int}C) = \emptyset \tag{16}$$

for any $u \in R_+^m$ with $0 < \|u\|_\gamma \le \delta$.

Remark 3.1. 1. It is not hard to show that the restriction $u \in R_+^m$ in the definition of the γ-rank uniform weak stability can be replaced by $u \in R^m$. This is also true for the γ-rank weak stability and γ-rank calmness in Definitions 3.4 and 3.7, respectively.

2. If $l = 1$ and $\gamma = 1$, then Definition 3.2 is equivalent to the stability of scalar optimization problems studied by Rosenberg [15]. (Any equality constraint $h(x) = 0$ with h being continuous can be equivalently written as the following inequality constraint: $|h(x)| \le 0$.) In the definition of γ-rank uniform weak stability of (MOP), the term "uniform" shows the difference from the usual stability in which $V(0)$ in (16) is replaced by a specific point of $V(0)$ and the fact that different points of $V(0)$ may have different M's in (16), and the term "weak" is used in contrast to the stability of (MOP) defined in [19, Definition 6.13, p. 182].

3. Let $0 < \gamma_1 < \gamma_2$. It is not hard to see that if (MOP) is γ_2-rank uniformly weakly stable, then it is also γ_1-rank uniformly weakly stable.

THEOREM 3.3. *If* (MOP) *is γ-rank uniformly weakly stable, then $\exists d^* \in R_+^m$ such that when $d - d^* \in R_+^m$,*

$$V(0) \subseteq q_\gamma(d), \tag{17}$$

where $q_\gamma(d) = \inf_{x \in X} L_\gamma(x, d)$. The converse is also true.

Proof. We begin by proving the first half of this theorem.

If $V(0) = \emptyset$, then the conclusion holds automatically. Now we assume that $V(0) \neq \emptyset$.

Let $\eta(z) = \min_{1\leq i\leq l} z_i \ \forall z = (z_1, \ldots, z_l) \in R^l$. We show by contradiction that $\eta(V(0)) = \{\eta(z) : z \in V(0)\}$ is bounded from above by some $M' > 0$. Otherwise, $\exists z_k \in V(0)$ such that $z_k \to +\infty$. Since $V(0) \neq \emptyset$, it follows that for any $\delta > 0$, $X(u_\delta) \supset X(0) = X_0 \neq \emptyset$, where $u_\delta = (0, 0, \ldots, 0, \delta) \in R^m_+$. Suppose that $x_0 \in X_0 \subset X(u_\delta)$. Then by Lemma 3.1 $\exists z_\delta \in V(u_\delta)$ such that

$$z_\delta \leq_C f(x_0).$$

Hence,

$$(z_\delta - z_k)/\|u_\delta\|^\gamma_\gamma \leq_C (f(x_0) - z_k)/\|u_\delta\|^\gamma_\gamma \to -\infty \text{ as } n \to \infty,$$

which contradicts (16) because $\delta > 0$ can be arbitrarily small.

Suppose that $\exists d_k = (d_{k,1}, \ldots, d_{k,m}) \to +\infty$ and $z_k \in V(0)$ such that $z_k \notin \inf_{x\in X} L_\gamma(x, d_k)$.

By $z_k \in V(0)$, it follows that $\exists x^j_k$ such that $g(x^j_k) \leq 0$ and $f(x^j_k) \to z_k$ as $j \to \infty$.

It follows from $z_k \notin \inf_{x\in X} L_\gamma(x, d_k)$ that $\exists x'_k \in X$ such that

$$L_\gamma(x'_k, d_k) \leq_{C\setminus\{0\}} z_k.$$

That is,

$$\sum_{i=1}^{l} \left[f^\gamma_i(x'_k) + \sum_{j=1}^{m} (d^\gamma_{k,j} g^{+\gamma}_j(x'_k)) \right]^{1/\gamma} e_i \leq_{C\setminus\{0\}} z_k. \tag{18}$$

Using (18), we deduce that $\max_{1\leq j\leq m} g_j(x'_k) > 0$ since $z_k \in V(0)$.

(18) also implies that

$$\sum_{j=1}^{m} d^\gamma_{k,j} g^{+\gamma}_j(x'_k) \leq (z_k)^\gamma_i - f^\gamma_i(x'_k) \leq (z_k)^\gamma_i, \quad i = 1, \ldots, l, \tag{19}$$

where $(z_k)_i$ denotes the ith component of vector z_k.

That is, $[\sum_{j=1}^m d^\gamma_{k,j} g^{+\gamma}_j(x'_k)]^{1/\gamma} \leq \eta(z_k) \leq M'$.

It follows that $g^+_j(x'_k) \to 0$ $(j = 1, \ldots, m)$ as $n \to +\infty$.

Now let $u_{k,j} = g^+_j(x'_k)$ and $u_k = (u_{k,1}, \ldots, u_{k,m})$. Clearly, $\|u_k\|_\gamma > 0$ and $\|u_k\|_\gamma \to 0$. It follows from (19) that $\|u_k\|^\gamma_\gamma \min_{1\leq j\leq m} d^\gamma_{k,j} \leq (z_k)^\gamma_i - f^\gamma_i(x'_k)$. By Lemma 3.1, we deduce that $\exists v_k \in V(u_k)$ such that $v_k \leq_C f(x'_k)$. By the mean-value theorem, we have $(z_k)^\gamma_i - (v_k)^\gamma_i = k(s_k)^{\gamma-1}_i((z_k)_i - (v_k)_i)$, where $(s_k)_i \in ((v_k)_i, (z_k)_i)$. Therefore, it follows from (19) that

$$\|u_k\|^\gamma_\gamma \min_{1\leq j\leq m} d^\gamma_{k,j} \leq k(s_k)^{\gamma-1}_i((z_k)_i - (v_k)_i) \leq \gamma(v_k)^{\gamma-1}_i((z_k)_i - (v_k)_i) \text{ if } \gamma \leq 1; \tag{20}$$

$$\|u_k\|^\gamma_\gamma \min_{1\leq j\leq m} d^\gamma_{k,j} \leq \gamma {M'}^{\gamma-1}((z_k)_i - (v_k)_i) \text{ if } \gamma > 1. \tag{21}$$

Since $\inf_{x\in X} f_i(x) > 0 \ \forall i$, it follows that

$$\min_{1\leq i\leq m} (v_k)_i \geq m_2 > 0. \tag{22}$$

Let $M'' = \max\{{M'}^{\gamma-1}, m_2^{\gamma-1}\}$. The combination of (20), (21), and (22) yields that

$$\|u_k\|_\gamma^\gamma \min_{1\le j\le m} d_{k,j}^\gamma \le \gamma M''((z_k)_i - (v_k)_i),$$

i.e.,

$$\frac{(v_k)_i - (z_k)_i}{\|u_k\|_\gamma^\gamma} \le -\frac{\min\limits_{1\le j\le m} d_{k,j}^\gamma}{\gamma M''},$$

which contradicts (16). Thus (17) holds.

Now we prove the second half of the theorem by contradiction.

Suppose that $\exists u_k = (u_{k,1}, \dots, u_{k,m}) \in R_+^m$ with $u_k \to 0^+$ and $z_k \in V(u_k), v_k \in V(0)$ such that

$$(z_k - v_k)/\|u_k\|_\gamma^\gamma \to -\infty \text{ as } k \to +\infty,$$

where the virtual element $-\infty$ is such that for any $\alpha \in R_+^1, -\infty \le_{\mathrm{int}C} -\alpha e$. Then $\exists x_k \in X$ with $g_j(x_k) \le u_{k,j} \forall j$ such that

$$(f(x_k) - v_k)/\|u_k\|_\gamma^\gamma \to -\infty \quad \text{as } k \to +\infty. \tag{23}$$

By the assumption of the theorem, $\exists d^* = (d_1^*, \dots, d_m^*) \in R_+^m$ such that when $d - d^* \in R_+^m$, $v_k \in \inf_{x\in X} L_\gamma(x, d)$. Therefore,

$$L_\gamma(x_k, d^*) \not\le_{C\setminus\{0\}} v_k. \tag{24}$$

We assume that $i^* \in \{1, \dots, l\}$ is such that

$$\left[f_{i^*}^\gamma(x_k) + \sum_{j=1}^m {d_j^*}^\gamma {g_j^+}^\gamma(x_k)\right]^{1/\gamma} \ge (v_k)_{i^*}.$$

Namely,

$$f_{i^*}^\gamma(x_k) - (v_k)_{i^*}^\gamma \ge -\sum_{j=1}^m {d_j^*}^\gamma {g_j^+}^\gamma(x_k). \tag{25}$$

It follows from (23) and (24) that $\max_{1\le j\le m} g_j(x_k) > 0$. So from (25) we deduce that

$$f_{i^*}^\gamma(x_k) - (v_k)_{i^*}^\gamma \ge -\max_{1\le j\le m} {d_j^*}^\gamma \|u_k\|_\gamma^\gamma.$$

That is,

$$[(v_k)_{i^*}^\gamma - f_{i^*}^k(x_k)]/\|u_k\|_\gamma^\gamma \le \max_{1\le j\le m} {d_j^*}^\gamma. \tag{26}$$

Since

$$(v_k)_{i^*}^\gamma - f_{i^*}^\gamma(x_k) = \gamma s_k^{\gamma-1}((v_k)_{i^*} - f_{i^*}(x_k)), \quad s_k \in (f_{i^*}(x_k), (v_k)_i),$$

it follows from the assumption on f that $\exists a > 0$ such that

$$(v_k)_{i^*}^\gamma - f_{i^*}^k(x_k) \ge \gamma a((v_k)_{i^*} - f_{i^*}(x_k)). \tag{27}$$

Equations (26) and (27) yield that

$$[f_{i^*}(x_k) - (v_k)_{i^*}]/\|u_k\|_\gamma^\gamma \geq - \max_{1\leq j\leq m} {d_j^*}^\gamma/(ka),$$

which contradicts (23). The proof is complete. □

Remark 3.2. When $l = 1, m = 1$, Theorem 3.3 reduces to Theorem 7.2 in [18].

DEFINITION 3.4. (i) *Let $z^* \in V(0)$. The problem* (MOP) *is said to be γ-rank weakly stable at z^* if there exist positive real numbers δ_{z^*} and M_{z^*} such that*

$$\left[\frac{V(u) - z^*}{\|u\|_\gamma^\gamma} + M_{z^*}e\right] \bigcap (-\text{int}C) = \emptyset$$

for any $u \in R_+^m$ with $0 < \|u\|_\gamma \leq \delta_{z^*}$.

(ii) *The problem* (MOP) *is said to be γ-rank weakly stable if it is γ-rank weakly stable at every $z^* \in V(0)$.*

Remark 3.3. 1. It is clear that if (MOP) is γ-rank uniformly weakly stable, then (MOP) is γ-rank weakly semistable.

2. It is not hard to check that if $f(X(u))$ is externally stable for any $u \in R_+^m$, then the stability of (MOP) defined in [19, Definition 6.1.3, p. 182] implies the 1-rank weak stability of (MOP).

The proof of the next theorem is similar to that of Theorem 3.3 and is thus omitted.

THEOREM 3.5. *Let $z^* \in V(0)$. Then* (MOP) *is γ-rank weakly stable at z^* if and only if there exists a $d^* \in R_+^m$ such that $z^* \in q_\gamma(d)$ whenever $d - d^* \in R_+^m$.*

COROLLARY 3.6. (MOP) *is γ-rank weakly stable if and only if for every z^* there exists a $d^* \in R_+^m$ such that $z^* \in q_\gamma(d)$ whenever $d - d^* \in R_+^m$.*

Remark 3.4. The following simple example shows that (MOP) is 1-rank weakly stable but not 1-rank uniformly weakly stable.

Example 3.1. Let $n = 1$, $l = 2$, $X = R^1$, and $m = 1$. Let $f(x) = (\exp(-x^{1/2}), \exp(-x^{1/2}))$ if $x > 0$; $f(x) = (\exp(x), \exp(-x))$ if $x \leq 0$. Let $g(x) = x\ \forall x \in R^1$. It is easy to check that $V(0) = \{(\exp(x), \exp(-x)) : x \leq 0\}$ and

$$V(u) = \{(\exp(-u^{1/2}),\ \exp(-u^{1/2}))\} \bigcup \{(\exp(x), \exp(-x)) : x < -u^{1/2}\}\ \forall u > 0.$$

It is elementary to prove that (MOP) is 1-rank weakly stable but not 1-rank uniformly weakly stable. By Corollary 3.6, we know that for every $z^* \in V(0)$ there exists $d^* \geq 0$ such that $z^* \in \inf_{x\in R^1}(f(x) + dg^+(x)e)$, where $d \geq d^*$. On the other hand, by Theorem 3.3, we deduce that there exists no $d^* \geq 0$ such that $V(0) \subseteq \inf_{x\in R^1}(f(x) + dg^+(x)e)$, whenever $d \geq d^*$.

DEFINITION 3.7. *Let $x^* \in LWE(0)$. We say that* (MOP) *is γ-rank calm at x^* if there exists $M > 0$ such that for any $u_k = (u_{k,1}, \ldots, u_{k,m}) \in R_+^m$ with $\|u_k\|_\gamma \to 0^+$ (namely, $\|u_k\|_\gamma > 0$ and $\|u_k\|_\gamma \to 0$), for any x_k satisfying $g_j(x_k) \leq u_{k,j}, j = 1, \ldots, m$ and $x_k \to x^*$, there holds*

$$\frac{f(x_k) - f(x^*)}{\|u_k\|_\gamma^\gamma} + Me \notin -C \quad \forall n.$$

Remark 3.5. 1. If $l = 1, \gamma = 1$, then this definition is equivalent to the calmness at a point of a scalar optimization problem (see, e.g., [15, 4]). If $l > 1, \gamma = 1$, then this definition is equivalent to the weak calmness at a point of the multiobjective optimization problem (MOP) defined in [16].

2. If $0 < \gamma_1 < \gamma_2$, then (MOP) is γ_2-rank calm at a point x^*, which implies that it is γ_1-rank calm at x^*.

The following local exact penalization result can also be similarly proved as Theorem 3.3.

THEOREM 3.8. *Let $0 < \gamma < +\infty$. The following statements hold.*

(i) *Assume that x^* is a locally weak efficient solution to* (MOP) *and* (MOP) *is γ-rank calm at x^*. Then there exist $\delta > 0$ and $d^* \in R^m_+$ such that x^* is also a weak efficient solution to the problem* $\min_{x\in X_\delta} L_\gamma(x, d)$, *for any d satisfying $d - d^* \in R^m_+$, where $X_\delta = \{x \in X : \|x - x^*\| \le \delta\}$.*

(ii) *If $x^* \in X_0$ and there exist $d^* \in R^m_+$ and $\delta > 0$ such that x^* is a locally weak efficient solution to the problem* $\min_{x\in X} L_\gamma(x, d^*)$, *then $x^* \in LWE(0)$ and* (MOP) *is γ-rank calm at x^*.*

The next theorem uses a well-known condition in the study of sensitivity of a constrained optimization problem (see, e.g., [12]), i.e., the compactness of the feasible set with a small perturbation. Under this condition, the set of efficient points of (MOP) and that of $L_\gamma(\cdot, d)$ are nonempty. The conclusion follows directly from Theorem 3.3.

THEOREM 3.9. *Assume that there exists $u^0 = (u^0_1, \dots, u^0_m) \in \operatorname{int} R^m_+$ with $\|u^0\| > 0$ sufficiently small such that $X_5 = \{x \in X : g_j(x) \le u^0_j \forall j\}$ is compact. If* (MOP) *is γ-rank uniformly weakly stable, then $\exists d^* \in R^m_+$ such that when $d - d^* \in R^m_+$,*

$$W(0) = f(E(0)) \subseteq \bar{q}_\gamma(d),$$

where $\bar{q}_\gamma(d)$ is the set of efficient points of $L_\gamma(\cdot, d)$ over X. The converse is also true.

The following theorem establishes a further relationship between the solutions of (MOP) and that of the penalty problems based on L_γ.

THEOREM 3.10. *Assume that $X_0 \ne \emptyset$ and $\exists d^* = (d^*_1, d^*_2, \dots, d^*_m) \in R^m_+$ such that for all d satisfying $d - d^* \in R^m_+$, $x^* \in X$ is an efficient solution of the problem* $\min_{x\in X} L_\gamma(x, d)$; *then x^* is an efficient solution of* (MOP).

Proof. Let x^* be an efficient solution of $\min_{x\in X} L_\gamma(x, d)$ for any d satisfying $d - d^* \in R^m_+$. Then we have

$$L_\gamma(x, d) - L_\gamma(x^*, d) \not\le_{C\setminus\{0\}} 0 \quad \forall x \in X, d \text{ satisfying } d - d^* \in R^m_+.$$

For any $x_0 \in X_0$, we have $L_\gamma(x_0, d) = f(x_0)\ \forall d \in R^m_+$ by Lemma 2.3. Thus,

$$f(x_0) - \sum_{i=1}^{l} \left[f_i^\gamma(x^*) + \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x^*) \right]^{1/\gamma} e_i \not\le_{C\setminus\{0\}} 0\ \forall x_0 \in X_0, d \text{ satisfying } d - d^* \in R^m_+. \tag{28}$$

We claim that $g^+_j(x^*) = 0\ \forall j$ (i.e., $x^* \in X_0$). Otherwise, $\sum_{j=1}^m g_j^{+\gamma}(x^*) > 0$.

It follows from (28) that there exists $i^* \in \{1, \dots, l\}$ such that

$$f^\gamma_{i^*}(x_0) - f^\gamma_{i^*}(x^*) \ge \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x^*) \ge \left(\min_{1\le j\le m} d_j^\gamma \right) \sum_{j=1}^{m} g_j^{+\gamma}(x^*).$$

Hence,

$$\max_{1\le i\le l} \{f^\gamma_i(x_0) - f^\gamma_i(x^*)\} \ge \sum_{j=1}^{m} d_j^\gamma g_j^{+\gamma}(x^*) \ge \left(\min_{1\le j\le m} d_j^\gamma \right) \sum_{j=1}^{m} g_j^{+\gamma}(x^*),$$

which is impossible if we let $d_j \to +\infty\ \forall j$. Therefore, $x^* \in X_0$. It follows directly from Lemma 2.3 and (28) that $x^* \in E(0)$, and the proof is complete. $\square$

What follows is a characterization of the γ-rank weak stability of (MOP) at a point $z^* \in V(0)$ in terms of the γ-rank stability of a scalar optimization problem (see below).

Let $z^* \in V(0)$. Recall $\xi(z) = \max_{1\le i\le l}\{z_i\}\ \forall z = (z_1, \ldots, z_l)$. Consider the following scalar optimization problem:

$$(\mathrm{P}(z^*)) \quad \inf_{x\in X} \xi(f(x) - z^*) \quad \text{s.t. } g_j(x) \le 0, \quad j = 1, \ldots, m,$$

and its perturbed problem,

$$(\mathrm{P}_u(z^*)) \quad \inf_{x\in X} \xi(f(x) - z^*) \quad \text{s.t. } g_j(x) \le u_j, \quad j = 1, \ldots, m$$

where $u = (u_1, \ldots, u_m) \in R^m_+$ is such that $\|u\|_\gamma > 0$ is sufficiently small.

Clearly, the optimal value of $(\mathrm{P}(z^*))$ is 0. We denote by $\pi(u)$ the optimal value of $(\mathrm{P}_u(z^*))$. $(\mathrm{P}(z^*))$ is said to be *γ-rank stable* if there exist positive numbers δ and M such that

$$\frac{\pi(u)}{\|u\|_\gamma^\gamma} \ge -M$$

for any $u \in R^m_+$ with $0 < \|u\|_\gamma \le \delta$.

Note that this notion of γ-rank stability of $(\mathrm{P}(z^*))$ is equivalent to the *stability* defined in [15] if $\gamma = 1$.

The following conclusion can be straightforwardly proved.

THEOREM 3.11. *Let $z^* \in V(0)$. Then* (MOP) *is γ-rank weakly stable at z^* if and only if $(P(z^*))$ is γ-rank stable.*

COROLLARY 3.12. (MOP) *is γ-rank weakly stable if and only if for any $z^* \in V(0)$, $(P(z^*))$ is γ-rank stable.*

Remark 3.6. As noted in [4, p. 238], for a scalar optimization problem, any constraint qualification (such as the Slater or Mangasarian–Fromowitz condition) which rules out abnormal Lagrangian multipliers at every optimum also guarantees a stronger version of stability of the optimization problem; that is, the optimal value function of $(\mathrm{P}_u(z^*))$ is locally Lipschitz at the origin of R^m.

In the following, we provide some criteria for the γ-rank calmness of (MOP) at a point.

Let $x^* \in LWE(0)$. Let $u \in R^m_+\backslash\{0\}$. We associate (MOP) with the following scalar optimization problem (P') and its perturbed problem (P'_u):

$$(P') \quad \inf_{x\in X} \xi(f(x) - f(x^*)) \quad \text{s.t. } g_j(x) \le 0, \quad j = 1, \ldots, m,$$

$$(P'_u) \quad \inf_{x\in X} \xi(f(x) - f(x^*)) \quad \text{s.t. } g_j(x) \le u_j, \quad j = 1, \ldots, m.$$

It is easy to see that x^* is also a local minimum to (P').

(P') is said to be *γ-rank calm* at x^* if there exists $M > 0$ such that for any $u_k = (u_{k,1}, \dots, u_{k,m}) \in R^m_+$ with $\|u_k\|_\gamma \to 0^+$, for any $x_k \to x^*$ with $g_j(x_k) \le u_{k,j}, \forall j$, we have

$$\xi(f(x_k) - f(x^*))/\|u_k\|^\gamma_\gamma \ge -M.$$

The following proposition establishes the relationship between the γ-rank calmness of (MOP) and that of (P').

PROPOSITION 3.13. *Let x^* be a locally weak efficient solution to* (MOP) *and $0 < \gamma < +\infty$. Then* (MOP) *is γ-rank calm at x^* if and only if (P') is γ-rank calm at x^*.*

A sufficient condition for the calmness of (MOP) *at a point is given in the following proposition.*

PROPOSITION 3.14. *Let $x^* \in X$ and $0 < \gamma < +\infty$. Assume that the following conditions hold:*

(i) *there exists $\lambda \in R^l_+ \backslash \{0\}$ such that x^* is a local minimum to*

$$(P_\lambda) \quad \inf_{x \in X} \quad \lambda^T f(x)$$
$$s.t. \quad g_j(x) \le 0, \quad j = 1, \dots, m;$$

(ii) (P_λ) *is γ-rank calm at x^*.*

Then (MOP) *is γ-rank calm at x^*.*

The following lemma follows from a statement in [4, p. 239].

LEMMA 3.15. *Let $\gamma \in (0, 1], f_i (i = 1, \dots, l), g_j (j = 1, \dots, m)$ be locally Lipschitz functions around a local minimum x^* to (P'). If (P') satisfies either of the following constraint qualifications:*

(i) *Mangasarian–Fromowitz-type constraint qualification: there exists $v \in T^C_X(x^*)$ such that $g^0_j(x^*; v) < 0 \ \forall j \in J(x^*)$, where $J(x^*) = \{j : g_j(x^*) = 0, j = 1, \dots, m\}$, $g^0_j(x^*; v)$ denotes the Clarke's generalized directional derivative of g_j at x^* in direction v, and $T^C_X(x^*)$ is the Clarke tangent cone of X at x^*,*

(ii) *Slater-type constraint qualification: if X is convex, g_j $(j = 1, \dots, m)$ is convex around x^* (i.e., $\exists \delta > 0$ such that g_j is convex on the set $X_\delta = \{x \in X : \|x - x^*\| \le \delta\}$), there exists $x_0 \in X_\delta$ such that $g_j(x_0) < 0 \ \forall j \in J(x^*)$,*

then (P') is 1-rank calm at x^*; therefore, it is γ-rank calm at x^*.

It follows from Lemma 3.15 and Proposition 3.13 that we have the following proposition.

PROPOSITION 3.16. *Let $f_i(i = 1, \dots, l), g_j(j = 1, \dots, m)$ be locally Lipschitz around a local efficient solution x^* of* (MOP) *and either of* (i) *and* (ii) *in Lemma* 3.15 *hold. Then* (MOP) *is γ-rank calm at x^*.*

Finally, we note that if f is locally Lipschitz and all the constraint functions $g_j, j = 1, \dots, m$, are affine and X is a polyhedron, then (P') is (1-rank) calm at any of its local minima (see [22], for instance). Thus, by Proposition 3.13, (MOP) is γ-rank calm at any of its local efficient solutions ($\gamma \in (0, 1]$).

4. Saddle points of nonlinear Lagrangian functions. In this section, we consider the saddle point problem of the nonlinear Lagrangian.

Let p be an increasing function defined on $C \times R^m$ (or $C \times R^m_+$) enjoying properties (A) and (B) and let the nonlinear Lagrangian L be defined by (1) (or (14)).

DEFINITION 4.1. *The point $(x^*, d^*) \in X \times R^m_+$ is called a saddle point of the nonlinear Lagrangian L if*

(i) $L(x, d^*) - L(x^*, d^*) \not\leq_{C\setminus\{0\}} 0 \; \forall x \in X$;
(ii) $L(x^*, d) - L(x^*, d^*) \not\geq_{C\setminus\{0\}} 0 \; \forall d \in R^m_+$.

It should be noted that a saddle point may not exist even if all the conditions of Theorem 2.12 *hold (see Example* 2.4 *due to Proposition* 4.2*).*

The following proposition presents the relationship among a saddle point of L, an efficient solution of (MOP), and an efficient solution of (DMOP) in the sense of maximum.

PROPOSITION 4.2. *The point* $(x^*, d^*) \in X \times R^m_+$ *is a saddle point of* L *if and only if* x^* *is an efficient solution of* (MOP)*,* $f(x^*) \in q(d^*)$*, and* d^* *is an efficient solution to* (DMOP)*.*

In the following, we compare the Lagrangian function defined analogously as in [19, pp. 185–187] with a special class of nonlinear Lagrangian functions. Then we provide sufficient conditions for the existence of a saddle point of this special class of nonlinear Lagrangian functions.

As in [19], we define a Lagrangian function as follows:

$$L'(x, d) = f(x) + \sum_{j=1}^{m} d_j g_j(x) e,$$

where the dual variable $d = (d_1, \dots, d_m) \in R^m_+, x \in X$.

Analogous to Definition 4.1, we can define a saddle point of L'.

It is clear that the following inequality holds:

$$\left(\sum_{i=1}^{m} b_i^{\gamma}\right)^{1/\gamma} \geq \sum_{i=1}^{m} b_i \quad \forall b_i \geq 0, \gamma \in (0, 1]. \tag{29}$$

Let $\gamma \in (0, 1]$. Consider the following class of nonlinear Lagrangian functions:

$$L_\gamma(x, d) = \sum_{i=1}^{l} \left[f_i^{\gamma}(x) + \sum_{j=1}^{m} d_j^{\gamma} g_j^{+\gamma}(x) \right]^{1/\gamma} e_i,$$

where $x \in X, d = (d_1, \dots, d_m) \in R^m_+$. It follows from (29) that

$$L_\gamma(x, d) \geq_C f(x) + \sum_{j=1}^{m} d_j g_j^+(x) e \geq_C L'(x, d) \quad \forall x \in X, \; d \in R^m_+. \tag{30}$$

This inequality allows us to establish the following conclusion.

PROPOSITION 4.3. *Assume that* $\gamma \in (0, 1]$*. Any saddle point of* L' *is also a saddle point of* L_γ*.*

The following theorem follows from Theorem 3.5 and Proposition 4.2.

THEOREM 4.4. *Assume that* $\gamma \in (0, 1]$ *and* (MOP) *is* 1*-rank weakly stable. Then* $x^* \in X$ *is an efficient solution of* (MOP) *if and only if there exists* $d^* \in R^m_+$ *such that* (x^*, d^*) *is a saddle point of* L_γ*.*

5. Conclusions. In this paper, we introduced nonlinear Lagrangian functions and nonlinear penalty functions for constrained multiobjective optimization problems. We obtained weak and strong duality and saddle point results based on nonlinear Lagrangian functions. We also studied the relationship between the γ-rank weak stability and the exact penalization for inequality constrained multiobjective optimization problems, and the relationship between the γ-rank calmness and the local exact penalization.

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