# THE ANDERSON MODEL OF LOCALIZATION：A CHALLENGE FOR MODERN EIGENVALUE METHODS＊ 

ULRICH ELSNER ${ }^{\ddagger}$ ，VOLKER MEHRMANN ${ }^{\ddagger}$ ，FRANK MILDE ${ }^{\S}$ ，RUDOLF A．RÖMER ${ }^{\S}$ ， and MiCHAEL SCHREIBER§


#### Abstract

We present a comparative study of the application of modern eigenvalue algorithms to an eigenvalue problem arising in quantum physics，namely，the computation of a few interior eigenvalues and their associated eigenvectors for the large，sparse，real，symmetric，and indefinite matrices of the Anderson model of localization．We compare the Lanczos algorithm in the 1987 implementation of Cullum and Willoughby with the implicitly restarted Arnoldi method coupled with polynomial and several shift－and－invert convergence accelerators as well as with a sparse hy－ brid tridiagonalization method．We demonstrate that for our problem the Lanczos implementation is faster and more memory efficient than the other approaches．This seemingly innocuous problem presents a major challenge for all modern eigenvalue algorithms．


Key words．Eigenvalue，eigenvector problems，Lanczos algorithm，Arnoldi algorithm，An－ derson model of localization，shift－and－invert，polynomial convergence accelerators

AMS subject classifications．65F15，65F50，82B44，65F10
1．Introduction．In this paper we present a comparative study of the applica－ tion of modern eigenvalue algorithms to an eigenvalue problem arising in quantum physics．The task is to compute a few（ $5-10$ ）interior eigenvalues and the associ－ ated eigenvectors of a family of structured large，sparse，real，symmetric，indefinite matrices．The off－diagonal elements are equal to the off－diagonal elements of the 7 －point central difference approximation to the three－dimensional Poisson equation on the unit cube with periodic boundary conditions．The matrices differ from each other only in the diagonal entries，which are suitably chosen random numbers．

Previously this problem was often solved by using the 1987 Cullum and Wil－ loughby implementation of the Lanczos algorithm［6，7］，in the following called CWI．But in the last 10 years several new eigenvalue methods have been developed and implemented as software packages，that seem，at least at first glance，more appropriate than CWI，see，e．g．，the recent survey and comparison given in 19 ． We apply these new codes to the described family of matrices and check whether they are faster and more memory efficient than CWI．To our surprise，none of the tested codes is consistently better than CWI．As we show below，we find only a single new code which is at least as fast as CWI．But this code needs two orders of magnitude more memory than CWI．We therefore believe that the described family of matrices will present an important new benchmark example and will hopefully lead to modifications and improvements for the current methods．

The paper is organized as follows．In $\S 2$ we describe the underlying quantum physics problem，i．e．，the Anderson model of localization，and introduce the pa－ rameters used in our study．In $\S$ 圂 we briefly review the Cullum／Willoughby version of the Lanczos method that has been previously used in the simulations for this model．We then give in $\S 母$ a brief survey of more recent eigenvalue methods．In $\S$ 国we present comparative results for the different methods and show that CWI is faster and needs less memory than all other approaches．

2．The Anderson model of localization．The Anderson model of local－ ization［1］is a convenient model for the investigation of electronic properties of disordered systems．Although it represents a severe simplification of amorphous

[^0]materials and alloys, it has nevertheless become a paradigmatic model and is currently widely used in the theoretical description of quantum mechanical effects of disorder such as, e.g., spatial localization of electronic wave functions with increasing strength of disorder and the corresponding metal-insulator transitions 18, 21, 35 . The quantum mechanical problem is represented by a Hamilton operator in the form of a real symmetric matrix $A$ and the quantum mechanical wave functions are simply the eigenvectors of $A$, i.e., finite vectors $x$ with real entries. E.g., for a simple cubic lattice with $M=N \times N \times N$ sites, we have to solve the eigenvalue equation $A x=\lambda x$, which is given in site representation as
\[

$$
\begin{align*}
x_{i-1, j, k}+x_{i+1, j, k}+x_{i, j-1, k}+x_{i, j+1, k}+x_{i, j, k-1}+x_{i, j, k+1} & +\varepsilon_{i, j, k} x_{i, j, k}  \tag{2.1}\\
& =\lambda x_{i, j, k}
\end{align*}
$$
\]

with $i, j, k$ denoting the cartesian coordinates of a site. The off-diagonal entries of $A$ correspond to hopping probabilities of the electrons from one site to a neighboring site. For simplicity, we have set them all to unity in (2.1). The disorder is encoded in the random potential site energies $\varepsilon_{i, j, k}$ on the diagonal of the matrix $A$. We consider only the case of $\varepsilon_{i, j, k}$ being uniformly distributed in the interval $[-w / 2,+w / 2]$. This is a common simplification, usually used in the studies of the Anderson model of localization with typical values of $w$ ranging from 1 to 30 . The boundary conditions are usually taken to be periodic, but hard wall and helical 28] boundary conditions are sometimes also used. According to the Gersgorin circle theorem (13] every such matrix $A$ has eigenvalues in the interval $[-w / 2-6,+w / 2+6]$. Possible generalizations of the Anderson model include anisotropic [21] or even random hopping [9] and various choices of the distribution function of the site energies [14. However, the graph of the matrix remains the same.

Although the above matrix seems to be fairly simple, the intrinsic physics is surprisingly rich. For small disorder $(w \ll 16.5)$, the eigenvectors are extended, i.e., $x_{i, j, k}$ is fluctuating from site to site but the envelope $|x|$ is approximately a non-zero constant. For large disorder $(w \gg 16.5)$, all eigenvectors are localized, i.e., the envelope $\left|x_{n}\right|$ of the $n$th eigenstate may be approximately written as $\exp [-\mid \vec{r}-$ $\vec{r}_{n} \mid / l_{n}(w)$ ] with $\vec{r}=(i, j, k)^{T}$ and $l_{n}(w)$ denoting the localization length of the eigenstate at the specified strength $w$ of the disorder. In Fig. 2.1, we show examples of such states for the Anderson model in one spatial dimension. Since extended states can contribute to electron transport, whereas localized states cannot, the Anderson model thus describes a metal-to-insulator transition: In three-dimensional samples at $w=w_{c} \approx 16.5$, the extended states at $\lambda \approx 0$ vanish and no current can flow. The eigenvector properties are also connected with the statistical properties of the spectrum $\sigma(A)$ of $A$. In the extended regime one finds level repulsion, while in the localized regime the eigenvalues are uncorrelated resulting in level clustering. These results agree quantitatively with random matrix theory [18]. Directly at $w_{c}$ there is a so-called critical regime where the eigenvectors are multifractal entities [21, 29] showing characteristic fluctuations of the amplitude on all length scales. In order to numerically distinguish these three regimes, namely localized, critical and extended behavior, one needs to (i) go to extremely large system sizes and (ii) average over many different realizations of the disorder, i.e., compute eigenvalues or -vectors for many matrices with different diagonals.

In the present paper we concentrate on the computation of a few eigenvalues and corresponding eigenvectors for the physically most interesting case of critical disorder $w_{c}$ and in the center of $\sigma(A)$, i.e., at $\lambda=0$, for system sizes as large as possible. In Fig. 2.2, we show a histogram of $\sigma(A)$ for different disorders. Note the high density of states at $\lambda=0$ in all cases. Therefore we have the further numerical challenge of distinguishing clearly the eigenstates in this high density region.


FIG. 2.1. Extended (dashed line) and localized (thick solid line) eigenstate for a single realization of the Anderson model in one spatial dimension with $N=200$ sites and periodic boundary conditions. For the localized eigenstate, we also show the exponential envelope with localization length $l \approx 12$ (thin lines) according to $\S$.


Fig. 2.2. Histogram $n(\lambda)$ of eigenvalues for a single system with $N^{3}=48^{3}$ sites and $w=10$ $(\square), 16.5(\circ)$, and $20(\diamond)$. The bin width is 0.05 . The lines are obtained by consecutively averaging 20 bins.
3. The Lanczos algorithm and the Cullum/Willoughby implementation. As outlined in the last section, each of the matrices $A$ is sparse, symmetric and indefinite. Furthermore, the matrix-vector multiplication $A x$ can be written explicitly as in (2.1) and is thus easily implemented. An ideal candidate for an algorithm taking advantage of nearly all these properties is the Lanczos algorithm 13. This algorithm iteratively generates a sequence of orthogonal vectors $v_{i}, i=1, \ldots, K$, such that $V_{K}^{T} A V_{K}=T_{K}$, with $V=\left\{v_{1}, v_{2}, \cdots, v_{K}\right\}$ and $T_{K}$ a symmetric tridiagonal $K \times K$ matrix. One obtains the recursion

$$
\begin{equation*}
\beta_{i+1} v_{i+1}=A v_{i}-\alpha_{i} v_{i}-\beta_{i} v_{i-1} \tag{3.1}
\end{equation*}
$$

where $\alpha_{i}=v_{i}^{T} A v_{i}$ and $\beta_{i+1}=v_{i+1} A v_{i}$ are the diagonal and subdiagonal entries of $T_{K}, v_{0}=0$ and $v_{1}$ is an arbitrary starting vector. For $K=M$ in exact arithmetic this is an orthogonal transformation to tridiagonal form that needs $M$ matrix-vector multiplications. The eigenvalues of the tridiagonal matrix $T_{K}$, also known as Ritz values, are then simply the eigenvalues of the matrix $A$ and the associated Ritz vectors yield the eigenvectors [6, 7, 13, 22, 25].

In finite precision arithmetic, however, the Lanczos vectors $v_{i}$ loose their orthogonality after a small number of Lanczos iterations. Consequently, there appear so called "spurious" or "ghost" eigenvalues in $\sigma\left(T_{K}\right)$, which do not belong to $\sigma(A)$.

There are several solutions to this problem: total reorthogonalization of all Lanczos vectors against each other, selective reorthogonalization 23, or distinguishing between good and spurious eigenvalues. While the reorthogonalization leads to an increase in memory requirements and computing time, since all or several of the $v_{i}$ need to be stored and reorthogonalized, the solution implemented in CWI [7] uses a simple and highly successful procedure to identify the spurious eigenvalues, thereby avoiding reorthogonalization. It thus only uses two Lanczos vectors in each iteration step and consequently the memory requirements are very small. An eigenvalue of $T_{K}$ is identified as being spurious if it is also an eigenvalue of the matrix $T_{K}^{\prime}$ which is constructed by deleting the first row and column of $T_{K}$. Still, the good eigenvalues produced may not yet have converged properly for a given $K$. So we further use the fact that good eigenvalues will be replicated in $\sigma\left(T_{K}\right)$ if $K$ is large enough. We only accept eigenvalues as being good eigenvalues after they have been replicated at least once in $\sigma\left(T_{K}\right)$. Hence we usually need at least $K \geq 2 M$. Finally, in order to obtain the eigenvectors corresponding to these good eigenvalues of $A$, all Lanczos vectors must be computed a second time, again doubling the computational effort.

The convergence of the Lanczos algorithm is very fast for the eigenvalues close to $\min \sigma(A)$ and $\max \sigma(A)$. This is especially true if these eigenvalues are well separated. However, for eigenvalues in the interior of $\sigma(A)$ and for eigenvalues which are not well separated the convergence is slow. Furthermore, the tridiagonal matrix $T_{K}$ becomes very large for an iteration in the interior of $\sigma(A)$. Nevertheless, the CWI has been used to study the Anderson model of localization even at $\lambda=0$ successfully for years [27, 29, 18, 14, 21, 35] and eigenstates for matrices with $N=100$ can be obtained within a few weeks of computing time [34].

We also remark that most of the computational effort in the Lanczos algorithm is spent on the iteration of (3.1), i.e., on matrix-vector multiplications and vector additions. These can be easily parallelized and thus the CWI is well suited for parallel architectures. For example, the eigenspectra presented in Fig. 2.2 have been obtained by such a parallel version of CWI running for about 60 hours for each realization using 16 processors of a Parsytec GCC Power Plus.
4. Modern approaches. Lately there has been much progress in eigenvalue methods mostly concentrating on non-symmetric matrices. We refer to 19 for a recent survey. The symmetric problem is usually assumed to be taken care of
implicitly. But although our symmetric eigenvalue problem is well-conditioned 13, the fact that the eigenvalues are clustered in the neighborhood of $\lambda=0$, our region of interest, creates difficulties for all numerical methods. Promising choices for possible replacements of the Cullum and Willoughby approach are the implicitly restarted Arnoldi method 19] and the hybrid tridiagonalization (HTD) algorithm of Cavers [4]. Another new approach is the Jacobi-Davidson method [3]. In the following we will pay special attention to the Arnoldi approach, since it allows the easy use of the shift-and-invert technique. We expect this to overcome the above mentioned clustering problem at $\lambda=0$.
4.1. Modifying the eigenproblem. The problem of slow convergence in the interior of $\sigma(A)$ can be overcome by computing eigenvalues and eigenvectors for a modified eigenvalue problem $f(A) x=f(\lambda) x$. The function $f$ is chosen such that the desired point $\lambda$ in $\sigma$ is mapped onto or close to the minimum or maximum of $\sigma(f)$. Furthermore, one should choose $f$, such that $\sigma(f)$ has well separated eigenvalues at $\min \sigma(f)$ and max $\sigma(f)$.

Among the many possible choices for $f(A)$, we shall consider in the following: (i) polynomial convergence accelerators, where $f(A)$ is chosen as a polynomial which has its maximum (or minimum) at $\lambda$. This moves $\lambda$ to max $\sigma(f(A))$ (or min $\sigma(f(A))$ resp.). We remark that occasionally these convergence accelerators are somewhat misleadingly called preconditioners. (ii) shift-and-invert with $f(A)=(A-\lambda I)^{-1}$ and $I$ the $M \times M$ identity matrix. This choice of $f$ requires the additional solution of a linear system with $A-\lambda I$ in each step of the eigenvalue iteration 13, 25]. For the solution of this linear system there are again two alternatives: (ii.a) direct sparse solvers for $A-\lambda I$. Unfortunately, not many direct solvers exist which can make efficient use of the sparseness for indefinite problems. (ii.b) iterative solvers using only matrix-vector multiplications with $A-\lambda I$. The iterative methods promise to make large matrix sizes possible, since they benefit in an optimal way from sparsity. Memory requirements and computational cost of a matrix-vector multiplication are proportional to the number of non-zeros and therefore proportional to $M$ for our present problem. However, we note that in our case $(A-\lambda I)$ is indefinite which will lead to slow convergence for most iterative solvers. Since the convergence of the iterative solver is dominated by the condition number of the linear system, one may employ preconditioners to accelerate its convergence 13].

All these approaches result in a competition between smaller numbers of Lanczos or Arnoldi iterations and increased costs for each such iteration step. For this reason it is not a priori clear whether they will indeed give a net reduction in computation time.
4.2. The implicitly restarted Arnoldi method. In a recent comparison [19] of different Arnoldi based packages ArPack 20 was found to be the fastest and most reliable of the compared codes.

When applied to symmetric eigenvalue problems, the main difference between the techniques in ArPack based on the Arnoldi iteration and the Lanczos iteration is an implicit restart technique. The Arnoldi method stores a number of Ritz vectors produced by the iterations and after a small number of steps initiates a restart which uses an implicit QR -algorithm for the small eigenvalue problem to create a new starting vector and to maintain orthogonality among the Ritz vectors. In contrast to the Lanczos algorithm more vectors have to be stored but spurious eigenvalues are avoided.

The ArPack implementation further allows the easy use of additional acceleration methods such as polynomial convergence acceleration and shift-and-invert as outlined above. So ArPack is probably the best choice for a replacement of CWI.
4.3. Other approaches. We have also studied the use of the HTD method [4] and the Jacobi/Davidson method [3]. The HTD method is a direct tridiagonalization method specifically designed for sparse matrices. This makes it an interesting approach for our purposes. However, it is not explicitly designed to compute only few interior eigenvalues and associated eigenvectors.

The Jacobi/Davidson method appears to be another promising future direction if it can be properly accelerated. The current version is designed for complex unsymmetric problems and to report comparative results would not be fair to this interesting new development. We intend to further study this method in the future.
5. Results. After a short discussion of the specific implementations and parameters, we now present the results of our comparison. The tests are performed on Hewlett-Packard HP9000 $735 / 125$ workstations for $N^{3} \leq 24^{3}$ and on a HP9000 K460 with the fast PA8000 processor for $N^{3} \geq 24^{3}$. The latter machine allows us to use up to 1.9 GB RAM and is about 3.5 times faster. In order to obtain a fair comparison we always require that the eigenresidual of the computed eigenvalue/eigenvector pair satisfies $|A x-\lambda x| \leq 10^{-8}$. The CPU times have been measured using the UnIX time command of the tcsh shell. Even for the largest system sizes considered, we have usually taken at least 5 different realizations of disorder and averaged the resulting CPU times. Sometimes, when the CPU times for a given algorithm fluctuate widely, we report the range of times instead of a simple average. We remark that the use of time introduces a further uncertainty into the results such that we always have an error of about $10 \%$. The random number generators used are ran2 from 24] and the rand command from Matlab.
5.1. The standard approach. For our particular problem we can reach $M=$ $80^{3}=512000$ with CPU times of about two weeks on the K460 machine using CWI. However, keeping in mind the configurational averaging necessitated by the underlying physical problem, a reasonable upper limit for the matrix size is $M=$ $50^{3}=125000$.

In Table 5.1 we show the CPU times obtained for CWI in the center and at the edge of $\sigma(A)$. Note that the computing times are nearly independent of the disorder parameter $W$, but, as expected, CWI is much faster at the edges of $\sigma(A)$ than at $\lambda=0$. In Table 5.2, we show the results for CWI at $\lambda=0$ in dependence on the matrix size $M$.

In the ArPack implementation of Lehoucq et al. 20] one has to set the parameter NCV which is the largest number of basis vectors that will be used in the implicitly restarted Arnoldi process. In normal mode and in the interior of $\sigma(A)$ we find that the actual value of NCV heavily influences computing time as shown in Table 5.3. This dependence on NCV becomes less pronounced for eigenvalues close to $\min \sigma(A)$ and max $\sigma(A)$ as shown in Table 5.4. However, since we do not know of any strategy to choose NCV optimally, this is a severe restriction of ArPACK. Furthermore, as shown in Table 5.1, ArPack, working in normal mode, is much slower than CWI both in the center and at the edge of $\sigma(A)$. It becomes too slow for practical use already at $M=12^{3}=1728$ as shown in Table 5.2.

In Tables 5.1 and 5.2 we also include CPU times for the HTD method. Note that we only show the CPU times needed to transform $A$ to tridiagonal form. Nevertheless, we find that HTD is much slower than CWI. We remark that when we use CWI to compute the full spectrum as in Fig. 2.2, it is still faster than HTD except for small system sizes $M \leq 12^{3}=1728$.
5.2. Polynomial convergence acceleration. As outlined above, polynomial convergence acceleration is usually a convenient choice to speed up the computation of eigenvalues and -vectors corresponding to a small region of $\sigma(A)$. Here, we test a polynomial provided by D. Sorensen, one of the authors of ArPack, and

| $M$ | $W$ | CWI | ARPACK | HTD |
| :---: | :---: | :---: | :---: | ---: |
| 1000 | 10.0 | 2.4 | $240-2200$ | 22 |
| 1000 | 16.5 | 2.5 | $230-1400$ | 22 |
| 1000 | 20.0 | 2.4 | $140-1300$ | 23 |
| 1728 | 10.0 | 7.9 | $1700-12000$ | 170 |
| 1728 | 16.5 | 7.8 | $410-12000$ | 170 |
| 1728 | 20.0 | 7.6 | $1100-20000$ | 160 |
| 4096 | 10.0 | 43 |  | 2600 |
| 4096 | 16.5 | 40 |  | 2500 |
| 4096 | 20.0 | 40 |  | 2500 |
| 1000 | 10.0 | 0.71 | 0.78 | 22 |
| 1000 | 16.5 | 0.77 | 0.85 | 22 |
| 1000 | 20.0 | 0.80 | 0.89 | 23 |
| 1728 | 10.0 | 0.94 | 1.5 | 170 |
| 1728 | 16.5 | 1.0 | 1.7 | 170 |
| 1728 | 20.0 | 1.1 | 1.8 | 160 |
| 13824 | 10.0 | 9.4 | 57 |  |
| 13824 | 16.5 | 9.2 | 57 |  |
| 13824 | 20.0 | 9.3 | 71 |  |

TABLE 5.1
$C P U$ times in seconds to compute 5 eigenvectors with $C W I, ~ A R P A C K, ~ a n d ~ H T D . ~ T h e ~ u p p e r ~$ part of the table corresponds to eigenvalues in the interior of $\sigma(A)$ at $\lambda \approx 0$, the lower part corresponds to the 5 largest eigenvalues.

| $M$ | CWI | CWI+ <br> conv. acc. | ARPACK | ArPACK+ <br> conv. acc. | HTD |
| ---: | ---: | ---: | :---: | :---: | ---: |
| 1000 | 2.5 | 5.6 | $230-1400$ | 4.9 | 22 |
| 1728 | 7.8 | 11 | $410-12000$ | 10 | 170 |
| 4096 | 40 | 59 |  | 66 | 2500 |
| 13824 | 770 | 1700 |  | 1700 |  |
| 13824 | 220 |  |  | 550 |  |
| 27000 | 1000 |  |  | 3500 |  |
| 91125 | 20000 |  |  |  |  |
| 110592 | 35000 |  |  |  |  |

TABLE 5.2
CPU times in seconds to compute at $w=16.5$ the eigenvectors corresponding to the 5 eigenvalues closest to $\lambda=0$ with CWI, CWI with Chebyshev-polynomial acceleration, ArPack in normal mode, ARPACK with Chebyshev-polynomial acceleration and HTD for various matrix sizes $M$. The CPU times in the upper (lower) part of the table have been measured on the HP 735 (HP K460).
C. Sun 31]. It is based on a Chebyshev-type polynomial given by the following recursion:

$$
\begin{array}{ll}
p_{1}(x) & =1 \\
p_{2}(x) & =a+b x^{2}  \tag{5.1}\\
p_{n+1}(x) & =2\left(a+b x^{2}\right) p_{n}-p_{n-1}
\end{array}
$$

where $a=\left(x_{1}^{2}+x_{2}^{2}\right) /\left(x_{1}^{2}-x_{2}^{2}\right)$ and $b=2 /\left(x_{2}^{2}-x_{1}^{2}\right)$. Also, $p_{n}$ is symmetric with a local maximum $p_{n}(0)>1$ at zero, $\left|p_{n}(x)\right| \leq 1$ in the intervals $\left[x_{1}, x_{2}\right]$ and $\left[-x_{2},-x_{1}\right]$, and $p_{n}$ grows rapidly for $|x|>x_{2}$ as shown, e.g., in Fig. 5.1 for $n=20$. In general, one would like to have $x_{1}$ and $x_{2}$ chosen automatically in order to obtain a suitable function $f$ as described in $\S 4$. In all our present calculations we use $n=50$ with $\left(x_{1}\right)^{2}=0.005$ and $\left(x_{2}\right)^{2}=1.1[\max \sigma(A)]^{2}$ according to 31]. This polynomial convergence acceleration speeds up ArPack immensely as one can see in Table 5.2. Since $\lambda=0$ is now mapped to $\max \sigma\left(p_{50}(A)\right)$, the actual value of NCV is less important. We found NCV $=50$ to be a good choice to make the execution times faster, although this requires more memory. Still CPU times are about a factor of

|  | NCV |  |  |
| ---: | ---: | ---: | ---: |
|  | 90 | 110 | 130 |
| 1 | 2577 | 4337 | 9455 |
| 2 | 2011 | 2267 | 3455 |
| 3 | 4190 | 1935 | 3635 |
| 4 | 411 | 755 | 1204 |
| 5 | 2435 | 1811 | 11620 |
| 6 | 8188 | 4704 | 2506 |

Table 5.3
$C P U$ times in seconds to compute 5 eigenvectors with ArPack for 6 different diagonals and 3 choices of NCV for $M=1728$ and $w=16.5$ at $\lambda=0$.

|  | NCV |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | ---: | ---: |
|  | $w=10$ |  | $w=16.5$ |  | $w=20$ |  |
|  | 15 | 20 | 15 | 20 | 15 | 20 |
| 1 | 41 | 44 | 63 | 62 | 56 | 58 |
| 2 | 65 | 65 | 43 | 47 | 49 | 55 |
| 3 | 57 | 60 | 54 | 64 | 44 | 47 |
| 4 | 62 | 62 | 44 | 49 | 49 | 55 |
| 5 | 72 | 76 | 72 | 68 | 76 | 71 |
| 6 | 44 | 45 | 56 | 62 | 87 | 85 |
| 7 | 81 | 70 | 72 | 88 | 108 | 90 |
| 8 | 52 | 60 | 49 | 51 | 70 | 66 |
| 9 | 42 | 48 | 58 | 68 | 99 | 75 |

Table 5.4
$C P U$ times in seconds to compute the eigenvectors corresponding to the 5 largest eigenvalues of A with ArPack for 9 different diagonals and 2 choices of $N C V$ with $M=13824$.


Fig. 5.1. Chebyshev polynomial $p_{20}(x)$ with $x_{1}=1$ and $x_{2}=10$.

| $M$ | CWI | ArPack+ <br> conv. acc. | HTD | ARPACK+ |  |  |  |  |
| ---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  | BKP | SuperLU | MA27 | MA27+HB |  |
| 1000 | 0.24 | 0.6 | 1.4 | 7.4 | 10 | 7.8 | 0.8 | 0.4 |
| 1728 | 0.43 | 1.2 | 6.7 | 18 | 22 | 19 | 2.2 | 0.9 |
| 4096 | 1.0 | 2.3 | 13 | 80 | 81 | 92 | 8.6 | 2.9 |
| 13824 | 3.4 | 6.6 |  |  |  |  | 70 | 22 |
| 27000 | 6.5 | 15 |  |  |  |  | 200 | 68 |
| 91125 | 22 |  |  |  |  |  | 1300 | 500 |
| 110592 | 27 |  |  |  |  |  | 2600 | 600 |

TABLE 5.5
Memory requirements in $M B$ to compute at $w=16.5$ the eigenvectors corresponding to the 5 eigenvalues closest to $\lambda=0$ for the different diagonalizers (Names as in the text, HB indicates hard wall boundary conditions)

| $M$ | CWI | ARPACK+ |  |  |  |  |
| :---: | ---: | ---: | ---: | :---: | :---: | :---: |
|  |  | LU | BKP | SuperLU | MA27 | MA27+HB |
| 1000 | 2.5 | 39 | 74 | 8.8 | 1.3 | 0.88 |
| 1728 | 7.8 | 150 | 300 | 28 | 5.0 | 2.0 |
| 4096 | 40 | 1200 | 1900 | 220 | 39 | 9.8 |
| 13824 | 770 |  |  |  | 740 | 140 |
| 13824 | 220 |  |  |  | 260 | 58 |
| 27000 | 1000 |  |  |  | 1300 | 250 |
| 91125 | 20000 |  |  |  | 19000 | 4900 |

CPU times in seconds to compute at $w=16.5$ the eigenvectors corresponding to the 5 eigenvalues closest to $\lambda=0$ with shift-and-invert ARPACK and different direct solvers. For easier comparison, we also include CWI. The CPU times in the upper (lower) part of the table have been measured on the HP 735 (HP K460).
two larger than for CWI without any convergence acceleration. Unfortunately, CWI itself is not made faster by the use of this accelerator as also shown in Table 5.2. Although the number of Lanczos vectors needed to achieve convergence is reduced remarkably, the additional computational effort now required for every Lanczos step becomes very large. At the end one needs even slightly more matrix-vector multiplications than without convergence acceleration.
5.3. Shift-and-invert with direct solvers. We now discuss the use of the shift-and-invert mode of ARPACK together with a direct solver for the linear system $(A-\lambda I) y=b$. We first note that although our matrix $A$ is symmetric, it is not positive definite and thus we cannot use a sparse Cholesky decomposition. Unfortunately, there are only few packages available for sparse symmetric indefinite problems [8]. Therefore we also investigated several packages for general sparse matrices.

Meschach [32] is a freely available mathematical package written in C. There are three sparse factorization methods implemented in Meschach: Cholesky, LU, and Bunch-Kaufmann-Parlett (BKP). Cholesky factorization does not work due to the indefiniteness of $A$. LU and BKP are supposed to take advantage of the sparseness of our problem. However, we find that they have huge memory requirements of the order of $M^{2}$ as shown in Table 5.5. So they are inapplicable for large system sizes. And even for small systems they turn out to be much too slow as shown in Table 5.6.

The Harwell Subroutine Library (16] contains the sparse symmetric indefinite solver MA27. As shown in Table 5.6, ArPack with MA27 is about as fast as CWI. In fact, MA27 seems to become faster than CWI for $M \geq 45^{3}=91125$. Unfortunately we could not test this because of the huge memory requirements of MA27 as shown in Table 5.5.

| $M$ | CWI | ARPACK+ |  |  |
| ---: | ---: | ---: | ---: | ---: |
|  |  | QMRX | QMRL | CPX |
| 1000 | 2.5 | 68 | 93 | 85 |
| 1728 | 7.8 | 239 | 320 | 330 |
| 4096 | 40 | 1600 |  | 2300 |
| 13824 | 770 | 35000 |  |  |
| 13824 | 220 | 12000 |  |  |
| 27000 | 1000 |  |  |  |
| 91125 | 20000 |  |  |  |

TABLE 5.7
CPU times in seconds to compute at $w=16.5$ the eigenvectors corresponding to the eigenvalues closest to $\lambda=0$ with shift-and-invert ARPACK and the iterative solvers from QMRPACK. For easier comparison, we also include CWI. The CPU times in the upper (lower) part of the table have been measured on the HP 735 (HP K460).

A noteworthy fact is that MA27 is much better for hard wall boundary conditions (HB). This can be explained by the fact that the bandwidth of the matrix is $\mathcal{O}\left(N^{2}\right)$ instead of $\mathcal{O}\left(N^{3}\right)$ as for periodic boundary conditions. However, on physical grounds, a calculation with HB is expected to be much more influenced by the finite size of the cubes considered. So although we can obtain larger system sizes here, the results for the interesting physical quantities may not be as reliable. Nevertheless, ArPack with MA27 for matrices with HB is faster than CWI for matrices with periodic boundary conditions. MA27 with HB is also faster than CWI with HB, since for CWI there is only a negligible difference in computing time between HB and periodic boundary conditions coming from the matrix-vector multiplication.

SUPERLU is a package by Demmel et al. 8] doing a sparse LU decomposition. Compared with CWI and MA27, SuperLU is much slower as shown in Table 5.6 . Furthermore, it needs about one order of magnitude more memory than MA27 as shown in Table 5.5. SuperLU allows the input of different preorderings in addition to the default minimum-degree ordering. We have tested a symmetric minimum degree ordering from the Matlab program and a nested dissection ordering computed by the Chaco package [17]. For some choices of diagonals we derive small savings in run time and/or memory but these are not consistent, i.e., the same kind of ordering speeds up the program for one choice of $N$ and slows it down for $N+1$.
5.4. Shift-and-invert with iterative solvers. Considering the recent advances in iterative solvers, we initially hoped that ARPACK in shift-and-invert mode coupled with a modern iterative method for the solution of linear systems would be quite efficient. As we will show below, this is not the case.

The quasi-minimal-residual (QMR) technique should be one of the best iterative solvers for symmetric matrices that works using only matrix-vector multiplications if no preconditioning is used 12]. However, as shown in Fig. 2.2, our matrices are indefinite with a nearly symmetric eigenvalue distribution around zero. This results in a very bad iteration count of about $2 M$ for the solution of a single linear system of size $M$. The times and iteration numbers from three variants implemented in QMRPACK [11], namely, QMR based on three-term Lanczos with and without look-ahead (QMRL/QMRX) and QMR based on coupled two-term Lanczos without look-ahead (CPX) are not very different as shown in Table 5.7. For all three methods the iteration count is rather high. Consequently, we find that ArPack in shift-andinvert mode coupled with QMRPACK as iterative solver is about 20 times slower than CWI. The ArPack input parameter NCV was set to 15 . We also find that the implementation of QMR based on coupled two-term Lanczos with look-ahead does not converge for larger systems within 50000 iterations.

In order to check if other iterative methods are perhaps more efficient than QMR for our family of matrices, we have also tried several such iterative solvers

| $M$ | QMR | CGS | Bi-CG | Bi-CGSTAB |
| ---: | :---: | :---: | :---: | :---: |
| 512 | $796-999$ | $843-$ n.c. | $809-980$ | $1006-$ n.c. |
| 1000 | $1705-$ n.c. | $1762-$ n.c. | $1701-$ n.c. | n.c. |
| 1728 | $2918-$ n.c. | n.c. | $2932-$ n.c. | n.c. |
| 2744 | $4809-$ n.c. | n.c. | $4775-$ n.c. | n.c. |
| 4096 | $7270-$ n.c. | n.c. | $7401-$ n.c. | n.c. |


| $M$ | GMRES | GMRES(5N) | QMR+jac | QMR+tri |
| ---: | :---: | :---: | :---: | :---: |
| 512 | $511-512$ | n.c. | $813-$ n.c. | $295-569$ |
| 1000 | $995-1000$ | n.c. | n.c. | $1389-$ n.c. |
| 1728 | $1723-1728$ | n.c. | n.c. | n.c. |
| 2744 | $2736-2744$ | n.c. | n.c. | n.c. |
| 4096 |  | n.c. | n.c. | n.c. |

TABLE 5.8
Number of iterations needed in Matlab in order to solve the linear system $A y=b$. The abbreviations for the different algorithms are explained in the text. The runs are aborted when the number of iterations is more than $2 M$. This case of no convergence is indicated by " $n . c$.".
using the MatLab programming environment. In addition to QMR, we have considered the conjugate-gradient-squared method (CGS) [30], the BiConjugate-Gradient method (BiCG) [10, its stabilized variant (Bi-CGSTAB) 33 and the generalized-minimal-residual (GMRES(k)) method 26]. Furthermore, several general purpose preconditioners [2], i.e., the Jacobi (jac) preconditioner, the $\operatorname{ILU}(0)$ preconditioner and also the three main diagonals as the preconditioning matrix (tri) have been tested. Since the performance of MATLAB programs cannot directly be compared to compiled programs, we only give the iteration count of each algorithm. One such iteration requires at least one matrix-vector multiplication and two inner products and is thus at least as expensive as one Lanczos step. We always use the built-in implementations of these algorithms as in Matlab v5.1.

In Table 5.8, we show results obtained for various matrix sizes $M$. The ranges reflect the variations corresponding to 12 different realizations of disorder on the diagonal of the matrices. Note that for the same $M$ we use the same 12 diagonals for all algorithms. We always choose $x_{0}=0$ as initial vector. The iteration count represents the number of iterations needed to solve the matrix equation $A x=y$ up to a relative accuracy of $10^{-8}$. We always stop the algorithms if after $2 M$ iterations this accuracy has not been achieved. For practical restart values $k \leq 200$, $\operatorname{GMRES}(k)$ does not converge at all within our iteration limit. With no restarts, GMRES needed $M$ or slightly less iterations. But note that both memory and computing-time requirements for $M$ steps of pure GMRES exceed those of a nonsparse direct solver.

None of the tested preconditioners is consistently effective. The Jacobi preconditioner in fact increases the iteration count most of the time. The $\operatorname{ILU}(0)$ preconditioner returns a singular matrix and consequently appears inapplicable. The tridiagonal preconditioner is more effective, in some cases reducing the iteration count by up to $50 \%$. But again there are examples where it fails to do anything. We remark that in general the iteration count is consistent with the results from QMRPACK. To sum up, we find that all of these iterative algorithms do not perform better that the QMR algorithm and consequently are no real alternative.

Another idea is to work with the matrix $A^{2}$ instead of $A$. Since it is symmetric and positive definite, we can now use the conjugate gradient method. But this squares the condition number of the linear system, which is already usually very large for $A$ (15]. Hence more effort has to be invested into the development of a good preconditioner. We find for our matrices that while the iteration count is in general a bit less than for the methods mentioned above and the preconditioners
are more consistently effective we still need of the order of $M$ steps with at least two matrix-vector multiplications for the solution of one linear system. And since the shift-and-invert ARPACK still needs to solve several linear systems, all the iterative methods working on $A^{2}$ were no match for CWI.
6. Summary. We have tested several modern methods to compute a few inner eigenvectors of a very large sparse matrix corresponding to the Anderson model of localization motivated within theoretical physics. Particularly the implicitly restarted Arnoldi method in connection with polynomial convergence acceleration and in shift-and-invert mode with several direct and iterative solvers for systems of linear equations is compared to the Cullum/Willoughby implementation of the Lanczos method. Despite the recent progress in linear system solvers we find all considered modern methods to be inapplicable for very large system sizes, because either the computation times or the memory requirements are much to large. To sum up, we find that CWI Lanczos is currently still the most efficient method for the matrix type we are interested in. We emphasize that the CWI Lanczos, with our slight modifications as outlined in $\S 3$, is a reliable tool for our problem. In particular, the problem of spurious eigenvalues which plague the original Lanczos algorithm, can be handled safely.

Since large scale diagonalizations are widely used in theoretical physics - and also theoretical chemistry [5] - we would be happy to learn about any algorithm that does better than CWI for our matrices. We are especially interested in a preconditioner for the iterative methods which is suitably adapted to our problem. Certainly improved direct methods for our matrix type are also of great importance. We hope to have convinced the reader that it may be worthwhile to rethink seemingly easy problems like the present eigenproblem for real and symmetric matrices.

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## REFERENCES

[1] P. W. Anderson, Absence of diffusion in certain random lattices, Phys. Rev., 109 (1958), pp. 1492-1505.
[2] R. Barrett, M. Berry, T. Chan, J. Demmel, J. Donato, J. Dongarra, V. Eijkhout, R. Pozo, C. Romine, and H. van der Vorst, Templates for the solution of linear systems, SIAM Publications, Philadelphia, 1994.
[3] J. G. L. Booten, H. A. Van der Vorst, P. M. Meijer, and H. J. J. te Riele, A preconditioned Jacobi-Davidson method for solving large generalized eigenvalue problems, Tech. Rep. CWI-NM-R9414, Centrum voor Wiskunde en Informatica, Amsterdam, July 1994.
[4] I. A. Cavers, A hybrid tridiagonalization algorithm for symmetric sparse matrices, SIAM J. Matrix Anal. Appl., 15 (1994), pp. 1363-1380.
[5] R. Chen and H. Guo, Benchmark calculations of bound states of $\mathrm{HO}_{2}$ via basis Lanczos algorithm, Chem. Phys. Lett., 277 (1997), pp. 191-198.
[6] J. Cullum and R. A. Willoughby, Lanczos Algorithms for Large Symmetric Eigenvalue Computations, Volume 1: Theory, Birkhäuser, Boston, 1985.
[7] ——, Lanczos Algorithms for Large Symmetric Eigenvalue Computations, Volume 2: Programs, Birkhäuser, Boston, 1985. http://www.netlib.org/lanczos/.
[8] J. Demmel, J. Gilbert, and X. Li, SuperLU Users' Guide, 1997. http://www.netlib.org/scalapack/prototype/superlu_ug.ps.
[9] A. Eilmes, R. A. Römer, and M. Schreiber, The Anderson model of localization with random hopping, Eur. Phys. J. B, 1 (1998). accepted for publication.
[10] R. Fletcher, Conjugate gradient methods for indefinite systems, in Proc. Dundee Biennal Conf. Numerical Analysis 1974, G. Watson, ed., Springer, Heidelberg, 1975, pp. 73-89.
[11] R. Freund and N. Nachtigal, QMRPACK. http://www.netlib.org/linalg/.
[12] ——, QMR: A quasi-minimal residual method for non-hermitian linear systems, Numer. Math., 60 (1991), pp. 315-339.
[13] G. H. Golub and C. F. v. Loan, Matrix Computations, Johns Hopkins University Press, Baltimore and London, 3rd ed., 1996.
[14] H. Grussbach and M. Schreiber, Determination of the mobility edge in the Anderson model of localization in three dimensions by multifractal analysis, Phys. Rev. B, 57 (1995), pp. 663-666.
[15] W. HackBUSCH, Iterative Solution of Large Sparse Systems of Equations, vol. 95 of Applied Mathematical Sciences, Springer, Berlin, 1994.
[16] Harwell subroutine library. http://www.dci.clrc.ac.uk/Activity.asp?HSL.
[17] B. HEndrickson and R. Leland, The Chaco User's Guide Version 2, Sandia National Laboratories, Albuquerque, 1995. ftp://ftp.cs.sandia.gov/pub/papers/bahendr/guide.ps.Z.
[18] E. Hofstetter and M. Schreiber, Statistical properties of the eigenvalue spectrum of the three-dimensional Anderson hamiltonian, Phys. Rev. B, 48 (1993), pp. 16979-16985.
[19] R. B. Lehouce and J. A. Scott, An evaluation of software for computing eigenvalues of sparse nonsymmetric matrices, Tech. Rep. MCS-P547-1195, Argonne National Laboratory, Argonne, 1996. ftp://info.mcs.anl.gov/pub/tech_reports/reports/P547.ps.Z. submitted to ACM Trans. Math. Software.
[20] R. B. Lehoucq, D. Sorensen, and C. Yang, ArPack users guide: Solution of large scale eigenvalue problems by implicitly restarted Arnoldi methods, tech. rep., Rice University, Department of Computational and Applied Mathematics, Houston, 1996. http://www.caam.rice.edu/software/ARPACK/ug.ps.
[21] F. Milde, R. A. Römer, and M. Schreiber, Multifractal analysis of the metal-insulator transition in anisotropic systems, Phys. Rev. B, 55 (1997), pp. 9463-9469.
[22] B. N. Parlett, The Symmetric Eigenvalue Problem, Prentice-Hall, Englewood Cliffs, 1980.
[23] B. N. Parlett and D. S. Scott, The Lanczos algorithm with selective orthogonalization, Math. Comp., 33 (1979), pp. 217-238.
[24] W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, Numerical Recipes in FORTRAN, Cambridge University Press, Cambridge, 2nd ed., 1992.
[25] Y. SAAD, Numerical Methods for Large Eigenvalue Problems: Theory and Algorithms, Wiley, New York, 1992.
[26] Y. SaAd and M. Schultz, GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems, SIAM J. Sci. Statist. Comput., 7 (1986), pp. 856-869.
[27] M. Schreiber, Fractal eigenstates in disordered systems, Physica A, 167 (1990), pp. 188-198.
[28] M. Schreiber, Multifractal characteristics of electronic wave functions in disordered systems, in Computational Physics, K. H. Hoffmann and M. Schreiber, eds., Springer, Berlin, 1996, pp. 147-165.
[29] M. Schreiber and H. Grussbach, Multifractal wave functions at the Anderson transition, Phys. Rev. Lett., 67 (1991), pp. 607-610.
[30] P. Sonneveld, CGS, a fast Lanczos-type solver for nonsymmetric problems, SIAM J. Sci. Statist. Comput., 12 (1989), pp. 36-52.
[31] D. Sorensen and C. Sun, private communications.
[32] D. E. Stewart and Z. Leyk, Meschach. http://www.netlib.org/c/meschach/.
[33] H. A. Van der Vorst, Bi-CGSTAB: A fast and smooth converging variant of Bi-CG for the solution of non-symmetric linear systems, SIAM J. Sci. Statist. Comput., 12 (1992), pp. 613-644.
[34] I. K. Zharekeshev and B. Kramer, private communications.
[35] —, Asymptotics of universal probability of neighboring level spacings at the Anderson transition, Phys. Rev. Lett., 79 (1997), pp. 717-720.


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    $\ddagger$ Fakultät für Mathematik，Technische Universität，D－09107 Chemnitz，Germany
    §Institut für Physik，Technische Universität，D－09107 Chemnitz，Germany

