SIAM J. SCI. COMPUT.
Vol. 23, No. 2, pp. 430–441

AN AGGREGATION-BASED DOMAIN DECOMPOSITION PRECONDITIONER FOR GROUNDWATER FLOW*

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Abstract. We consider theoretical and computational issues associated with an aggregation-based domain decomposition preconditioner applied to a Bi-CGSTAB iterative solver used to solve both Laplace's equation and an important nonlinear model from hydrology used to simulate unsaturated flow, Richards' equation. Theoretical results for Laplace's equation provide estimates of the condition number and the rate of convergence for a two-level Schwarz domain decomposition preconditioner. Computational results for Laplace's equation and Richards' equation show excellent scalability, although no theory is yet available to support the results for the difficult nonlinear problem.

Key words. domain decomposition, Newton–Krylov–Schwarz methods, Richards' equation, nonlinear equations, aggregation

AMS subject classifications. 65H10, 65N55, 76S05, 76T05, 86A05

PII. S1064827500372274

1. Introduction.

In this paper we report on a scalability study and derive condition number estimates for a two-level additive Schwarz preconditioner. In our previous work we have applied this preconditioner to a set of problems in hydrology [8, 9, 7]. We build the coarse mesh problem with aggregation, an approach used in both algebraic multigrid [2, 21] and domain decomposition methods [11, 12, 13, 3]. In the case of minimal overlap, we implement this as a simple unweighted sum of nodal values. This implementation permits a simple construction of a two-level preconditioner on unstructured grids.

Our methods were developed for use in the adaptive hydrology model (ADH) [18], a production code being developed at the U.S. Army Engineer Research and Development Center. The use of aggregation arose from necessity. In the applications reported in [8, 9, 7] the subdomains were irregular, and a coarse mesh based on "hat functions" over the subdomains was impractical. For the same reason, we needed minimal overlap between subdomains. Unlike the method from [5], we do not need to create a coarse mesh geometry or use geometric information about the subdomains. Neither theoretical analysis of this algorithm applied to any problem nor in-depth scalability studies have been performed to date to our knowledge.

The overall goal of this work is to advance our understanding of this aggregation-based domain decomposition method. The specific objectives are (1) to develop estimates of the condition number as a function of coarse and fine grid size for Laplace's

*Received by the editors May 18, 2000; accepted for publication (in revised form) February 21, 2001; published electronically July 10, 2001. This research was supported in part by Army Research Office contract DAAD19-99-1-0186, U.S. Army contract DACA39-95-K-0098, National Science Foundation grants DMS-9700569 and DMS-0070641, a Cray Research Corporation Fellowship, and a Department of Education GAANN fellowship. Computing activity was partially supported by an allocation from the North Carolina Supercomputing Center.

http://www.siam.org/journals/sisc/23-2/37227.html

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equation; (2) to evaluate computationally the scalability of this algorithm for Laplace's equation and to compare to theoretical estimates; and (3) to apply this algorithm to a difficult and important problem in hydrology, the flow of water in the unsaturated zone modeled by Richards' equation [16]. The first two objectives will advance and test the current theory of this algorithm for a simple equation, while the latter objective will provide some evidence of the applicability of this algorithm to a challenging class of applications for which improved solution algorithms are badly needed. We consider the theoretical analysis of this algorithm applied to Richards' equation and other nonlinear models as open issues and beyond the scope of this effort.

The use of smoothed aggregation elements in an additive Schwarz scheme was originally proposed in [2, 3]. In these papers this idea was tested on a variety of elliptic problems with complicated geometries. Construction of coarse mesh problems by aggregation was critical to the results reported in [8, 9, 7], where computations were done on unstructured grids in three space dimensions. Our coarse mesh basis functions are nonsmoothed, which means that the square of the H^1 bound on the basis functions is $O(\frac{H^{d-1}}{\delta})$ instead of $O(H^{d-1})$. Note that we do, for generality, include the possibility of overlap $\delta > h$. The method in [2, 3], however, assumes a physical overlap of $O(h)$ and obtains the benefits of overlap (i.e., bounds on the condition number independent of H and h) by smoothing the basis functions. These differences lead to a change in the energy of the coarse mesh operator so that $|Qu|^2_{H^1} \leq C\frac{H}{\delta}|u|^2_{H^1}$. Thus instead of having a condition number bound by a constant we get a condition bound that is $O(\frac{H^2}{\delta^2})$.

In our previous work [8, 9, 7], we used nonsmoothed aggregation elements with minimal overlap and exact subdomain solves. Similarly, in section 3 the overlap is h, where h is the fine mesh length scale, and we solve the subdomain problems exactly with sparse Gaussian elimination. The convergence results in section 2 allow for more flexibility in overlap and subdomain solvers.

The analysis in section 2 uses the standard finite element framework from [17, 23]. The preconditioner also works well in the context of finite differences, however, as the example in section 3.2 illustrates.

Richards' equation [16] is a model of flow through unsaturated porous media. In this paper we consider the head-based form of the equation and for a homogeneous media in two space dimensions,

$$\left[\frac{\partial\theta}{\partial\psi} + \frac{S_s}{\theta_s}\theta\right]\frac{\partial\psi}{\partial t} = \nabla\cdot\left[K_s k_r \nabla\left(\psi + z\right)\right], \tag{1.1}$$

where ψ is the pressure head, θ is the volume fraction of the wetting phase, and k_r is the relative permeability of the wetting phase. The remaining terms are scalar coefficients given in Table 1.1, along with their values for the test problem. θ and k_r are functions of ψ given by

$$\theta = (\theta_s - \theta_r)(1 + |\alpha\hat{\psi}|^n)^{-m} + \theta_r, \tag{1.2}$$

$$k_r = (1 + |\alpha\hat{\psi}|^n)^{-m/2}[1 - |\alpha\hat{\psi}|^{n-1}(1 + |\alpha\hat{\psi}|^n)^{-m}]^2, \text{ and} \tag{1.3}$$

$$\hat{\psi} = \min(\psi, 0), \tag{1.4}$$

where $m = 1 - 1/n$. These functions are derived from the van Genuchten [20] and Mualem [14] empirical relations among pressure, saturation, and relative permeability. We discretize (1.1) in space with finite differences and integrate the resulting system of differential algebraic equations in time with the fixed leading coefficient backward

TABLE 1.1
Richards' equation parameters.

Description	Symbol	Value
Saturated volume fraction	θ_s	3.01×10^{-1}
Residual volume fraction	θ_r	9.30×10^{-2}
Specific storage	S_s	1.00×10^{-6} (1/m)
Hydraulic conductivity	K_s	5.04×10^{0} (m/day)
Mean pore size	α	5.47×10^{0} (1/m)
Pore size uniformity	n	4.26×10^{0}

difference formulas of orders one to five [4, 10]. Within the implicit temporal integration is a Newton iteration, and we solve the nonsymmetric linear equation for the Newton step with a preconditioned Bi-CGSTAB [19] linear iteration.

2. Theory for an elliptic model problem. The convergence theory in this paper is for the weak form of an elliptic boundary value problem with Dirichlet boundary conditions on a domain $\Omega \subset R^d$ with boundary Γ, with spatial dimension d. We will restrict our attention to piecewise linear nodal finite element spaces.

The goal is to find $u \in V$ such that

$$a(u, v) = l(v) \text{ for all } v \in V, \tag{2.1}$$

where a is a strongly elliptic bilinear form on V, l is a linear functional on V, and V is an appropriate function space.

We let $V^h \subset V$ be the appropriate space of piecewise linear functions. The approximating problem at level h is to find $u^h \in V^h$ such that

$$a(u^h, v) = l(v) \text{ for all } v \in V^h. \tag{2.2}$$

The problem (2.2) is equivalent to a linear system

$$Au^h = f \tag{2.3}$$

on V^h, where $a(u, v) = (Au, v)$ for all $u, v \in V^h$. Here $(\cdot, \cdot)$ is the l^2 inner product.

Schwarz preconditioners are designed to accelerate Krylov space iterative methods for the solution of (2.3).

2.1. One-level method. We begin with the one-level additive Schwarz preconditioner. We divide Ω into subdomains $\{\Omega_j\}_{j=1}^J$ with an overlap width of δ and assume that $\bigcup_{j=1}^J \Omega_j = \Omega$.

Let R_j be the restriction map from an element of V^h to the subspace V_j of functions in V^h with support on Ω_j. Let

$$A_j = R_j A R_j^T$$

be the subdomain operator. We assume that A_j is nonsingular on V_j and define

$$B_j = R_j^T \tilde{A}_j^{-1} R_j,$$

where $\tilde{A}_j$ is an approximation of A_j. The one-level additive Schwarz preconditioner is

$$M = \sum_{j=1}^{J} B_j. \tag{2.4}$$

2.2. Two-level method. The two-level additive Schwarz method adds a coarse mesh term

$$B_0 = R_0^T \tilde{A}_0^{-1} R_0$$

to the one-level preconditioner. Here $\tilde{A}_0$ is an approximation of A_0. We let V^H denote the space of coarse mesh basis functions. If the coarse mesh restriction map R_0 and the coarse mesh operator A_0 are well designed, the condition number of MA is significantly reduced.

One way to define a coarse mesh problem is to discretize the continuous problem on a coarser mesh. There are a few difficulties associated with forming the coarse problem this way. First, for unstructured meshes, such as the ones considered in [7, 9, 8], the interpolation operators between the fine mesh and the coarse mesh are difficult to define. Second, a coarse mesh must be generated, stored, and parallelized. Finally, the PDE must be discretized and recomputed on the coarse mesh.

Alternatively, the discretization of the coarse mesh operator may be defined in terms of the existing fine mesh discretization. A Galerkin or variational coarse grid correction uses the fine grid matrix to obtain the coarse grid operator as $A_0 = R_0 A R_0^T$, where R_0^T is the interpolation operator from the coarse mesh function space, and R_0 is the restriction operator. If the coarse mesh basis functions are obtained from the fine mesh basis functions, then the coarse mesh space V^H is contained in the fine grid space V^h.

In this section we use the aggregation-based basis from [2, 3, 7, 8, 9], where one coarse mesh basis function is defined for each subdomain as the sum of the fine mesh basis functions for that subdomain.

To set the notation that we will need in section 2.4, let the expansion of a function $u \in V^h$ in the finite element basis be

$$u = \sum_l u_l \psi_l, \tag{2.5}$$

where the ψ_l's are the nodal basis functions for the fine mesh. A function $u_C \in V^H$ can be represented on the coarse mesh space as

$$u_C = \sum_K u_{C_K} \Psi_K, \tag{2.6}$$

where the Ψ_K's are the basis functions for the coarse mesh space. Since $V^H \subset V^h$, Ψ_K can be written as

$$\Psi_K = \sum_l R_{Kl} \psi_l. \tag{2.7}$$

The index K represents the subdomain number. The value of $R_{Kl} \geq 0$ is constrained by the conditions that $\nabla \Psi_K = O(\delta)$ and the requirement of the theory that the coarse mesh basis functions provide a partition of unity, i.e.,

$$\sum_K \Psi_K = 1.$$

In the case of minimal overlap, where $\delta = O(h)$, $R_{Kl} = 1$ if the support of the fine mesh basis function ψ_l is contained in subdomain K; otherwise, $R_{Kl} = 0$.

Further expanding the representation of u_C gives

$$\begin{aligned} u_C &= \sum_K u_{C_K} \Psi_K, \\ &= \sum_K u_{C_K} \sum_j R_{Kl} \psi_l, \\ &= \sum_l \left(\sum_K u_{C_K} R_{Kl} \right) \psi_l, \\ &= \sum_l \left(R^T u_C \right)_l \psi_l. \end{aligned}$$

Thus R^T is the operator which interpolates from the coarse mesh to the fine mesh. Any function on the coarse mesh can be represented solely in terms of the already existing fine mesh functions, making the formulation of a separate coarse mesh unnecessary.

2.3. Condition number estimate. In Assumption 2.1 we make precise the idea that H is the characteristic diameter of a subdomain. In Assumption 2.2 we make precise the overlap δ between the subdomains and the properties of the coarse mesh basis functions. These assumptions are based on assumptions given in [2, 3], but ours differ in the fact that an overlap parameter of δ is considered and the square of the energy of our coarse mesh basis function is bound by $\frac{H^{d-1}}{\delta}$ instead of H^{d-2}, where d is the dimension of the problem.

ASSUMPTION 2.1.

1. *There is $C > 0$ such that $diam(\Omega_j) \leq CH$ for all $j = 1, \ldots, J$.*
2. *There is $c > 0$ such that for all $x \in \Omega$ there exists $j \geq 1$ such that $x \in \Omega_j$ and*

$$\operatorname{dist}(x, \partial\Omega_j \backslash \partial\Omega) \geq c\delta.$$

3. *There are $C_R, C_1, C_2 > 0$ such that for all $x \in \Omega$ the ball*

$$B(x, C_R H) = \{y \in \Omega : \operatorname{dist}(y, x) \leq C_R H\}$$

intersects at most $C_1 + C_2^d$ subdomains Ω_j (i.e., an object of diameter $O(H)$ intersects at most $O(1)$ subdomains Ω_i).
4. *$\mu(\Omega_j) \geq CH^d$, $j = 1, \ldots, J$.*

In Assumption 2.1, μ denotes the Lebesgue measure.

ASSUMPTION 2.2. *Assume the basis functions Ψ_i of the coarse space satisfy the following.*

1. $|\Psi_i|^2_{H^1(\Omega)} \leq C\frac{H^{d-1}}{\delta}$,
   $\|\Psi_i\|^2_{L^2} \leq CH^d$.
2. *There is a domain $\Omega^{int} \subset \Omega$ such that $\sum_i \Psi_i(x) = 1$ for every $x \in \Omega^{int}$ and* $\operatorname{dist}(x, \partial\Omega) \leq C\delta$ *for every $x \in \Omega \backslash \Omega^{int}$.*
3. $\operatorname{supp}(\Psi_i) \subset \bar{\Omega}_i$.

In section 2.4 we prove the following theorem.

THEOREM 2.1. *Let $V^h = \sum_{j=0}^J V_j \subset C(\bar{\Omega})$, and let Assumptions 2.1 and 2.2 hold. Assume that there is $\omega \geq 1$ such that*

$$(v, v) \leq (\tilde{A}_j^{-1} A_j v, v) \leq \omega(v, v)$$

for all $v \in V^h$ *and* $0 \le j \le J$. *Let*

$$M = \sum_{j=0}^{J} B_j; \tag{2.8}$$

then there is $C > 0$, *independent of* H *and* h *such that*

$$\kappa(MA) \le C\omega(1 + (H/\delta)^2). \tag{2.9}$$

2.4. Convergence theory. The bound on the condition number for the system preconditioned using two-level additive Schwarz methods is given in [23, 22]. We use this theory to calculate our bound.

THEOREM 2.2. *Let* K_0 *be a positive constant so that, for any* $v \in V^h$, *there exists a decomposition* $v = \sum_{i=0}^{J} v_i$ *such that* $v_i \in V_i$ *and*

$$\sum_{i=0}^{J} (A_i v_i, v_i) \le K_0 (Av, v). \tag{2.10}$$

Let

$$K_1 = \max_{1 \le j \le J} \sum_{i=1}^{J} \varepsilon_{ij}, \tag{2.11}$$

where, for $1 \le i, j \le J$, $\varepsilon_{ij} = 0$ *if* $V_i \perp V_j$; $\varepsilon_{ij} = 1$ *otherwise. Then*

$$\kappa (MA) \le \omega K_0 (1 + K_1), \tag{2.12}$$

where

$$\omega = \max_{0 \le j \le J} \lambda_{\max} \left(\tilde{A}_j^{-1} A_j \right).$$

We assume that the energy norm is equivalent to the H^1 seminorm, and we can therefore replace (2.10) by

$$\sum_{i=0}^{J} |u_i|^2_{H^1(\Omega)} \le K_0 |u|^2_{H^1(\Omega)}. \tag{2.13}$$

Our estimate for K_0 will be based on (2.13).

The value of K_1 is an indicator of the number of subdomains which contain any given point in Ω; we assume our partition is such that $K_1 = O(1)$. We solve the subdomain problems exactly in our numerical results, so $\tilde{A}_j = A_j$ for all j and $\omega = 1$ in section 3. Thus our condition number estimate is based on the estimate of K_0, which we obtain using Lemmas 2.3 and 2.4.

We define the coarse mesh projection $Q : V^h \to V^H$ by

$$Qu = \sum_{i=1}^{J} \alpha_i \Psi_i, \quad \alpha_i = \alpha_i (u) = \frac{1}{\mu (\Omega_i)} \int_{\Omega_i} u (x) \, dx,$$

where $u \in V^h$.

The value of K_0 depends on the bound of the energy of Qu and on the L^2 bound of the error in the coarse mesh operator, i.e., $u - Qu$. These bounds are provided in Corollary 2.3. This analysis is a modification of the results in [2, 3] for nonsmoothed aggregates.

Here, as in the remainder of this section, C is a constant that is independent of H and h. C may increase as the analysis progresses.

COROLLARY 2.3. *If Assumptions* 2.1 *and* 2.2 *hold, then*

$$\|u - Qu\|_{L^2}^2 \leq CH^2 \, |u|_{H^1}^2 \,, \tag{2.14}$$

$$|Qu|_{H^1}^2 \leq C \frac{H}{\delta} \, |u|_{H^1}^2 \,. \tag{2.15}$$

Proof. We give here the main results where our work differs from that in [2, 3], and we refer the reader to the details of the proof in those papers.

Since we have $\|\Psi_i\|_{L^2}^2 \leq CH^d$ by construction of our coarse mesh basis functions, we get

$$\|Qu\|_{L^2(\Omega)}^2 \leq C \, \|u\|_{L^2(\Omega)}^2 \,.$$

Therefore, by Poisson's inequality [6],

$$\|u - Qu\|_{L^2(\Omega)} \leq \|u\|_{L^2(\Omega)} + \|Qu\|_{L^2(\Omega)} \leq C \, \|u\|_{L^2(\Omega)} \leq CH \, |u|_{H^1(\Omega)} \,. \tag{2.16}$$

Since by construction we also have $|\Psi_i|_{H^1}^2 \leq C \frac{H^{d-1}}{\delta}$, we obtain

$$|Qu|_{H^1(\Omega)}^2 \leq C \tfrac{H}{\delta} \, |u|_{H^1(\Omega)}^2 \,. \qquad \square \tag{2.17}$$

We use these bounds in the following lemma.

LEMMA 2.4. *Under Assumptions* 2.1 *and* 2.2*, for every finite element function* $u \in V^h$*, there exists a decomposition* $\{u_i\}_{i=0}^J$*,* $u_i \in V_i$*, such that*

$$u = \sum_{i=0}^{J} u_i, \tag{2.18}$$

$$\sum_{i=0}^{J} |u_i|_{H^1(\Omega)}^2 \leq C \left(1 + \frac{H}{\delta}\right)^2 |u|_{H^1(\Omega)}^2 \,. \tag{2.19}$$

Proof. Define the fine mesh pointwise projection $I_h : C(\Omega) \to V^h$ by

$$I_h \, (u) = \sum_{l=1}^{n} u \, (x_l) \, \psi_l,$$

where $\{\psi_l\}_{l=1}^n$ is the finite element basis on the fine mesh, and $\{x_l\}_{l=1}^n$ are the fine mesh nodal points. Let u be partitioned such that

$$u_0 = Qu \text{ and } u_i = I_h \left(\theta_i (u - Qu)\right),$$

where, as in [17], $\{\theta_i\}$ is a smooth partition of unity such that

$$\begin{aligned} \theta_i &= 1 \text{ if } x \in \Omega_i \cup \Omega^{int}, \, x \notin \Omega_j \text{ for } j \neq i, \text{ and} \\ \theta_i &= 0 \text{ if } x \notin \Omega_i. \end{aligned}$$

Hence $|\nabla\theta_i| \leq \frac{1}{\delta}$. Clearly, by construction, for all $u \in V^h$,

$$\sum_{i=0}^{J} u_i = u.$$

The standard arguments [17] imply that

$$\sum_{i=0}^{J} |u_i|^2_{H^1(\Omega)} \leq C \left(\frac{1}{\delta^2} \|u - Qu\|^2_{L^2(\Omega)} + |u - Qu|^2_{H^1(\Omega)} + |Qu|^2_{H^1(\Omega)} \right). \tag{2.20}$$

We complete the proof by applying Lemma 2.3 to estimate $|Qu|^2_{H^1(\Omega)}$ and $\|u - Qu\|^2_{L^2(\Omega)}$. □

If we let

$$K_0 = C \left(1 + \frac{H}{\delta} \right)^2,$$

we obtain the result in Theorem 2.1.

3. Numerical results. In the two examples we used minimal overlap. Hence the coarse mesh problem is constructed by simply adding the matrix contributions for the nodes in a subdomain. The matrix representation for the restriction operator R_0 given minimal overlap h is

$$R_0 = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & & & & & & & & & \end{bmatrix},$$

where the length of 1's in row i is determined by the number of unknowns in subdomain i.

Subdomain solves were exact, meaning that $\tilde{A}_j = A_j$ for all j and, therefore, that $\omega = 1$.

3.1. Laplace equation. In this section we consider the simple test problem

$$\nabla^2 u = 0 \tag{3.1}$$

on the unit square $[0,1] \times [0,1]$ with zero Dirichlet boundary conditions. We use the function identically equal to one on the mesh as the initial iterate for a preconditioned conjugate gradient iteration. We terminated the iterations when the residual had been reduced by a factor of 10^{-4}. This example fully conforms to the theory from section 2.4.

In Table 3.1 we report results for a piecewise linear finite element discretization of (3.1). From the theory, one would expect the condition number κ to increase by a factor of 4 as either H is doubled or h is halved and to remain constant along the diagonals where H/h is constant. If, as is the case with the unpreconditioned problem, the iteration count is $O(\sqrt{\kappa})$, we would expect the iteration count to double if either H is held fixed and h is reduced by a factor of two or h is held fixed and H is increased by a factor of two.

If the iteration count is proportional to the square root of the condition number, as it is in the unpreconditioned case, then Table 3.1 shows that the growth in the number of iterations for fixed H and decreasing h is slower than predicted by the theory and is consistent with an $O(H/h)$ growth in the condition number. The rate of reduction in the iteration count as h is held fixed and as H is decreased is smaller still.

TABLE 3.1
Finite element discretization of Laplace's equation.

H \ h	1/64	1/128	1/256
1/4	37	51	68
1/8	32	44	61
1/16	26	36	49
1/32		26	37

3.2. Richards' equation. The test domain is the unit square $[0, 1m] \times [0, 1m]$ with boundary and initial conditions

$$
\begin{aligned}
\psi(x,0) &= 0.0, && x \in [0,1], && t > 0,\\
\psi(x,1) &= 0.1, && x \in [1/3, 2/3], && t > 0,\\
\frac{\partial \psi}{\partial z}(x,1) &= -1.0, && x \in [0,1/3) \cup (2/3,1], && t > 0,\\
\frac{\partial \psi}{\partial x}(x,z) &= 0.0, && x = 0,1 \quad z \in [0,1], && t > 0,\\
\psi(x,z) &= -z, && x,z \in [0,1] \times [0,1], && t = 0.
\end{aligned} \tag{3.2}
$$

3.2.1. Finite difference discretization with minimal overlap. We discretized (1.1) by applying cell-centered finite differences to the spatial operator, thereby yielding the system of differential-algebraic equations,

$$
\begin{aligned}
F_{i,j}(t,\psi,\frac{\partial \psi}{\partial t}) = \quad & \left(\frac{d\theta}{d\psi}_{i,j} + \frac{S_s}{\theta_s}\theta_{i,j} \right) \frac{\partial \psi_{i,j}}{\partial t}\\
& - \frac{1}{\Delta z^2}\left[K_{i+\frac{1}{2},j}(\psi_{i+1,j} - \psi_{i,j}) - K_{i-\frac{1}{2},j}(\psi_{i,j} - \psi_{i-1,j}) \right]\\
& - \frac{1}{\Delta z}(K_{i+\frac{1}{2},j} - K_{i-\frac{1}{2},j})\\
& - \frac{1}{\Delta x^2}\left[K_{i,j+\frac{1}{2}}(\psi_{i,j+1} - \psi_{i,j}) - K_{i,j-\frac{1}{2}}(\psi_{i,j} - \psi_{i,j-1}) \right],
\end{aligned} \tag{3.3}
$$

where $i = 0, \ldots, N-1$, $j = 0, \ldots, N-1$, $\Delta z = \Delta x = 1/N$, and

$$K_{i\pm\frac{1}{2},j} = \left[(K_s k_r)_{i\pm1,j} + (K_s k_r)_{i,j} \right]/2, \tag{3.4}$$

$$K_{i,j\pm\frac{1}{2}} = \left[(K_s k_r)_{i,j\pm1} + (K_s k_r)_{i,j} \right]/2. \tag{3.5}$$

The semidiscrete system was integrated in time over [0, 0.0149 days]. Order and step-size were selected via local truncation error estimates, and the local truncation error tolerance was set to $10\Delta x^2$, thereby balancing temporal and spatial truncation error. A secondary effect of this choice is that the number of iterations needed for convergence grows more slowly as Δx is reduced than would be the case for a steady-state problem. This effect is clearly visible in all the results reported in Tables 3.2, 3.3, and 3.4.

At a given step t_{n+1}, the application of the integration method yielded a nonlinear system of the form

$$F[t_{n+1}, \psi_{n+1}, g(\psi_{n+1})] = G(\psi_{n+1}) = 0,$$

where $g(\psi)$ is a the backward difference formula for $\partial\psi/\partial t$. We solved the nonlinear system with an inexact Newton iteration that terminated when the 2-norm of the nonlinear residual was reduced by a factor of 10^{-5}.

TABLE 3.2
Richards' equation iteration statistics, 2-level Schwarz.

$H\backslash h$	1/16	1/32	1/64	1/128	1/256
1/8	7	8	9	12	15
1/16		7	9	11	14
1/32			7	9	11
1/64				7	9
1/128					7

TABLE 3.3
Richards' equation iteration statistics, 1-level Schwarz.

$H\backslash h$	1/16	1/32	1/64	1/128	1/256
1/8	5	6	6	6	6
1/16	6	7	7	7	7
1/32		10	10	10	10
1/64			15	14	14
1/128				21	20
1/256					29

TABLE 3.4
Richards' equation iteration statistics, no preconditioner.

h	1/16	1/32	1/64	1/128	1/256
	31	56	84	129	193

At each Newton iteration we obtained the Newton step δ^{m+1}, by solving the linear system

$$\left[\frac{\partial G}{\partial \psi}(\psi_{n+1}^m)\right]\delta^{m+1} = -G(\psi_{n+1}^m)$$

with scaled, preconditioned BiCGstab. The scaling was obtained from the integration method's weighted root mean squared norm [15, 10, 1, 4]. In real applications, such a scaling would allow termination of the linear iteration according to tolerances specified by the integration scheme. However, for this test we computed the l^2 norm of the true linear residual for each linear iteration and terminated the linear iteration when that norm had been reduced by a factor of 10^{-7}. We did this both to more accurately estimate the effects of the preconditioner and to insure that errors in the Newton step were insignificant with respect to the Newton iteration and integration. The choices of termination criteria for the linear and nonlinear solvers imply that the time steps that the code uses are independent of the choice of preconditioner.

The preconditioner was two-level additive Schwarz using (3.1). The subdomains had the minimal overlap of $\Delta x = 1/N$. Table 3.2 gives the average BiCGstab iterations per Newton iteration for two-level additive Schwarz. The iteration count is constant as H and h are reduced simultaneously, as was the case with the simple example from section 3.1.

The iteration counts in Table 3.2 increase more slowly than the theory would predict if, as would be the case with a discretized elliptic problem and conjugate gradient iteration, the iteration count increased as the square of the condition number. In that case, if the condition number is $O(1 + H^2/h^2)$, then the iteration count would double as either H doubled or as h was halved.

For comparison we include similar statistics for a one-level additive Schwarz preconditioner in Table 3.3 and for no preconditioning in Table 3.4. Point Jacobi pre-

conditioning is the $H = h$ diagonal in Table 3.3. Neither of these scale well as H and h are reduced together.

4. Conclusions. In the context of Richards' equation, a model for subsurface flow through the unsaturated zone, we have demonstrated the effectiveness of the two-level additive Schwarz preconditioner using nonsmoothed aggregates for the coarse space on a temporally dependent, nonlinear problem. Our convergence estimates for a simpler model problem are consistent with the observations for Richards' equation, a problem to which the theory does not apply.

Acknowledgments. We wish to thank Marian Brezina, Jim Jones, Van Henson, David Keyes, Andrea Toselli, Mary Wheeler, Carol Woodward, and Jun Zou for many helpful discussions and Dave Klepacki, Matt Matthews, Mark Reed, Eric Sills, and Bob Walkup for their help with the computations. We are particularly grateful to our collaborators at the U.S. Army Engineer Research and Development Center: Charlie Berger, Jackie Hallberg, Stacy Howington, John Peters, Joe Schmidt, and Alan Stagg. We also wish to thank the referees for their helpful suggestions.

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