Simultaneous Avoidance of Large Squares and Fractional Powers in Infinite Binary Words

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Abstract

In 1976, Dekking showed that there exists an infinite binary word that contains neither squares yy with $|y| \ge 4$ nor cubes xxx. We show that 'cube' can be replaced by any fractional power > 5/2. We also consider the analogous problem where '4' is replaced by any integer. This results in an interesting and subtle hierarchy.

1 Introduction

A square is a nonempty word of the form yy, as in the English word murmur. It is easy to see that every word of length ≥ 4 constructed from the symbols 0 and 1 contains a square, so it is impossible to avoid squares in infinite binary words. However, in 1974, Entringer, Jackson, and Schatz [3] proved the surprising fact that there exists an infinite binary word containing no squares yy with $|y| \geq 3$. Further, the bound 3 is best possible.

A cube is a nonempty word of the form xxx, as in the English sort-of-word shshsh. An overlap is a word of the form axaxa, where a is a single letter and x is a (possibly empty) word, as in the French word entente. Dekking [2] showed that there exists an infinite binary word that contains neither squares yy with $|y| \ge 4$ nor cubes xxx. Furthermore, the bound 4 is best possible. He also proved that every overlap-free word contains arbitrarily large squares.

These two results suggest the following natural question: for each length $l \geq 1$, determine the fractional exponent p (if it exists) such that

(a) there is no infinite binary word simultaneously avoiding squares yy with $|y| \ge l$ and fractional powers x^e with $e \ge p$;

(b) there is an infinite binary word simultaneously avoiding squares yy with $|y| \ge l$ and fractional powers x^e with e > p?

Here we say a word w is an e'th power (e rational) if there exist words $y, y' \in \Sigma^*$ such that $w = y^n y'$, and y' is a prefix of y with n + |y'|/|y| = e. For example, the English word abracadabra is an $\frac{11}{7}$ -power. We say a word avoids p powers if it contains no subword of the form y^e with $e \geq p$. We say a word avoids p^+ powers if it contains no subword of the form y^e with e > p.

In this paper we completely resolve this question. It turns out there is a rather subtle hierarchy depending on l. The results are summarized in Table 1.

minimum length l	avoidable	unavoidable
of square avoided	power	power
2	none	all
3	3+	3
4, 5, 6	$(5/2)^+$	5/2
≥ 7	$(7/3)^+$	7/3

Figure 1: Summary of Results

More precisely, we have

Theorem 1

- (a) There are no infinite binary words that avoid all squares yy with $|y| \geq 2$.
- (b) There are no infinite binary words that simultaneously avoid all squares yy with $|y| \ge 3$ and cubes xxx.
- (c) There is an infinite binary word that simultaneously avoids all squares yy with $|y| \ge 3$ and 3^+ powers.
- (d) There is an infinite binary word that simultaneously avoids all squares yy with $|y| \ge 4$ and $\frac{5}{2}^+$ powers.
- (e) There are no infinite binary words that simultaneously avoid all squares yy with $|y| \ge 6$ and $\frac{5}{2}$ powers.
- (f) There is an infinite binary word that simultaneously avoids all squares yy with $|y| \ge 7$ and $\frac{7}{3}$ powers.
- (g) For all $t \ge 1$, there are no infinite binary words that simultaneously avoid all squares yy with $|y| \ge t$ and $\frac{7}{3}$ powers.

The result (a) is originally due to Entringer, Jackson, and Schatz [3]. The result (b) is due to Dekking [2]. The result (g) appears in a recent paper of the author and J. Karhumäki [5]. We mention them for completeness. The remaining results are new.

2 Proofs of the negative results

We say a word avoids (l, p) if it simultaneously avoids squares yy with $|y| \ge l$ and p powers. The negative results (a), (b), and (e) can be proved purely mechanically. The idea is as follows. Given l and p, we can create a tree T = T(l, p) of all binary words avoiding (l, p) as follows: the root of T is labeled ϵ . If a node is labeled w and avoids (l, p), then it is an internal node with two children, where the left child is labeled w0 and the right child is labeled w1. If it does not avoid (l, p), then it is an external node (or "leaf").

It is now easy to see that no infinite word avoiding (l, p) exists if and only if T(l, p) is finite. In this case, a breadth-first search will suffice to resolve the question. Furthermore, certain parameters of T(l, p) correspond to information about the finite words avoiding (l, p):

- the number of leaves n is one more than the number of internal nodes, and so n-1 represents the total number of finite words avoiding (l, p);
- if the height of the tree (i.e., the length of the longest path from the root to a leaf) is h, then h is the smallest integer such that there are no words of length $\geq h$ avoiding (l,p);
- the internal nodes at depth h-1 gives the all words of maximal length avoiding (l,p).

The following table lists (l, p, n, h, t, S), where

- l = |y|, where one is trying avoiding yy;
- p, the fractional exponent one is trying to avoid;
- n, the number of leaves of T(l, p);
- h, the height of the tree T(l, p).
- t, the number of internal nodes at depth h-1 in the tree.
- S, the set of labels of the internal nodes at depth h-1 that start with 0. (The other words can be obtained simply by interchanging 0 and 1.)

For completeness, we give the results for the optimal exponents for $2 \le l \le 7$. As mentioned above, the case l = 2 is due to Entringer, Jackson, and Schatz [3] and the case l = 3 is due to Dekking [2].

l	p	n	h	t	S
2	∞	478	19	2	{010011000111001101}
3	3	578	30	2	{001100101001101101100101001100}
4	5/2	6860	84	4	$\{001011010010110010011011001011010011011$
					001100100110100101100100110110010110110
5	5/2	15940	93	2	$\{001001011001101001011001001101100101101$
6	5/2	15940	93	2	$\{001001011001101001011001001101100101101$
7	7/3	3548	43	2	$\{00101100101101001100101100110100101100101$

Figure 2: Proofs of the negative results (a), (b), (e)

3 Proof of (c)

In this section we prove that there is an infinite binary word that simultaneously avoids yy with $|y| \ge 3$ and 3^+ powers.

We introduce the following notation for alphabets: $\Sigma_k := \{0, 1, \dots, k-1\}$. Let the morphism $f: \Sigma_3^* \to \Sigma_2^*$ be defined as follows.

 $\begin{array}{cccc} 0 & \to & 0010111010 \\ 1 & \to & 0010101110 \\ 2 & \to & 0011101010 \end{array}$

We will prove

Theorem 2 If w is any squarefree word over Σ_3 , then f(w) avoids yy with $|y| \geq 3$ and 3^+ powers.

Proof. We argue by contradiction. Let $w = a_1 a_2 \cdots a_n$ be a squarefree string such that f(w) contains a square yy with $|y| \geq 3$, i.e., f(w) = xyyz for some $x, z \in \Sigma_2^*$, $y \in \{\Sigma_2^{\geq 3}$. Without loss of generality, assume that w is a shortest such string, so that $0 \leq |x|, |z| < 20$.

Case 1: $|y| \le 20$. In this case we can take $|w| \le 5$. To verify that f(w) has no squares yy with $|y| \ge 3$, it therefore suffices to check each of the 30 possible words $w \in \Sigma_2^5$.

Case 2: |y| > 20. First, we establish the following result.

Lemma 3 (a) (inclusion property) Suppose f(ab) = tf(c)u for some letters $a, b, c \in \Sigma_2$ and strings $t, u \in \Sigma_2^*$. Then this inclusion is trivial (that is, $t = \epsilon$ or $u = \epsilon$).

(b) (interchange property) Suppose there exist letters a, b, c and strings s, t, u, v such that f(a) = st, f(b) = uv, and f(c) = sv. Then either a = c or b = c.

Proof.

- (a) A short computation verifies there are no a, b, c for which the equality f(ab) = tf(c)u holds nontrivially.
- (b) This can also be verified with a short computation. If $|s| \ge 6$, then no two distinct letters share a prefix of length 6. If $|s| \le 5$, then $|t| \ge 5$, and no two distinct letters share a suffix of length 5.

Once Lemma 3 is established, the rest of the argument is fairly standard. It can be found, for example, in [5], but for completeness we repeat it here.

For i = 1, 2, ..., n define $A_i = f(a_i)$. Then if f(w) = xyyz, we can write

$$f(w) = A_1 A_2 \cdots A_n = A'_1 A''_1 A_2 \cdots A_{j-1} A'_j A''_j A_{j+1} \cdots A_{n-1} A'_n A''_n$$

where

$$A_{1} = A'_{1}A''_{1}$$

$$A_{j} = A'_{j}A''_{j}$$

$$A_{n} = A'_{n}A''_{n}$$

$$x = A'_{1}$$

$$y = A''_{1}A_{2} \cdots A_{j-1}A'_{j} = A''_{j}A_{j+1} \cdots A_{n-1}A'_{n}$$

$$z = A''_{n},$$

where $|A_1''|, |A_j''| > 0$. See Figure 3.

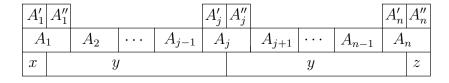


Figure 3: The string xyyz within f(w)

If $|A_1''| > |A_j''|$, then $A_{j+1} = f(a_{j+1})$ is a subword of $A_1''A_2$, hence a subword of $A_1A_2 = f(a_1a_2)$. Thus we can write $A_{j+2} = A_{j+2}'A_{j+2}''$ with

$$A_1''A_2 = A_i''A_{j+1}A_{j+2}'.$$

See Figure 4.

$$y = \begin{bmatrix} A_1'' & A_2 & & & \\ A_j'' & A_{j+1} & A_{j+2}' & \cdots & A_{n-1} & A_n' \end{bmatrix}$$

$$y = \begin{bmatrix} A_1'' & A_2 & & & \\ A_j' & A_{j+1} & A_{j+2}' & \cdots & A_{n-1} & A_n' \end{bmatrix}$$

Figure 4: The case $|A_1''| > |A_j''|$

But then, by Lemma 3 (a), either $|A''_j| = 0$, or $|A''_1| = |A''_j|$, or A'_{j+2} is a not a prefix of any f(d). All three conclusions are impossible.

If $|A''_1| < |A''_j|$, then $A_2 = f(a_2)$ is a subword of $A''_j A_{j+1}$, hence a subword of $A_j A_{j+1} = f(a_j a_{j+1})$. Thus we can write $A_3 = A'_3 A''_3$ with

$$A_1'' A_2 A_3' = A_j'' A_{j+1}.$$

See Figure 5.

$$y = \begin{vmatrix} A''_1 & A_2 & A'_3 \\ y = & A''_j & A_{j+1} \end{vmatrix} \cdots \begin{vmatrix} A_{j-1} & A'_j \\ A_{n-1} & A'_n \end{vmatrix}$$

Figure 5: The case $|A_1''| < |A_i''|$

By Lemma 3 (a), either $|A_1''| = 0$ or $|A_1''| = |A_j''|$ or A_3' is not a prefix of any f(d). Again, all three conclusions are impossible.

Therefore $|A_1''| = |A_j''|$. Hence $A_1'' = A_j''$, $A_2 = A_{j+1}$, ..., $A_{j-1} = A_{n-1}$, and $A_j' = A_n'$. Since h is injective, we have $a_2 = a_{j+1}, \ldots, a_{j-1} = a_{n-1}$. It also follows that |y| is divisible by 10 and $A_j = A_j' A_j'' = A_n' A_1''$. But by Lemma 3 (b), either (1) $a_j = a_n$ or (2) $a_j = a_1$. In the first case, $a_2 \cdots a_{j-1} a_j = a_{j+1} \cdots a_{n-1} a_n$, so w contains the square $(a_2 \cdots a_{j-1} a_j)^2$, a contradiction. In the second case, $a_1 \cdots a_{j-1} = a_j a_{j+1} \cdots a_{n-1}$, so w contains the square $(a_1 \cdots a_{j-1})^2$, a contradiction.

It now follows that if w is squarefree then f(w) avoids squares yy with $|y| \geq 3$.

It remains to see that f(w) avoids 3^+ powers. If f(w) contained x^e for some fractional exponent e > 3, then it would contain x^2 , so from above we have $|x| \le 2$. Thus it suffices to show that f(w) avoids the words 0000, 1111, 0101010, 1010101. This can be done by a short computation.

Corollary 4 There is an infinite binary word avoiding squares yy with $|y| \geq 3$ and 3^+ powers.

Proof. As is very well-known, there are infinite squarefree words over Σ_3 [7, 1]. Take any such word \mathbf{w} (for example, the fixed point of the morphism $2 \to 210$, $1 \to 20$, $0 \to 1$), and apply the map f. The resulting word $f(\mathbf{w})$ avoids $(3, 3^+)$.

It may be of some interest to explain how the morphism f was discovered. We iteratively generated all words of length $1, 2, 3, \ldots$ (up to some bound) that avoid $(3, 3^+)$. We then guessed such words were the image of a k-uniform morphism applied to a squarefree word over Σ_3 . For values of $k = 2, 3, \ldots$, we broke up each word into contiguous blocks of size k, and discarded any word for which there were more than 3 blocks. For certain values of k, this procedure eventually resulted in 0 words fitting the criteria. At this point we knew a k-uniform morphism cannot work, so we increased k and started over. Eventually a k was

found for which the number of such words appeared to increase without bound. We then examined the possible sets of 3 k-blocks to see if any satisfied the requirements of Lemma 3. This gave our candidate morphism f.

Theorem 5 Let A_n denote the number of binary words of length n avoiding yy with $|y| \ge 3$ and 3^+ powers. Then $A_n = \Omega(1.01^n)$ and $A_n = O(1.49^n)$.

Proof. Grimm [4] has shown there are $\Omega(\lambda^n)$ squarefree words over Σ_3 , where $\lambda = 1.109999$. Since the map f is 10-uniform, it follows that $A_n = \Omega(\lambda^{n/10}) = \Omega(1.01^n)$.

For the upper bound, we reason as follows. The set of binary words of length n avoiding yy with $|y| \geq 3$ and 3^+ powers is a subset of the set of binary words avoiding 0000 and 1111. The number A'_n of words avoiding 0000 and 1111 satisfies the linear recurrence $A'_n = A'_{n-1} + A'_{n-2} + A'_{n-3}$ for $n \geq 4$. It follows that $A'_n = O(\alpha^n)$, where α is the largest zero of $x^3 - x^2 - x - 1$, the characteristic polynomial of the recurrence. Here $\alpha < 1.84$, so $A_n = O(1.84^n)$.

This reasoning can be extended using a symbolic algebra package such as Maple. Noonan and Zeilberger [6] have written a Maple package DAVID_IAN that allows one to specify a list L of forbidden words, and computes the generating function enumerating words avoiding members of L. We used this package for a list L of 62 words of length \leq 12:

$$0000, 1111, \dots, 1110101111010$$

obtaining a characteristic polynomial of degree 67 with dominant zero $\doteq 1.4895$.

4 Proof of (e)

In this section we prove that there is an infinite binary word that simultaneously avoids yy with $|y| \ge 4$ and $\frac{5}{2}^+$ powers.

Let $g_1: \Sigma_8^* \to \Sigma_2^*$ be defined as follows.

- $0 \rightarrow 0011010010110$
- $1 \rightarrow 0011010110010$
- $2 \rightarrow 0011011001011$
- $3 \rightarrow 0100110110010$
- $4 \rightarrow 0110100101100$
- $5 \rightarrow 1001101011001$
- $6 \rightarrow 1001101100101$
- $7 \rightarrow 1010011011001$

Let $g_2: \Sigma_4^* \to \Sigma_8^*$ be defined as follows.

 $0 \rightarrow 03523503523453461467$

 $1 \rightarrow 03523503523453467167$

 $2 \rightarrow 16703523503523461467$

 $3 \rightarrow 03523503523461467167$

Let $g_3: \Sigma_3^* \to \Sigma_4^*$ be defined as follows.

 $0 \rightarrow 010203$

 $1 \rightarrow 010313$

 $2 \rightarrow 021013$

Finally, define $g: \Sigma_3^* \to \Sigma_2^*$ by $g = g_1 \circ g_2 \circ g_3$. Note that g is 1560-uniform. We will prove

Theorem 6 If w is any squarefree word over Σ_3 , then g(w) avoids yy with $|y| \ge 4$ and $\frac{5}{2}^+$ powers.

Proof. The proof is very similar to the proof of Theorem 2, and we indicate only what must be changed.

First, it can be checked that Lemma 3 also holds for the morphism g.

As before, we break the proof up into two parts: the case where g(w) = xyyz for some y with $4 \le |y| \le 2 \cdot 1560$, and the case where g(w) = xyyz for some y with $|y| \ge 2 \cdot 1560$. The former can be checked by examining the image of the 30 squarefree words in Σ_3^5 under g. The latter is handled as we did in the proof of Theorem 2. We checked these conditions with programs written in Pascal; these are available from the author on request.

Corollary 7 There is an infinite binary word avoiding squares yy with $|y| \ge 4$ and $\frac{5}{2}^+$ powers.

It may be of some interest to explain how the morphisms g_1, g_2, g_3 , were discovered.

We used a procedure analogous to that described above in Section 3. However, since it was not feasible to generate all words avoiding $(4, \frac{5}{2}^+)$ and having at most 3 contiguous blocks of length 1560, we increased the alphabet size and and tried various k-blocks until we found a combination of alphabet size and block size for which the number of words appeared to increase without bound. We then obtained a number of possible candidates for blocks.

Next, we determined the necessary avoidance properties of the blocks given by images of letters under g_1 . For example, $g_1(0)$ cannot be followed by $g_1(1)$, because this results in the subword 000, which is a 3rd power (and 3 > 2.5). The blocks that must be avoided include all words with squares, and

01, 02, 04, 05, 06, 07, 10, 12, 13, 17, 20, 21, 24, 25, 26, 27, 30, 31, 32, 36, 37, 40, 41, 42, 43, 47,

51, 54, 56, 57, 60, 62, 63, 64, 65, 72, 73, 74, 75, 76, 034, 145, 153, 161, 353, 450, 452, 535, 615, 616,714, 715, 2346703, 5234670, 5234671, 53467035, 6703523461, 2346146703503, 5234614670350This list was computed purely mechanically, and it is certainly possible that this list is not

exhaustive.

We now iterated our guessing procedure, looking for a candidate uniform morphism that creates squarefree words avoiding the patterns in the list above. This resulted in the 20uniform morphism q_2 .

We then computed the blocks that must be avoided for g_2 . This was done purely mechanically. Our procedure suggested that arbitrarily large blocks must be avoided, but luckily they (apparently) had a simple finite description: namely, we must avoid 12, 23, 32, and blocks of the form 2x0x1 and 3x1x0 for all nonempty words x, in addition to words with squares.

We then iterated our guessing procedure one more time, looking for a candidate uniform morphism that avoids these patterns. This gave us the morphism q_3 .

Of course, once the morphism $g = g_1 \circ g_2 \circ g_3$ is discovered, we need not rely on the list of avoidable blocks; we can take the morphism as given and simply verify the properties of inclusion and interchange as in Lemma 3.

Theorem 8 Let B_n denote the number of binary words of length n avoiding yy with $|y| \geq 4$ and $\frac{5}{2}^+$ powers. Then $B_n = \Omega(1.000066^n)$ and $B_n = O(1.122^n)$.

Proof. The proof is analogous to that of Theorem 5. We use the fact that g is 1560-uniform, which, combined with the result of Grimm [4], gives the bound $1.109999^{1/1560} \doteq 1.000066899$

For the upper bound, we again use the Noonan-Zeilberger Maple package. We used the 54 patterns corresponding to words of length ≤ 20 . This gave us a polynomial of degree 27 with dominant zero $\doteq 1.12123967$.

Proof of (f) 5

In this section we prove that there is an infinite binary word that simultaneously avoids yywith $|y| \ge 7$ and $\frac{7}{3}$ powers. Let $h_1: \Sigma_5^* \to \Sigma_2^*$ be defined as follows.

 $0 \rightarrow 00110100101100$

 $\rightarrow 00110100110010$

 $\rightarrow 01001100101100$

 \rightarrow 10011011001011

11010011011001

Let $h_2: \Sigma_3^* \to \Sigma_5^*$ be defined as follows.

 $0 \rightarrow 032303241403240314$

 $1 \rightarrow 032314041403240314$

 $2 \rightarrow 032414032303240314$

Finally, define $h: \Sigma_3^* \to \Sigma_2^*$ by $h = h_1 \circ h_2$. Note that h is 252-uniform. We will prove

Theorem 9 If w is any squarefree word over Σ_3 , then h(w) avoids yy with $|y| \geq 7$ and $\frac{7}{3}$ powers.

Proof. Again, the proof is quite similar to that of Theorem 2. We leave it to the reader to verify that the inclusion and interchange properties hold for h, and that the image of all the squarefree words of length ≤ 5 are free of squares yy with |y| < 7 and $\frac{7}{3}$ powers.

Corollary 10 There is an infinite binary word avoiding squares yy with $|y| \geq 7$ and $\frac{7}{3}$ powers.

The morphisms h_1, h_2 were discovered using the heuristic procedure mentioned in Section 3. The avoiding blocks for h_1 were heuristically discovered to include

$$01, 02, 10, 12, 13, 20, 21, 34, 42, 43, 304, 23031, 24041, 231403141, 232403241$$

as well as blocks containing any squares. Then h_2 was constructed to avoid these blocks.

Theorem 11 Let C_n denote the number of binary words of length n avoiding yy with $|y| \ge 7$ and $\frac{7}{3}$ powers. Then $C_n = \Omega(1.0004^n)$ and $C_n = O(1.162^n)$.

Proof. The proof is very similar to that of Theorems 5 and 8.

For the lower bound, note that h is 252-uniform. This, combined with the bound of Grimm [4], gives a lower bound of $\Omega(\lambda^n)$ for all $\lambda < 1.109999^{1/252} \doteq 1.0004142$.

For the upper bound, we again used the Noonan-Zeilberger Maple package. We avoided 58 words of length \leq 20. This resulted in a polynomial of degree 26, with dominant zero \doteq 1.1615225.

6 Enumeration results

In this section we provide a table of the first values of the sequences A_n , B_n , and C_n , defined in Sections 3, 4, and 5, for $1 \le n \le 25$.

n	A_n	B_n	C_n
0	1	1	1
1	2	2	2
2	4	4	4
3	8	6	6
4	14	10	10
5	26	16	14
6	42	24	20
7	68	36	30
8	100	46	38
9	154	64	50
10	234	74	64
11	356	88	86
12	514	102	108
13	768	114	136
14	1108	124	164
15	1632	140	196
16	2348	160	226
17	3434	178	264
18	4972	198	322
19	7222	212	384
20	10356	230	436
21	14962	256	496
22	21630	294	578
23	31210	342	674
24	44846	366	754
25	64584	392	850

Figure 6: Values of $A_n,\,B_n,\,C_n,\,0\leq n\leq 25$

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References

[1] J. Berstel. Axel Thue's Papers on Repetitions in Words: a Translation. Number 20 in Publications du Laboratoire de Combinatoire et d'Informatique Mathématique. Université du Québec à Montréal, February 1995.

- [2] F. M. Dekking. On repetitions of blocks in binary sequences. J. Combin. Theory. Ser. A 20 (1976), 292–299.
- [3] R. C. Entringer, D. E. Jackson, and J. A. Schatz. On nonrepetitive sequences. *J. Combin. Theory. Ser. A* **16** (1974), 159–164.
- [4] U. Grimm. bounds Improved on the number ternary squarefree words. J. Integer Sequences (2001),01.2.7(electronic). http://www.math.uwaterloo.ca/JIS/VOL4/GRIMM/words.html
- [5] J. Karhumäki and J. Shallit. Polynomial versus exponential growth repetition-free binary words, 2003. Submitted. Preprint in available http://www.arxiv.org/abs/math.CO/0304095.
- [6] J. Noonan and D. Zeilberger. The Goulden-Jackson cluster method: extensions, applications and implementations. *J. Differ. Equations Appl.* **5** (1999), 355–377.
- [7] A. Thue. Über unendliche Zeichenreihen. Norske vid. Selsk. Skr. Mat. Nat. Kl. 7 (1906), 1–22. Reprinted in Selected Mathematical Papers of Axel Thue, T. Nagell, editor, Universitetsforlaget, Oslo, 1977, pp. 139–158.