DISOCCLUSION BY JOINT INTERPOLATION OF VECTOR FIELDS AND GRAY LEVELS*

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Abstract. In this paper we study a variational approach for filling-in regions of missing data in $2D$ and $3D$ digital images. Applications of this technique include the restoration of old photographs and removal of superimposed text like dates, subtitles, or publicity, or the zooming of images. The approach presented here, initially introduced in [12], is based on a joint interpolation of the image gray-levels and gradient/isophotes directions, smoothly extending the isophote lines into the holes of missing data. The process underlying this approach can be considered as an interpretation of the Gestaltist's principle of good continuation. We study the existence of minimizers of our functional and its approximation by smoother functionals. Then we present the numerical algorithm used to minimize it and display some numerical experiments.

Key words. Disocclusion, Elastica, BV functions, Interpolation, Variational approach, Γ-convergence

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1. Introduction. Filling-in missing data in digital images has a number of fundamental applications. Between them, we can mention the removal of scratches in old photographs and films, the removal of superimposed text like dates, subtitles, or publicity from a photograph, or the recovery of pixel blocks corrupted during binary transmission. The basic idea is to fill-in the gap of missing data in a form that it is non-detectable by an ordinary observer. This process has received different names in the literature, as disocclusion [52, 54], filling-in [12], or inpainting [19] (*inpainting* is the name used in art restoration [73, 34, 46]).

Since the early days of art and photography, inpainting has been done by professional artists. Imitating their performance with semi-automatic digital techniques is currently an active area of research. Most of the efforts have been directed either to the recovery of the textured part of the image or to the recovery of its geometry. Several succesful algorithms exist for the recovery of textures. The basic idea in them is to select a texture (tipically modeled as a Markov random field) and synthesize it inside the region to be filled-in (the hole) [40, 39, 32, 66]. The recovery of the geometric part of the image in a hole, or region where the data is missing, was first formulated by S. Masnou and J.M. Morel [54] as a variational problem, trying to interpolate the data in the hole. They were inspired by the work of D. Mumford, M. Nitzberg and T. Shiota [59] on image segmentation with depth. The same approach was followed in [12] and in the work of T. Chan and J. Shen [28]. A different approach, based on the transportation of information along the isophotes of the image, was proposed in [19]. We shall review in more detail this geometric approach below. A related and important area of research is the restoration of damaged films. The basic idea here

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is to use information from past and future frames to restore the current one, e.g., [43, 48, 49], an approach that can not be used when dealing with still images.

Let us review in some detail the variational approaches used for filling-in the missing information in a region of the image. A pioneering contribution in the recovery of plane image geometry is due to D. Mumford, M. Nitzberg and T. Shiota [59]. They were not directly concerned with the problem of recovering the missing parts of the image, instead, they addressed the problem of segmenting the image into objects which should be ordered according to their depth in the scene. The segmentation functional should be able to find which are the occluding and the occluded objects while finding the occluded boundaries. For that they relied in a basic principle of Gestalt's psychology: our visual system is able to complete partially occluded boundaries and the completion tends to respect the *principle of good continuation* [44]. When an object occludes another the occluding and occluded boundaries form a particular configuration, called T-junction, which is the point where the visible part of the boundary of the occluded object terminates. Then our visual system smoothly continues the occluded boundary between T-junctions. In [59], the authors proposed an energy functional to segment a scene which took into account the depth of the objects in the scene and the energy of the occluded boundaries between T-junctions. They assumed that the completion curves should be as short as possible and should respect the principle of good (smooth) continuation. Thus, to define the energy of the missing curve they had to give a mathematical formulation of the above principles. Given two T-junction points p and q and the tangents τ_p and τ_q to the respective terminating edges, they proposed as smooth continuation curve Euler's elastica, i.e., the curve minimizing the energy

$$\int_C (\alpha + \beta k^2) ds \tag{1.1}$$

where the minimum is taken among all curves C joining p and q with tangents τ_p and τ_q, respectively, k denotes the curvature of C, ds its arc length, and α, β are positive constants. Let us mention that Euler's elastica has been frequently used in computer vision ([41],[50],[64], [69],[70],[71],[75],[76],[74]) and a beautiful account on it can be found in [58].

In an important contribution to the question Masnou and Morel [52, 54, 53] proposed a variational formulation for the recovery of the missing parts of a grey level two-dimensional image and they referred to this interpolation process as *disocclusion*, since the missing parts can be considered as occlusions hiding the part of the image we want to recover. Their energy functional was also based on the elastica and we shall review it in some detail.

An image is usually modeled as a function defined in a bounded domain $D \subseteq R^N$ (tipically $N = 2$ for usual snapshots, $N = 3$ for medical images or movies) with values in R^k ($k = 1$ for grey level images, or $k = 3$ for color images). For simplicity, we shall consider only the case of grey level images. Any real image is determined in a unique way by its upper (or lower) level sets $X_\lambda u := \{x \in D : u(x) \geq \lambda\}$ ($X'_\lambda u := \{x \in D : u(x) \leq \lambda\}$). Indeed we have the reconstruction formula

$$u(x) = \sup\{\lambda \in R : x \in X_\lambda u\}. \tag{1.2}$$

The basic postulate of Mathematical Morphology prescribes that the geometric information of the image u is contained in the family of its level sets [62], [38], or in a more

local formulation, in the family of connected components of the level sets of u [62, 63], [24]. We shall refer to the family of connected components of the upper level sets of u as the topographic map of u. In case that u is a function of bounded variation in $D \subseteq R^2$, i.e., $u \in BV(D)$ [2, 35, 77], its topographic map has a description in terms of Jordan curves [3]. With an adequate definition of connected components, the essential boundary of a connected component of a rectifiable subset of R^2 consists, modulo an $\mathcal{H}^1$ null set, of an exterior Jordan curve and an at most countable family of interior Jordan curves which may touch in a set of $\mathcal{H}^1$-null Hausdorff measure [3]. Since almost all level sets $X_\lambda u$ of a function u of bounded variation are rectifiable sets, its essential boundary, $\partial^* X_\lambda u$, consists of a family of Jordan curves called the level lines of u. Thus, the topographic map of u can be described in terms of Jordan curves. In this case, the monotone family of upper level sets $X_\lambda u$ suffices to have the reconstruction formula (1.2) which holds almost everywhere [38].

Let D be a square in R^2 and $\tilde{\Omega}$ be an open bounded subset of D with Lipschitz continuous boundary. Suppose that we are given an image $u_0 : D \setminus \overline{\tilde{\Omega}} \to [a, b]$, $0 \leq a < b$. Using the information of u_0 on $D \setminus \overline{\tilde{\Omega}}$ we want to reconstruct the image u_0 inside $\tilde{\Omega}$. We shall call $\tilde{\Omega}$ the *hole* or *gap*. We shall assume that the function u_0 is a function of bounded variation in $D \setminus \overline{\tilde{\Omega}}$. Then the topographic structure of the image u_0 outside $\tilde{\Omega}$ is given by a family of Jordan curves. Generically, by slightly increasing the hole, we may assume that, for almost all levels λ, the level lines of $X_\lambda u_0$ transversally intersect the boundary of the hole in a finite number of points [52]. Let us call $\Lambda \subseteq R$ the family of such levels. As formulated by Masnou [52, 54, 53], the disocclusion problem consists in reconstructing the topographic map of u_0 inside $\tilde{\Omega}$. Given $\lambda \in \Lambda$ and two points $p, q \in X_\lambda u_0 \cap \partial\tilde{\Omega}$ whose tangent vector at the level line $X_\lambda u_0$ is τ_p and τ_q, respectively, the optimal completion curve proposed in [52, 54] is a curve Γ contained in $\tilde{\Omega}$ minimizing the criterion

$$\int_\Gamma (\alpha + \beta |k|^p) d\mathcal{H}^1 + (\tau_p, \tau_\Gamma(p)) + (\tau_q, \tau_\Gamma(q)) \tag{1.3}$$

where k denotes the curvature of Γ, $\tau_\Gamma(p)$ and $\tau_\Gamma(q))$ denote the tangents to Γ at the points p and q, respectively, and $(\tau_p, \tau_\Gamma(p))$, $(\tau_q, \tau_\Gamma(q))$ denote the angle formed by the vectors τ_p and $\tau_\Gamma(p)$, and, respectively, for q. Here α, β are positive constants, and $p \geq 1$. The optimal disocclusion is obtained by minimizing the energy functional

$$\int_{-\infty}^{+\infty} \sum_{\Gamma \in F_\lambda} \Big(\int_\Gamma (\alpha + \beta |k|^p) d\mathcal{H}^1 + (\tau_p, \tau_\Gamma(p)) + (\tau_q, \tau_\Gamma(q)) \Big) \, d\lambda \tag{1.4}$$

where F_λ denotes the family of completion curves associated to the level set $X_\lambda u_0$. As we noted above, the family F_λ is generically finite, thus the sum in (1.4) is generically finite. In [52, 55] the authors proved that for each $p \geq 1$ there is an optimal disocclusion in $\tilde{\Omega}$ and proposed an algorithm based on dynamic programming to find optimal pairings between compatible points in $\partial X_\lambda u_0 \cap \partial\Omega$ for $p = 1$, curves which are straight lines, thus finding in this case the minimum of (1.4) [52, 53]. In [4] the authors interpreted the disocclusion problem in a slightly different way. First, they observed that by computing the criterion $\int_\Gamma (\alpha + \beta |k|^p) d\mathcal{H}^1$ not only on the completion curve but also in a small piece of the associated level line outside $\tilde{\Omega}$, the criterion (1.4) can be written as

$$\int_{-\infty}^{+\infty} \sum_{\Gamma \in F_\lambda} \Big(\int_\Gamma (\alpha + \beta |k|^p) d\mathcal{H}^1 \Big) \, d\lambda \tag{1.5}$$

where now the curves in F_λ are union of a completion curve and a piece of level line of u_0 in $\Omega \setminus \tilde{\Omega}$ for a domain $\Omega \supset \tilde{\Omega}$. This requires that the level lines of u_0 are essentially in $W^{2,p}$ in $\Omega \setminus \tilde{\Omega}$. Then, at least for C^2 functions u, (1.5) can be written as

$$\int_\Omega |\nabla u|(\alpha + \beta |div \, \frac{\nabla u}{|\nabla u|}|^p) \, dx \tag{1.6}$$

with the convention that the integrand is 0 when $|\nabla u| = 0$. In [4], the authors considered this functional when the image domain D and the hole $\tilde{\Omega}$ are subsets in R^N whit $N \geq 2$ and they studied the relaxed functional, proving that it coincides with

$$\int_R \int_{\partial [u \geq t]} (\alpha + \beta |H_{[u \geq t]}|^p) \, d\mathcal{H}^{N-1} \, dt \tag{1.7}$$

for functions $u \in C^2(\Omega)$, $N \geq 2$, $p > N-1$, and $H_{[u \geq t]}$ denotes the mean curvature of $[u \geq t]$. Moreover they proved that the functional in (1.7) is lower semicontinuous on $u \in L^1(D) \cap C^2(\Omega)$, extending the result in [14]. Moreover they obtained a regularity result for the level lines of the optimal disocclusion [4].

In [28] the authors proposed a direct numerical approach to the solutions of (1.6). The authors also compared it with previous curvature driven diffusion and Total Variation based inpaintings [27, 26]. Their analysis in [27] showed that a curvature term was necessary to have a connectivity principle [27]. In addition, they considered the interpolation and filling-in in the presence of noise, an important additional contribution. We shall later comment on this work after we introduce our approach.

In [12, 13] the authors proposed to fill-in the hole $\tilde{\Omega}$ using both the gray level and the vector field of tangents (or normals) to the level lines of the image outside the hole. Let Ω be an open subset of D with Lipschitz boundary such that $\overline{\tilde{\Omega}} \subset\subset \Omega$. The band around $\tilde{\Omega}$ will be the set $B = \Omega \setminus \overline{\tilde{\Omega}}$. To fill-in the hole $\tilde{\Omega}$ we shall use the information of u_0 contained in B, mainly the gray level and the vector field of normals (or tangents) to the level lines of u_0 in B. As in the previous approach we attempt to continue the level sets of u_0 in B inside $\tilde{\Omega}$ taking into account the principle of good continuation. The energy functional proposed in [12, 13] was a function of two relied variables : a vector field θ which represents the directions of the level lines of u, and the gray level u. Both u and θ were constrained in the band B by their known values there. Thus the authors proposed to minimize the functional

$$\begin{aligned}
&\text{Minimize} \int_\Omega |div(\theta)|^p (\gamma + \beta |\nabla k * u|) dx \\
&|\theta| \leq 1, \, \| \, u \, \| \leq M \\
&|Du| - \theta \cdot Du = 0 \, \text{in} \, \Omega \\
&u = u_0 \text{ in } B, \\
&\theta \cdot \nu^\Omega|_{\partial\Omega} = \theta_0 \cdot \nu^\Omega|_{\partial\Omega},
\end{aligned} \tag{1.8}$$

where $p > 1$, $\gamma > 0$, $\beta \geq 0$, k denotes a regularizing kernel of class C^1 such that $k(x) > 0$ a.e., $M = \|u_0\|_{L^\infty(B)}$, θ_0 is any vector field in $D \setminus \tilde{\Omega}$ such that $\theta_0 \cdot Du_0 = |Du_0|$,

and ν^Ω denotes the outer unit normal to Ω. The convolution of Du with the kernel k in (1.8) is necessary to be able to prove the existence of a minimum of (1.8). The authors proved that the functional is coercive and admits a minimum in a suitable class of admissible functions.

Our purpose in this paper is to study a variant of problem (1.8), namely

$$\begin{aligned} &\text{Minimize} \int_\Omega |div(\theta)|^p(\gamma+\beta|\nabla k * u|)\,dx + \\ &\qquad + \alpha\int_\Omega |Du| - \alpha\int_{\partial\Omega} g_0 u + \lambda\int_B |u-u_0|^q dx \\ &|\theta|\le 1, \quad |Du| - \theta\cdot Du = 0 \,\text{in}\,\Omega, \\ &\theta\cdot\nu^\Omega|_{\partial\Omega} = g_0, \end{aligned} \tag{1.9}$$

where $g_0 = \theta_0\cdot\nu^\Omega|_{\partial\Omega}$, $\gamma,\alpha,\lambda>0$, $\beta\ge 0$, $q\ge 1$, and θ_0 is a vector field in $D\setminus\tilde\Omega$ such that $\theta_0\cdot Du_0 = |Du_0|$ and the trace $\theta_0\cdot\nu^\Omega|_{\partial\Omega}$ exists (for instance, if $div\,\theta_0\in L^p(D\setminus\tilde\Omega)$). We have relaxed the condition $u=u_0$ in B to the integral term $\int_B |u-u_0|^q\,dx$. This is the main difference with (1.8) besides the inclusion of the total variation on the functional. One of the main reasons for this is the possibility to approximate (1.9) by the problems

$$\begin{aligned} &\text{Minimize} \int_\Omega \Big|div\Big(\frac{Du}{\sqrt{\epsilon^2+|Du|^2}}\Big)\Big|^p(\gamma+\beta|\nabla k * u|)dx + \\ &\qquad + \alpha\int_\Omega |Du| - \alpha\int_{\partial\Omega} g_0 u + \lambda\int_B |u-u_0|^q dx \\ &\frac{Du}{\sqrt{\epsilon^2+|Du|^2}}\cdot\nu^\Omega = \frac{Du_0}{\sqrt{\epsilon^2+|Du_0|^2}}\cdot\nu^\Omega, \end{aligned} \tag{1.10}$$

in the sense that the minimizers of (1.10) converge (modulo a subsequence) to a minimum of (1.9) $\epsilon\to 0+$. For that, we shall prove first the existence of minimizers for both problems and study the two operators $div\Big(\frac{Du}{|Du|}\Big)$ and $div\,\Big(\frac{Du}{\sqrt{\epsilon^2+|Du|^2}}\Big)$ which appear in (1.9) and (1.10) respectively. Notice that the convergence of minima of (1.10) to minima of (1.9) establishes a connection between the numerical appproach of T. Chan and J Shen [28] and ours. After this mathematical discussion we shall present the numerical algorithm used to minimize (1.9) and we shall display some numerical experiments in $2D$ and $3D$ dimensions. Finally, let us say that both names disocclusion and inpainting have been used in the literature; since our approach is again a relaxed formulation of the elastica we shall refer to it as a disocclusion model.

All the above models were based on the elastica, a different approach has been taken in [19, 20] where the authors proposed a propagation model which, when written as a partial differential equation, coincides with Navier-Stokes equation for an incompressible fluid in 2D [20] plus a term of mean curvature diffusion. In this model, the image is identified with the stream function. The Navier-Stokes term (written in terms of the stream function) seems to propagate the information along the isophotes and sends the information inside the hole while the curvature diffusion tends to straight up the isophotes.

In a recent work [21], M. Bertalmio, L. Vese, G. Sapiro and S. Osher proposed a model for separate geometry and texture inpainting. The authors propose to use a recently introduced model for structure and texture separation by L. Vese and S. Osher [72] in combination with any successful disocclusion/inpainting and texture synthesis models to fill-in the information on the hole. The authors used the texture synthesis model given in [32] and the Navier-Stokes inpainting model [20, 19] in their experiments but they suggest that any other successful model could be used instead with the same purpose. The strategy of separating both geometry and texture seems to improve the existing models [21].

Let us explain the plan of the paper. Section 2 contains some basic definitions about functions of bounded variation and vector fields. We include also some preliminary results about the operator $div\left(\frac{Du}{|Du|}\right)$ with Neumann type boundary conditions which will permit us to introduce our model. In Section 3 we introduce the energy functional we shall use for disocclusion (1.9) which can be also interpreted as a relaxation of the Elastica. In Section 4 we prove the existence of minimizers of (1.9). In Section 5 we prove the convergence (after subsequence extraction) of the minima of the functionals (1.10) to a minimum of the functional in problem (1.9). For that we need to prove the closedness and maximal accretivity of the operator $div\,\frac{\nabla u}{\sqrt{\epsilon^2+|\nabla u|^2}}$ in L^p for any $p \geq 1$. Section 6 contains a review of some regularity results for quasi minimizers of the perimeter which apply to prove the regularity of the level sets of the solution of (1.9). Section 7 contains the numerical experiments and a description of the algorithm we used for them.

2. Preliminaries. Let us first recall the definition of BV functions and total variation. Let Q be an open subset of R^N. A function $u \in L^1(Q)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Q is called a function of bounded variation. The class of such functions will be denoted by $BV(Q)$. Thus $u \in BV(Q)$ if and only if there are Radon measures $\mu_1, \ldots, \mu_N$ defined in Q with finite total mass in Q and

$$\int_Q u D_i \varphi dx = -\int_Q \varphi d\mu_i \tag{2.1}$$

for all $\varphi \in C_0^\infty(Q)$, $i = 1, \ldots, N$. Thus the distributional gradient of u is a vector valued measure with finite total variation

$$\| Du \| = \sup\{\int_Q u\, div\, \varphi\, dx : \varphi \in C_0^\infty(Q, R^N), |\varphi(x)| \leq 1 \text{ for } x \in Q\}. \tag{2.2}$$

The space $BV(Q)$ is endowed with the norm

$$\| u \|_{BV} = \| u \|_{L^1(Q)} + \| Du \| . \tag{2.3}$$

As usual, we shall denote by $\mathcal{H}^{N-1}$ the $N-1$ dimensional Haussdorf measure in R^N. The Lebesgue measure in R^N will be denoted by λ_N.

We say that a measurable set $E \subseteq Q$ has *finite perimeter* in Q if its indicator function $\chi_E \in BV(Q)$. If $u \in BV(Q)$ almost all its level sets $[u \geq \lambda] = \{x \in Q : u(x) \geq \lambda\}$ are sets of finite perimeter. For sets of finite perimeter E one can define the essential boundary $\partial^* E$, which is rectifable with finite $\mathcal{H}^{N-1}$ measure, and compute the normal to the level set at $\mathcal{H}^{N-1}$ almost all points of $\partial^* E$. Thus at almost all

points of almost all level sets of $u \in BV(Q)$ we may define a normal vector $\theta(x)$ which coincides $|Du|$-a.e. with the Radon-Nikodym derivative of the measure Du with respect to $|Du|$, hence it formally satisfies $\theta \cdot Du = |Du|$ (and also $|\theta| \leq 1$ a.e.) [2], 3.9. For further information concerning functions of bounded variation we refer to [2, 35, 77].

Next, we shall give a sense to the integrals of bounded vector fields with divergence in L^p integrated with respect to the gradient of a BV function. For that, we shall need some results from [11] (see also [47] and [31]). Let Q be an open bounded subset of R^N with Lipschitz continuous boundary. Let $p \geq 1$ and $p' \geq 1$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Following [11], let

$$X(Q)_p = \{z \in L^\infty(Q, R^N) : div(z) \in L^p(Q)\}. \tag{2.4}$$

If $z \in X(Q)_p$ and $w \in BV(Q) \cap L^{p'}(Q)$ we define the functional $(z, Dw) : C_0^\infty(Q) \to R$ by the formula

$$< (z, Dw), \varphi >= -\int_Q w\, \varphi\, div(z)\, dx - \int_Q w\, z \cdot \nabla\varphi\, dx. \tag{2.5}$$

Then (z, Dw) is a Radon measure in Q,

$$\int_Q (z, Dw) = \int_Q z \cdot \nabla w\, dx \tag{2.6}$$

for all $w \in W^{1,1}(Q) \cap L^p(Q)$, and

$$\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw| \tag{2.7}$$

for any Borel set $B \subseteq Q$. By writing

$$z \cdot D^s u := (z, Du) - (z \cdot \nabla u)\ d\lambda_N, \tag{2.8}$$

we see that $z \cdot D^s u$ is a bounded measure, furthermore, in [47], it is proved that $z \cdot D^s u$ is absolutely continuous with respect to $|D^s u|$ (and thus is singular) and

$$|z \cdot D^s u| \leq \|z\|_\infty |D^s u|. \tag{2.9}$$

If no confusion arises, we shall indifferently write $z \cdot Dw$ or (z, Dw) for $z \in X(Q)_p$, $w \in BV(Q) \cap L^{p'}(Q)$.

In [11], a weak trace on ∂Q of the normal component of $z \in X(Q)_p$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(Q)_p \to L^\infty(\partial Q)$ such that

$$\|\gamma(z)\|_\infty \leq \|z\|_\infty$$

$$\gamma(z)(x) = z(x) \cdot \nu^Q(x) \quad \text{for all } x \in \partial Q \text{ if } z \in C^1(\overline{Q}, R^N),$$

where $\nu^Q(x)$ denotes the outer unit normal at $x \in \partial Q$. We shall denote $\gamma(z)(x)$ by $z \cdot \nu^Q(x)$. Moreover, the following *Green's formula*, relating the function $z \cdot \nu^Q$ and the measure (z, Dw), for $z \in X(Q)_p$ and $w \in BV(Q) \cap L^{p'}(Q)$, is established:

$$\int_Q w\ div(z)\ dx + \int_Q (z, Dw) = \int_{\partial Q} z \cdot \nu^Q w\ d\mathcal{H}^{N-1}. \tag{2.10}$$

For any $p \geq 1$, let us define the space

$$\mathcal{E}_p(Q) = \{(u,z) : u \in BV(Q), z \in X(Q)_p, |z| \leq 1, (z, DT_k(u)) = |DT_k(u)|, \forall k > 0\}.$$

Since $BV(Q) \subseteq L^{N/(N-1)}(Q)$, if $p \geq N$, then

$$\mathcal{E}_p(Q) = \{(u,z) : u \in BV(Q), z \in X(Q)_p, |z| \leq 1, (z, Du) = |Du|\}.$$

We may define the corresponding space if we require that $z \in L^\infty(Q, R^N)$ and $div(z)$ is a finite Radon measure in Q, but this space, more adapted to the decription of images will not be used in the sequel.

2.1. Definition of the operator $div\,(\frac{Du}{|Du|})$ with Neumann type boundary conditions. Assume that Ω is an open bounded set whose boundary is of class C^1. Let $g \in L^\infty(\partial\Omega)$ be such that $\|g\|_\infty < 1$. Let $T_k(r) = [k - (k - |r|)^+]sign_0(r)$, $k \geq 0$, $r \in R$. Later we shall need to consider a more general set of truncature functions, concretely, the set $\mathcal{P}$ of all nondecreasing Lipschitz-continuous fuctions $p : R \to R$, such that $p'(s) \in \{0, 1\}$ and $\{r \in R \ : \ p'(r) = 1\} = \cup_{j=1}^m]a_j, b_j[$, $a_1 < b_1 < a_2 < \ldots < a_m < b_m$.

We need to consider the function space

$$TBV(\Omega) := \{u \in L^1(\Omega) \ : \ T_k(u) \in BV(\Omega), \ \ \forall\ k > 0\}.$$

Notice that the function space $TBV(\Omega)$ is closely related with the space $GBV(\Omega)$ of generalized functions of bounded variation introduced by E. Di Giorgi and L. Ambrosio ([30], see also [2]).

We shall define the operator $div\,(\frac{Du}{|Du|})$ with Neumann type boundary conditions in the space $L^1(\Omega)$. For that we shall use the graph notation, usual in the theory of accretive operators [18],[17]. Let $\mathcal{B}$ be the following operator in $L^1(\Omega) \times L^1(\Omega)$:

$$(u,v) \in \mathcal{B} \iff u, v \in L^1(\Omega), u \in TBV(\Omega)$$

$$\text{and there exists } \ z \in X(\Omega)_1 \ \text{ such that:}$$

$$-v = div\ z \qquad \text{in } \ \mathcal{D}'(\Omega), \tag{2.11}$$

$$z \cdot DT_k(u) = |DT_k(u)| \qquad \forall k > 0, \tag{2.12}$$

$$z \cdot \nu^\Omega = g \quad \mathcal{H}^{N-1} - \text{a.e.} \tag{2.13}$$

Let us define

$$\mathcal{B}_{(1,p)} = \mathcal{B} \cap (L^1(\Omega) \times L^p(\Omega)) \quad \text{and} \quad \mathcal{B}_p = \mathcal{B} \cap (L^p(\Omega) \times L^p(\Omega)).$$

Note that, since $BV(\Omega) \subseteq L^p(\Omega)$ for all $p \leq \frac{N}{N-1}$, we have that $\mathcal{B}_{(1,p)} \cap (BV(\Omega) \times L^p(\Omega)) = \mathcal{B}_p \cap (BV(\Omega) \times L^p(\Omega))$. We shall prove below that $\mathcal{B}_{(1,p)}$ and $\mathcal{B}_p$ are completely accretive.

The domain of $\mathcal{B}_{(1,p)}$, denoted by $Dom\,\mathcal{B}_{(1,p)}$ is the set of functions of $L^1(\Omega)$ such that there is a function $v \in L^p(\Omega)$ such that $(u,v) \in \mathcal{B}_{(1,p)}$. Let $\mathcal{B}_{(1,p)}u := \{v \in$

$L^p(\Omega) : (u,v) \in \mathcal{B}_{(1,p)}\}$, if $u \in Dom\,\mathcal{B}_{(1,p)}$, otherwise $\mathcal{B}_{(1,p)}u = \emptyset$. In a similar way we define the domain of $\mathcal{B}_p$ and $\mathcal{B}_p u$. It follows easily from the definition that the sets $\mathcal{B}_{(1,p)}u$, and $\mathcal{B}_p u$ are convex for any $u \in L^1(\Omega)$, respectively, for any $u \in L^p(\Omega)$. As a consequence of Proposition 2.2, the sets $\mathcal{B}_{(1,p)}u$, and $\mathcal{B}_p u$ are closed in $L^1(\Omega)$ and $L^p(\Omega)$, respectively. Let $\omega \in C(\overline{\Omega})$, $\omega(x) > 0$ for all $x \in \overline{\Omega}$. Let us consider the space $L^p(\Omega,\omega) = L^p(\Omega)$, endowed with the norm

$$\|u\|_{p,\omega} = \Big(\int_\Omega |u(x)|^p \omega(x)\,dx\Big)^{1/p}.$$

Let $u \in Dom\,\mathcal{B}_{(1,p)}$. Then, if $p > 1$, there is a unique vector $v \in \mathcal{B}_{(1,p)}u$ of minimal norm in $L^p(\Omega,\omega)$. We denote this vector by $\mathcal{B}^\omega_{(1,p)}u$. In a similar way we define $\mathcal{B}^\omega_p u$ for any $u \in Dom\,\mathcal{B}_p$.

Let us recall the following result whose proof can be found in [15].

PROPOSITION 2.1. *Let $g \in L^\infty(\partial\Omega)$, $\|g\|_\infty < 1$. Let $\Phi : L^2(\Omega) \to (-\infty,+\infty]$ be given by*

$$\Phi(u) = \begin{cases} \int_\Omega |Du| - \int_{\partial\Omega} gu & \text{if} \quad u \in BV(\Omega) \cap L^2(\Omega) \\ \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus (BV(\Omega) \cap L^2(\Omega)). \end{cases}$$

Then $\partial\Phi = \mathcal{B}_2$.

We note that the functional Φ is lower semicontinuous [37] (this result is also true if $\|g\|_\infty = 1$ [57]).

Recall that an operator $\mathcal{A}$ in $L^p(\Omega)$ is called completely accretive if

$$\int_\Omega (v_2 - v_1)p(u_2 - u_1)\,dx \geq 0, \qquad \text{for any } (u_1,v_1),(u_2,v_2) \in \mathcal{A} \text{ and any } p \in \mathcal{P}.$$

If $\mathcal{A}$ is completely accretive, then it is also accretive in $L^p(\Omega)$, i.e.

$$\int_\Omega (v_2 - v_1)\beta_p(u_2 - u_1)\,dx \geq 0, \qquad \text{for any } (u_1,v_1),(u_2,v_2) \in \mathcal{A},$$

where $\beta_p(r) = |r|^{p-1} sign_0(r)$, $sign_0(r) = sign(r)$ if $r \neq 0$, $sign_0(0) = 0$.

PROPOSITION 2.2. *The operator $\mathcal{B}_p$ is closed and completely accretive in $L^p(\Omega)$. Moreover, if $(u_n,v_n) \in \mathcal{B}_p$, $u_n \to u$ in $L^1(\Omega)$, $v_n \to v$ weakly in $L^1(\Omega)$, then $(u,v) \in \mathcal{B}_p$. Similar assertions hold for $\mathcal{B}_{1,p}$. In particular, $\mathcal{B}_p u$ (resp. $\mathcal{B}_{(1,p)}u$) is closed in $L^p(\Omega)$ for any $u \in Dom\,\mathcal{B}_p$ (resp. $u \in Dom\,\mathcal{B}_{(1,p)}$).*

Proof. Let $(u_1, -div\,\theta_1),(u_2,-div\,\theta_2) \in \mathcal{B}_p$, $p(r) \in \mathcal{P}$. Then

$$\begin{aligned} &-\int_\Omega (div\,\theta_1 - div\,\theta_2)p(T_k(u_1) - T_k(u_2))\,dx = \\ &= \int_\Omega (\theta_1 - \theta_2)\cdot Dp(T_k(u_1) - T_k(u_2)) - \int_{\partial\Omega} (\theta_1\cdot\nu^\Omega - \theta_2\cdot\nu^\Omega)p(T_k(u_1) - T_k(u_2)) \\ &= \int_\Omega (\theta_1 - \theta_2)\cdot Dp(T_k(u_1) - T_k(u_2)). \end{aligned}$$

Since $\theta_1 \cdot DT_k(u_1) = |DT_k(u_1)|$, $\theta_2 \cdot DT_k(u_2) = |DT_k(u_2)|$, $|\theta_1 \cdot DT_k(u_2)| \leq |DT_k(u_2)|$, and $|\theta_2 \cdot DT_k(u_1)| \leq |DT_k(u_1)|$, and following the steps in [7] we deduce that the right

hand side of the above expression is positive (see also the proof of Proposition 5.11 for the details in a similar computation). Letting $k \to \infty$ we obtain that $\mathcal{B}_p$ is completely accretive. The same proof gives us that $\mathcal{B}_{(1,p)}$ is completely accretive.

Now, let $(u_n, v_n) \in \mathcal{B}_p$ be such that $u_n \to u$ in $L^1(\Omega)$, $v_n \to v$ weakly in $L^1(\Omega)$. Let $f_n = u_n + v_n \to f = u + v$ weakly in $L^1(\Omega)$. Let $\theta_n \in L^\infty(\Omega, R^N)$, $|\theta_n| \leq 1$ be such that $\theta_n \cdot DT_k(u_n) = |DT_k(u_n)|$, $\theta_n \cdot \nu^\Omega = g$, and $v_n = -div\,\theta_n$. By extracting a subsequence, if necessary, we may assume that $\theta_n \to \theta$ weakly* in $L^\infty(\Omega, R^N)$ and $v = -div\,\theta$. Let φ be a smooth function in Ω, continuous up to $\partial\Omega$. We multiply $u_n - div\,\theta_n = f_n$ by φ and integrate by parts to obtain

$$\int_\Omega u_n\varphi + \int_\Omega \theta_n \cdot \nabla\varphi - \int_{\partial\Omega} \theta_n \cdot \nu^\Omega \varphi = \int_\Omega f_n\varphi. \tag{2.14}$$

Letting $n \to \infty$ and using that $\theta_n \cdot \nu^\Omega = g$, we obtain

$$\int_\Omega u\varphi + \int_\Omega \theta \cdot \nabla\varphi - \int_{\partial\Omega} g\varphi = \int_\Omega f\varphi. \tag{2.15}$$

Integrating by parts the second term of the above equality, we get

$$\int_\Omega u\varphi - \int_\Omega \operatorname{div}\theta\,\varphi + \int_{\partial\Omega} (\theta \cdot \nu^\Omega - g)\varphi = \int_\Omega f\varphi. \tag{2.16}$$

Now, using equation $u - div\,\theta = f$, it follows that

$$\int_{\partial\Omega} (\theta \cdot \nu^\Omega - g)\varphi = 0$$

for all test functions φ. This implies that

$$\theta \cdot \nu^\Omega = g \quad \text{on } \partial\Omega.$$

To prove that $\theta \cdot DT_k(u) = |DT_k(u)|$, we observe that

$$\begin{aligned}
\int_\Omega |DT_k(u)| - \int_{\partial\Omega} gT_k(u) &\leq \liminf_n \int_\Omega |DT_k(u_n)| - \int_{\partial\Omega} gT_k(u_n) \\
&= \liminf_n \int_\Omega \theta_n \cdot DT_k(u_n) - \int_{\partial\Omega} gT_k(u_n) \\
&= \liminf_n -\int_\Omega \operatorname{div}\theta_n\, T_k(u_n) + \int_{\partial\Omega} \theta_n \cdot \nu^\Omega T_k(u_n) - \int_{\partial\Omega} gT_k(u_n) \\
&= \liminf_n -\int_\Omega \operatorname{div}\theta_n\, T_k(u_n) = -\int_\Omega \operatorname{div}\theta\, T_k(u) \\
&= \int_\Omega \theta \cdot DT_k(u) - \int_{\partial\Omega} gT_k(u) \leq \int_\Omega |DT_k(u)| - \int_{\partial\Omega} gT_k(u).
\end{aligned}$$

We conclude that $\theta \cdot DT_k(u) = |DT_k(u)|$ for all $k > 0$. We have proved that $(u, v) \in \mathcal{B}_p$. The proofs for $\mathcal{B}_{(1,p)}$ are the same and we omit the details.□

3. Joint interpolation of vector fields and gray values. Let $u_0 : D \to R$ be an image defined on a domain D of R^N, $N \geq 2$, which we may suppose to be a hyperrectangle. Let $\Omega, \tilde{\Omega}$ be two open bounded domains in R^N with Lipschitz boundary and suppose that $\overline{\tilde{\Omega}} \subset\subset \Omega \subset\subset D$. To simplify our presentation we shall

FIG. 3.1. *The hole and the band*

assume that Ω does not touch the boundary of the image domain D. Let $B := \Omega \setminus \overline{\tilde{\Omega}}$. The set B will be called the band around $\tilde{\Omega}$ (see Figure 3.1). Suppose that a function u_0 is given in $D \setminus \tilde{\Omega}$ which, for the moment being, we shall assume to be smooth (later we shall assume that u_0 is of bounded variation, i.e., $u_0 \in BV(D \setminus \overline{\tilde{\Omega}})$). Let θ_0 be the vector field of directions of the gradient of u_0 on $D \setminus \tilde{\Omega}$, i.e., θ_0 is a vector field with values in R^2 satisfying $\theta_0(x) \cdot \nabla u_0(x) = |\nabla u_0(x)|$ and $|\theta_0(x)| \leq 1$. We shall assume that $\theta_0(x)$ has a trace on $\partial\Omega$.

We pose the image disocclusion problem in the following form: Can we extend (in a reasonable way) the pair of functions (u_0, θ_0) from the band $\Omega \setminus \overline{\tilde{\Omega}}$ to a pair of functions (u, θ) defined inside $\tilde{\Omega}$? Of course, we will have to precise what we mean by a reasonable way. We shall discuss and analyze a variant of the variational formulation of the disocclusion problem introduced in [12] and study its approximation with more regular functionals which have a direct interpretation.

The data u_0 is given on the band B and we should constrain the solution u to be near the data on B. The vector field θ should satisfy $\theta \cdot \nu^\Omega = \theta_0 \cdot \nu^\Omega$, $|\theta| \leq 1$ on Ω and should be related to u by the constraint $\theta \cdot Du = |Du|$, i.e., we should impose that θ is related to the vector field of directions of the gradient of u. The condition $|\theta(x)| \leq 1$ should be interpreted as a relaxation of this. Indeed, it may happen that $\theta(x) = 0$ (flat regions) and then we cannot normalize the vector field to a unit vector (the ideal case would be that $\theta = \frac{Du}{|Du|}$, u being a smooth function with $Du(x) \neq 0$ for all $x \in \Omega$). Finally, we should impose that the vector field θ_0 in $D \setminus \overline{\Omega}$ is smoothly continued by θ inside Ω. Note that if θ represents the directions of the normals to the level lines of u, i.e., of the hypersurfaces $u(x) = \lambda$, $\lambda \in R$, then $div(\theta)$ represents its mean curvature. We shall impose the smooth continuation of the levels lines of u_0 inside Ω by requiring that $div\,\theta \in L^p(\Omega)$.

After this discussion, we can make precise the functional analytic model for u and θ. We shall assume that $u_0 \in BV(D \setminus \overline{\tilde{\Omega}})$, and $\theta_0 : D \setminus \tilde{\Omega} \to R^N$ is the vector field of directions of the gradient of u_0, i.e., a vector field $\theta_0 \in L^\infty(D \setminus \tilde{\Omega}, R^N)$, such that $|\theta_0| \leq 1$ and

$$\begin{aligned} &div\,\theta_0 \in L^p(B) \\ &\theta_0 \cdot Du_0 = |Du_0| \quad \text{as measures in } B \text{ (therefore, a.e.).} \end{aligned} \tag{3.1}$$

We shall assume in the rest of the paper that Ω is a domain of class C^1. Let $g_0 = \theta_0 \cdot \nu^\Omega$. We shall assume that $\|g_0\|_\infty < 1$. This assumption will be used in Section 5 to prove the convergence (after subsequence extraction) of the minima of the functionals (1.10)

to a minimum of the functional in problem (1.9). Notice that it does not permit the level lines of the topographic map of the image to be tangent to the boundary of the hole $\tilde{\Omega}$, and to ensure it we may slightly change the topographic map by replacing the level lines near the tangent one by a constant gray level, which gives us more freedom to choose the vector field θ_0.

We define the space

$$\mathcal{E}_p(\Omega, B, \theta_0) = \{(u,\theta) \in \mathcal{E}_p(\Omega),\, u|_B \in L^q(B),\, \theta \cdot \nu^\Omega = g_0 \text{ on } \partial\Omega \,\}$$

If $(u,\theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$ we define

$$\begin{aligned} E_p(u,\theta) = &\int_{\tilde{\Omega}} |div(\theta)|^p(\gamma + \beta|\nabla k * u|)dx \\ &+ \alpha \int_{\tilde{\Omega}} |Du| - \alpha \int_{\partial\Omega} g_0 u + \lambda \int_B |u - u_0|^q \, dx \end{aligned} \tag{3.2}$$

where $\gamma, \alpha, \lambda > 0$, $\beta \geq 0$, $p > 1$, $q \geq 1$, and k denotes a regularizing kernel of class C^1 such that $k(x) > 0$ a.e..

We propose to interpolate the pair (θ, u) in Ω by solving the minimization problem

$$\begin{aligned} &\text{Minimize } E_p(u,\theta) \\ &(u,\theta) \in \mathcal{E}_p(\Omega, B, \theta_0) \end{aligned} \tag{3.3}$$

We shall prove that this functional is coercive and admits a minimum in the class of functions described above if $p > 1$. The case $p = 1$ is is particularly interesting but is not covered by our results. The functional can be interpreted as a formulation of the principle of good continuation and amodal completion as formulated in the Gestalt's theory of vision. We shall prove that the minima of functionals (1.10) converge (modulo a subsequence) to a minimum of (3.2). Before going into the proofs let us discuss in more detail the main features of the model.

Remark. Observe that for any $1 < p < \infty$ we have

$$(u,\theta) \in \mathcal{E}_p(\Omega, B, \theta_0) \text{ if and only if } (u, -div\,\theta) \in \mathcal{B}_{(1,p)},\, u \in BV(\Omega),\, u|_B \in L^q(B)$$

and, if $p \leq \frac{N}{N-1}$, we may write $\mathcal{B}_p$ instead of $\mathcal{B}_{(1,p)}$.

One of the key tricks above is the band around the hole. The band is of local character but in principle it could be extended to all the known part of the image. Obviously, what happens at distant parts can be independent or not from what happens at the hole, but, in our construction below, we suppose that only a narrow band around the hole influences what happens inside the hole. Could we fill-in the hole without the band ? To discuss this suppose that we are given the image of Figure 3.2, left, which is a gray band on a black background partially occluded by a square $\tilde{\Omega}$. We suppose that the sides of the square hole $\tilde{\Omega}$ are orthogonal to the level lines of the original image. In these conditions, the normal component of the vector field θ_0 outside $\tilde{\Omega}$ is null at $\partial\tilde{\Omega}$. Thus if the boundary data is just $\theta_0 \cdot \nu^{\tilde{\Omega}}|_{\partial\tilde{\Omega}}$, we would have that $\theta_0 \cdot \nu^{\tilde{\Omega}}|_{\partial\tilde{\Omega}} = 0$. In particular, the vector field $\theta = 0$ satisfies this condition. If we are not able to propagate θ inside $\tilde{\Omega}$ this may become an unpleasant situation, since this would mean that we do no propagate the values of u at the boundary. If

FIG. 3.2. *Left: a strip with a hole. Middle: image disocclusion obtained using Total Variation. Right: Image disocclusion obtained using functional (3.2).*

we write the functional (3.2) with $\theta = 0$, $\alpha = 1$, it turns out to be the Total Variation [61]. The decision of extending the gray band or filling-in the hole with the black gray level would be taken as a function of the perimeter of the discontinuities of the function in the hole. Then the result of interpolating Figure 3.2, left, using Total Variation would be that of Figure 3.2, middle, and not the one in Figure 3.2, right, because the interpolating lines in Figure 3.2, middle, are shorter than the ones in Figure 3.2, right. To overcome this situation we introduce the band around the hole. The introduction of the band permits us to effectively incorporate in the functional the information given by the data u_0 and the vector field θ outside $\tilde{\Omega}$. In Figure 3.2, middle, we display the result of the interpolation with $\theta = 0$ on $\tilde{\Omega}$. In Figure 3.2, right, we display the result of the interpolation using (3.2), which takes into account the band B and computes the vector field θ in Ω.

The following remarks contain heuristic arguments which may help to understand our choice.

Remarks.

1. If u is the characteristic function of the region enclosed by a curve C then the terms

$$\beta \int_\Omega |div(\theta)|^p |Du| + \alpha \int_\Omega |Du| \tag{3.4}$$

are related to $\int_C (\alpha + \beta|\kappa|^p) ds$, where κ is the Euclidean curvature (of the level-sets). If $p = 2$, this coincides with Euler's elastica,

$$\int_C (\alpha + \beta\kappa^2) ds, \quad \alpha, \beta > 0. \tag{3.5}$$

Euler's elastica (3.5) was proposed in [59] as a technique for removing occlusions with the goal of image segmentation, since this criterion yields smooth, short, and not too curvy curves. In terms of characteristic functions, Euler's elastica can be written as

$$\int |\nabla u| \left(\alpha + \beta \left| div \left(\frac{\nabla u}{|\nabla u|} \right) \right|^2 \right). \tag{3.6}$$

In [14], it was shown that the elastica functional is not lower semicontinuous. As shown in [4], the functional proposed by Masnou and Morel [52, 53, 54]

can be interpreted as a relaxation of it, since it integrates functionals like the elastica along the level lines of the function u. Our functional can be also considered as a relaxed formulation of the energy of the elastica. For that, we introduced θ as a independent variable, and we tried to couple it to u by imposing that $\theta \cdot Du = |Du|$. This restriction could be also incorporated as a penalization term. Finally, let us say that for mathematical reasons we have convolved the Du term of (3.4) to be able to prove the existence of a minimum for (3.2). From a theoretical point of view, this may invalidate our previous comments. But, from a practical point of view, it gives a weight to the curve of discontinuities of the image.

2. The constant γ has to be > 0. Otherwise we do not get an L^p bound on $div(\theta)$. Both coefficients γ and β are required to be > 0 (even if the existence of minimizers can be proved for $\beta = 0$). Indeed, in a heuristic way, if we do not compute θ in a proper way, in an image like Figure 3.2, θ could be zero except on some curves. Then $\theta = 0$ a.e. on B (or on Ω) and a term like

$$\int_\Omega |div(\theta)|^p dx \tag{3.7}$$

would produce a null value since $div(\theta) = 0$. On the other hand, the term corresponding to the coefficient $\beta > 0$ would integrate a power of the curvature on the level line corresponding to the boundary of the object and it would guarantee that the functional is not null.

3. Given the image u_0, to construct a vector field θ_0 in a Lipschitz domain Q such that $div(\theta_0) \in L^p(Q)$ and $\theta_0 \cdot Du_0 = |Du_0|$ we may use the equation

$$\begin{aligned} &\frac{\partial u}{\partial t} = div\left(\frac{Du}{|Du|}\right) \qquad \text{in } (0,\infty)\times Q \\ &u(0,x) = u_0(x) \qquad \text{for } x \in Q, \end{aligned} \tag{3.8}$$

with Neumann or Dirichlet boundary conditions. As it is shown in [7, 8], this equation permits a regularization of the vector field of directions of the gradient of u, i.e., there is a vector field z, $|z| \le 1$, such that $u_t = div(z)$ and $\int_Q z \cdot Du = \int_Q |Du|$. Moreover, for each $t > 0$, $div(z(t)) \in L^p(Q)$ if $u_0 \in L^p(Q)$ for all $p \ge 1$. This regularized vector field has a normal trace at the boundary of Q.

4. We have incorporated the constraint that u is near the data u_0 in B via the penalty term $\int_B |u - u_0|^q$. This type of approach was followed in the work of T. Chan and J. Shen mentioned before [26]. Similarly, we could add a penalty term to constraint θ to be near θ_0 inside B.

5. In practice, functional (3.2) is used to interpolate shapes, i.e., to interpolate level sets. The image is decomposed into upper level sets $[u_0 \ge \lambda]$, which are interpolated using (3.2) to produce the level sets $X_\lambda u$ of a function u, which is reconstructed inside Ω by using the reconstruction formula

$$u(x) = \sup\{\lambda : x \in X_\lambda u\}.$$

To guarantee that the reconstructed level sets correspond to the level sets of a function u, they should satisfy that $X_{\lambda+1}u \subseteq X_\lambda u$. In practice, we force our solution to satisfy this property.

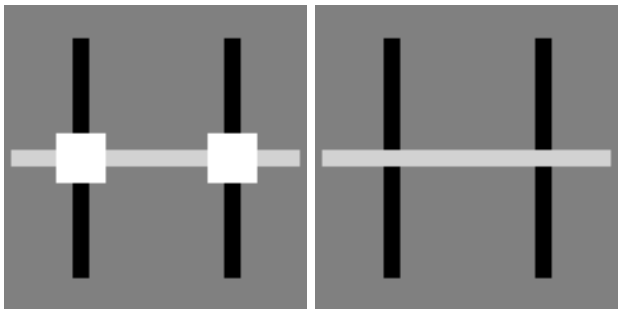

FIG. 3.3. *Left: a double cross with holes. Right: reconstructed image using functional (3.2). Observe that due to our choice of upper level sets to decompose and reconstruct the image, the white bar goes above the black ones.*

In principle, our functional (3.2) could be used directly to interpolate functions. But, discontinuities of the image have a contribution to the energy which is proportional to the jump. This gives different weights to discontinuities of different sizes and, as a consequence, they are not treated in the same manner. This is not reasonable if we want to interpolate the shapes of the image, independently of their contrast. When taking level sets, we treat all shapes equally, and the parameters of the functional should only weight geometric quantities (like length, total curvature) and decide which interpolation is taken as a function of them. This approach is less diffusive than directly interpolating the gray levels. Theorem 4.1 in Section 4 proves the existence of minimizers for our model and can be applied to both cases, binary and gray level images. In Section 5 we shall approximate (3.2) by functionals (1.10) which have a direct geometric interpretation.

6. The choice made in Remark 5 of decomposing the image u_0 into upper level sets, interpolating them and reconstructing the function u, introduces a lack of symmetry. Indeed, we are giving more weight to upper level sets than to lower level sets. This can be seen in Figure 3.3. Figure 3.3, left, displays the image to be interpolated. It is clear that several reasonable solutions are possible and no one of them is preferable to the others. The choice we made gives Figure 3.3, right, as solution, favoring that the object whose level is 210 goes above the object whose level is 0. But, in that case, the "true" information is lacking and we selected one of the possible reasonable solution.

4. Existence of minimizers. Let us first recall a couple of inequalities that we shall use at several places below. We are assuming that Ω is an open bounded set in R^N such that $\partial\Omega$ is of class C^1, and $g \in L^\infty(\Omega)$ with $\|g\|_\infty < 1$.

For $x \in \partial\Omega$ we define

$$q(x) = \limsup_{r\to 0+} \Big\{ \frac{\int_{\partial\Omega} \chi_A \, d\mathcal{H}^1}{P(A,\Omega)} : A \subseteq B(x,r),\ |A| > 0 \Big\}. \tag{4.1}$$

Observe that, since $\partial\Omega$ is of class C^1, we have $q(x) = 1$ for all $x \in \partial\Omega$ [37].

Since $\| gq \|_\infty =: 1 - 2\sigma < 1$, there is a constant c depending on σ, g, Ω, such that

$$\Big| \int_{\partial\Omega} gw \Big| \le (1-\sigma) \int_\Omega |Dw| + c \int_\Omega |w| \tag{4.2}$$

for all $w \in BV(\Omega)$ ([37], Lemma 1.2).

On the other hand, by [37], Lemma 2.2, there is some $\epsilon_0 > 0$ such that for each $\delta > 0$ we may find a constant $c(\delta) > 0$ such that

$$\Big|\int_{\partial\Omega} gw\Big| \leq (1-\epsilon_0)\int_{S_\delta} |Dw| + c(\delta)\int_{S_\delta} |w| \tag{4.3}$$

for all $w \in BV(\Omega)$, where $S_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$.

THEOREM 4.1. *If $p > 1$, $q \geq 1$, $\gamma, \alpha, \lambda > 0$, and $\beta \geq 0$, then there is a minimum $(u, \theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$ for the problem (3.3).*

Proof. Let (u_n, θ_n) be a minimizing sequence for $E_p(u, \theta)$ in $\mathcal{E}_p(\Omega, B, \theta_0)$. Since $E_p(u_n, \theta_n)$ is bounded, we obtain that

$$\int_\Omega |div(\theta_n)|^p, \quad \int_\Omega |Du_n| - \int_{\partial\Omega} gu_n, \quad \text{and} \int_B |u_n - u_0|^q dx$$

are bounded. Then, using (4.3), we have

$$\int_\Omega |Du_n| \leq C + \int_{\partial\Omega} gu_n \leq C + (1-\epsilon_0)\int_{S_\delta} |Du_n| + c(\delta)\int_{S_\delta} |u_n|$$

for some constants $C, c(\delta), \epsilon_0 > 0$. Taking δ small enough so that $S_\delta \subseteq B$ we have

$$\epsilon_0 \int_\Omega |Du_n| \leq C + c(\delta)\int_B |u_n| \leq C'$$

for some constant $C' > 0$. Since $|\theta_n| \leq 1$, we have that θ_n is weakly* relatively compact in $L^\infty(\Omega, R^N)$ and we may assume that $\theta_n \to \theta$ weakly* in $L^\infty(\Omega, R^N)$, and in $div\,\theta_n \to div\,\theta$ weakly in $L^p(\Omega)$. On the other hand, since $\int_\Omega |Du_n|$, and $\int_B |u_n - u_0|$ are bounded, by extracting a subsequence we may assume that u_n converges to some function $u \in BV(\Omega)$ in $L^r(\Omega)$ for all $r \in [1, \frac{N}{N-1})$. In particular, we have that $\nabla k * u_n \to \nabla k * u$ uniformly in Ω, and we obtain

$$\int_\Omega |div(\theta)|^p(\gamma + \beta|\nabla k * u|)dx \leq \liminf_n \int_\Omega |div(\theta_n)|^p(\gamma + \beta|\nabla k * u_n|)dx,$$

$$\int_\Omega |Du| - \int_{\partial\Omega} g_0 u \leq \liminf_n \int_\Omega |Du_n| - \int_{\partial\Omega} g_0 u_n,$$

and

$$\int_B |u - u_0|^q dx \leq \liminf_n \int_B |u_n - u_0|^q.$$

On the other hand, by Proposition 2.2, we know that $(u, -div\,\theta) \in \mathcal{B}_{(1,p)}$, hence $(u, \theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$. Thus, collecting all these facts, we have proved that $E_p(u, \theta) \leq \liminf_n E_p(u_n, \theta_n)$ and the pair (u, θ) is a minimum of E_p in $\mathcal{E}_p(\Omega, B, \theta_0)$.□

Note that if $p > N$, we have $\theta \cdot Du = |Du|$.

5. Approximation of the functional (3.2). Our purpose in this Section is to prove that from any sequence $\{u_\epsilon\}_\epsilon$ of mimima of the the family of functionals (1.10) (as ϵ varies) we may extract a subsequence converging to a minimum of the functional (3.2).

5.1. The operator $div\,\frac{\nabla u}{\sqrt{\epsilon^2+|\nabla u|^2}}$ in L^p. We shall follow the techniques introduced in [10, 9]. We shall need the function space $TBV(\Omega)$ and to give a sense to the Radon-Nikodym derivative ∇u (with respect to the Lebesgue measure) of a function $u \in TBV(\Omega)$. In [16] a similar problem is treated to give sense to the derivative of functions for which their truncatures are in a Sobolev space (in their notation, for functions in $\mathcal{T}^{1,p}_{loc}(\Omega)$, $p \geq 1$). Using chain's rule for BV-functions (see for instance [2]), with a similar proof to the one given in Lemma 2.1 of [16], we obtain the following result.

LEMMA 5.1. *For every $u \in TBV(\Omega)$ there exists a unique measurable function $v : \Omega \to R^N$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|<k\}} \qquad \lambda_N - a.e \tag{5.1}$$

Thanks to this result we define ∇u for a function $u \in TBV(\Omega)$ as the unique function v which satisfies (5.1). This notation will be used throughout in the sequel.

Let $a_\epsilon(\xi) = \frac{\xi}{\sqrt{\epsilon^2+|\xi|^2}}$, $h_\epsilon(\xi) = a_\epsilon(\xi)\cdot\xi$, $f_\epsilon(\xi) = \sqrt{\epsilon^2+|\xi|^2}$, $\xi \in R^N$. If $u \in TBV(\Omega)$ we may define $a_\epsilon(\nabla u)$. Observe that $\| a_\epsilon(\nabla u) \|_\infty \leq 1$.

Let $p \geq 1$, $\epsilon > 0$. Recall that we are assuming that Ω is an open bounded subset of R^N with C^1 boundary. Let $g \in L^\infty(\partial\Omega)$. Assume that $\| g \|_\infty < 1$. Let us define the operator $\mathcal{A}^{\epsilon,p}_g$ on $L^p(\Omega)\times L^p(\Omega)$:

$$(u,v) \in \mathcal{A}^{\epsilon,p}_g \iff u, v \in L^p(\Omega), u \in TBV(\Omega)$$

$$\text{and } a_\epsilon(\nabla u) \in X(\Omega)_p \text{ satisfies :}$$

$$-v = div\ a_\epsilon(\nabla u) \qquad \text{in } \mathcal{D}'(\Omega), \tag{5.2}$$

$$a_\epsilon(\nabla u)\cdot D^s p(u) = |D^s p(u)| \qquad \forall p \in \mathcal{P}, \tag{5.3}$$

$$a_\epsilon(\nabla u)\cdot \nu^\Omega = g \quad \mathcal{H}^{N-1} - \text{a.e.} \tag{5.4}$$

The domain of $\mathcal{A}^{\epsilon,p}_g$ is the set of functions $u \in L^p(\Omega)$ such that there is a function $v \in L^p(\Omega)$ such that $(u,v) \in \mathcal{A}^{\epsilon,p}_g$. Let $\mathcal{A}^{\epsilon,p}_g u := \{v \in L^p(\Omega) : (u,v) \in \mathcal{A}^{\epsilon,p}_g\}$.

LEMMA 5.2. *Let $u \in Dom\,\mathcal{A}^{\epsilon,p}_g$. Then we have*

$$f_\epsilon(Dp(u)) = \frac{\epsilon^2}{\sqrt{\epsilon^2+|\nabla p(u)|^2}} + a_\epsilon(\nabla u)\cdot Dp(u) \tag{5.5}$$

for any $p(r) \in \mathcal{P}$.

To prove Lemma 5.2, let us prove first the following Lemma.

LEMMA 5.3. *Let $Q \subseteq R^N$ be an open set. Let $u \in BV(Q)$, $a_\epsilon(\nabla u) = \frac{\nabla u}{\sqrt{\epsilon^2+|\nabla u|^2}}$. Then $a_\epsilon(\nabla u)$ is characterized as the unique vector field $\theta_\epsilon \in L^\infty(Q, R^N)$ such that $\|\theta_\epsilon\|_\infty \leq 1$ and*

$$\sqrt{\epsilon^2+|\nabla u|^2} - \epsilon\sqrt{1-|\theta_\epsilon|^2} = \theta_\epsilon\cdot\nabla u \quad a.e. \tag{5.6}$$

Lemma 5.3 follows as a consequence of the corresponding scalar result.

LEMMA 5.4. *Let* $f(v) = \sqrt{\epsilon^2 + |v|^2}$, $v \in R^N$. *If* $f^*(\xi) = \sup\{< v, \xi > -f(v) : v \in R^N\}$, $\xi \in R^N$, *then* $f^*(v) = -\epsilon\sqrt{1 - |\xi|^2}$ *if* $|\xi| \leq 1$, $f^*(\xi) = +\infty$ *if* $|\xi| > 1$. *We have*

(i) if $v \in R^N$ *and* $\xi = \frac{v}{\sqrt{\epsilon^2+|v|^2}}$ *then*

$$f(v) + f^*(\xi) = \xi \cdot v \tag{5.7}$$

(ii) if $\xi, v \in R^N$, $|\xi| \leq 1$, *satisfy (5.7) then* $\xi = \frac{v}{\sqrt{\epsilon^2+|v|^2}}$.

Proof. We leave the proof of (i) and the computation of f^* as an exercice. If $v, \xi \in R^N$ satisfy (5.7) then $f^*(\xi) < \infty$ and, thus, $|\xi| \leq 1$. If $v = 0$, then $\xi = 0$. Thus we may assume that $v \neq 0$. Let $P_v = \frac{v \otimes v}{|v|^2}$. Observe that

$$f(v) + f^*(P_v(\xi)) \leq f(v) + f^*(\xi) =< \xi, v >=< P_v(\xi), v >\leq f(v) + f^*(P_v(\xi))$$

the last inequality being always true. Thus $f^*(P_v(\xi)) = f^*(\xi)$, and we obtain that $|P_v(\xi)| = |\xi|$, which in turn implies that

$$| < \xi, v > | = |\xi||v|.$$

If $\xi = 0$, then $v = 0$. Thus we may assume that $\xi \neq 0$. Then we deduce that $\xi = \lambda v$, $\lambda \neq 0$. Introducing this value in (5.7) we obtain

$$\lambda = \pm \frac{1}{\sqrt{\epsilon^2 + |v|^2}}.$$

Finally, a direct computation shows that only the positive sign permits to satisfy (5.7).□

Proof of Lemma 5.2. Let $p(r) \in \mathcal{P}$, and denote $A = \{r : p'(r) = 1\}$, $B = R \setminus A$. Recall that $f_\epsilon(Dp(u)) = \sqrt{\epsilon^2 + |\nabla p(u)|^2} + |D^s p(u)|$. Using (5.6) we may write

$$\begin{aligned}
\sqrt{\epsilon^2 + |\nabla p(u)|^2} &= \sqrt{\epsilon^2 + p'(u)|\nabla u|^2} = \sqrt{\epsilon^2 + |\nabla u|^2}\chi_A(u) + \epsilon\chi_B(u) \\
&= \epsilon\sqrt{1 - |a_\epsilon(\nabla u)|^2}\chi_A(u) + a_\epsilon(\nabla u) \cdot \nabla u\, p'(u) + \epsilon\chi_B(u) \\
&= \epsilon\sqrt{1 - |a_\epsilon(\nabla u)|^2}\chi_A(u) + \epsilon\chi_B(u) + a_\epsilon(\nabla u) \cdot \nabla p(u) \\
&= \frac{\epsilon^2}{\sqrt{\epsilon^2 + |\nabla p(u)|^2}} + a_\epsilon(\nabla u) \cdot \nabla p(u).
\end{aligned}$$

Now, taking into account (5.3) we have

$$\begin{aligned}
f_\epsilon(Dp(u)) &= \sqrt{\epsilon^2 + |\nabla p(u)|^2} + |D^s p(u)| \\
&= \frac{\epsilon^2}{\sqrt{\epsilon^2 + |\nabla p(u)|^2}} + a_\epsilon(\nabla u) \cdot \nabla p(u) + a_\epsilon(\nabla u) \cdot D^s p(u) \\
&= \frac{\epsilon^2}{\sqrt{\epsilon^2 + |\nabla p(u)|^2}} + a_\epsilon(\nabla u) \cdot Dp(u).
\end{aligned}$$

□

LEMMA 5.5. *Let $(u_n, v_n) \in \mathcal{A}_g^{\epsilon,p}$ be such that $u_n \to u$ in $L^1(\Omega)$, v_n is bounded in $L^1(\Omega)$, and*

$$\|Dp(u_n)\| \tag{5.8}$$

is bounded for all $p \in \mathcal{P}$. Assume that $a_\epsilon(\nabla u_n) \to a_\epsilon(\nabla u)$ in the weak topology of $L^\infty(\Omega, R^N)$. Then*

$$\nabla u_n \to \nabla u \quad a.e.\ in\ \Omega. \tag{5.9}$$

Proof. Observe that we have the the following strict monotonicity condition on a_ϵ:

$$(a_\epsilon(\eta) - a_\epsilon(\xi)) \cdot (\eta - \xi) > 0 \quad \text{if} \ \ \xi \neq \eta. \tag{5.10}$$

Let us prove that $\{\nabla u_n\}$ is a Cauchy sequence in measure. To do that, we follow the same technique as in [22] (see also [10]). Let $t, \varepsilon > 0$. For $a > 1$, we set

$$C(x,a,t) := \inf\{(a_\epsilon(\xi) - a_\epsilon(\eta)) \cdot (\xi - \eta) \ : \ \|\xi\| \le a,\ \|\eta\| \le a,\ \|\xi - \eta\| \ge t\}.$$

Having in mind that the function $\xi \mapsto a_\epsilon(\xi)$ is continuous for almost all $x \in \Omega$, and the set $\{(\xi,\eta) \ : \ \|\xi\| \le a,\ \|\eta\| \le a,\ \|\xi - \eta\| \ge t\}$ is compact, the infimum in the definition of $C(x,a,t)$ is a minimum. Hence by (5.10), it follows that

$$C(x,a,t) > 0 \quad \text{for almost all} \ \ x \in \Omega. \tag{5.11}$$

For $n, m \in N$, and any $k > 0$, we have

$$\begin{aligned}
&\{\|\nabla u_n - \nabla u_m\| > t\} \subset \{\|\nabla T_a u_n\| \ge a^2\} \cup \{\|\nabla T_a u_m\| \ge a^2\} \\
&\cup \{|u_n| \ge a\} \cup \{|u_m| \ge a\} \cup \{|u_n - u_m| \ge k^2\} \cup \{C(x,a^2,t) \le k\} \\
&\cup \{|u_n - u_m| < k^2, |u_n| < a, |u_m| < a, C(x,a^2,t) \ge k, \\
&\|\nabla T_a u_n\| \le a^2, \|\nabla T_a u_m\| \le a^2, \|\nabla u_n - \nabla u_m\| > t\}.
\end{aligned} \tag{5.12}$$

Since $\{u_n\}$ is bounded in $L^1(\Omega)$ we can choose a large enough in order to have

$$\lambda_N\big(\{|u_n| \ge a\} \cup \{|u_m| \ge a\}\big) \le \frac{\varepsilon}{5} \qquad \text{for all} \ \ n, m \in N. \tag{5.13}$$

Similarly, by (5.8), we can choose a large enough in order to have

$$\lambda_N\big(\{\|\nabla T_a u_n\| \ge a^2\} \cup \{\|\nabla T_a u_m\| \ge a^2\}\big) \le \frac{\varepsilon}{5} \qquad \text{for all} \ \ n, m \in N. \tag{5.14}$$

Fixing a satisfying (5.13) and (5.14), by (5.11), taking k small enough, we have

$$\lambda_N\big(\{C(x,a^2,t) \le k\}\big) \le \frac{\varepsilon}{5}. \tag{5.15}$$

On the other hand, since $v_n = -div\, a_\epsilon(\nabla u_n)$, using Green's formula, for any $j > 0$ we have

$$\int_\Omega (a_\epsilon(\nabla u_n) - a_\epsilon(\nabla u_m), DT_r(T_j(u_n) - T_j(u_m))) = \int_\Omega (v_n - v_m) T_r(T_j(u_n) - T_j(u_m))\, dx$$

$$+\int_{\partial\Omega}(a_\epsilon(\nabla u_n)\cdot\nu^\Omega - a_\epsilon(\nabla u_m)\cdot\nu^\Omega)T_r(T_j(u_n)-T_j(u_m))\,d\mathcal{H}^{N-1}\le 2Qr,\quad \forall\, n,m\in N$$

where Q denotes a bound for $\|v_n\|_1$. Now,

$$\int_\Omega (a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m), DT_r(T_j(u_n)-T_j(u_m)))$$

$$=\int_\Omega (a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot\nabla T_r(T_j(u_n)-T_j(u_m))\,dx$$

$$+\int_\Omega (a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot D^sT_r(T_j(u_n)-T_j(u_m)).$$

Moreover, by chain's rule for BV functions [2], there exists a positive function η such that

$$\int_\Omega (a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot D^sT_r(T_j(u_n)-T_j(u_m))$$

$$=\int_\Omega \eta[(a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot D^s(T_j(u_n)-T_j(u_m))]$$

$$=\int_\Omega \eta[|D^sT_j(u_n)|-a_\epsilon(\nabla u_m)\cdot D^sT_j(u_n)+|D^sT_j(u_m)|-a_\epsilon(\nabla u_m)\cdot D^sT_j(u_m)]\ge 0,$$

the last term being positive because, by (2.9), we have $|D^sT_j(u_n)|-a_\epsilon(\nabla u_m)\cdot D^sT_j(u_n)\ge 0$ and the analogous inequality with n and m interchanged. Therefore, we obtain

$$\int_\Omega (a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot\nabla T_r(T_j(u_n)-T_j(u_m))\,dx\le 2Qr\quad\forall j>0. \tag{5.16}$$

If

$$S:=\{|u_n-u_m|<k^2, |u_n|<a, |u_m|<a, C(x,a^2,t)\ge k,$$

$$\|\nabla T_a u_n\|\le a^2, \|\nabla T_a u_m\|\le a^2, \|\nabla u_n-\nabla u_m\|>t\},$$

since $\nabla T_a u_n=\nabla u_n$ a.e in S, by (5.16), we get

$$\lambda_N(S)\le$$

$$\le \lambda_n(\{|u_n-u_m|<k^2, |u_n|<a, |u_m|<a, (a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot(\nabla u_n-\nabla u_m)\ge k\}$$

$$\le \frac{1}{k}\int_{|u_n-u_m|<k^2}(a_\epsilon(\nabla u_n)-a_\epsilon(\nabla u_m))\cdot(\nabla T_a(u_n)-\nabla T_a(u_m))\,dx\le 2Qk.$$

Hence, for k small enough, we have

$$\lambda_N(S)\le\frac{\varepsilon}{5}. \tag{5.17}$$

Since a and k have already been choosen, if n_0 is large enough, we have for $n, m \geq n_0$ the estimate $\lambda_n(\{|u_n - u_m| \geq k^2\}) \leq \frac{\varepsilon}{5}$. Now, using (5.12), (5.13), (5.14), (5.15) and (5.17), it follows that

$$\lambda_N(\{\|\nabla u_n - \nabla u_m\| > t\}) \leq \varepsilon \qquad \text{for } \ n, m \geq n_0.$$

Consequently, $\{\nabla u_n\}$ is a Cauchy sequence in measure. Then, up to extraction of a subsequence, we have convergence a.e., and we can say that there exists a measurable function F, such that

$$\nabla u_n \to F \quad \text{ a.e. in } \Omega. \tag{5.18}$$

Now, $a_\epsilon(\nabla u_n) \rightharpoonup a_\epsilon(\nabla u)$ in the weak* topology of $L^\infty(\Omega, R^N)$, and by (5.18), $a_\epsilon(\nabla u_n) \to a_\epsilon(F)$ a.e. in Ω. Hence, $a_\epsilon(F) = a_\epsilon(\nabla u)$ a.e. in Ω. Therefore, by (5.10), we deduce that

$$\nabla u_n \to \nabla u \quad \text{ a.e. in } \Omega.$$

□

PROPOSITION 5.6. *The operator $\mathcal{A}_g^{\epsilon,p}$ is closed in $L^p(\Omega) \times L^p(\Omega)$. Even more, if $(u_n, v_n) \in \mathcal{A}_g^{\epsilon,p}$ and $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ weakly in $L^p(\Omega)$, then $(u, v) \in \mathcal{A}_g^{\epsilon,p}$.*

Proof. Let $(u_n, v_n) \in \mathcal{A}_g^{\epsilon,p}$ be such that $u_n \to u$ in $L^p(\Omega)$ and $v_n \to v$ weakly in $L^p(\Omega)$. Let us prove that $(u, v) \in \mathcal{A}_g^{\epsilon,p}$. Since $(u_n, v_n) \in \mathcal{A}_g^{\epsilon,p}$, we have $a_\epsilon(\nabla u_n) \in X(\Omega)_p$ satisfying

$$-v_n = div\ a_\epsilon(\nabla u_n) \qquad \text{in } \ \mathcal{D}'(\Omega), \tag{5.19}$$

$$a_\epsilon(\nabla u_n) \cdot D^s p(u_n) = |D^s p(u_n)| \qquad \forall\ p \in \mathcal{P}, \tag{5.20}$$

$$a_\epsilon(\nabla u_n) \cdot \nu^\Omega = g \quad \mathcal{H}^{N-1} - \text{a.e.} \tag{5.21}$$

Then, given $p(r) \in \mathcal{P}$, we have

$$\begin{aligned}\int_\Omega v_n p(u_n)\ dx &= \int_\Omega (a_\epsilon(\nabla u_n), Dp(u_n)) - \int_{\partial\Omega} a_\epsilon(\nabla u_n) \cdot \nu^\Omega p(u_n)\ dH^{N-1} \\ &= \int_\Omega a_\epsilon(\nabla u_n) \cdot \nabla p(u_n)\ dx + \int_\Omega |D^s p(u_n)| - \int_{\partial\Omega} g p(u_n)\ dH^{N-1}.\end{aligned}$$

Applying estimate (4.2) to $p(u_n)$ we have

$$\int_{\partial\Omega} g p(u_n)\ dH^{N-1} \leq (1-\sigma) \int_\Omega |Dp(u_n)| + c \int_\Omega |p(u_n)|$$

for some $\sigma > 0$ and some constant $c > 0$. Since $\frac{x^2}{\sqrt{\epsilon^2 + |x|^2}} \geq |x| - \epsilon$, we have

$$\begin{aligned}\|Dp(u_n)\| &= \int_\Omega |\nabla p(u_n)|\ dx + \int_\Omega |D^s p(u_n)| \\ &\leq \int_\Omega a_\epsilon(\nabla p(u_n)) \cdot \nabla p(u_n)\ dx + \int_\Omega |D^s p(u_n)| + \epsilon \lambda_N(\Omega) \\ &\leq \int_\Omega v_n p(u_n)\ dx + \int_{\partial\Omega} g p(u_n)\ dH^{N-1} + \epsilon \lambda_N(\Omega) \\ &\leq \| p \|_\infty \| v_n \|_1 + (1-\sigma)\|Dp(u_n)\| + c\|u_n\|_1 + \epsilon \lambda_N(\Omega).\end{aligned}$$

Thus

$$\|Dp(u_n)\| \le C \qquad \forall\ n \in N, \tag{5.22}$$

for some constant C. Therefore, $p(u) \in BV(\Omega)$ for any $p(r) \in \mathcal{P}$. On the other hand, since

$$\|a_\epsilon(\nabla u_n)\|_\infty \le 1,$$

we may assume that $a_\epsilon(\nabla u_n) \rightharpoonup z$ in the weak* topology of $L^\infty(\Omega, R^N)$ with $\|z\|_\infty \le 1$. Moreover, since $v_n \to v$ weakly in $L^p(\Omega)$, we have that $v = -div(z)$ in $\mathcal{D}'(\Omega)$. By the definition of the weak trace on $\partial\Omega$ of the normal component of z, it is easy to see that

$$a_\epsilon(\nabla u_n)\cdot \nu^\Omega \rightharpoonup z\cdot\nu^\Omega \qquad \text{weakly}^* \ \text{ in } \ L^\infty(\partial\Omega), \tag{5.23}$$

and therefore $z\cdot\nu^\Omega = g$. On the other hand,

$$\begin{aligned} &\lim_{n\to\infty}\left(\int_\Omega h_\epsilon(Dp(u_n)) - \int_{\partial\Omega} gp(u_n)\ dH^{N-1}\right) = \lim_{n\to\infty}\int_\Omega v_n p(u_n)\,dx \\ &= \textstyle\int_\Omega vp(u)\ dx = -\int_\Omega div(z)p(u) = \int_\Omega (z, Dp(u)) - \int_{\partial\Omega} gp(u)\ dH^{N-1}. \end{aligned} \tag{5.24}$$

It is not difficult to prove that

$$\lim_{n\to\infty}\int_\Omega (a_\epsilon(x,\nabla u_n) - a_\epsilon(x,\nabla p(u_n)))\cdot\nabla p(u)\ dx = 0$$

for all $p(r) \in \mathcal{P}$. Consequently,

$$\lim_{n\to\infty}\int_\Omega a_\epsilon(x,\nabla p(u_n))\cdot\nabla p(u_n)\ dx = \int_\Omega z\cdot\nabla p(u)\ dx \quad \forall\ p\in\mathcal{P}. \tag{5.25}$$

Let us now prove the convergence of the energies. We consider the energy functional $\Psi_g : L^1(\Omega) \to [0,+\infty]$ defined by

$$\Psi_g(v) := \begin{cases} \displaystyle\int_\Omega f_\epsilon(Dv) - \int_{\partial\Omega} gv\ dH^{N-1} & \text{if} \quad v \in BV(\Omega) \\ +\infty & \text{if} \quad v \in L^1(\Omega)\setminus BV(\Omega). \end{cases}$$

The functional Ψ_g is convex. As a consequence of Giusti's result [37], Proposition 2.1, if $w_n \in BV(\Omega)$ is bounded in $L^{N/N-1}(\Omega)$ and $w_k \to w$ in $L^1(\Omega)$, $w \in BV(\Omega)$, then

$$\Psi_g(w) \le \liminf_k \Psi_g(w_k).$$

Using the convexity of f_ϵ we have

$$\begin{aligned} \Psi_g(p(u_n)) &= \int_\Omega f_\epsilon(\nabla p(u_n))\ dx + \int_\Omega |D^s p(u_n)| - \int_{\partial\Omega} gp(u_n)\ dH^{N-1} \\ &\le \int_\Omega f_\epsilon(\nabla p(u))\ dx + \int_\Omega a_\epsilon(\nabla p(u_n))\cdot\nabla p(u_n)\ dx \\ &- \int_\Omega a_\epsilon(\nabla p(u_n))\cdot\nabla p(u)\ dx + \int_\Omega a_\epsilon(\nabla u_n)\cdot D^s p(u_n) - \int_{\partial\Omega} gp(u_n)\ dH^{N-1} \end{aligned}$$

$$
\begin{aligned}
&= \int_\Omega f_\epsilon(\nabla p(u))\ dx + \int_\Omega (a_\epsilon(\nabla u_n), Dp(u_n)) \\
&- \int_\Omega a_\epsilon(\nabla p(u_n)) \cdot \nabla p(u)\ dx - \int_{\partial\Omega} g p(u_n)\ dH^{N-1} \\
&= \int_\Omega f_\epsilon(\nabla p(u))\ dx - \int_\Omega div(a_\epsilon(\nabla u_n)) p(u_n)\ dx \\
&- \int_\Omega a_\epsilon(\nabla p(u_n)) \cdot \nabla p(u)\ dx
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality, and using (5.25), we obtain

$$
\begin{aligned}
\limsup_{n\to\infty} \Psi_g(p(u_n)) &\le \int_\Omega f_\epsilon(\nabla p(u))\ dx - \int_\Omega div(z) p(u)\ dx - \int_\Omega z \cdot \nabla p(u)\ dx \\
&= \int_\Omega f_\epsilon(\nabla p(u))\ dx + \int_\Omega (z, Dp(u)) - \int_\Omega z \cdot \nabla p(u)\ dx \\
&- \int_{\partial\Omega} g p(u)\ dH^{N-1} \\
&= \int_\Omega f_\epsilon(\nabla p(u))\ dx + \int_\Omega z \cdot D^s p(u) - \int_{\partial\Omega} g p(u)\ dH^{N-1}.
\end{aligned}
$$

Now, according to (2.9), we have

$$\int_\Omega z \cdot D^s p(u) \le \int_\Omega |D^s p(u)|,$$

hence,

$$\limsup_n \Psi_g(p(u_n)) \le \int_\Omega f_\epsilon(\nabla p(u))\, dx + \int_\Omega |D^s p(u)|\, dx - \int_{\partial\Omega} g p(u)\, dH^{N-1} = \Psi_g(p(u)),$$

and, having in mind the lower-semicontinuity result for Ψ_g, this yields

$$\lim_{n\to\infty} \Psi_g(p(u_n)) = \Psi_g(p(u)). \tag{5.26}$$

Now, let us prove that

$$z(x) = a_\epsilon(\nabla u(x)) \qquad \text{a.e. } x \in \Omega. \tag{5.27}$$

Let $0 \le \phi \in C_0^1(\Omega)$ and $g \in C^1(\overline{\Omega})$. We observe that

$$
\begin{aligned}
&\int_\Omega \phi[(a_\epsilon(\nabla u_n), Dp(u_n - g)) - a_\epsilon(\nabla g) Dp(u_n - g)] = \\
&\int_\Omega \phi[a_\epsilon(\nabla u_n) - a_\epsilon(\nabla g)) \cdot \nabla p(u_n - g)]\ dx + \int_\Omega \phi[a_\epsilon(\nabla u_n) - a_\epsilon(\nabla g)] \cdot D^s p(u_n - g)).
\end{aligned}
$$

Since both terms at the right hand side of the above expression are positive, we have

$$\int_\Omega \phi[(a_\epsilon(\nabla u_n), Dp(u_n - g)) - a_\epsilon(\nabla g) Dp(u_n - g)] \ge 0.$$

Since

$$
\begin{aligned}
\int_\Omega \phi(a_\epsilon(\nabla u_n), Dp(u_n - g)) &= - \int_\Omega div(a_\epsilon(\nabla u_n)) \phi p(u_n - g)\ dx \\
&- \int_\Omega p(u_n - g) a_\epsilon(\nabla u_n) \cdot \nabla \phi\ dx,
\end{aligned}
$$

we get

$$\lim_{n\to\infty}\int_\Omega \phi(a_\epsilon(\nabla u_n), Dp(u_n-g)) =$$

$$= -\int_\Omega div(z)\phi p(u-g)\ dx - \int_\Omega p(u-g) z\cdot\nabla\phi\ dx = \int_\Omega \phi(z, Dp(u-g)).$$

On the other hand,

$$\lim_{n\to\infty}\int_\Omega \phi\, a_\epsilon(\nabla g) Dp(u_n-g) = \int_\Omega \phi\, a_\epsilon(\nabla g) Dp(u-g).$$

Consequently, we obtain

$$\int_\Omega \phi[(z, Dp(u-g)) - a_\epsilon(\nabla g) Dp(u-g)] \geq 0, \qquad \forall\ 0\leq\phi\in C_0^1(\Omega).$$

Thus the measure $(z, Dp(u-g)) - a_\epsilon(\nabla g) Dp(u-g) \geq 0$. Then its absolutely continuous part

$$(z - a_\epsilon(\nabla g))\cdot\nabla p(u-g)) \geq 0 \quad \text{a.e. in } \Omega.$$

Hence,

$$(z - a_\epsilon(\nabla g))\cdot\nabla (u-g)) \geq 0 \quad \text{a.e. in } \Omega.$$

Since we may take a countable set dense in $C^1(\overline{\Omega})$, we have that the above inequality holds for all $x\in\hat\Omega$, where $\hat\Omega\subset\Omega$ is such that $\lambda_N(\Omega\setminus\hat\Omega)=0$, and all $g\in C^1(\overline{\Omega})$. Now, fixed $x\in\hat\Omega$ and given $\xi\in R^N$, there is $g\in C^1(\overline{\Omega})$ such that $\nabla g(x)=\xi$. Then

$$(z(x) - a_\epsilon(\xi))\cdot(\nabla u(x) - \xi) \geq 0, \quad \forall\ \xi\in R^N.$$

These inequalities imply (5.27) by an application of the Minty-Browder's method in R^N. Since $v = -div(z)$ in $\mathcal{D}'(\Omega)$, by (5.27) we get

$$v = -div\, a_\epsilon(\nabla u) \quad \text{in } \mathcal{D}'(\Omega).$$

By (5.21), (5.23) and (5.27), we also have

$$a_\epsilon(\nabla u)\cdot\nu^\Omega = g \quad \text{a.e. on } \partial\Omega.$$

By Lemma 5.5 we have that $\nabla u_n\to\nabla u$ a.e.. Then, it is easy to check that $\nabla p(u_n)\to\nabla p(u)$ a.e.. Then using (5.5), we obtain

$$\begin{aligned}\int_\Omega h_\epsilon(Dp(u)) - \int_{\partial\Omega} gp(u) &= \int_\Omega f_\epsilon(Dp(u)) - \int_\Omega \frac{\epsilon^2}{\sqrt{\epsilon^2+|\nabla p(u)|^2}}\,dx - \int_{\partial\Omega} gp(u)\\ &= \lim_n \int_\Omega f_\epsilon(Dp(u_n)) - \int_\Omega \frac{\epsilon^2}{\sqrt{\epsilon^2+|\nabla p(u_n)|^2}}\,dx - \int_{\partial\Omega} gp(u_n)\\ &= \lim_n \int_\Omega h_\epsilon(Dp(u_n)) - \int_{\partial\Omega} gp(u_n).\end{aligned}$$

Together with (5.24) this gives

$$\int_\Omega h_\epsilon(Dp(u)) = \int_\Omega (z, Dp(u)). \tag{5.28}$$

Now, (5.28) can be written as

$$\int_\Omega a_\epsilon(\nabla u)\cdot\nabla p(u)\ dx + \int_\Omega |D^s p(u)| = \int_\Omega z\cdot\nabla p(u)\ dx + \int_\Omega z\cdot D^s p(u). \tag{5.29}$$

Using (5.27), we deduce that

$$\int_\Omega z\cdot D^s p(u) = \int_\Omega |D^s p(u)|. \tag{5.30}$$

Now, since by (2.9) the singular parts satisfy the inequality

$$|z\cdot D^s p(u)| \le |D^s p(u))| \qquad \text{as measures in } \Omega, \tag{5.31}$$

using (5.31) and (5.30) we deduce that

$$z\cdot D^s p(u) = |D^s p(u)|, \tag{5.32}$$

which, by (5.27), is the same as $a_\epsilon(\nabla u)\cdot D^s p(u) = |D^s p(u)|$.□

PROPOSITION 5.7. *The operator $\mathcal{A}_g^{\epsilon,p}$ is an m-accretive operator in $L^p(\Omega)$, i.e., for any $f\in L^p(\Omega)$ there is a unique solution $u\in L^p(\Omega)$ of*

$$u + \mathcal{A}_g^{\epsilon,p} u \ni f, \tag{5.33}$$

and we have the estimate

$$\| u-\tilde u\|_p \le \| f-\tilde f\|_p\ . \tag{5.34}$$

for any two solutions $u,\tilde u$ of (5.33) corresponding to the right hand side $f,\tilde f\in L^p(\Omega)$.

Proof. Since we are assuming that $\| g\|_\infty < 1$ and $\partial\Omega$ is of class C^1, then $q(x)=1$ for all $x\in\Omega$, and

$$\| gq\|_\infty < 1 \tag{5.35}$$

holds. This implies that assumptions (1.3) and (1.4) in [37] hold. As a consequence, if $f\in W^{1,\infty}(\Omega)$, there is a unique solution u is $C^{2,\alpha}(\Omega)$ ([36], [37]) of

$$\begin{cases} u - div\ \dfrac{Du}{\sqrt{\epsilon^2+|Du|^2}} = f \quad \text{in } \Omega \\ a_\epsilon(\nabla u)\cdot\nu^\Omega = g \quad \text{in } \partial\Omega \end{cases} \tag{5.36}$$

We shall only need that $u\in W^{1,1}(\Omega)$. Then u is a solution of (5.33). The accretivity estimate (5.34) when $f,\tilde f\in W^{1,\infty}(\Omega)$ follows by a simple integration by parts.

Now, let $f\in L^p(\Omega)$ and let $f_n\in W^{1,\infty}(\Omega)$ be such that $\| f-f_n\|_p\to 0$. Let $(u_n,v_n)\in\mathcal{A}_g^{\epsilon,p}$ be such that $u_n+v_n=f_n$. By the accretivity estimate we have that

$$\| u_n-u_m\|_p \le \| f_n-f_m\|_p\ .$$

Thus (u_n, v_n) converge in $L^p(\Omega)^2$ to a pair of functions $(u, v) \in L^p(\Omega)^2$. By Proposition 5.6, we have that $(u, v) \in \mathcal{A}_g^{\epsilon,p}$. The accretivity estimate (5.34) for general $f, \tilde{f} \in L^p(\Omega)$ follows as in [10] by using the doubling variables technique of Kruzhkov and refer to it for details (for limit solutions it would be a consequence of the same estimate for any $f, \tilde{f} \in W^{1,\infty}(\Omega)$).□

If $f \in L^p(\Omega)$ with $p \geq 2$, we may prove that $u \in BV(\Omega)$. This implies the following result.

PROPOSITION 5.8. *If $p \geq 2$, then $(u, v) \in \mathcal{A}_g^{\epsilon,p}$ if and only if $u, v \in L^p(\Omega)$, $u \in BV(\Omega)$, and $a_\epsilon(\nabla u) \in X(\Omega)_p$ satisfies (5.2),(5.4) and*

$$a_\epsilon(\nabla u) \cdot D^s u = |D^s u|. \tag{5.37}$$

5.2. Some approximation results. Let g and Φ be as in Proposition 2.1.

PROPOSITION 5.9. *Let $\rho(x) \in C(\overline{\Omega})$, $\rho(x) \geq \rho_0 > 0$, $\mu > 0$. For any $f \in L^2(\Omega)$ there is a unique solution $u \in L^2(\Omega)$ of the equation*

$$\frac{1}{\rho(x)} u + \mu \partial\Phi(u) \ni \frac{1}{\rho(x)} f, \tag{5.38}$$

in other words, there exists $z \in L^\infty(\Omega, R^N)$, $\|z\|_\infty \leq 1$ such that $z \cdot Du = |Du|$, $z \cdot \nu^\Omega = g$, and

$$\frac{1}{\rho(x)} u - \mu \, div \, z = \frac{1}{\rho(x)} f. \tag{5.39}$$

Proof. Let us consider the variational problem

$$\min_{w \in BV(\Omega)} D(w), \qquad D(w) := \int_\Omega |Dw| + \frac{1}{2\mu} \int_\Omega (w - f)^2 \frac{1}{\rho(x)} \, dx - \int_{\partial\Omega} gw \tag{5.40}$$

We observe that the functional D is convex and L^1–lower semicontinuous [37]. Moreover, since $\| g \|_\infty < 1$ and $\partial\Omega$ is of class C^1, using the results of Giusti [37] we get that D is coercive. Therefore it attains its minimum, which is also unique. Hence $u = \arg\min D$ if and only if $0 \in \partial D(u)$. Since $\partial D(u) = \partial\Phi(u) + \frac{1}{\mu\rho(x)}(u - f)$, (5.38) follows. By Proposition 2.1, (5.38) and (5.39) are equivalent.□

PROPOSITION 5.10. *Let $f \in L^2(\Omega) \cap Dom\, \mathcal{B}_p$, $\rho(x) \in C(\overline{\Omega})$, $\rho(x) \geq \rho_0 > 0$. For each $\mu > 0$, let u_μ be the unique solution of*

$$\frac{1}{\rho(x)} u + \mu \partial\Phi(u) \ni \frac{1}{\rho(x)} f, \tag{5.41}$$

and let $v_\mu \in \partial\Phi(u_\mu)$ be such that

$$\frac{1}{\rho(x)} u_\mu + \mu v_\mu = \frac{1}{\rho(x)} f. \tag{5.42}$$

Then

$$\Big(\int_\Omega \frac{|u_\mu|^p}{\rho(x)} \, dx \Big)^{1/p} \leq \Big(\int_\Omega \frac{|f|^p}{\rho(x)} \, dx \Big)^{1/p} + \mu C \Big(\int_\Omega \frac{1}{\rho} \, dx \Big)^{1/p} \tag{5.43}$$

$$\|v_\mu\|_{p,\rho^{(p-1)}} \le \|\mathcal{B}_p^{\rho^{(p-1)}} f\|_{p,\rho^{(p-1)}}, \tag{5.44}$$

$$v_\mu \to \mathcal{B}_p^{\rho^{(p-1)}} f \qquad \text{in } L^p(\Omega) \text{ as } \mu \to 0,$$

and

$$\int_\Omega |DT_k(u_\mu)| - \int_{\partial\Omega} gT_k(u_\mu) \to \int_\Omega |DT_k(u)| - \int_{\partial\Omega} gT_k(u) \tag{5.45}$$

as $\mu \to 0$*, for all* $k > 0$*.*

Proof. To prove (5.43), let $\beta_\epsilon : R \to R$ be a C^1 function, $\beta_\epsilon'(r) > 0$ for all $r \in R$ such that $\beta_\epsilon(r) = |r|^{p-1} sign(r)$ for $|r| \ge \epsilon$. Let $z_\mu \in L^\infty(\Omega, R^N)$, $\|z_\mu\|_\infty \le 1$ be such that $v_\mu = -div\, z_\mu$. Multiplying (5.42) by $\beta_\epsilon(T_k(u_\mu))$ and integrating by parts, we obtain

$$\begin{aligned}\int_\Omega u_\mu \beta_\epsilon(T_k(u_\mu)) \frac{1}{\rho(x)}\,dx + \mu \int_\Omega |D\beta_\epsilon(T_k(u_\mu))| &= \int_\Omega f\beta_\epsilon(T_k(u_\mu)) \frac{1}{\rho(x)}\,dx \\ &+ \mu \int_\Omega g\beta_\epsilon(T_k(u_\mu)) \le \int_\Omega |f||\beta_\epsilon(T_k(u_\mu))| \frac{1}{\rho(x)}\,dx \\ &+ \mu(1-\sigma)\int_\Omega |D\beta_\epsilon(T_k(u_\mu))| + \mu C \int_\Omega |\beta_\epsilon(T_k(u_\mu))| \frac{1}{\rho(x)}\,dx\end{aligned}$$

where we have used (4.2), $\|g\|_\infty = 1 - 2\sigma$, and $C > 0$ is a positive constant. Using Hoelder's inequality we obtain

$$\int_\Omega \frac{u_\mu \beta_\epsilon(T_k(u_\mu))}{\rho(x)}\,dx \le \Big(\Big(\int_\Omega \frac{|f|^p}{\rho(x)}\,dx\Big)^{1/p} + \mu C\Big(\int_\Omega \frac{dx}{\rho}\Big)^{1/p}\Big)\Big(\int_\Omega \frac{|\beta_\epsilon(T_k(u_\mu))|^{p'}}{\rho(x)}\,dx\Big)^{1/p'}$$

where p' denotes the conjugate exponent of p. Letting $\epsilon \to 0$ we obtain

$$\int_\Omega \frac{|T_k(u_\mu)|^p}{\rho(x)}\,dx \le \Big(\Big(\int_\Omega \frac{|f|^p}{\rho(x)}\,dx\Big)^{1/p} + \mu C\Big(\int_\Omega \frac{1}{\rho}\,dx\Big)^{1/p}\Big)\Big(\int_\Omega \frac{|T_k(u_\mu)|^p}{\rho(x)}\,dx\Big)^{1/p'},$$

i.e.,

$$\Big(\int_\Omega \frac{|T_k(u_\mu)|^p}{\rho(x)}\,dx\Big)^{1/p} \le \Big(\int_\Omega \frac{|f|^p}{\rho(x)}\,dx\Big)^{1/p} + \mu C\Big(\int_\Omega \frac{1}{\rho}\,dx\Big)^{1/p}.$$

Letting $k \to \infty$ we obtain (5.43).

To prove (5.44), let us write $v = \mathcal{B}_p^{\rho^{(p-1)}} f$. Recall that $\beta_p(r) = |r|^{p-1} sign_0(r)$. By Proposition 2.2 we have the inequalities

$$\int_\Omega (v_\mu - v)\beta_p(u_\mu - f)\,dx \ge 0$$

Using (5.42) we may write

$$\int_\Omega |u_\mu - f|^p \frac{1}{\rho(x)}\,dx \le -\mu \int_\Omega v\beta_p(u_\mu - f)\,dx,$$

or,

$$\int_\Omega |v_\mu|^p \rho(x)^{p-1}\,dx \le -\int_\Omega v\beta_p(\frac{u_\mu - f}{\mu})\,dx = -\int_\Omega v\beta_p(v_\mu)\rho(x)^{p-1}\,dx.$$

Using Holder's inequality, we obtain

$$\int_\Omega |v_\mu|^p \rho(x)^{p-1}\,dx \le \Big(\int_\Omega |v|^p \rho(x)^{p-1}\,dx\Big)^{1/p}\Big(\int_\Omega |v_\mu|^p \rho(x)^{p-1}\,dx\Big)^{(p-1)/p}$$

which implies (5.44).

To prove the last assertion, observe that from estimate (5.44) we have that v_μ is bounded in $L^p(\Omega)$ and we may assume that $v_\mu \to v^*$ weakly in $L^p(\Omega)$ for some $v^* \in L^p(\Omega)$. On the other hand, since $u_\mu - f = -\mu\rho v_\mu$ we deduce that $u_\mu \to f$ in $L^p(\Omega)$. Since $(u_\mu, v_\mu) \in \mathcal{B}_p$, by Proposition 2.2 we obtain that $(f, v^*) \in \mathcal{B}_p$. Now

$$\|v^*\|_{(p,\rho^{(p-1)})} \le \liminf_\mu \|v_\mu\|_{(p,\rho^{(p-1)})} \le \|\mathcal{B}_p^{\rho^{(p-1)}} f\|_{(p,\rho^{(p-1)})}$$

This implies that $v^* = v = \mathcal{B}_p^{\rho^{(p-1)}} f$, and $\|v_\mu\|_{(p,\rho^{(p-1)})} \to \|v\|_{(p,\rho^{(p-1)})}$. Since $p > 1$, this implies that $v_\mu \to v$ in $L^p(\Omega, \rho^{(p-1)})$, hence, also in $L^p(\Omega)$. The proof of (5.45) follows as in the proof of Proposition 2.2.□

PROPOSITION 5.11. *Let $(u, v) \in \mathcal{B}_p$, i.e., there is $\theta \in L^\infty(\Omega, R^N)$, $|\theta| \le 1$, such that $\theta \cdot DT_k(u) = |DT_k(u)|$ for all $k > 0$, $v = -div\,\theta$ and $\theta \cdot \nu^\Omega = g$. Let $f = u + v$. Let u_ϵ be the unique solution of*

$$\begin{aligned} &U - div\, a_\epsilon(\nabla U) = f \quad \text{in } \Omega, \\ &a_\epsilon(\nabla U)\cdot \nu^\Omega = g \quad \text{in } \partial\Omega. \end{aligned} \tag{5.46}$$

Then $u_\epsilon \to u$, $div\, a_\epsilon(\nabla u_\epsilon) \to v$ in $L^p(\Omega)$ and $\int_\Omega |DT_k(u_\epsilon)| - \int_{\partial\Omega} g u_\epsilon \to \int_\Omega |DT_k(u)| - \int_{\partial\Omega} gu$ as $\epsilon \to 0$, for any $k > 0$. Moreover, if $f \in L^\infty(\Omega)$, we have that u_ϵ is bounded in $L^\infty(\Omega)$

Proof. Let us prove some estimates. Let $\beta(r)$ be a smooth strictly increasing function such that $\beta(r) = |r|^{p-1} sign(r)$ for r large enough. Multiply (5.46) by $\beta(T_k(u_\epsilon))$ and integrate on Ω. Using (4.2) we obtain

$$\begin{aligned} \int_\Omega u_\epsilon \beta(T_k(u_\epsilon)) + \int_\Omega a_\epsilon(\nabla u_\epsilon) D\beta(T_k(u_\epsilon)) &= \int_\Omega f\beta(T_k(u_\epsilon)) + \int_{\partial\Omega} g\beta(T_k(u_\epsilon)) \\ &\le \|f\|_p \|\beta(T_k(u_\epsilon)))\|_{p'} + c\int_\Omega |D\beta(T_k(u_\epsilon))| + C\int_\Omega |\beta(T_k(u_\epsilon))| \end{aligned}$$

for some $c < 1$. Then

$$\begin{aligned} \int_\Omega u_\epsilon \beta(T_k(u_\epsilon)) + (1-c)\int_\Omega |D\beta(T_k(u_\epsilon))| &\le \|f\|_p \|\beta(T_k(u_\epsilon)))\|_{p'} \\ &+ C\int_\Omega |\beta(T_k(u_\epsilon))| + \epsilon \int_\Omega |\beta'(u_\epsilon)| \end{aligned}$$

Since the terms at the right hand side involving $\beta(T_k(u_\epsilon))$ and $\beta'(T_k(u_\epsilon))$ can be controlled by the first term of the left hand side, letting $k \to \infty$ in the above inequality we obtain that u_ϵ is bounded in $L^p(\Omega)$ and $\beta(T_k(u_\epsilon))$, hence also $\beta(u_\epsilon)$, is bounded in $BV(\Omega)$ (independently of k and $\epsilon > 0$). Note that, if $f \in L^\infty(\Omega)$, the L^p estimate on u_ϵ is independent of p, and we obtain an estimate for $\|u_\epsilon\|_\infty$ which is independent of ϵ.

Let us now prove that $u_\epsilon \to u$ as $\epsilon \to 0$. For that we write (5.46) in the form

$$u_\epsilon - u - div\,(a_\epsilon(\nabla u_\epsilon) - \theta) = 0, \tag{5.47}$$

we multiply it by $\beta(T_k(u_\epsilon) - T_k(u))$, and integrate it on Ω. We obtain

$$\int_\Omega (u_\epsilon - u)\beta(T_k(u_\epsilon) - T_k(u)) + \int_\Omega (a_\epsilon(\nabla u_\epsilon) - \theta) \cdot D\beta(T_k(u_\epsilon) - T_k(u)) = 0.$$

Let us now prove that

$$(a_\epsilon(\nabla u_\epsilon) - \theta) \cdot D\beta(T_k(u_\epsilon) - T_k(u)) + \epsilon\beta'(T_k(u_\epsilon) - T_k(u)) \geq 0. \tag{5.48}$$

Since

$$a_\epsilon(\nabla u_\epsilon) \cdot DT_k(u_\epsilon) = a_\epsilon(\nabla u_\epsilon) \cdot \nabla T_k(u_\epsilon) + |D^s T_k(u_\epsilon)| \geq |DT_k(u_\epsilon)| - \epsilon,$$

$$\theta \cdot DT_k(u_\epsilon) \leq |DT_k(u_\epsilon)|$$

and

$$(a_\epsilon(\nabla u_\epsilon) - \theta) \cdot DT_k(u) \leq 0$$

we have that

$$(a_\epsilon(\nabla u_\epsilon) - \theta) \cdot D(T_k(u_\epsilon) - T_k(u)) + \epsilon \geq 0. \tag{5.49}$$

Let $\Theta(a_\epsilon(\nabla u_\epsilon) - \theta, D(T_k(u_\epsilon) - T_k(u)))$ be the Radon-Nikodym derivative of $(a_\epsilon(\nabla u_\epsilon) - \theta) \cdot D(T_k(u_\epsilon) - T_k(u))$ with respect to the measure $|D(T_k(u_\epsilon) - T_k(u))|$. We may write (5.49) as

$$\Theta(a_\epsilon(\nabla u_\epsilon) - \theta, DT_k(u_\epsilon) - T_k(u))|D(T_k(u_\epsilon) - T_k(u))| + \epsilon \geq 0. \tag{5.50}$$

Using Volpert's chain rule for BV functions ([2]) there is a function $\tilde{\beta}'(x)$, measurable both with respect to the measure $D(T_k(u_\epsilon) - T_k(u))$ and with respect to the Lebesgue measure, such that

$$D\beta(T_k(u_\epsilon) - T_k(u)) = \tilde{\beta}'(x)D(T_k(u_\epsilon) - T_k(u)).$$

Multiplying (5.50) by $\tilde{\beta}'(x)$ we obtain

$$\Theta(a_\epsilon(\nabla u_\epsilon) - \theta, DT_k(u_\epsilon) - DT_k(u))|D\beta(T_k(u_\epsilon) - T_k(u))| + \epsilon\tilde{\beta}'(x) \geq 0. \tag{5.51}$$

Since, by Anzellotti's results in [11], we have

$$\Theta(a_\epsilon(\nabla u_\epsilon) - \theta, DT_k(u_\epsilon) - u) = \Theta(a_\epsilon(\nabla u_\epsilon) - \theta, D\beta(T_k(u_\epsilon) - T_k(u)))$$

almost everywhere with respect to the measure $|D\beta(T_k(u_\epsilon) - T_k(u))|$, we conclude that

$$(a_\epsilon(\nabla u_\epsilon) - \theta) \cdot D\beta_p(T_k(u_\epsilon) - T_k(u)) + \epsilon\tilde{\beta}'(x) \geq 0. \tag{5.52}$$

Since $\tilde{\beta}'(x) = \beta'(T_k(u_\epsilon(x)) - T_k(u(x)))$ a.e., we deduce (5.48).

Using (5.48) we have

$$\int_\Omega (u_\epsilon - u)\beta(T_k(u_\epsilon) - T_k(u)) \le \epsilon \int_\Omega \beta'(T_k(u_\epsilon(x)) - T_k(u(x)))$$

Letting $k \to \infty$ and $\epsilon \to 0$ in this order we conclude that $u_\epsilon \to u$ in $L^p(\Omega)$. From (5.47) we deduce that

$$div\, a_\epsilon(\nabla u_\epsilon) \to div\, \theta \quad \text{in } L^p(\Omega)$$

as $\epsilon \to 0$.

Let us finally prove that

$$\int_\Omega |DT_k(u_\epsilon)| - \int_\Omega gT_k(u_\epsilon) \to \int_\Omega |DT_k(u)| - \int_\Omega gT_k(u)$$

as $\epsilon \to 0$. Indeed, using (5.5) with $p(r) = T_k(r)$, we have

$$\begin{aligned}
\int_\Omega |DT_k(u)| - \int_{\partial\Omega} gT_k(u) &=\le \liminf_\epsilon \int_\Omega |DT_k(u_\epsilon)| - \int_{\partial\Omega} gT_k(u_\epsilon) \\
&= \liminf_\epsilon \epsilon|\Omega| + \int_\Omega a_\epsilon(\nabla u_\epsilon)\cdot DT_k(u_\epsilon) - \int_{\partial\Omega} gT_k(u_\epsilon) \\
&= \liminf_\epsilon \int_\Omega a_\epsilon(\nabla u_\epsilon)\cdot DT_k(u_\epsilon) - \int_{\partial\Omega} gT_k(u_\epsilon) \\
&= \liminf_\epsilon -\int_\Omega div\, a_\epsilon(\nabla u_\epsilon)\, T_k(u_\epsilon) + \int_{\partial\Omega} (a_\epsilon(\nabla u_\epsilon)\cdot \nu^\Omega - g)\, T_k(u_\epsilon) \\
&= -\int_\Omega div\, \theta\, T_k(u) = \int_\Omega \theta \cdot DT_k(u) - \int_{\partial\Omega} gT_k(u) \\
&\le \int_\Omega |DT_k(u)| - \int_{\partial\Omega} gT_k(u)
\end{aligned}$$

□

5.3. A convergence result for the mimima of (1.10). Let $\gamma, \alpha, \lambda > 0$, $\beta \ge 0$, $p > 1$, $q \ge 1$, and $\epsilon > 0$. We shall use the same assumptions on u_0 and θ_0 as in Section 3. Recall that $g_0 = \theta_0 \cdot \nu^\Omega$, and we are assuming that $\|g_0\|_\infty < 1$. Let

$$\mathcal{A}(G_\epsilon) = \{u \in BV(\Omega) :\, u \in Dom\mathcal{A}^{\epsilon,1}_{g_0},\ div\, a_\epsilon(\nabla u_\epsilon) \in L^p(\Omega)\,,\, u|_B \in L^q(B)\}.$$

If $u \in \mathcal{A}(G_\epsilon)$ we define

$$\begin{aligned}
G_\epsilon(u) = &\textstyle\int_\Omega |div\, a_\epsilon(\nabla u)|^p(\gamma + \beta|\nabla k * u|)dx \\
&+ \alpha \int_\Omega |Du| - \alpha \int_\Omega g_0 u + \lambda \int_B |u - u_0|^q\, dx,
\end{aligned} \tag{5.53}$$

if $u \in L^1(\tilde\Omega) \setminus \mathcal{A}(G_\epsilon)$ we set

$$G_\epsilon(u) = +\infty.$$

Following the proof of Theorem 4.1 we prove:

PROPOSITION 5.12. *The functional G_ϵ attains its infimum at some $(u,\theta) \in \mathcal{A}(G_\epsilon)$.*

Let us extend $E_p(u,\theta)$ by writing $E_p(u,\theta) = +\infty$ if $(u,\theta) \in L^1(\tilde{\Omega}) \times L^\infty(\tilde{\Omega}, R^N) \subseteq \mathcal{E}_p(\Omega, B, \theta_0)$ with $\|\theta\|_\infty \leq 1$.

THEOREM 5.13. *For each $\epsilon > 0$, let u_ϵ be a minimum of (5.53). Then there exists a subsequence, call it again u_ϵ, such that $u_\epsilon \to u$ in $L^r(\Omega)$ for all $r \in [1, \frac{N}{N-1})$ and $a_\epsilon(\nabla u_\epsilon) \to \theta$ weakly* in $L^\infty(\Omega, R^N)$ where $(u,\theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$ is a minimum of E_p.*

Proof. Let $(u_\epsilon, \theta_\epsilon)$ be a minimum of G_ϵ on $\mathcal{A}(G_\epsilon)$. We may assume that

$$\int_\Omega |div\, \theta_\epsilon|^p, \ \int_\Omega |Du_\epsilon| - \int_{\partial\Omega} g u_\epsilon \quad \text{and} \int_B |u_\epsilon - u_0|^q \, dx$$

are bounded independently of $\epsilon > 0$. Indeed, to justify it it suffices to construct a sequence $U_\epsilon \in Dom\, G_\epsilon$ such that $G_\epsilon(U_\epsilon)$ is bounded independently of ϵ. For that, by Proposition 5.9 with $\rho(x) = 1$, $f = 0$ and $g = g_0$, there exists $(u, v = -div(z)) \in \mathcal{B}_p$ such that $u + v = 0$. Let U_ϵ be the unique solution of (5.46) with $f = 0$, $g = g_0$. By Proposition 5.11 we know that $G_\epsilon(U_\epsilon)$ converges to $E(u,z)$, hence it is bounded. Notice that, as in the proof of Theorem 4.1, we have that $\int_\Omega |Du_\epsilon|$ is bounded.

Thus, by extracting a subsequence, if necessary, we may assume that θ_ϵ converges to some function $\tilde{\theta}$ weakly* in $L^\infty(\Omega, R^N)$, with $|\tilde{\theta}| \leq 1$, $div\, \tilde{\theta} \in L^p(\Omega)$ and $div\, \theta_\epsilon \to div\, \tilde{\theta}$ weakly in $L^p(\Omega)$. We may also assume that $u_\epsilon \to u$ in $L^r(\Omega)$ for any $r \in [1, \frac{N}{N-1})$ for some function $\tilde{u} \in BV(\Omega)$.

Let $f := \tilde{u} - div\, \tilde{\theta}$, and $f_\epsilon := u_\epsilon - div\, \theta_\epsilon$. Proceeding as in the proof of Proposition 2.2 we deduce that $\tilde{\theta} \cdot \nu^\Omega = g_0$ on $\partial\Omega$. Proceeding as in the proof of Proposition 5.11 we prove that $\tilde{\theta} \cdot DT_k(\tilde{u}) = |DT_k(\tilde{u})|$ for all $k > 0$. We have proved that $(\tilde{u}, \tilde{\theta}) \in \mathcal{E}_p(\Omega, B, \theta_0)$.

Let us prove that $(\tilde{u}, \tilde{\theta})$ is a minimum of E. For that, let $(u,\theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$. Observe that $(T_k(u), \theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$ for any $k > 0$. Since $E(T_k(u), \theta) \to E(u,\theta)$ it will be sufficient to prove that

$$E(\tilde{u}, \tilde{\theta}) \leq E(u,\theta)$$

for any $(u,\theta) \in \mathcal{E}_p(\Omega, B, \theta_0)$ with $u \in L^\infty(\Omega)$. Thus, let us assume that $u \in L^\infty(\Omega)$. Let θ^* be such that $-div\, \theta^* = \mathcal{B}_p^{\omega(u)} u$ where $\omega(u) = \gamma + \beta|\nabla k * u|$. Since $E(u,\theta^*) \leq E(u,\theta)$, we may assume that $\theta = \theta^*$, hence $-div\, \theta = \mathcal{B}_p^{\omega(u)} u$. Now, we apply Proposition 5.10 with $f = u$, $g = g_0$, $\mu = \frac{1}{n}$, and $\rho(x)^{p-1} = \omega(u)$ to obtain a sequence $(u_n, \theta_n) \in \mathcal{B}_p$ such that u_n is bounded in $L^\infty(\Omega)$, and $div\, \theta_n \to div\, \theta$ in $L^p(\Omega)$, $\int_\Omega |Du_n| - \int_{\partial\Omega} g_0 u_n \to \int_\Omega |Du| - \int_{\partial\Omega} g_0 u$ as $n \to \infty$ (hence $E(u_n, \theta_n) \to E(u,\theta)$). Since $div\, \theta_n = n \frac{u_n - u}{\rho(x)}$ we obtain that $div\, \theta_n \in L^\infty(\Omega)$. Now, we apply Proposition 5.11 to $f_n := u_n - div\, \theta_n$ and we obtain functions $u_{\epsilon,n} \in \mathcal{A}_{g_0}^{\epsilon,p}$ which are bounded in $L^\infty(\Omega)$ independently of ϵ and $G(u_{\epsilon,n}) \to E(u_n, \theta_n)$ as $\epsilon \to 0$. Since $G(u_\epsilon) \leq G(u_{\epsilon,n})$ for each ϵ and each n, we have

$$E(\tilde{u}, \tilde{\theta}) \leq \liminf_\epsilon G(u_\epsilon) \leq \liminf_\epsilon G(u_{\epsilon,n}) = E(u_n, \theta_n)$$

for all n. Letting $n \to \infty$ we obtain $E(\tilde{u}, \tilde{\theta}) \leq E(u,\theta)$. We conclude that $(\tilde{u}, \tilde{\theta})$ is a minimum of (3.2). □

6. Regularity. The following Theorem collects some results which have been proved in the literature. For a proof we refer to [1, 5] and the references therein.

THEOREM 6.1. *Let $u \in BV(\Omega)$, $z \in X(\Omega)_p$, $N \leq p \leq \infty$, be such that $(z, Du) = |Du|$. Then for almost all levels $t \in R$, the sets $E_t = [u > t]$ satisfy*

(i) if $N < p < \infty$ ($p = \infty$) then the reduced boundary $\partial^* E_t$ is relatively open in ∂E_t and is a hypersurface of class $C^{1,\alpha}$ for any $\alpha < \frac{p-N}{2p}$ (resp., for any $\alpha < 1$). Moreover the closed set $\Sigma(E_t) = \partial E_t \setminus \partial^* E_t$ is empty if $N < 8$, discrete if $N = 8$ and has Haussdorff dimension not greater that $N - 8$ if $N > 8$.

(ii) if $p = N$, there is a closed set $\Sigma(E_t)$ of Haussdorff dimension not greater than $N - 8$ such that $\partial E_t \setminus \Sigma(E_t)$ is a $(N - 1)$ dimensional manifold of class $C^{0,\alpha}$ for all $\alpha < 1$.

If $N = 2$, these results can be further precised. If $p = 2$, then ∂E_t is locally parameterizable with a bilipschitz map (a Lipschitz map with a Lipschitz inverse). If $p = \infty$, ∂E_t is of class $C^{1,1}$.

Proof. Let t be such that $[u > t]$ has finite perimeter in Ω and $z \cdot D\chi_{[u>t]} = |D\chi_{[u>t]}|$ (in particular, for almost every t). Let $x \in \Omega$, F be a finite perimeter set such that $F \triangle [u > t] \subset\subset B_\rho(x) \subseteq \Omega$. Then

$$\int_{[u>t]\cap\Omega} div z - \int_{E\cap\Omega} div z \leq P(E, \Omega) - P([u > t], \Omega). \tag{6.1}$$

We have

$$P([u > t], \Omega) \leq P(F, \Omega) - \int_{[u>t]\Delta F} div\, z$$

and, thus, also

$$\begin{aligned} P([u > t], B_\rho(x)) &\leq P(F, B_\rho(x)) - \int_{[u>t]\Delta F} div\, z \\ &\leq P(F, B_\rho(x)) + \|div\, z\|_{L^N(B_\rho(x))} |[u > t]\Delta F|^{\frac{N}{N-1}} \\ &\leq P(F, B_\rho(x)) + \omega_N^{\frac{p-N}{N}} \|div\, z\|_{L^p(B_\rho(x))} \rho^{2\alpha} |[u > t]\Delta F|^{\frac{N}{N-1}} \end{aligned}$$

with $\alpha = \frac{p-N}{2p}$. This permits to prove that there is a constant $C(N)$ such that

$$|[u > t]\Delta F| \leq C(N) P([u > t], B_\rho(x)),$$

hence

$$P([u > t], B_\rho(x)) \leq \frac{1}{1 - \eta(\rho)} P(F, B_\rho(x))$$

where $\eta(\rho) = C(N)\omega_N^{\frac{p-N}{N}} \|div\, z\|_{L^p(B_\rho(x))} \rho^{2\alpha}$. The above inequality may be written as

$$P([u > t], B_\rho(x)) \leq (1 + \omega(\rho)) P(F, B_\rho(x))$$

where $\omega(\rho) = \frac{\eta(\rho)}{1-\eta(\rho)}$. In other words, $[u > t]$ is a quasi minimizer of the perimeter. The study of the regularity of quasiminimizers of the perimeter can be found in [1, 5] and the references therein. □

7. Algorithm and numerical experiments. To minimize the functional (3.2) we use the steepest descent method. If we denote the energy by $\tilde{E}(\theta, u)$, the steepest descent equations are

$$\theta_t = -\nabla_\theta \tilde{E}(\theta, u) \tag{7.1}$$

and

$$u_t = -\nabla_u \tilde{E}(\theta, u) \tag{7.2}$$

in $(0,\infty)\times\Omega$ supplemented with the corresponding boundary data and initial conditions. The constraints on (θ, u) can be incorporated either by penalization or by projecting onto them after each time step. Indeed, we have tested both methods method in an implicit (also in a explicit) in time discretization of (7.1), (7.2). Let us explain in some detail the implicit in time implementation of (7.1), (7.2) with the constraint $\theta \cdot Du = |Du|$ incorporated by penalization. Thus we take

$$\begin{aligned}\tilde{E}(u,\theta) &= \int_\Omega |div(\theta)|^p(\gamma+\beta|\nabla k * u|)dx + \eta\int_\Omega (|Du| - \theta\cdot Du) + \\ &\quad +\alpha\int_\Omega |Du| - \alpha\int_{\partial\Omega} g_0 u + \lambda\int_B (u-u_0)^2 \\ &= \int_\Omega |div(\theta)|^p(\gamma+\beta|\nabla k * u|)dx + (\alpha+\eta)\int_\Omega |Du| + \eta\int_\Omega div\,\theta\, u + \\ &\quad -(\alpha+\eta)\int_{\partial\Omega} g_0 u + \lambda\int_B (u-u_0)^2\end{aligned} \tag{7.3}$$

which corresponds to the energy (3.2) with plus a penalization term for the constraint that $\theta\cdot Du = |Du|$. To simplify our notation, let us write $g(\theta) = \beta|div(\theta)|^p$, $h(u) = \gamma + \beta|\nabla k * u|$. Then

$$\nabla_\theta \tilde{E}(\theta, u) = -p\nabla\big[h(u)|div(\theta)|^{p-2}div(\theta)\big] - \eta Du = 0 \tag{7.4}$$

and

$$\begin{aligned}\nabla_u \tilde{E}(\theta,u) = -div\left(k*\left(g(\theta)\tfrac{\nabla k*u}{|\nabla k*u|}\right)\right) - (\eta+\alpha)\,div\left(\tfrac{Du}{|Du|}\right) + \\ +\eta\,div\,\theta + 2\lambda(u-u_0)\chi_B = 0.\end{aligned} \tag{7.5}$$

To solve equations (7.1) and (7.2), we use a implicit discretization in time. To be precise, we write

$$\nabla_\theta \tilde{E}(\theta,\theta',u,v) = -p\nabla\big[h(u)(\epsilon+|div(\theta')|^{p-2})div(\theta)\big] - \eta Du = 0 \tag{7.6}$$

and

$$\begin{aligned}\nabla_u \tilde{E}(\theta,\theta',u,v) = -div\left(k*\left(g(\theta)\tfrac{\nabla k*u}{\sqrt{\epsilon+|\nabla k*v|^2}}\right)\right) - (\eta+\alpha)\,div\left(\tfrac{\nabla u}{\sqrt{\epsilon+|\nabla v|^2}}\right) \\ +\eta\,div(\theta) + 2\lambda(u-u_0)\chi_B = 0.\end{aligned} \tag{7.7}$$

Then, we use the discretization in time given by

$$\theta^{n+1} - \theta^n = \Delta t \nabla_\theta \tilde{E}(\theta^{n+1}, \theta^n, u^n, u^n), \tag{7.8}$$

and

$$u^{n+1} - u^n = \Delta t \nabla_u \tilde{E}(\theta^{n+1}, \theta^{n+1}, u^{n+1}, u^n). \tag{7.9}$$

Finally, we make the change of variables $\xi^{n+1} = \theta^{n+1} - \theta^n$, $v^{n+1} = u^{n+1} - u^n$ and we have

$$\xi^{n+1} = \Delta t \nabla_\theta \tilde{E}(\xi^{n+1} + \theta^n, \theta^n, u^n, u^n), \tag{7.10}$$

$$v^{n+1} = \Delta t \nabla_u \tilde{E}(\theta^{n+1}, \theta^{n+1}, v^{n+1} + u^n, u^n). \tag{7.11}$$

In practice we solve equations (7.10),(7.11) in Ω with the boundary conditions

$$\theta^{n+1} \cdot \nu^\Omega = g_0$$

$$k * \left(g(\theta^{n+1}) \frac{\nabla k * u^{n+1}}{\sqrt{\epsilon + |\nabla k * u^n|^2}} \right) \cdot \nu^\Omega + (\eta + \alpha) \left(\frac{\nabla u^{n+1}}{\sqrt{\epsilon + |\nabla u^n|^2}} \right) \cdot \nu^\Omega = (\alpha + \eta) g_0$$

Now, since $\theta^n \cdot \nu^\Omega|_{\partial\Omega} = \theta^{n+1} \cdot \nu^\Omega|_{\partial\Omega}$ then we may write the first of the above boundary conditions as

$$\xi^{n+1} \cdot \nu^\Omega = 0.$$

On the other hand we approximate the second of the above boundary conditions by

$$k * \left(g(\theta^{n+1}) \frac{\nabla k * v^{n+1}}{\sqrt{\epsilon + |\nabla k * u^n|^2}} \right) \cdot \nu^\Omega + (\eta + \alpha) \left(\frac{\nabla v^{n+1}}{\sqrt{\epsilon + |\nabla u^n|^2}} \right) \cdot \nu^\Omega = 0$$

Then we may use a conjugate gradient method to solve (7.10) and (7.11). The constraint $|\theta| \leq 1$ is incorporated by renormalizing θ^n (when $|\theta_n| > 1$) after each time step. In spite of the penalization term, the relationship $|Du| = \theta \cdot Du$ is lost and we reinforce it after a certain number of time steps.

We can also incorporate the constraint that $|Du| = \theta \cdot Du$ by projecting onto it after each time step. Indeed, we have implemented this in both in a time implicit and explicit discretization of equations (7.1), (7.2). After each time step of θ and u we redefine

$$\theta(i,j) = \frac{\theta(i,j) + \tilde{\eta} Du(i,j)}{max(1, |\theta(i,j) + \tilde{\eta} Du(i,j)|)}$$

for some $\tilde{\eta} > 0$. As it has been shown in [42] this is a good way of imposing that $|\theta| \leq 1$ and $|Du| = \theta \cdot Du$. We have found quite similar results using both described methods.

In our experiments, we take k a Gaussian kernel with small variance, say one or two pixels. In practice, one can also dismiss the kernel k. The initial conditions are

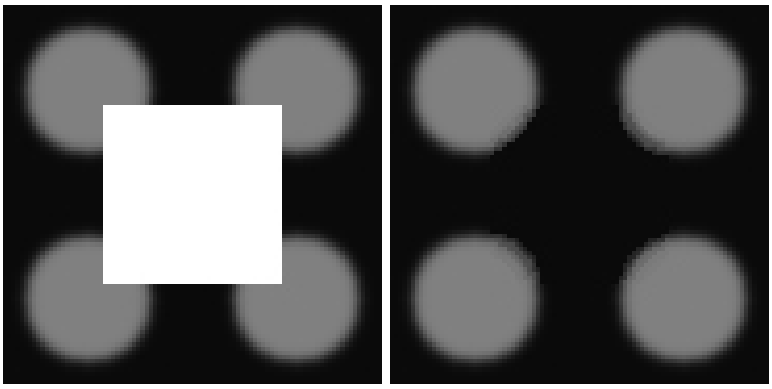

FIG. 7.1. *Left: Four circles. Right: Reconstructed image.*

ad-hoc interpolations, for instance, we can take $u = u_0$ on B, and on $\tilde{\Omega}$, take u equal to the average value of u_0 in B, θ inside Ω being the direction of the gradient of u. One can also take a geodesic propagation inside Ω of the values of u_0 in B, with θ being again the direction of the gradient of u.

In the experiments below, this algorithm is used to interpolate level sets, following the approach in [52], [54]. The image in B is decomposed into level sets and we get a family of binary images $u_{0\lambda} = \chi_{[u_0 \geq \lambda]}$, $\lambda = 0, 1, 2, ..., 255$. These functions are interpolated inside Ω and we obtain a family of level sets $X_\lambda u$. Then the function u is reconstructed using the reconstruction formula

$$u(x) = \sup\{\lambda \in \{0, 1, ..., 255\} : x \in X_\lambda u\}.$$

As observed in Remark 5 of Section 2.2, we force our solution to satisfy the monotonicity property of the level sets, i.e., that $X_{\lambda+1} u \subseteq X_\lambda u$. This could also be imposed in the initialization of the level set $X_\lambda u$ and maintained at each iteration of the algorithm by taking the supremum of the current solution with the characteristic function of $X_\lambda u$. With the level set approach, we diminish the diffusive effects of the above algorithm and we better capture the shapes and discontinuities on the interpolated image.

7.1. Experimental results: examples in 2D. In the following experiments we show the results of the joint interpolation of gray level and the vector field of directions using functional (3.2). The experiments have been done with $p = 1$ and/or $p = 2$. The results are quite similar and, unless explicitely stated, we display the results obtained with $p = 1$.

Figure 7.1 displays an image made of four circles covered by a square (left image) and the result of the interpolation (right image) obtained with $p = 2$. Figure 7.2.a is a detail of the mouth of Lena with a hole. Figures 7.2.b displays the result of the interpolation using (3.2). Figure 7.2.c shows the result of interpolating the hole of Figure 7.2.a by using a simple algorithm: the value of pixels at distance k from the boundary is the average of its neighboring pixels at distance $k-1$ from the boundary. In Figure 7.2.b we see the effect of continuing the level lines along the mouth, which is not the case in Figure 7.2.c. Figure 7.3.a is an image of Einstein smoking with a pipe. In Figure 7.3.b we have represented a hole covering the region of the pipe. In Figure 7.3.c we display the result of interpolating the hole of Figure 7.3.b using (3.2).

FIG. 7.2. *a) Left: Detail of the mouth of Lena. b) Middle: Reconstructed mouth using (3.2). c) Right: Result of interpolating the hole in a) by means of a propagation of neighbouring values.*

FIG. 7.3. *a) Left: Einstein with a pipe. b) Middle: Einstein with a mask on the pipe representing the hole. c) Right: Result of interpolation using (3.2)*

FIG. 7.4. *Removing the text on an image. a) Left: original image, b) Right: reconstructed image.*

Figure 7.4.a displays an image with text to be removed. Figure 7.4.b displays the corresponding reconstructed result.

7.2. Experimental results: the 3D case . Figure 7.5 displays a portion of a sphere with a hole of size $30 \times 30 \times 30$ seen from two different points of view and its corresponding reconstruction using functional (3.2) with $N = 3$. We see in the frontal view of the hole that the reconstruction is somewhat flat at the center, we have checked that this defect would not appear in a hole of size $20 \times 20 \times 20$. Figure 7.6.a displays a 3D image with a hole at the lower part of the right hand corner. Figure 7.6.b displays its reconstruction. The object appearing behind the queue was present

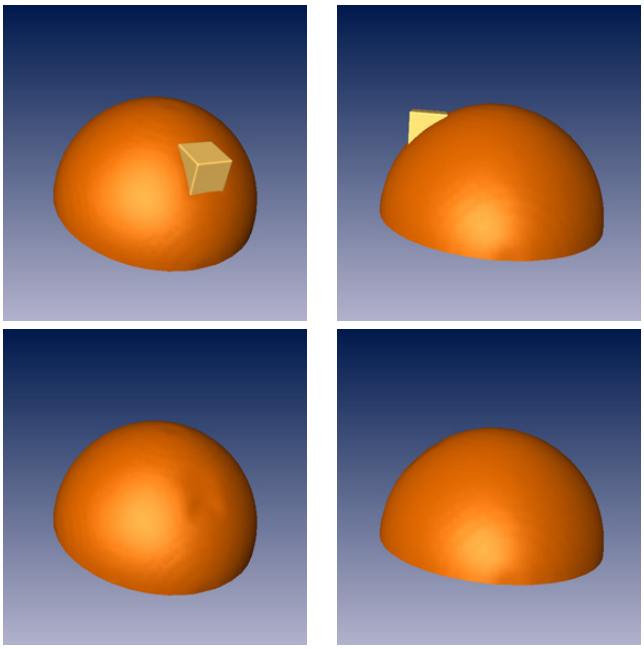

FIG. 7.5. *This Figure displays a piece of a sphere with a hole of size* $30 \times 30 \times 30$ *from two different point of views and the corresponding reconstruction*

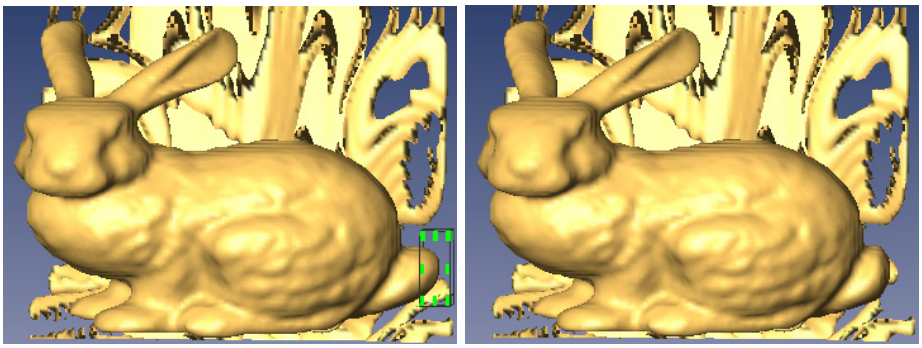

FIG. 7.6. *This is a experiment in 3D. a) Left: original image with the hole (the lower right box). b) Right: reconstructed image, the object appearing was hidden by the queue.*

in the top image, but it was hidden by it. The queue has been extended outside about 3 pixels and looks more flat. These two images have been rendered using the AMIRA Visualization and Modeling System [6].

Figure 7.7 displays six consecutive slices of a CT image with a scratch covering two frames. Figure 7.8 displays the corresponding result obtained using functional (3.2) in the 3D case.

Figure 7.9 displays nine frames of the Foreman video sequence. There are two scratches in the sequence, one of them extends along 6 frames (from frame 3 to frame 8) of the sequence while the second one is located at the ear in frame 6. We shall refer to them as right and left scratches respectively. Figure 7.10 displays the reconstructed

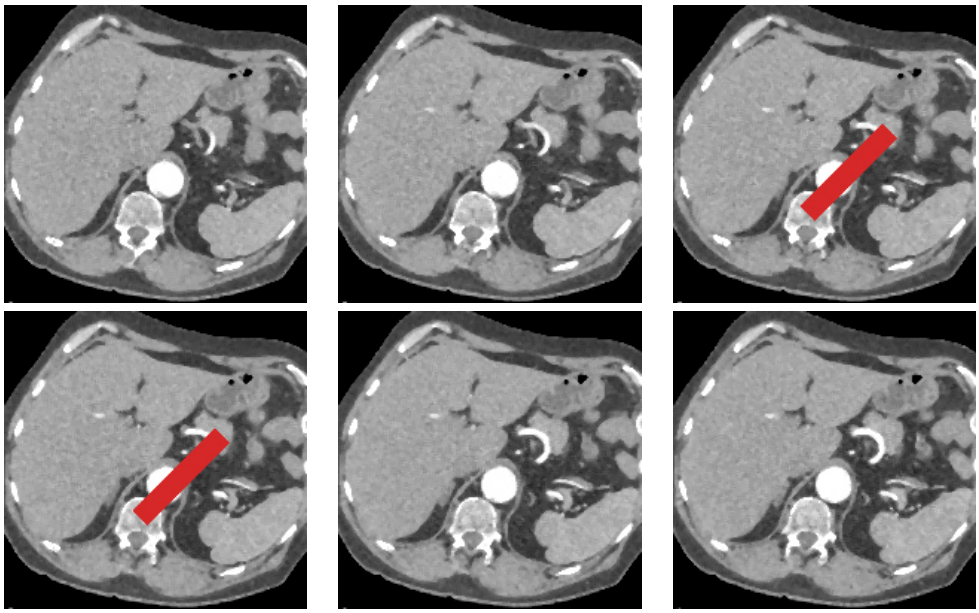

FIG. 7.7. *This Figure displays six consecutive slices of a CT image with a scratch covering two frames.*

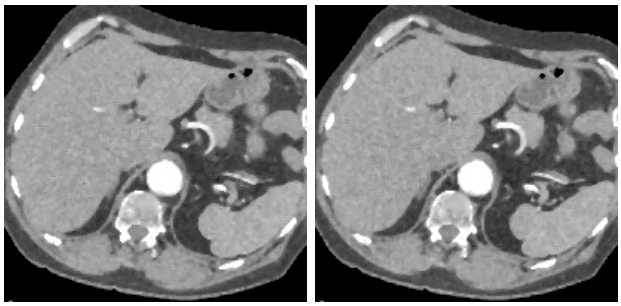

FIG. 7.8. *Interpolation of the scratch in Fig. 7.7.*

frames 3 to 8 obtained using functional (3.2) with $N = 3$. As we see the quality of the results depends on the scratch. The reconstruction of the right scratch which extends along 6 frames has some defects in frame 5, 6, and 7 along the face boundary. The reconstruction of the ear in frame 6 is too smooth and lacks of a textured appearance. In Figure 7.11 we have displayed the reconstruction of the ear in frame 6 of Figure 7.9 obtained by averaging frames 5 and 7 at the hole pixels, and we display a zoom of the result. Figure 7.12 displays the results of interpolating the right scratches of frames 5, 6 and 7 of Figure 7.9 (not ear's scratch) frame by frame using functional (3.2) with $N = 2$. These experiments show the complexity of using functional (3.2) with $N = 3$ for video sequences. Indeed, this functional is not adapted to video. Tipically, reconstructing video scratches requires an estimation of the optical flow and will be considered elsewhere [25]. On the other hand it may happen that in some cases it is preferable to use a internal frame reconstruction than using the estimation of the optical flow.

FIG. 7.9. *9 consecutive frames of the Foreman sequence with several scratches.*

FIG. 7.10. *The results of interpolating the scratches of Figure 7.9. We displayed only frames 3 to 8 of the Foreman sequence.*

7.3. Experimental results: 2D zoom. Figure 7.13 displays a rose and a detail of it. Figure 7.14.a displays the result of zooming by a factor of 8 the rose detail of Figure 7.13. This zoom has been obtained using functional (3.2) with $N = 2$.

FIG. 7.11. *Left: Result of reconstructing ear's hole in frame 6 of Figure 7.9 by averaging previous and next frame in the hole. Right: a zoom of the result.*

FIG. 7.12. *The results of interpolating the scratches of frames 5, 6 and 7 of Figure 7.9 frame by frame using functional (3.2) with* $N = 2$.

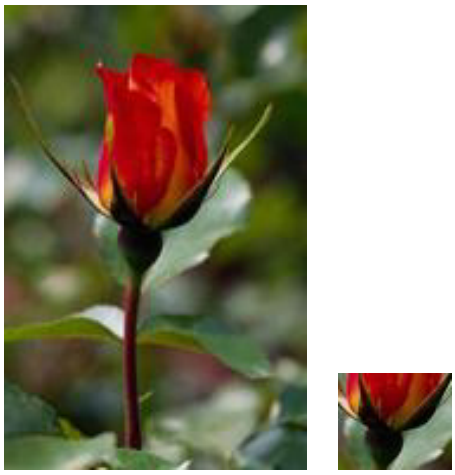

FIG. 7.13. *a) Left: a rose image. b) Right: a detail of it.*

Figure 7.14.b displays the zoom of factor 8 of the same detail obtained using Bicubic Photoshop interpolation. Comparing both Figures we observe a better preservation of geometry in Figure 7.14.a, but a similar result could be obtained by using Total Variation zooming [51].

8. Concluding remarks. In this paper we have proposed a variational approach based on the energy functional (3.2) for filling-in regions of missing data in still 2D and 3D images. The basic idea is to smoothly extend inside the hole both the vector field obtained from the image gradient and the corresponding gray values. We have proved

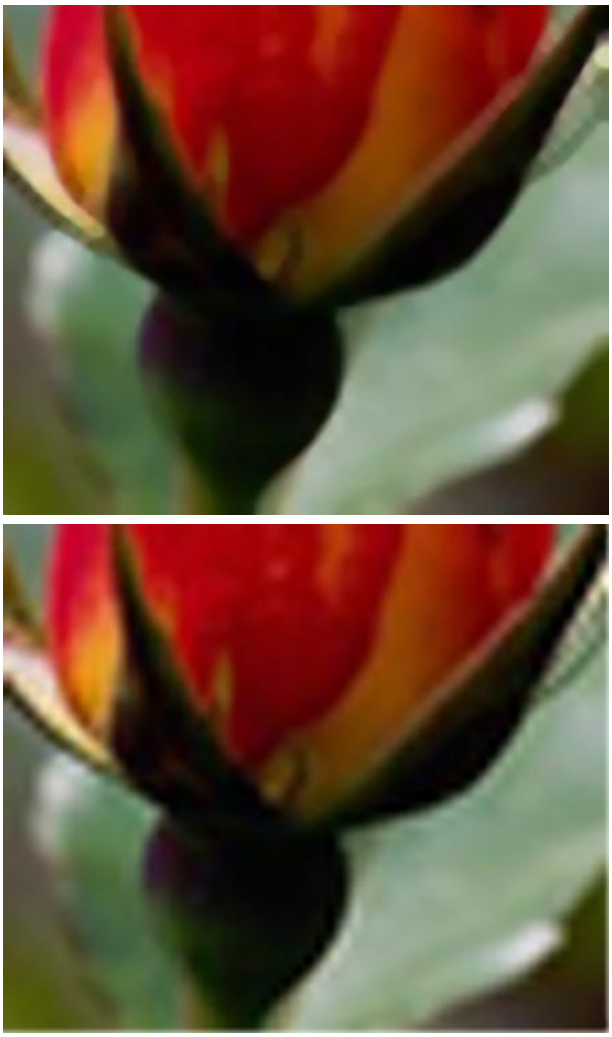

FIG. 7.14. *a) Left: Zoom by a factor of* 8 *of the rose detail of Figure 7.13 obtained using functional (3.2) with* $N = 2$. *b) Right: a zoom of factor 8 obtained using Bicubic Photoshop interpolation. We observe a better preservation of geometry in Figure a), but a similar result could be obtained using the Total Variation zooming prposed in [51].*

existence of minimizers for functional (3.2) and proved that it can be approximated by functionals (5.53). We have displayed some numerical experiments for 2D and 3D images, and we have discussed its limitations in the case of video sequences.

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