# Simultaneous Avoidance of Large Squares and Fractional Powers in Infinite Binary Words 

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#### Abstract

In 1976, Dekking showed that there exists an infinite binary word that contains neither squares $y y$ with $|y| \geq 4$ nor cubes $x x x$. We show that 'cube' can be replaced by any fractional power $>5 / 2$. We also consider the analogous problem where ' 4 ' is replaced by any integer. This results in an interesting and subtle hierarchy.


## 1 Introduction

A square is a nonempty word of the form $y y$, as in the English word murmur. It is easy to see that every word of length $\geq 4$ constructed from the symbols 0 and 1 contains a square, so it is impossible to avoid squares in infinite binary words. However, in 1974, Entringer, Jackson, and Schatz [3] proved the surprising fact that there exists an infinite binary word containing no squares $y y$ with $|y| \geq 3$. Further, the bound 3 is best possible.

A cube is a nonempty word of the form $x x x$, as in the English sort-of-word shshsh. An overlap is a word of the form axaxa, where $a$ is a single letter and $x$ is a (possibly empty) word, as in the French word entente. Dekking [2] showed that there exists an infinite binary word that contains neither squares $y y$ with $|y| \geq 4$ nor cubes $x x x$. Furthermore, the bound 4 is best possible. He also proved that every overlap-free word contains arbitrarily large squares.

These two results suggest the following natural question: for each length $l \geq 1$, determine the fractional exponent $p$ (if it exists) such that
(a) there is no infinite binary word simultaneously avoiding squares $y y$ with $|y| \geq l$ and fractional powers $x^{e}$ with $e \geq p$;
(b) there is an infinite binary word simultaneously avoiding squares $y y$ with $|y| \geq l$ and fractional powers $x^{e}$ with $e>p$ ?

Here we say a word $w$ is an $e^{\prime}$ th power (e rational) if there exist words $y, y^{\prime} \in \Sigma^{*}$ such that $w=y^{n} y^{\prime}$, and $y^{\prime}$ is a prefix of $y$ with $n+\left|y^{\prime}\right| /|y|=e$. For example, the English word abracadabra is an $\frac{11}{7}$-power. We say a word avoids $p$ powers if it contains no subword of the form $y^{e}$ with $e \geq p$. We say a word avoids $p^{+}$powers if it contains no subword of the form $y^{e}$ with $e>p$.

In this paper we completely resolve this question. It turns out there is a rather subtle hierarchy depending on $l$. The results are summarized in Table 1.

| minimum length $l$ <br> of square avoided | avoidable <br> power | unavoidable <br> power |
| :---: | :---: | :---: |
| 2 | none | all |
| 3 | $3^{+}$ | 3 |
| $4,5,6$ | $(5 / 2)^{+}$ | $5 / 2$ |
| $\geq 7$ | $(7 / 3)^{+}$ | $7 / 3$ |

Figure 1: Summary of Results
More precisely, we have

## Theorem 1

(a) There are no infinite binary words that avoid all squares yy with $|y| \geq 2$.
(b) There are no infinite binary words that simultaneously avoid all squares yy with $|y| \geq 3$ and cubes xxx.
(c) There is an infinite binary word that simultaneously avoids all squares yy with $|y| \geq 3$ and $3^{+}$powers.
(d) There is an infinite binary word that simultaneously avoids all squares yy with $|y| \geq 4$ and $\frac{5}{2}^{+}$powers.
(e) There are no infinite binary words that simultaneously avoid all squares yy with $|y| \geq 6$ and $\frac{5}{2}$ powers.
(f) There is an infinite binary word that simultaneously avoids all squares yy with $|y| \geq 7$ and $\frac{7}{3}^{+}$powers.
(g) For all $t \geq 1$, there are no infinite binary words that simultaneously avoid all squares yy with $|y| \geq t$ and $\frac{7}{3}$ powers.

The result (a) is originally due to Entringer, Jackson, and Schatz [3]. The result (b) is due to Dekking [2]. The result (g) appears in a recent paper of the author and J. Karhumäki [5]. We mention them for completeness. The remaining results are new.

## 2 Proofs of the negative results

We say a word avoids $(l, p)$ if it simultaneously avoids squares $y y$ with $|y| \geq l$ and $p$ powers.
The negative results (a), (b), and (e) can be proved purely mechanically. The idea is as follows. Given $l$ and $p$, we can create a tree $T=T(l, p)$ of all binary words avoiding $(l, p)$ as follows: the root of $T$ is labeled $\epsilon$. If a node is labeled $w$ and avoids $(l, p)$, then it is an internal node with two children, where the left child is labeled $w 0$ and the right child is labeled $w 1$. If it does not avoid ( $l, p$ ), then it is an external node (or "leaf").

It is now easy to see that no infinite word avoiding $(l, p)$ exists if and only if $T(l, p)$ is finite. In this case, a breadth-first search will suffice to resolve the question. Furthermore, certain parameters of $T(l, p)$ correspond to information about the finite words avoiding $(l, p)$ :

- the number of leaves $n$ is one more than the number of internal nodes, and so $n-1$ represents the total number of finite words avoiding $(l, p)$;
- if the height of the tree (i.e., the length of the longest path from the root to a leaf) is $h$, then $h$ is the smallest integer such that there are no words of length $\geq h$ avoiding $(l, p)$;
- the internal nodes at depth $h-1$ gives the all words of maximal length avoiding $(l, p)$.

The following table lists ( $l, p, n, h, t, S$ ), where

- $l=|y|$, where one is trying avoiding $y y$;
- $p$, the fractional exponent one is trying to avoid;
- $n$, the number of leaves of $T(l, p)$;
- $h$, the height of the tree $T(l, p)$.
- $t$, the number of internal nodes at depth $h-1$ in the tree.
- $S$, the set of labels of the internal nodes at depth $h-1$ that start with 0 . (The other words can be obtained simply by interchanging 0 and 1.)

For completeness, we give the results for the optimal exponents for $2 \leq l \leq 7$. As mentioned above, the case $l=2$ is due to Entringer, Jackson, and Schatz [3] and the case $l=3$ is due to Dekking [2].

| $l$ | $p$ | $n$ | $h$ | $t$ | $S$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2 | $\infty$ | 478 | 19 | 2 | \{010011000111001101\} |
| 3 | 3 | 578 | 30 | 2 | \{00110010100110101100101001100\} |
| 4 | 5/2 | 6860 | 84 | 4 | $\{0010110100101100100110110010110100110110010011010010110010011011001010100110110011$, 00110010011010010110010011011001011010011011001001101001011001001101100101101001011\} |
| 5 | 5/2 | 15940 | 93 | 2 | $\{00100101100110100101100100110110010110100110110010011010010110010011011001011010011001011011\}$ |
| 6 | 5/2 | 15940 | 93 | 2 | \{001001011001101000101100100110110010110100110110000011010010110010011011001011010011001011011\} |
| 7 | 7/3 | 3548 | 43 | 2 | \{001011001011010011001011001101001011001011\} |

Figure 2: Proofs of the negative results (a), (b), (e)

## 3 Proof of (c)

In this section we prove that there is an infinite binary word that simultaneously avoids $y y$ with $|y| \geq 3$ and $3^{+}$powers.

We introduce the following notation for alphabets: $\Sigma_{k}:=\{0,1, \ldots, k-1\}$.
Let the morphism $f: \Sigma_{3}^{*} \rightarrow \Sigma_{2}^{*}$ be defined as follows.

$$
\begin{aligned}
& 0 \rightarrow 0010111010 \\
& 1 \rightarrow 0010101110 \\
& 2 \rightarrow 0011101010
\end{aligned}
$$

We will prove
Theorem 2 If $w$ is any squarefree word over $\Sigma_{3}$, then $f(w)$ avoids yy with $|y| \geq 3$ and $3^{+}$ powers.

Proof. We argue by contradiction. Let $w=a_{1} a_{2} \cdots a_{n}$ be a squarefree string such that $f(w)$ contains a square $y y$ with $|y| \geq 3$, i.e., $f(w)=x y y z$ for some $x, z \in \Sigma_{2}^{*}, y \in\left\{\Sigma_{2}^{\geq 3}\right.$. Without loss of generality, assume that $w$ is a shortest such string, so that $0 \leq|x|,|z|<20$.

Case 1: $|y| \leq 20$. In this case we can take $|w| \leq 5$. To verify that $f(w)$ has no squares $y y$ with $|y| \geq 3$, it therefore suffices to check each of the 30 possible words $w \in \Sigma_{2}^{5}$.

Case 2: $|y|>20$. First, we establish the following result.
Lemma 3 (a) (inclusion property) Suppose $f(a b)=t f(c) u$ for some letters $a, b, c \in \Sigma_{2}$ and strings $t, u \in \Sigma_{2}^{*}$. Then this inclusion is trivial (that is, $t=\epsilon$ or $u=\epsilon$ ).
(b) (interchange property) Suppose there exist letters $a, b, c$ and strings $s, t, u, v$ such that $f(a)=s t, f(b)=u v$, and $f(c)=s v$. Then either $a=c$ or $b=c$.

## Proof.

(a) A short computation verifies there are no $a, b, c$ for which the equality $f(a b)=t f(c) u$ holds nontrivially.
(b) This can also be verified with a short computation. If $|s| \geq 6$, then no two distinct letters share a prefix of length 6. If $|s| \leq 5$, then $|t| \geq 5$, and no two distinct letters share a suffix of length 5 .

Once Lemma 3 is established, the rest of the argument is fairly standard. It can be found, for example, in [5], but for completeness we repeat it here.

For $i=1,2, \ldots, n$ define $A_{i}=f\left(a_{i}\right)$. Then if $f(w)=x y y z$, we can write

$$
f(w)=A_{1} A_{2} \cdots A_{n}=A_{1}^{\prime} A_{1}^{\prime \prime} A_{2} \cdots A_{j-1} A_{j}^{\prime} A_{j}^{\prime \prime} A_{j+1} \cdots A_{n-1} A_{n}^{\prime} A_{n}^{\prime \prime}
$$

where

$$
\begin{aligned}
A_{1} & =A_{1}^{\prime} A_{1}^{\prime \prime} \\
A_{j} & =A_{j}^{\prime} A_{j}^{\prime \prime} \\
A_{n} & =A_{n}^{\prime} A_{n}^{\prime \prime} \\
x & =A_{1}^{\prime} \\
y & =A_{1}^{\prime \prime} A_{2} \cdots A_{j-1} A_{j}^{\prime}=A_{j}^{\prime \prime} A_{j+1} \cdots A_{n-1} A_{n}^{\prime} \\
z & =A_{n}^{\prime \prime},
\end{aligned}
$$

where $\left|A_{1}^{\prime \prime}\right|,\left|A_{j}^{\prime \prime}\right|>0$. See Figure 3,


Figure 3: The string $x y y z$ within $f(w)$
If $\left|A_{1}^{\prime \prime}\right|>\left|A_{j}^{\prime \prime}\right|$, then $A_{j+1}=f\left(a_{j+1}\right)$ is a subword of $A_{1}^{\prime \prime} A_{2}$, hence a subword of $A_{1} A_{2}=$ $f\left(a_{1} a_{2}\right)$. Thus we can write $A_{j+2}=A_{j+2}^{\prime} A_{j+2}^{\prime \prime}$ with

$$
A_{1}^{\prime \prime} A_{2}=A_{j}^{\prime \prime} A_{j+1} A_{j+2}^{\prime}
$$

See Figure 4


Figure 4: The case $\left|A_{1}^{\prime \prime}\right|>\left|A_{j}^{\prime \prime}\right|$

But then, by Lemma 3 (a), either $\left|A_{j}^{\prime \prime}\right|=0$, or $\left|A_{1}^{\prime \prime}\right|=\left|A_{j}^{\prime \prime}\right|$, or $A_{j+2}^{\prime}$ is a not a prefix of any $f(d)$. All three conclusions are impossible.

If $\left|A_{1}^{\prime \prime}\right|<\left|A_{j}^{\prime \prime}\right|$, then $A_{2}=f\left(a_{2}\right)$ is a subword of $A_{j}^{\prime \prime} A_{j+1}$, hence a subword of $A_{j} A_{j+1}=$ $f\left(a_{j} a_{j+1}\right)$. Thus we can write $A_{3}=A_{3}^{\prime} A_{3}^{\prime \prime}$ with

$$
A_{1}^{\prime \prime} A_{2} A_{3}^{\prime}=A_{j}^{\prime \prime} A_{j+1}
$$

See Figure 5


Figure 5: The case $\left|A_{1}^{\prime \prime}\right|<\left|A_{j}^{\prime \prime}\right|$
By Lemma 3 (a), either $\left|A_{1}^{\prime \prime}\right|=0$ or $\left|A_{1}^{\prime \prime}\right|=\left|A_{j}^{\prime \prime}\right|$ or $A_{3}^{\prime}$ is not a prefix of any $f(d)$. Again, all three conclusions are impossible.

Therefore $\left|A_{1}^{\prime \prime}\right|=\left|A_{j}^{\prime \prime}\right|$. Hence $A_{1}^{\prime \prime}=A_{j}^{\prime \prime}, A_{2}=A_{j+1}, \ldots, A_{j-1}=A_{n-1}$, and $A_{j}^{\prime}=A_{n}^{\prime}$. Since $h$ is injective, we have $a_{2}=a_{j+1}, \ldots, a_{j-1}=a_{n-1}$. It also follows that $|y|$ is divisible by 10 and $A_{j}=A_{j}^{\prime} A_{j}^{\prime \prime}=A_{n}^{\prime} A_{1}^{\prime \prime}$. But by Lemma (b), either (1) $a_{j}=a_{n}$ or (2) $a_{j}=a_{1}$. In the first case, $a_{2} \cdots a_{j-1} a_{j}=a_{j+1} \cdots a_{n-1} a_{n}$, so $w$ contains the square $\left(a_{2} \cdots a_{j-1} a_{j}\right)^{2}$, a contradiction. In the second case, $a_{1} \cdots a_{j-1}=a_{j} a_{j+1} \cdots a_{n-1}$, so $w$ contains the square $\left(a_{1} \cdots a_{j-1}\right)^{2}$, a contradiction.

It now follows that if $w$ is squarefree then $f(w)$ avoids squares $y y$ with $|y| \geq 3$.
It remains to see that $f(w)$ avoids $3^{+}$powers. If $f(w)$ contained $x^{e}$ for some fractional exponent $e>3$, then it would contain $x^{2}$, so from above we have $|x| \leq 2$. Thus it suffices to show that $f(w)$ avoids the words $0000,1111,0101010,1010101$. This can be done by a short computation.

Corollary 4 There is an infinite binary word avoiding squares yy with $|y| \geq 3$ and $3^{+}$ powers.

Proof. As is very well-known, there are infinite squarefree words over $\Sigma_{3}$ [7, 1]. Take any such word $\mathbf{w}$ (for example, the fixed point of the morphism $2 \rightarrow 210,1 \rightarrow 20,0 \rightarrow 1$ ), and apply the map $f$. The resulting word $f(\mathbf{w})$ avoids $\left(3,3^{+}\right)$.

It may be of some interest to explain how the morphism $f$ was discovered. We iteratively generated all words of length $1,2,3, \ldots$ (up to some bound) that avoid $\left(3,3^{+}\right)$. We then guessed such words were the image of a $k$-uniform morphism applied to a squarefree word over $\Sigma_{3}$. For values of $k=2,3, \ldots$, we broke up each word into contiguous blocks of size $k$, and discarded any word for which there were more than 3 blocks. For certain values of $k$, this procedure eventually resulted in 0 words fitting the criteria. At this point we knew a $k$-uniform morphism cannot work, so we increased $k$ and started over. Eventually a $k$ was
found for which the number of such words appeared to increase without bound. We then examined the possible sets of $3 k$-blocks to see if any satisfied the requirements of Lemma 3 This gave our candidate morphism $f$.

Theorem 5 Let $A_{n}$ denote the number of binary words of length $n$ avoiding yy with $|y| \geq 3$ and $3^{+}$powers. Then $A_{n}=\Omega\left(1.01^{n}\right)$ and $A_{n}=O\left(1.49^{n}\right)$.

Proof. Grimm [4] has shown there are $\Omega\left(\lambda^{n}\right)$ squarefree words over $\Sigma_{3}$, where $\lambda=1.109999$. Since the map $f$ is 10 -uniform, it follows that $A_{n}=\Omega\left(\lambda^{n / 10}\right)=\Omega\left(1.01^{n}\right)$.

For the upper bound, we reason as follows. The set of binary words of length $n$ avoiding $y y$ with $|y| \geq 3$ and $3^{+}$powers is a subset of the set of binary words avoiding 0000 and 1111. The number $A_{n}^{\prime}$ of words avoiding 0000 and 1111 satisfies the linear recurrence $A_{n}^{\prime}=$ $A_{n-1}^{\prime}+A_{n-2}^{\prime}+A_{n-3}^{\prime}$ for $n \geq 4$. It follows that $A_{n}^{\prime}=O\left(\alpha^{n}\right)$, where $\alpha$ is the largest zero of $x^{3}-x^{2}-x-1$, the characteristic polynomial of the recurrence. Here $\alpha<1.84$, so $A_{n}=O\left(1.84^{n}\right)$.

This reasoning can be extended using a symbolic algebra package such as Maple. Noonan and Zeilberger [6] have written a Maple package DAVID_IAN that allows one to specify a list $L$ of forbidden words, and computes the generating function enumerating words avoiding members of $L$. We used this package for a list $L$ of 62 words of length $\leq 12$ :

$$
0000,1111, \ldots, 111010111010
$$

obtaining a characteristic polynomial of degree 67 with dominant zero $\doteq 1.4895$.

## 4 Proof of (e)

In this section we prove that there is an infinite binary word that simultaneously avoids $y y$ with $|y| \geq 4$ and $\frac{5}{2}^{+}$powers.

Let $g_{1}: \Sigma_{8}^{*} \rightarrow \Sigma_{2}^{*}$ be defined as follows.

$$
\begin{aligned}
& 0 \rightarrow 0011010010110 \\
& 1 \rightarrow 0011010110010 \\
& 2 \rightarrow 0011011001011 \\
& 3 \rightarrow 0100110110010 \\
& 4 \rightarrow 0110100101100 \\
& 5 \rightarrow 1001101011001 \\
& 6 \rightarrow 1001101100101 \\
& 7 \rightarrow 1010011011001
\end{aligned}
$$

Let $g_{2}: \Sigma_{4}^{*} \rightarrow \Sigma_{8}^{*}$ be defined as follows.

$$
\begin{aligned}
& 0 \rightarrow 03523503523453461467 \\
& 1 \rightarrow 03523503523453467167 \\
& 2 \rightarrow 16703523503523461467 \\
& 3 \rightarrow 03523503523461467167
\end{aligned}
$$

Let $g_{3}: \Sigma_{3}^{*} \rightarrow \Sigma_{4}^{*}$ be defined as follows.

$$
\begin{aligned}
& 0 \rightarrow 010203 \\
& 1 \rightarrow 010313 \\
& 2 \rightarrow 021013
\end{aligned}
$$

Finally, define $g: \Sigma_{3}^{*} \rightarrow \Sigma_{2}^{*}$ by $g=g_{1} \circ g_{2} \circ g_{3}$. Note that $g$ is 1560 -uniform. We will prove

Theorem 6 If $w$ is any squarefree word over $\Sigma_{3}$, then $g(w)$ avoids yy with $|y| \geq 4$ and $\frac{5}{2}^{+}$ powers.

Proof. The proof is very similar to the proof of Theorem 2, and we indicate only what must be changed.

First, it can be checked that Lemma 3 also holds for the morphism $g$.
As before, we break the proof up into two parts: the case where $g(w)=x y y z$ for some $y$ with $4 \leq|y| \leq 2 \cdot 1560$, and the case where $g(w)=x y y z$ for some $y$ with $|y| \geq 2 \cdot 1560$. The former can be checked by examining the image of the 30 squarefree words in $\Sigma_{3}^{5}$ under $g$. The latter is handled as we did in the proof of Theorem 2. We checked these conditions with programs written in Pascal; these are available from the author on request.

Corollary 7 There is an infinite binary word avoiding squares yy with $|y| \geq 4$ and $\frac{5}{2}^{+}$ powers.

It may be of some interest to explain how the morphisms $g_{1}, g_{2}, g_{3}$, were discovered.
We used a procedure analogous to that described above in Section 3. However, since it was not feasible to generate all words avoiding $\left(4, \frac{5}{2}^{+}\right)$and having at most 3 contiguous blocks of length 1560 , we increased the alphabet size and and tried various $k$-blocks until we found a combination of alphabet size and block size for which the number of words appeared to increase without bound. We then obtained a number of possible candidates for blocks.

Next, we determined the necessary avoidance properties of the blocks given by images of letters under $g_{1}$. For example, $g_{1}(0)$ cannot be followed by $g_{1}(1)$, because this results in the subword 000 , which is a 3 rd power (and $3>2.5$ ). The blocks that must be avoided include all words with squares, and

$$
01,02,04,05,06,07,10,12,13,17,20,21,24,25,26,27,30,31,32,36,37,40,41,42,43,47 \text {, }
$$

$51,54,56,57,60,62,63,64,65,72,73,74,75,76,034,145,153,161,353,450,452,535,615,616$, $714,715,2346703,5234670,5234671,53467035,6703523461,2346146703503,5234614670350$
This list was computed purely mechanically, and it is certainly possible that this list is not exhaustive.

We now iterated our guessing procedure, looking for a candidate uniform morphism that creates squarefree words avoiding the patterns in the list above. This resulted in the $20-$ uniform morphism $g_{2}$.

We then computed the blocks that must be avoided for $g_{2}$. This was done purely mechanically. Our procedure suggested that arbitrarily large blocks must be avoided, but luckily they (apparently) had a simple finite description: namely, we must avoid $12,23,32$, and blocks of the form $2 x 0 x 1$ and $3 x 1 x 0$ for all nonempty words $x$, in addition to words with squares.

We then iterated our guessing procedure one more time, looking for a candidate uniform morphism that avoids these patterns. This gave us the morphism $g_{3}$.

Of course, once the morphism $g=g_{1} \circ g_{2} \circ g_{3}$ is discovered, we need not rely on the list of avoidable blocks; we can take the morphism as given and simply verify the properties of inclusion and interchange as in Lemma 3

Theorem 8 Let $B_{n}$ denote the number of binary words of length $n$ avoiding yy with $|y| \geq 4$ and $\frac{5}{2}^{+}$powers. Then $B_{n}=\Omega\left(1.000066^{n}\right)$ and $B_{n}=O\left(1.122^{n}\right)$.

Proof. The proof is analogous to that of Theorem 5. We use the fact that $g$ is 1560 -uniform, which, combined with the result of Grimm [4], gives the bound $1.109999^{1 / 1560} \doteq 1.000066899$.

For the upper bound, we again use the Noonan-Zeilberger Maple package. We used the 54 patterns corresponding to words of length $\leq 20$. This gave us a polynomial of degree 27 with dominant zero $\doteq 1.12123967$.

## 5 Proof of (f)

In this section we prove that there is an infinite binary word that simultaneously avoids $y y$ with $|y| \geq 7$ and $\frac{7}{3}^{+}$powers.

Let $h_{1}: \Sigma_{5}^{*} \rightarrow \Sigma_{2}^{*}$ be defined as follows.

$$
\begin{aligned}
& 0 \rightarrow 00110100101100 \\
& 1 \rightarrow 00110100110010 \\
& 2 \rightarrow 01001100101100 \\
& 3 \rightarrow 10011011001011 \\
& 4 \rightarrow 11010011011001
\end{aligned}
$$

Let $h_{2}: \Sigma_{3}^{*} \rightarrow \Sigma_{5}^{*}$ be defined as follows.

$$
\begin{aligned}
& 0 \rightarrow 032303241403240314 \\
& 1 \rightarrow 032314041403240314 \\
& 2 \rightarrow 032414032303240314
\end{aligned}
$$

Finally, define $h: \Sigma_{3}^{*} \rightarrow \Sigma_{2}^{*}$ by $h=h_{1} \circ h_{2}$. Note that $h$ is 252 -uniform. We will prove

Theorem 9 If $w$ is any squarefree word over $\Sigma_{3}$, then $h(w)$ avoids yy with $|y| \geq 7$ and $\frac{7^{+}}{}{ }^{+}$ powers.

Proof. Again, the proof is quite similar to that of Theorem 2. We leave it to the reader to verify that the inclusion and interchange properties hold for $h$, and that the image of all the squarefree words of length $\leq 5$ are free of squares $y y$ with $|y|<7$ and $\frac{7}{3}^{+}$powers.

Corollary 10 There is an infinite binary word avoiding squares $y y$ with $|y| \geq 7$ and $\frac{7}{3}^{+}$ powers.

The morphisms $h_{1}, h_{2}$ were discovered using the heuristic procedure mentioned in Section 3. The avoiding blocks for $h_{1}$ were heuristically discovered to include

$$
01,02,10,12,13,20,21,34,42,43,304,23031,24041,231403141,232403241
$$

as well as blocks containing any squares. Then $h_{2}$ was constructed to avoid these blocks.
Theorem 11 Let $C_{n}$ denote the number of binary words of length $n$ avoiding yy with $|y| \geq 7$ and $\frac{7}{3}^{+}$powers. Then $C_{n}=\Omega\left(1.0004^{n}\right)$ and $C_{n}=O\left(1.162^{n}\right)$.

Proof. The proof is very similar to that of Theorems 5 and 8 ,
For the lower bound, note that $h$ is 252 -uniform. This, combined with the bound of Grimm [4], gives a lower bound of $\Omega\left(\lambda^{n}\right)$ for all $\lambda<1.109999^{1 / 252} \doteq 1.0004142$.

For the upper bound, we again used the Noonan-Zeilberger Maple package. We avoided 58 words of length $\leq 20$. This resulted in a polynomial of degree 26 , with dominant zero $\doteq 1.1615225$.

## 6 Enumeration results

In this section we provide a table of the first values of the sequences $A_{n}, B_{n}$, and $C_{n}$, defined in Sections 3, 4, and 5, for $1 \leq n \leq 25$.

| $n$ | $A_{n}$ | $B_{n}$ | $C_{n}$ |
| ---: | ---: | ---: | ---: |
| 0 | 1 | 1 | 1 |
| 1 | 2 | 2 | 2 |
| 2 | 4 | 4 | 4 |
| 3 | 8 | 6 | 6 |
| 4 | 14 | 10 | 10 |
| 5 | 26 | 16 | 14 |
| 6 | 42 | 24 | 20 |
| 7 | 68 | 36 | 30 |
| 8 | 100 | 46 | 38 |
| 9 | 154 | 64 | 50 |
| 10 | 234 | 74 | 64 |
| 11 | 356 | 88 | 86 |
| 12 | 514 | 102 | 108 |
| 13 | 768 | 114 | 136 |
| 14 | 1108 | 124 | 164 |
| 15 | 1632 | 140 | 196 |
| 16 | 2348 | 160 | 226 |
| 17 | 3434 | 178 | 264 |
| 18 | 4972 | 198 | 322 |
| 19 | 7222 | 212 | 384 |
| 20 | 10356 | 230 | 436 |
| 21 | 14962 | 256 | 496 |
| 22 | 21630 | 294 | 578 |
| 23 | 31210 | 342 | 674 |
| 24 | 44846 | 366 | 754 |
| 25 | 64584 | 392 | 850 |

Figure 6: Values of $A_{n}, B_{n}, C_{n}, 0 \leq n \leq 25$

## 7 Acknowledgments

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