# New Results on the Minimum Amount of Useful Space 

Zuzana Bednárová* and Viliam Geffert*<br>Department of Computer Science, P. J. Safárik University, Košice, Slovakia<br>\{zuzana.bednarova, viliam.geffert\}@upjs.sk<br>Klaus Reinhardt<br>Wilhelm-Schickard-Institut für Informatik, University of Tübingen, Germany and Institut für Informatik, University of Halle, Germany<br>klaus.reinhardt@uni-tuebingen.de

Abuzer Yakarylmaz ${ }^{\dagger}$
National Laboratory for Scientific Computing, Petrópolis, RJ, Brazil
abuzer@lncc.br
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#### Abstract

We present several new results on minimal space requirements to recognize a nonregular language: (i) realtime nondeterministic Turing machines can recognize a nonregular unary language within weak $\log \log n$ space, (ii) $\log \log n$ is a tight space lower bound for accepting general nonregular languages on weak realtime pushdown automata, (iii) there exist unary nonregular languages accepted by realtime alternating one-counter automata within weak $\log n$ space, (iv) there exist nonregular languages accepted by two-way deterministic pushdown automata within strong $\log \log n$ space, and, (v) there exist unary nonregular languages accepted by two-way one-counter automata using quantum and classical states with middle $\log n$ space and bounded error.


Keywords: pushdown automata; counter automata; nondeterminism; alternation; quantum computation; unary languages; nonregular languages.

## 1. Introduction

The minimum amount of useful "resources" which are necessary for a finite automaton to recognize a nonregular language is one of the fundamental research directions. Many different "resources" have been introduced, e.g., access to the input tape (realtime, one-way, or two-way), computational mode of the model (deterministic, nondeterministic, alternating, probabilistic, or quantum), type of the working memory (counter, pushdown store, or tape),

[^0]etc. Moreover, unary languages need a special attention, since they may require resources that are different from those for languages built over general (or binary) alphabets. We shall focus on the minimum amount of useful space and present some new results.

First, we show that realtime nondeterministic Turing machines (NTMs) can recognize unary nonregular languages in weak $O(\log \log n)$ space. 1 Second, if the worktape is replaced by a pushdown store - which gives realtime nondeterministic pushdown automata (PDAs) - we obtain the same result on the binary alphabet. Third, we show that their deterministic counterparts, two-way deterministic PDAs, recognize nonregular languages with strong $O(\log \log n)$ space. These bounds are tight, matching the lower bound for twoway alternating Turing machines (ATMs) [17]. In the unary case, we know that one-way nondeterministic PDAs recognize regular languages only, but their alternating counterparts can simulate any ATM that uses $\Theta(n)$ space [7]. Such power does not seem to hold if we replace the pushdown store with a counter. Fourth, we show that realtime alternating onecounter automata recognize some nonregular unary languages with weak $O(\log n)$ space. (By "space" we mean the value of the counter, rather than the length of its binary representation.) Here we also present a trade-off to alternation depth. Fifth, we show that a two-way deterministic one-counter automaton (2DCA) with two qubits (2QCCA) can recognize a nonregular unary language by using $O(\log n)$ space on its counter for accepted inputs. Without qubits and with $o(n)$ space, 2DCAs recognize only regular unary languages [ $\underline{9}$ ].

Our results are presented in Section 2, with a discussion of the known results. We also identify some new directions and formulate a few open questions. The reader is assumed to be familiar with the classical computational models and so we provide only the definition for 2DCAs using a fixed-size quantum memory (in Section 3). The proofs are put in the remaining sections. We refer the reader to [29] for short and to [23] for complete references on quantum computation.

## 2. Our Results and New Directions

### 2.1. Deterministic, Nondeterministic, and Alternating Machines

It is known that (i) no weak $o(\log \log n)$ space bounded alternating two-way Turing machine (TM) can recognize a nonregular language and (ii) there exists a unary nonregular language recognized by a deterministic two-way TM in strong $O(\log \log n)$ space $[1, \underline{17}, \underline{30}]$. For oneway TMs, the bounds are given in Table 1 , taken from a recent paper by Yakarylmaz and Say [36], in which it was shown that all these bounds are tight for almost all realtime TMs. However, for realtime nondeterministic and alternating TMs accepting unary nonregular languages, it was left open whether the double logarithmic lower bounds are tight. We solve this problem positively (bold entries in Table 1), so now we have a complete picture for TMs:

Theorem 1. There exists a unary nonregular language accepted by a realtime nondeterministic Turing machine having a single worktape in weak $O(\log \log n)$ space.

In our construction, we use a working alphabet with more than 2 symbols (except for the blank symbol) and so it is still open whether we can obtain the same result with a binary working alphabet.

[^1]Table 1: Minimum space used by one-way TMs for recognizing nonregular languages.

|  | General input alphabet |  |  | Unary input alphabet |  |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | Strong | Middle | Weak | Strong | Middle | Weak |
| Deterministic TM | $\log n$ | $\log n$ | $\log n$ | $\log n$ | $\log n$ | $\log n$ |
| Nondeterministic TM | $\log n$ | $\log n$ | $\log \log n$ | $\log n$ | $\log n$ | $\log \log \boldsymbol{n}$ |
| Alternating TM | $\log n$ | $\log \log n$ | $\log \log n$ | $\log n$ | $\log n$ | $\log \log \boldsymbol{n}$ |

By using a TM with a restricted access to working tape, we obtain a pushdown automaton (PDA). It is known that no weak $o(n)$ space bounded one-way deterministic PDA can recognize a nonregular language [15] and that realtime deterministic PDAs can recognize $\left\{a^{n} b^{n} \mid n \geq 0\right\}$ in strong $O(n)$ space. For one-way nondeterministic PDAs, a weak $O(\log n)$ space algorithm was given for a nonregular language in [26]. We improve this to weak $O(\log \log n)$ space. For this purpose, we introduce, in Section $\underline{5}$, a new language called REI (used also in Section 8).

Theorem 2. Realtime nondeterministic PDAs can recognize nonregular language $\underline{R E I}$ with weak $O(\log \log n)$ space.

Since a pushdown automaton is a special case of the Turing machine, this bound is tight also for any kind of alternating PDAs, by [17, 30]. On the other hand, we do not know the tight strong/middle space bounds for one-way/realtime nondeterministic and alternating PDAs recognizing nonregular languages.

In the unary case, one-way nondeterministic PDAs cannot recognize nonregular languages [16]. Realtime alternating one-counter automata (CAs), on the other hand, can recognize some unary nonregular languages even in weak $O(\log n)$ space (counter value). Here we shall use the following two unary languages:

$$
\underline{\mathrm{UPOWER}}=\left\{a^{2^{n}} \mid n \geq 0\right\} \quad \text { and } \quad \underline{\mathrm{UPOWER}+}=\left\{a^{2^{n}+4 n-4} \mid n \geq 3\right\}
$$

Theorem 3. Realtime alternating $C A s$ can recognize nonregular UPOWER+ in weak $O(\log n)$ space.

The tight space bounds for realtime/one-way alternating counter automata recognizing nonregular unary/binary languages are still not known.

In Section 6, we present a one-way algorithm for UPOWER and then our realtime algorithm for UPOWER+. Both algorithms have a linear alternation depth on accepted inputs. In Section 7, we consider the existence of a shorter alternation depth and present a realtime algorithm for UPOWER (a slightly modified version of the algorithm provided by Ďuriš [8]) with alternation depth bounded by $O(\log n)$, but it needs a linear counter. Moreover, we show that if the counter is replaced by a pushdown store, we have only a single alternation, using linear space.

In the case of two-way PDAs, we have tight bounds:
Theorem 4. Two-way deterministic PDAs can recognize REI in strong $O(\log \log n)$ space.
In [9], it was shown that any unary language recognized by a two-way deterministic PDA using $o(n)$ space is regular. Moreover, two-way deterministic CAs can recognize nonregular unary UPOWER with $O(n)$ space. Therefore, linear space is a tight bound for both two-way
deterministic PDAs and CAs, but we do not know whether nondeterministic or random choices can help for unary languages.

Another interesting direction is to identify the tight bounds for one-way/realtime multicounter/pushdown automata. Yakaryilmaz and Say [36] showed that realtime deterministic automata with $k$ counters can recognize some nonregular languages in middle $O\left(n^{1 / k}\right)$ space, where $k>1$. The same result can be obtained by bounded-error probabilistic one-counter automata, but the error bound increases in $k$.

### 2.2. Probabilistic and Quantum Machines

Clearly, probabilistic models are special cases of their quantum counterparts. In the unbounded error case, realtime probabilistic finite automata (PFAs) can recognize unary nonregular languages [24], so let us consider the bounded error case. One-way PFAs with bounded-error recognize only regular languages [25]. Two-way PFAs can recognize some nonregular languages but only with exponential expected time [12, 10]. With an arbitrarily small (not constant) space, two-way probabilistic TMs can recognize nonregular languages in polynomial time [14], but one-way probabilistic TMs do not recognize nonregular languages in space $o(\log \log n)[13,19]$.

Two-way quantum finite automata (QFAs), on the other hand, can recognize some nonregular languages in polynomial time [3]. If the input head is quantum, i.e., it can be in a superposition of several places on the input tape, then one-way QFAs can recognize some nonregular languages in linear time $[21, \underline{2}, \underline{34}, \underline{31}]$. But, it is not known whether twoway QFAs can recognize any nonregular unary language with bounded-error (which is not possible for 2PFAs [18]).

We shall show that two-way QFAs with a classical counter - or 2DCAs with a fixedsize quantum memory (2QCCAs) - can recognize a nonregular unary language with space smaller than required by the deterministic 2DCAs.

Theorem 5. The unary nonregular language UPOWER can be recognized by a 2QCCA with bounded-error, using middle $O(\log n)$ space on its counter.

One-way probabilistic PDAs cannot recognize nonregular unary languages with boundederror [20] but the question is open for their quantum counterpart. On the other hand, using middle $O(\log n)$ space, realtime bounded-error probabilistic PDAs can recognize the language $\left\{b_{1} a b_{2}^{\mathrm{R}} a b_{3} a b_{4}^{\mathrm{R}} a \cdots a b_{2 k-1} a b_{2 k}^{\mathrm{R}} \mid k>0\right\}$, where $\alpha^{\mathrm{R}}$ denotes the reversal of a string $\alpha$ and $b_{i}$ the binary representation of $i$. Currently, we do not know any better result and whether quantumness helps.

## 3. Definitions

We use three different modes of space usage [30]: (i) Strong space $s(n)$ refers to the space used by the machine along all computation paths on all inputs of length $n$, (ii) middle space to the space used along all computation paths on accepted inputs, and (iii) weak space to an accepting path using minimum space.

A one-way machine model a restricted two-way variant never moving the input head to the left. A realtime machine a restricted one-way variant in which the input head can

A superoperator $\mathcal{E}=\left\{E_{1}, \ldots, E_{k}\right\}$ is composed of some operation elements $E_{i}$ satisfying

$$
\sum_{i=1}^{k} E_{i}^{\dagger} E_{i}=I
$$

where $k>0$ is a constant, and the indices are the measurement outcomes. When the superoperator is applied to the quantum register in a state $|\psi\rangle$, i.e., $\mathcal{E}(|\psi\rangle)$, we obtain the measurement outcome $i \in\{1, \ldots, k\}$ with probability $p_{i}=\left\langle\widetilde{\psi}_{i} \mid \widetilde{\psi}_{i}\right\rangle$, where $\left|\widetilde{\psi}_{i}\right\rangle$, the unconditional state vector, is calculated as $\left|\widetilde{\psi}_{i}\right\rangle=E_{i}|\psi\rangle$. Note that using unconditional state vector simplifies calculations in many cases. If the outcome $i$ is observed with $p_{i}>0$, the new state of the system, which is obtained by normalizing $\left|\widetilde{\psi}_{i}\right\rangle$, is given by $\left|\psi_{i}\right\rangle=\left|\widetilde{\psi_{i}}\right\rangle / \sqrt{p_{i}}$. Moreover, as a special operator, the quantum register can be initialized to a predefined quantum state. This initialization operator has only one outcome.

Figure 1: The details of superoperators [33].
stay on the same symbol only a fixed number of steps. (Actually, the realtime machines presented in this paper never wait on the same symbol.)

A two-way one-counter automaton with quantum and classical states (2QCCA) [32] is a two-way one-counter automaton having a constant-size quantum register. Without the counter, we obtain a two-way finite automaton with quantum and classical states (2QCFA) [3]. In the original definition, the automaton can apply unitary and measurement operators to its quantum part. Here we allow to apply a superoperator (see Figure 1), a generalization of classical and unitary operators including measurement. In general, this does not change the computational power of 2QCFAs and 2QCCAs [3]. We only do not know whether the original models are less powerful than the ones with superoperators if we use only rational amplitudes. (All quantum algorithms presented here use rational superoperators.)

A 2QCCA is an 8 -tuple $\mathcal{P}=\left(S, Q, \Sigma, \delta, s_{\mathrm{I}}, s_{\mathrm{A}}, s_{\mathrm{R}}, q_{\mathrm{I}}\right)$, where $S$ denotes the set of classical states, $s_{\mathrm{I}}, s_{\mathrm{A}}, s_{\mathrm{R}} \in S$ (with $s_{\mathrm{A}} \neq s_{\mathrm{R}}$ ) the initial, accepting, and rejecting states, $Q$ the set of quantum states, $q_{\mathrm{I}} \in Q$ the initial quantum state, $\Sigma$ the input alphabet (not containing $\Phi, \$$, the left and right endmarkers), and $\delta$ the transition function composed of $\delta_{\mathrm{q}}$ and $\delta_{\mathrm{c}}$ governing the quantum and classical parts, respectively.

Such machine $\mathcal{P}$ starts with the given input $w \in \Sigma^{*}$ enclosed in between $\phi$ and $\$$, the input head placed on $\phi$, in the state $\left(\left|q_{\mathrm{I}}\right\rangle, s_{\mathrm{I}}\right)$, and zero in the counter. Now, if $\mathcal{P}$ is in a state $(|\psi\rangle, s)$ with the input head on a symbol $a$ and with $\theta \in\{"=0 ", " \neq 0 "\}$, the next step consists of the following quantum and classical transitions. First, $\delta_{\mathrm{q}}(s, a, \theta)$ determines a superoperator which is applied to the quantum register and some classical outcome $\tau$ is observed. The quantum part of the state is updated to $\left|\psi_{\tau}\right\rangle$. After that, if $\delta_{\mathrm{c}}(s, a, \theta, \tau)=\left(s^{\prime}, d, c\right)$, the classical part of the state changes to $s^{\prime}$ and the input head and the counter value are updated with respect to $d \in\{\leftarrow, \downarrow, \rightarrow\}$ and $c \in\{-1,0,+1\}$. $\mathcal{P}$ accepts or rejects when it enters $s_{\mathrm{A}}$ or $s_{\mathrm{R}}$, respectively.

## 4. Realtime Nondeterministic Turing Machine - Theorem 1

The following function will play an important role in our considerations:

$$
f(n)=\text { the smallest positive integer not dividing } n
$$

We take $f(0)=+\infty$ (i.e., undefined): there is no positive integer not dividing 0 . We shall use the fact that $f(n)$ can be written down with $O(\log \log n)$ bits $[1,11, \underline{30}$. However, we
shall need to be more precise about the constants hidden in the big- $O$ notation: ${ }^{2}$
Lemma 6. $f(n)<2 \cdot \log n$, for each $n \geq 3$.
Proof: (i) Consider first the case of $n \geq 363$. For the given $n$, let $f(n)=1+m$ be the smallest positive integer not dividing $n$. Consequently, each $k \in\{1, \ldots, m\}$ divides $n$. Thus, $p^{\left\lfloor\log _{p} m\right\rfloor}$ must divide $n$ for each prime $p$, since $1 \leq p^{\left\lfloor\log _{p} m\right\rfloor} \leq m$. But then $n$ is a common multiple of all values $p^{\left\lfloor\log _{p} m\right\rfloor}$. Second, for primes satisfying $p \leq m$, we have $\log _{p} m \geq 1$, and hence also $\left\lfloor\log _{p} m\right\rfloor \geq 1$. Combining this, we get:

$$
\begin{aligned}
n & \geq \prod_{p \leq m} p^{\left\lfloor\log _{p} m\right\rfloor}=\prod_{p \leq m}\left(p^{\left\lfloor\log _{p} m\right\rfloor+\left\lfloor\log _{p} m\right\rfloor}\right)^{1 / 2} \geq \prod_{p \leq m}\left(p^{1+\left\lfloor\log _{p} m\right\rfloor}\right)^{1 / 2} \\
& >\prod_{p \leq m}\left(p^{\log _{p} m}\right)^{1 / 2}=\prod_{p \leq m} m^{1 / 2}=m^{1 / 2 \cdot \pi(m)},
\end{aligned}
$$

where all products are taken over primes $p \leq m$ and $\pi(m)$ denotes the total number of primes smaller than of equal to $m$. From [28], we know that $\pi(x)>\frac{x}{\ln x}$, for each real $x \geq 17$. Using this, we have the following two subcases:

If $m \geq 17$, we can bound $n$ from below as follows:

$$
n>m^{1 / 2 \cdot \pi(m)}>m^{1 / 2 \cdot m / \ln m}=e^{\ln m \cdot 1 / 2 \cdot m / \ln m}=e^{1 / 2 \cdot m}
$$

and hence $\frac{m}{2}<\ln n$. Now, using $1<0.6 \cdot \log 4,4<n$, and $2 \cdot \ln 2<1.4$, we get

$$
\begin{aligned}
f(n) & =1+m<0.6 \cdot \log 4+2 \cdot \ln n<0.6 \cdot \log n+2 \cdot \ln 2 \cdot \frac{\ln n}{\ln 2} \\
& <0.6 \cdot \log n+1.4 \cdot \log n=2 \cdot \log n .
\end{aligned}
$$

Conversely, if $m<17$, that is, if $m \leq 16$, we can use the fact that $\frac{17}{2}<\log 363 \leq \log n$. This gives:

$$
f(n)=1+m \leq 17<2 \cdot \log n .
$$

(ii) It only remains to prove the statement of the lemma for $n<363$. However, it is a routine task to compute the table of values $\frac{f(n)}{\log n}$, for $n=362, \ldots, 3$, and verify that each of these values is smaller than 2 .

Consider now the following unary language:

$$
\underline{\mathrm{AM}}=\left\{1^{n} \mid \underline{f(n)} \text { is not equal to a power of } 2\right\} .
$$

Historically, the complement of AM was the first known unary nonregular language accepted with only $O(\log \log n)$ space, by a strongly bounded two-way deterministic Turing machine [1]. Later, in [6] (see also [22]), it was shown that AM can be accepted by a weakly bounded one-way nondeterministic machine, with $O(\log \log n)$ space again. The machine in [6] is based on the observation that, for each $n>0$,

- $1^{n} \in$ AM if and only if there exist two positive integers $k$ and $i$ satisfying $2^{i}<k<2^{i+1}$, such that $n \bmod k \neq 0$ and $n \bmod 2^{i}=0$.
- Moreover, for $1^{n} \in \operatorname{AM}$, the membership can be certified by taking $k=f(n)$ and $2^{i}=2^{\lfloor\log k\rfloor}$. This gives $2^{i}<k<2 \cdot \log n$, by Lemma $\underline{6}$.

[^2]This machine accepts AM as follows: (i) Guess some $k, 2^{i}$ satisfying $2^{i}<k<2^{i+1}$. (ii) Traversing across the input, count $r_{1}=n \bmod k$ and $r_{2}=n \bmod 2^{i}$. That is, at each input tape position, execute $r_{1}:=\left(r_{1}+1\right) \bmod k$ and $r_{2}:=\left(r_{2}+1\right) \bmod 2^{i}$. (iii) If the end of the input is reached with $r_{1} \neq 0$ and $r_{2}=0$, accept.

The values $k, 2^{i}, r_{1}, r_{2}$ are stored in binary in four worktape tracks, "one above another", using some $\ell$ worktape cells. ${ }^{3}$ Since $2^{i}, r_{1}, r_{2}$ are all smaller than $k$ and we can choose $k=f(n)$, bounded by $2 \cdot \log n$, an accepting computation path using optimal amount of space works with $\ell \leq O(\log \log n)$ worktape cells. ${ }^{4}$

### 4.1. A Realtime Version - Main Idea

In what follows, suppose that $n>0$ and $f(n) \geq 17$. (We shall later see how to avoid this assumption.) But then $n$ must be a common multiple of $\{1, \ldots, 16\}$ different from zero, and hence this assumption can be formulated as follows:

$$
\begin{equation*}
f(n) \geq 17 \quad \text { and } \quad n \geq 720720 . \tag{1}
\end{equation*}
$$

Now, we would like to accept AM without stationary moves on the input. First, take the machine with the sweeping worktape head, discussed above. In this machine, we modify every single operation so that we make the input head move one position forward. But then the progress in the input head movement becomes $2 \cdot(\ell+1)$ times faster than the progress in modular incrementing of $r_{1}, r_{2}$ on the worktape. Recall that our machine $\mathcal{A}$ requires exactly $2 \cdot(\ell+1)$ steps to execute $r_{1}:=\left(r_{1}+1\right) \bmod k, r_{2}:=\left(r_{2}+1\right) \bmod 2^{i}$, and to test whether $r_{1} \neq 0, r_{2}=0$, all this by one double-sweep, per each input tape position.

However, we can increment faster, by executing $r_{1}:=\left(r_{1}+\Delta\right) \bmod k$ and, simultaneously, $r_{2}:=\left(r_{2}+\Delta\right) \bmod 2^{i}$, where $\Delta>1$ is a value stored in binary in a separate worktape track. The modified machine $\mathcal{A}$ still uses a single double-sweep across the worktape, with exactly $2 \cdot(\ell+1)$ steps, during which the input head travels forward exactly $2 \cdot(\ell+1)$ positions. Thus, using $\Delta=2 \cdot(\ell+1)$, the progress in the input head movement agrees with the progress in modular incrementing. But then, for each input tape position $r$ that is an integer multiple of $2 \cdot(\ell+1)$, the machine gets to the position $r$ with the corresponding worktape tracks containing $r_{1}=r \bmod k$ and $r_{2}=r \bmod 2^{i}$. Moreover, at each such input position, the machine "knows" whether $r_{1} \neq 0$ and $r_{2}=0$, keeping this information in the finite state control. Thus, if $n$ is an integer multiple of $2 \cdot(\ell+1)$, we can correctly decide between acceptance and rejection. (This also requires to initialize the worktape with exactly $2 \cdot(\ell+1)$ steps, assigning initially $r_{1}:=(2 \cdot(\ell+1)) \bmod k$ and $\left.r_{2}:=(2 \cdot(\ell+1)) \bmod 2^{i}.\right)$

But a problem arises if $n$ is not an integer multiple of $2 \cdot(\ell+1)$. In this case, $\mathcal{A}$ gets to the end of the input at the moment when it is busy with computing in the middle of the worktape. Thus, the respective worktape tracks for $r_{1}, r_{2}$ contain some intermediate

[^3]

Figure 2: Initial content on the worktape, created in the course of one sweep to the left, followed by one sweep to the right, during which the input head travels exactly $2(\ell+1)$ positions. The lengths $j, i, \ell$ are guessed nondeterministically, and so are the bits in the track for $k$, for the bit positions $0, \ldots, i-1$, displayed here as " $0 / 1$ ". The condition $2^{j}=2 \cdot(\ell+1)$ is not granted, to be verified later.
data, from which $\mathcal{A}$ cannot quickly deduce whether $n \bmod k$ or $n \bmod 2^{i}$ is equal to zero. This problem is resolved by tuning up the size of the worktape more carefully. Namely, if $2 \cdot(\ell+1)=2^{j}$, for some power of two satisfying $2^{j} \leq 2^{i}$, the machine can get to the end of the input while working in the middle of the worktape only if $n$ is not an integer multiple of $2^{j}$. But then $n$ is not an integer multiple of $2^{i}$, and hence $n \bmod 2^{i} \neq 0$. In this case, $\mathcal{A}$ rejects.

Now we are ready to summarize all values, kept in six separate worktape tracks, as well as requirements on these values. (This can also be seen in Figure 2.)
$k, 2^{i}$ : guessed, so that $2^{i}<k<2^{i+1}$ and $k<2^{\ell}$. Also $\ell$, the length of the allocated worktape space, is guessed.
$2^{j}$ : guessed, so that $2^{j} \leq 2^{i}$ and $2^{j}=2 \cdot(\ell+1)$.
$r_{1}, r_{2}$ : initialized to $r_{1}=2^{j} \bmod k$ and $r_{2}=2^{j} \bmod 2^{i}$. (Clearly, this is equivalent to $r_{1}=(2 \cdot(\ell+1)) \bmod k$ and $\left.r_{2}=(2 \cdot(\ell+1)) \bmod 2^{i}.\right)$
$a$ : auxiliary track, initialized to $a=0$. (Used for verification, in Section 4.4.)
Before passing to implementation details, let us show that the above requirements are realistic. More precisely, for each $1^{n} \in \mathrm{AM}$, there exist $k, i, j, \ell$ satisfying not only $n \bmod k \neq 0$ and $n \bmod 2^{i}=0$, but also additional requirements imposed by the realtime processing. This can be achieved by using the following values:

$$
\begin{equation*}
k=f(n), i=\lfloor\log k\rfloor, j=2+\lfloor\log (2+i)\rfloor, \ell=2^{j-1}-1 . \tag{2}
\end{equation*}
$$

We have to show that $2^{i}<k<2^{i+1}, k<2^{\ell}, 2^{j} \leq 2^{i}$, and $2^{j}=2 \cdot(\ell+1)$, for each $1^{n} \in \mathrm{AM}$.
First, since $k=f(n)$ is not a power of 2 for $1^{n} \in$ AM, the value $\log k$ is not an integer. This gives $\lfloor\log k\rfloor<\log k<\lfloor\log k\rfloor+1$, and hence

$$
2^{i}=2^{\lfloor\log k\rfloor}<2^{\log k}=k=2^{\log k}<2^{\lfloor\log k\rfloor+1}=2^{i+1} .
$$

Second, using the above inequality and (2), we get

$$
k<2^{i+1}=2^{(2+i)-1}=2^{2^{\log (2+i)}-1}<2^{2^{1+\lfloor\log (2+i)\rfloor}-1}=2^{2^{j-1}-1}=2^{\ell} .
$$

Third, $k=f(n) \geq 17$, by the additional assumption (1). But then $i=\lfloor\log k\rfloor \geq 4$. Now, using $2+\lfloor\log (2+i)\rfloor \leq i$ for each $i \geq 4$, we have

$$
2^{j}=2^{2+\lfloor\log (2+i)\rfloor} \leq 2^{i} .
$$

Finally, in (2), we took $\ell=2^{j-1}-1$. Consequently,

$$
2^{j}=2 \cdot(\ell+1) .
$$

In addition, by the use of (2), Lemma $\underline{6}$, and the fact that $5<2 \cdot \log \log n$ for $n \geq 51$ (by (1), we actually have $n \geq 720720$ ), we obtain that

$$
\begin{align*}
\ell & =2^{j-1}-1=2^{1+\lfloor\log (2+i)\rfloor}-1 \leq 2^{1+\log (2+i)}-1=2 \cdot(2+i)-1=3+2 \cdot i \\
& =3+2 \cdot\lfloor\log k\rfloor \leq 3+2 \cdot \log k=3+2 \cdot \log f(n)<3+2 \cdot \log (2 \cdot \log n)  \tag{3}\\
& =5+2 \cdot \log \log n<4 \cdot \log \log n .
\end{align*}
$$

This clearly gives $O(\log \log n)$ weak space bound for $\mathcal{A}$.
We are now ready to provide implementation details, showing that the above values can be guessed, incremented, and tested for zero by single double-sweeps.

### 4.2. Initialization - Details

Here we show how the worktape is initialized in a single double-sweep, with exactly $2 \cdot(\ell+1)$ steps. We point out that even though we guess the values $k, i, j, \ell$ that satisfy (2), they are not guessed in the order in which they appeared in (2).

Sweep to the Left: Running the input head forward, $\mathcal{A}$ moves the worktape head from the right endmarker and rewrites the blank symbols so that the worktape becomes organized into six parallel tracks, ${ }^{5}$ containing binary written $k, 2^{i}, 2^{j}, r_{1}, r_{2}, a$. (See Figure 2.) This is done as follows. ${ }^{6}$

- In a loop, running through the bit positions $t=0, \ldots, j-1$, the machine assigns the $t^{\text {th }}$ bit in the track for $k$ (that is, the value $[k]_{t}$ ) by guessing, while all bits in the remaining tracks are filled with zeros.
- At the position $t=j$ (this moment is chosen nondeterministically), the bit $[k]_{t}$ is guessed while the bits in other tracks are set as follows: $\left[2^{i}\right]_{t}=0,\left[2^{j}\right]_{t}=\left[r_{1}\right]_{t}=$ $\left[r_{2}\right]_{t}=1$, and $[a]_{t}=0$.
- In a loop, running through the bit positions $t=j+1, \ldots, i-1$, the bit $[k]_{t}$ is guessed and all other bits are set to 0 .
- At the position $t=i$ (nondeterministically chosen), we set $[k]_{t}=\left[2^{i}\right]_{t}=1$, all other bits are set to 0 .
- In a loop, running through the bit positions $t=i+1, \ldots, \ell-1$ (for nondeterministically chosen $\ell$ ), all bits are set to 0 .
As a special case, $\mathcal{A}$ may guess $i=j$. In this case, the positions $i$ and $j$ overlap, the phase running through $j+1, \ldots, i-1$ is skipped, and the bits at the position $t=i=j$ are set as follows: $[k]_{t}=\left[2^{i}\right]_{t}=\left[2^{j}\right]_{t}=\left[r_{1}\right]_{t}=1$ and $\left[r_{2}\right]_{t}=[a]_{t}=0$.

[^4]From Figure 2, we see that the worktape now contains the following values:

$$
\begin{array}{rlrl}
k & =" 0^{\ell-i-1} 1 b_{i-1} \cdots b_{0} "<2^{i+1} \leq 2^{\ell}, & & \\
2^{i} & =" 0^{\ell-i-1} 10^{i} " \leq k, & & \\
2^{j} & =" 0^{\ell-j-1} 10^{j} " \leq 2^{i}, & & \text { provided that } 2^{i}<k, \\
r_{1} & =2^{j}=2^{j} \bmod k, & & \text { if } i>j, \\
r_{2} & =2^{j}=2^{j} \bmod 2^{i}, & & \\
r_{2} & =" 0^{\ell "}=0=2^{i} \bmod 2^{i}=2^{j} \bmod 2^{i}, & \text { if } i=j, \\
a & =0, & &
\end{array}
$$

for some nondeterministically chosen values $j \leq i<\ell$ and $b_{0}, \ldots, b_{i-1} \in\{0,1\}$.
Note that this initialization ensures automatically all requirements imposed on $k, i, j, \ell$, $r_{1}, r_{2}, a$, as listed in Section 4.1 (hence, no verification needed), except for: (i) Instead of $2^{i}<k<2^{i+1}$, we guarantee only $2^{i} \leq k<2^{i+1}$. The inequality $2^{i}<k$ will be verified on the way back to the right worktape endmarker. (ii) The condition $2^{j}=2 \cdot(\ell+1)$ is not granted, to be verified later (described in Section 4.4).

Sweep to the Right: Running the input head forward, $\mathcal{A}$ moves the worktape head back to the right endmarker and verifies whether $2^{i}<k$. Since $k$ and $2^{i}$ have the most significant bit at the same position $i$ and, in the track for $2^{i}$, this is the only bit set to 1 , it is sufficient to check whether the track for $k$ contains at least two 1 's. If $2^{i}<k$, the machine proceeds to testing membership of $1^{n}$ in AM, assuming that the condition $2^{j}=2 \cdot(\ell+1)$ is valid. If $2^{i}=k$, the machine rejects.

### 4.3. Modular Incrementing and Testing for Zero - Details

After initialization presented in the previous section, we increment $r_{1}, r_{2}$, by executing the statements $r_{1}:=\left(r_{1}+2^{j}\right) \bmod k, r_{2}:=\left(r_{2}+2^{j}\right) \bmod 2^{i}$, and check whether $r_{1} \neq 0, r_{2}=0$, all this in a single double-sweep, with $2 \cdot(\ell+1)$ steps. This is repeated until we get to the end of the input.

Sweep to the Left: At the beginning, $\mathcal{A}$ nondeterministically chooses

- between $r_{1}:=r_{1}+2^{j}$ or $r_{1}:=r_{1}+2^{j}-k$, and
- between $r_{2}:=r_{2}+2^{j}$ or $r_{2}:=r_{2}+2^{j}-2^{i}$.

These two choices are independent, hence, $\mathcal{A}$ updates $r_{1}, r_{2}$ in one of four modes. For the correct combination, we preserve $r_{1} \in\{0, \ldots, k-1\}$ and $r_{2} \in\left\{0, \ldots, 2^{i}-1\right\}$. The respective tracks are updated simultaneously, traversing from the right endmarker to the first blank, while running the input head forward.

To see details, consider, as an example, implementation of $r_{1}:=r_{1}+2^{j}-k$ by a single sweep. With access to $\left[r_{1}\right]_{t},\left[2^{j}\right]_{t},[k]_{t}$ at each bit position $t=0, \ldots, \ell-1$, we can combine the classical binary addition and subtraction into a single procedure, keeping a carry value $c_{t} \in\{-1,0,+1\}$ in the finite state control (instead of a carry bit, sufficient for one operation alone). Starting with $c_{t}=0$ for $t=0$, the new values of $\left[r_{1}\right]_{t}$ and $c_{t+1}$ are determined as follows:

$$
c_{t+1}:=\left\lfloor\left(\left[r_{1}\right]_{t}+\left[2^{j}\right]_{t}-[k]_{t}+c_{t}\right) / 2\right\rfloor ; \quad\left[r_{1}\right]_{t}:=\left(\left[r_{1}\right]_{t}+\left[2^{j}\right]_{t}-[k]_{t}+c_{t}\right) \bmod 2 .
$$

Finally, if $\mathcal{A}$ reaches the first blank symbol "\#" on the worktape with $c_{\ell}=-1$ in the finite state control, it rejects (wrong guess, resulting in negative $r_{1}$ ).

The other operations for updating $r_{1}, r_{2}$ are implemented analogically.

Sweep to the Right: Next, while running the input head forward, $\mathcal{A}$ moves the worktape head back to the right endmarker. During this process the following four conditions are tested simultaneously:

- $r_{1}<k$ and $r_{2}<2^{i}$. If any of these conditions is not valid, reject.
- $r_{1} \stackrel{?}{=} 0$ and $r_{2} \stackrel{?}{=} 0$. The outcome of these comparisons is kept in the finite state control when we get back to the right worktape endmarker.
As an example, to check whether $r_{1}<k$, scan the respective tracks for the first different bit. If $\left[r_{1}\right]_{t}<[k]_{t}$ at some position $t$, the remaining bits are ignored. Conversely, if $\left[r_{1}\right]_{t}>[k]_{t}$ or there is no difference at all, reject. The other comparisons, running simultaneously, are implemented in a similar way.

When the sweep has been completed, i.e., the worktape head is back at the right endmarker, $\mathcal{A}$ starts another double-sweep, to process the next $2 \cdot(\ell+1)$ input symbols. If, at this moment, we have reached the end of the input, $\mathcal{A}$ halts. This is done in an accepting or rejecting state, depending on whether $r_{1} \neq 0$ and $r_{1}=0$. Recall that this information is remembered in the finite state control.

If $\mathcal{A}$ hits the end of the input in the course of execution of this double-sweep, it rejects; this can happen only if the length of the input is not an integer multiple of $2 \cdot(\ell+1)$. Therefore, $n \bmod 2^{i} \neq 0$, by the reasoning from Section 4.1, based on the assumption that $2 \cdot(\ell+1)=2^{j}$ (not verified here).

### 4.4. Verifying the Size of the Allocated Space - Details

Now we shall verify the condition $\ell=2^{j-1}-1$, equivalent to $2^{j}=2 \cdot(\ell+1)$. The verification is activated after the initialization, presented in Section 4.2, and it runs in parallel with the "main" procedure counting in $r_{1}, r_{2}$, presented in Section 4.3. This causes no problems, since both routines move both the input and worktape heads in the same way. We shall keep data on the auxiliary track reserved for $a$, while the "main" procedure runs simultaneously on the first five tracks. (The track for $2^{j}$ is shared, however, both routines use it in a read-only way.) When the condition $\ell=2^{j-1}-1$ has been confirmed, the verification routine stops. From that moment on, the "main" procedure runs alone. If we find that $\ell \neq 2^{j-1}-1$, the computation of the "main" procedure is aborted and $\mathcal{A}$ rejects.

In the first double-sweep, write the bits $0,1,1$ in the auxiliary track, in that order, starting from the right endmarker and moving to the left. Then traverse across the worktape and return back. This initializes the track to $a=$ " 110 " $=6=2^{h}+h$, for $h=2$. If there is no room for storing " 110 ", reject. (If $1^{n} \in \mathrm{AM}$ and $k, i, j, \ell$ were guessed in accordance with (2), then, by (1), we have $\ell>\log k=\log f(n) \geq \log 17>4$.)

Now, in a loop, the machine modifies the auxiliary track as follows.
Sweep to the Left: Moving across the worktape, $\mathcal{A}$ increases $a$, that is, $a:=a+1$. Now the auxiliary track contains $a^{\prime}=2^{h}+(h+1)$. The bit positions for $2^{h}$ and $h+1$ do not overlap, since $\lfloor\log (h+1)\rfloor<h$, for each $h \geq 2$.

Sweep to the Right: On the way back, moving across the initial segment of zeros, $\mathcal{A}$ guesses the position $h+1$ standing in front of the leftmost 1, and sets this bit to $[a]_{h+1}=1$. Next, $\mathcal{A}$ verifies whether $[a]_{h}=1$ and clears this bit to $[a]_{h}=0$. (In case of a wrong guess, $\mathcal{A}$ rejects.) Thus, the leftmost 1 is shifted one position to the left; now the track contains $a^{\prime \prime}=2^{h+1}+(h+1)$. After that, $\mathcal{A}$ returns to the right endmarker and starts another sweep to the left, for the new value $h:=h+1$.

However, if there is no initial segment of zeros, i.e., if the leftmost 1 is at the position $\ell-1$, the machine proceeds in a different way. At this moment, we clearly have $h=\ell-1$, which gives $a^{\prime}=2^{h}+(h+1)=2^{\ell-1}+\ell$. Thus, by ignoring the leftmost bit $[a]_{\ell-1}=1$, the content of the auxiliary track becomes equal to $\ell$, the length of the worktape written in binary. $\mathcal{A}$ proceeds across the worktape to the right and compares $\ell$ with $2^{j-1}-1$. The comparison is based on the following two facts: First, $2^{j-1}-1=$ " $1^{j-1}$ ", binary written as a string not containing zeros. Second, the length of this string can be determined by looking into the track for $2^{j}$, containing " $0^{\ell-j-1} 10^{j}$ ". Thus, in a loop, running through the bit positions $t=\ell-1, \ell-2, \ldots$, the machine searches for the first $\left[2^{j}\right]_{t}=1$, checking also if $[a]_{t}=0$ in the meantime (not taking $[a]_{\ell-1} \neq 0$ into account). At the position $j$, the machine checks if $[a]_{j}=0$, and then if $[a]_{j-1}=0$. After that, in a loop, for $t=j-2, \ldots, 0$, the machine checks if $[a]_{t}=1$. If the value stored in the auxiliary track passes the test, we can confirm that $\ell=2^{j-1}-1$, the verification is over. Otherwise, we reject.

If $\mathcal{A}$ hits the end of the input before the verification is over, it rejects, overriding any potential acceptance made by the "main" procedure. To see that this is enough to fix the problem, consider how many steps are performed in the course of verification, i.e., how far we can get along the input tape.

First, the verification is activated after the initialization in Section 4.2, done in a single double-sweep. The verification routine uses also its own initial double-sweep, to write down " 110 " $=2^{2}+2$ in the auxiliary track. Finally, for $h=2, \ldots, \ell-1$, the track is updated, from $2^{h}+h$ to $2^{h+1}+(h+1)$. (This includes the last double-sweep making the final comparisons.) Since each double-sweep across the worktape takes exactly $2 \cdot(\ell+1)$ steps, the total number of steps is bounded by

$$
\begin{aligned}
V(n) & =[1+1+(\ell-2)] \times 2 \cdot(\ell+1)=2 \cdot \ell^{2}+2 \cdot \ell \leq 4 \cdot \ell^{2} \leq 4 \cdot(4 \cdot \log \log n)^{2} \\
& =64 \cdot(\log \log n)^{2}<n,
\end{aligned}
$$

using (르) and the fact that $64 \cdot(\log \log n)^{2}<n$, for each $n \geq 668$. But, under the assumption (1), we actually have $n \geq 720720$.

Summing up, if $1^{n} \in \mathrm{AM}$, then either (i) $k, i, j, \ell$ are guessed correctly, in accordance with (2), but then, by (1), there is enough time to finish verification, or (ii) $k, i, j, \ell$ are not guessed correctly, possibly violating (2), but then a premature rejection will do no harm. Clearly, for $1^{n} \notin \mathrm{AM}$, no kind of premature rejection can do harm, whatever happens.

### 4.5. The Finishing Touch

The above nondeterministic machine never accepts an input $1^{n} \notin \mathrm{AM}$, since there are no positive integers $k$ and $i$ satisfying $2^{i}<k<2^{i+1}$, such that $n \bmod k \neq 0$ and $n \bmod 2^{i}=0$. Conversely, for each $1^{n} \in \mathrm{AM}$, the machine has at least one accepting computation path, provided that $f(n) \geq 17$, the assumption introduced by (1).


Figure 3: The structure of $\omega$, where each counter representation $c_{k}$ consists of $O(\log k)$ bits, and each of these bits is associated with a subcounter of size $O(\log \log k)$ bits.

If $1^{n} \in$ AM but $f(n)<17$, we can still guess $k=f(n)$ with $2^{i}=2^{\lfloor\log k\rfloor}$ and prove the membership by verifying that $n \bmod k \neq 0$ and $n \bmod 2^{i}=0$. However, the above algorithm requires also $j, \ell$ satisfying additional conditions, not granted for $k<17$. But, for each fixed constant $k$, the task of verifying can be implemented as a finite state automaton counting the length of the input modulo $k \cdot 2^{\lfloor\log k\rfloor}$.

Thus, the updated machine $\mathcal{A}$ nondeterministically chooses from among (i) running the procedure presented above, (ii) counting the length of the input modulo $k \cdot 2^{\lfloor\log k\rfloor}$ in the finite state control, for some ${ }_{-}^{7} k \in\{13,11,9,7,5\}$. The machine starts in an accepting state, which ensures that $1^{0} \in \underline{A M}$ is accepted.

## 5. Realtime Nondeterministic PDA - Theorem $\underline{2}$

The language REI consists of inputs which are not prefixes of the infinite word

- $\omega=b c_{1} a c_{2}^{\mathrm{R}} b c_{2} a c_{3}^{\mathrm{R}} \cdots b c_{k} a c_{k+1}^{\mathrm{R}} b c_{k+1} a c_{k+2}^{\mathrm{R}} \cdots$, where
- $c_{k}=e b_{0} d b_{k, 0} d b_{0}^{\mathrm{R}} e b_{1} d b_{k, 1} d b_{1}^{\mathrm{R}} \cdots e b_{\lfloor\log k\rfloor} d b_{k,\lfloor\log k\rfloor} d b_{\lfloor\log k\rfloor}^{\mathrm{R}} e$ is a counter representation for $k$, augmented with subcounters,
- $b_{k, i} \in\{0,1\}$ is the $i^{\text {th }}$ bit in the binary representation of $k$, and $b_{i} \in\{0,1\}^{*}$ denotes the number $i$ written in binary, for $i \in\{0,1, \ldots,\lfloor\log k\rfloor\}$.
A realtime nondeterministic PDA accepts a word $w$ which is not a prefix of $\omega$ by guessing and verifying an error, which can be of the following kind (see also Figure 3):
(i) There is some error in the format, that is, the input $w$ is not a prefix of any word $\omega^{\prime} \in\left(b\left(e\{0,1\}^{*} d\{0,1\} d\{0,1\}^{*}\right)^{*} e a\left(e\{0,1\}^{*} d\{0,1\} d\{0,1\}^{*}\right)^{*} e\right)^{*}$.
(ii) The input $w$ does not begin with the counter representation for 1 , that is, with $b c_{1} a=b e 0 d 1 d 0 e a$, or $|w| \leq\left|b c_{1} a\right|$ but $w$ is not a prefix of $b c_{1} a$.
(iii) For some $k$, the counter representation $c_{k}$ does not begin with the first subcounter, i.e., with $b e b_{0} d=b e 0 d$. Symmetrically, we check whether $c_{k}^{\mathrm{R}}$ does not end by $d b_{0}^{\mathrm{R}} e b=d 0 e b$.
(iv) For some $k$ and $i$, the subcounter $b_{i}$ in $c_{k}$ is not correct, i.e., $c_{k}$ contains a defective part $d b_{i-1}^{\mathrm{R}} e b_{i^{\prime}} d$ or $e b_{i} d \xi d b_{i^{\prime}}^{\mathrm{R}} e$ with $i^{\prime} \neq i$ and $\xi \in\{0,1\}$. This can be recognized by using the pushdown store. Assuming that, within $c_{k}$, the $i^{\text {th }}$ subcounter is the smallest one with

[^5]this error, i.e., the subcounters $b_{0}, \ldots, b_{i-1}$ are correct, the pushdown space can be bounded by $O\left(\left|b_{i-1}\right|\right) \leq O\left(\left|b_{\lfloor\log k\rfloor}\right|\right)$, even though $b_{\lfloor\log k\rfloor}$ is actually not present. Symmetrically, we check whether $c_{k}^{\mathrm{R}}$ contains a defective $b_{i}^{\mathrm{R}}$.
(v) All subcounters in $w$ are correct but, for some $k$, the part between two consecutive $a$ 's is of the form $a c_{k}^{\mathrm{R}} b c_{k^{\prime}} a$, with $k^{\prime} \neq k$. This leaves us two subcases. First, $c_{k}^{\mathrm{R}}$ and $c_{k^{\prime}}$ do not agree in the highest subcounter. This is detected by loading the highest subcounter in $c_{k}^{\mathrm{R}}$ (the first one) into the pushdown store and check it against the highest subcounter in $c_{k^{\prime}}$ (the last one). Second, $k$ and $k^{\prime}$ differ in the $i^{\text {th }}$ bit, for some $i$. This is detected by guessing the position of $b_{k, i}$ in $c_{k}^{\mathrm{R}}$, pushing the following subcounter $b_{i}$ on the pushdown store, then guessing the corresponding position in $c_{k^{\prime}}$, verifying the subcounter value there, and checking that $b_{k^{\prime}, i} \neq b_{k, i}$.
(vi) All subcounters in $w$ are correct but, for some $k$, the part between two consecutive $b$ 's is of the form $b c_{k} a c_{k^{\prime}}^{\mathrm{R}} b$, with $k^{\prime} \neq k+1$. Here we have three subcases. First, the binary written $k$ is not of the form " $1 \ell$ " for some $\ell \geq 1$, but $c_{k}$ and $c_{k^{\prime}}^{\mathrm{R}}$ do not agree in the highest subcounter. Second, the binary written $k$ is of the form " 1 " (hence, the highest subcounter in $c_{k}$ is $b_{\ell-1}$ ), but the highest subcounter in $c_{k^{\prime}}^{\mathrm{R}}$ is not equal to $b_{\ell}^{\mathrm{R}}$. Both these subcases can be recognized similarly as in the previous case. Third, $k+1$ and $k^{\prime}$ differ in some bit. This is again recognized similarly as in the previous case: this time we verify that, for some $i$, by going from $c_{k}$ to $c_{k^{\prime}}^{\mathrm{R}}$, either the $i^{\text {th }}$ bit changed but the $(i-1)^{\text {st }}$ bit did not change from 1 to 0 , or the $i^{\text {th }}$ bit did not change but the $(i-1)^{\text {st }}$ bit changed from 1 to 0 (or $i=0$ ).

The errors described in the cases (i)-(iii) are detected by using the finite state control. In the cases (iv)-(vi), we only need to store one subcounter. Thus, the size of the pushdown store can be bounded by $O\left(\left|b_{[\log k\rfloor}\right|\right) \leq O(\log \log k)$. If $c_{k}$ is the smallest counter representation with an error, i.e., $c_{1}, \ldots, c_{k-1}$ are correct, we have indeed $k-1$ counters present along the input, and hence $k-1 \leq n$. Thus, the pushdown size $O(\log \log n)$ is sufficient to guess and verify the smallest occurring incorrectness.

## 6. Realtime Alternating Counter Automaton - Theorem $\underline{3}$

First, we give a one-way $\mathcal{A}$ for a finite variation of UPOWER which, in one step, either moves the input head or changes the value in the counter, but not both. Each such step is followed by exactly one " $\varepsilon$-step" neither moving the input head nor changing the counter. (But even $\varepsilon$-steps depend on whether the counter contains zero.)

The idea is to represent $k$, the current distance from the end of the input, by parallel processes of an alternating machine $\mathcal{A}$. Each process uses its counter to address only a single bit of $k$. Here we use two existential states $s_{0}, s_{1}: \mathcal{A}$ has an accepting alternating subtree in the configuration $\left(s_{v}, k, j\right)$ - corresponding to $s_{v} \in\left\{s_{0}, s_{1}\right\}$ with the head $k \geq 0$ positions away from the end of the input and a number $j \geq 0$ stored in the counter - if and only if $b_{k, j}$, the $j^{\text {th }}$ bit in the binary written $k$, is equal to $v$. (In $q_{v}$, only $\varepsilon$-steps are executed.)

The computation is based on the fact that $b_{k, j}$ depends only on $b_{k-1, j}, b_{k, j-1}$, and $b_{k-1, j-1}$. Namely, for each $j>0$ and $k>0$, using Boolean notation,

$$
\begin{aligned}
b_{k, j} & \equiv\left(\neg b_{k-1, j} \wedge \neg b_{k, j-1} \wedge b_{k-1, j-1}\right) \vee\left(b_{k-1, j} \wedge \neg b_{k-1, j-1}\right) \vee\left(b_{k-1, j} \wedge b_{k, j-1} \wedge b_{k-1, j-1}\right), \\
\neg b_{k, j} & \equiv\left(\neg b_{k-1, j} \wedge \neg b_{k-1, j-1}\right) \vee\left(\neg b_{k-1, j} \wedge b_{k, j-1} \wedge b_{k-1, j-1}\right) \vee\left(b_{k-1, j} \wedge \neg b_{k, j-1} \wedge b_{k-1, j-1}\right) .
\end{aligned}
$$

For $j=0$ with $k>0$, we have $b_{k, j} \equiv \neg b_{k-1, j}$ and, for $k=0, b_{k, j} \equiv 0$. Thus, in $s_{v} \in\left\{s_{0}, s_{1}\right\}$ and with the counter not containing zero, $\mathcal{A}$ guesses existentially which of the three clauses leads
to $b_{k, j}=v$. (For $s_{v}=s_{0}$, we have a fourth branch, guessing $k=0$. This branch switches to a state $\hat{s}$, described later.) Next, $\mathcal{A}$ branches universally to verify that all literals in the chosen clause are valid. This moves the input head and/or decreases the counter, ending in the state $s_{0}$ or $s_{1}$, depending on whether the given literal is negated. As an example, in $s_{1}$, $\mathcal{A}$ might guess, by one $\varepsilon$-step, that $b_{k, j}=1$ because of the clause $\left(\neg b_{k-1, j} \wedge \neg b_{k, j-1} \wedge b_{k-1, j-1}\right)$. After that, $\mathcal{A}$ branches to verify literals: the respective parallel path (a) moves the input head by going to $q_{0}$, (b) decreases the counter by going to $q_{0},(c)$ moves the input head, executes one $\varepsilon$-step, and then decreases the counter by going to $q_{1}$.

The case of $s_{v} \in\left\{s_{0}, s_{1}\right\}$ with the counter containing zero is similar, utilizing $b_{k, 0} \equiv \neg b_{k-1,0}$. (Also here we have a path to $\hat{s}$, guessing $k=0$.)

Finally, at the end of the input, $\mathcal{A}$ enters the state $\hat{s}$ by guessing the case of $k=0$. This can happen in the state $s_{0}$ only, since $b_{0, j} \equiv 0$. Now, in a loop, $\mathcal{A}$ decreases the counter and executes one $\varepsilon$-step. When the counter is cleared, $\mathcal{A}$ halts and accepts. (If, due to a wrong guess, $\mathcal{A}$ enters $\hat{s}$ with $k>0$, the computation is blocked in the middle of the input, and hence such path rejects.)

Given an input $a^{m}, \mathcal{A}$ verifies that $m=2^{n}$ for some $n \geq 2$, that is, whether (i) $b_{m, n}=1$, (ii) $b_{m, j}=0$ for each $j<n$, and (iii) $b_{k, n}=0$ for each $k<m$. Thus, $\mathcal{A}$ starts with a loop, in which it first increases the counter and then, by the next $\varepsilon$-step, it guesses existentially whether to exit. This chooses some $n-1 \geq 1$. After that, by increasing the counter once more, $\mathcal{A}$ branches universally to verify the conditions (i)-(iii). That is, $\mathcal{A}$ branches to (i) the state $s_{1}$, (ii) a universal loop consisting of one $\varepsilon$-step followed by one decreasing of the counter with branching to $s_{0}$, (iii) a universal loop consisting of one $\varepsilon$-step followed by one move of the input head with branching to $s_{0}$. (Both these loops halt in accepting states.)

This completes the construction of $\mathcal{A}$. For each $a^{m}$ with $m=2^{n}$ and $n \geq 2$, the accepting alternating subtree is unique, with all paths moving the input head $2^{n}$ times and changing the counter value exactly $2 n$ times. Each such step is followed by exactly one $\varepsilon$-step. Thus, by making the input head move in every step, we get a realtime $\mathcal{A}^{\prime}$ accepting $a^{m^{\prime}}$ with $m^{\prime}=2 \cdot 2^{n}+4 n=2^{n+1}+4 \cdot(n+1)-4$, which changes the accepted language from UPOWER $\backslash\left\{a^{1}, a^{2}\right\}$ to UPOWER + .

## 7. A Trade-Off to Alternation Depth

The machine in the proof of Theorem $\underline{3}$ uses a linear number of alternations. However, we can recognize UPOWER with only a logarithmic alternation depth, but using a counter of linear size. To make this algorithm easier to follow, we construct a realtime alternating $\mathcal{A}$ with a counter capable of containing also negative integers and, moreover, in one step, the counter can be updated by any $\Delta \in\{-3, \ldots,+3\}$, instead of $\Delta \in\{-1,0,+1\}$. ( $\mathcal{A}$ can be easily modified to meet the standard definition without changing the language, and hence this extension is not essential.)

First, along the input $a^{m}, \mathcal{A}$ existentially picks a position $j_{1}$, increasing the counter by 1 per each input symbol. Thus, the counter contains $j_{1}>0$ and the remaining part of the input is of length $m_{1}=m-j_{1}$. Then $\mathcal{A}$ branches universally:
(i) In the first branch, $\mathcal{A}$ verifies that $j_{1}=m_{1}$, i.e., that $j_{1}=m-j_{1}$, decreasing the counter by 1 per each symbol until it gets to the end of the input. Thus, this branch is
successful only if $2 j_{1}=m$, i.e., only if $j_{1}$ is the exact half of $m$.
(ii) In the second branch, assume that $j_{1}=m_{1}$ and $2 j_{1}=m$ since, for any other values, the outcome is overridden due to the first branch. Along $a^{m_{1}}, \mathcal{A}$ existentially picks a new position $j_{2}$, decreasing the counter by 3 per each input symbol. Now the counter contains $j_{1}-3 j_{2}<0$, the rest of the input is of length $m_{2}=m_{1}-j_{2}=j_{1}-j_{2}$. Then $\mathcal{A}$ makes a universal branching similar to the previous one:
(ii.i) In the first branch, $\mathcal{A}$ verifies that $-\left(j_{1}-3 j_{2}\right)=m_{2}$, i.e., that $-\left(j_{1}-3 j_{2}\right)=j_{1}-j_{2}$, increasing the counter by 1 per each symbol until it gets to the end of the input. Thus, this branch is successful only if $2 j_{2}=j_{1}$.
(ii.ii) In the second branch, assume that $-\left(j_{1}-3 j_{2}\right)=m_{2}$ and $2 j_{2}=j_{1}$, so we start with the counter containing $j_{1}-3 j_{2}=-j_{2}$ and the rest of the input of length $m_{2}=-\left(j_{1}-3 j_{2}\right)=j_{2}$. Now, along $a^{m_{2}}, \mathcal{A}$ existentially picks $j_{3}$, increasing the counter by 3 per each input symbol, so the counter contains $-j_{2}+3 j_{3}>0$, with the rest of the input of length $m_{3}=m_{2}-j_{3}=j_{2}-j_{3}$. Then $\mathcal{A}$ branches universally:

First, in (ii.ii.i), $\mathcal{A}$ verifies that $\left(-j_{2}+3 j_{3}\right)=m_{3}$, i.e., that $\left(-j_{2}+3 j_{3}\right)=j_{2}-j_{3}$, decreasing the counter by 1 per each symbol until the end of the input. This branch is successful only if $2 j_{3}=j_{2}$. Second, in parallel (ii.ii.ii), we assume $\left(-j_{2}+3 j_{3}\right)=m_{3}$ and $2 j_{3}=j_{2}$, so $\mathcal{A}$ starts with the counter containing $-j_{2}+3 j_{3}=j_{3}$ and the rest of the input of length $m_{3}=-j_{2}+3 j_{3}=j_{3}$. Now $\mathcal{A}$ proceeds in the same situation as in (ii), with $j_{3}=m_{3}$ instead of $j_{1}=m_{1} \ldots \ldots$

This is repeated until, for some $i$, there remains $j_{i}=1$ and a single input symbol, when $\mathcal{A}$ accepts. If $m$ is a power of 2 , we have an accepting computation subtree in which $j_{1}=\frac{m}{2}$, $j_{2}=\frac{m}{4}, \ldots, j_{i}=\frac{m}{2^{i}}, \ldots, j_{\log m}=\frac{m}{2^{\log m}}=1$, with the counter containing $+j_{1},-j_{2},+j_{3},-j_{4}, \ldots$ at the moment of universal choice. This values are unique, leading to a unique accepting alternating subtree for each accepted input, with a logarithmic number of alternations. If $m$ is not a power of 2 , there is no accepting subtree, since, for some $j_{i}, \mathcal{A}$ fails to find the exact half of the remaining input.

In a similar way, UPOWER can be recognized with only one alternation but using a linear pushdown store instead of a counter, as follows: For the given $a^{m}$, guessing existentially, the automaton loads some $w^{\mathrm{R}} c w^{\prime} \in b\{a, b\}^{*} c b\{a, b\}^{*}$ into the pushdown store and, branching universally, it verifies that $\left|w^{\prime}\right|=\left|a^{m}\right|, w^{\prime}=w, w_{2^{i}}=b$ for each $i \geq 0$ (provided that $2^{i} \leq m$ ), and that $w_{j}=a$ at all other positions. (Combining these conditions, we get that $m=2^{n}$, for some $n$.) The verification of $\left|w^{\prime}\right|=\left|a^{m}\right|$ just compares the lengths, by popping one symbol from the pushdown store and by moving one input position forward, until the symbol $c$ is popped. But the machine universally branches at each position in $w^{\prime}$, to do the following two checks:

First, the same letter from $\{a, b\}$ that appears $j$ positions away from $c$ in $w^{\prime}$ must also appear $j$ positions away from $c$ in $w^{\mathrm{R}}$. (This ensures $w^{\prime}=w$.) To guarantee equal distance in $w^{\prime}$ and $w^{\mathrm{R}}$, the parallel branch stops the input head movement until $c$ is popped out, after which the input head movement is synchronized with popping out again, until the end of the input is reached.

Second, the same letter from $\{a, b\}$ that appears $j$ positions away from $c$ in $w^{\prime}$ must also appear $2 j$ positions away from $c$ in $w^{\mathrm{R}}$. Moreover, in $w^{\mathrm{R}}$, the symbol $a$ must appear $2 j+1$ positions away. (This ensures all remaining conditions.) Checking this is similar to the previous test but, after popping $c$ out, the input head moves only one position forward per two pushdown symbols popped out.

```
if \(m=0\) then reject else if \(m=1\) then accept
loop
    for \(i:=1\) to \(m\) do
        run \(\mathcal{P}\) on the input \(w^{\prime}=a^{i} b^{m}\)
                if \(\mathcal{P}\) accepts \(w^{\prime}\) then exit for
                if \(\mathcal{P}\) rejects \(w^{\prime}\) and \(i=m\) then reject
    end for
    accept with a probability \(p\) satisfying \(0<p \leq\left(\frac{1}{9}\right)^{m}\)
end loop
```

Figure 4: A pseudo-code for $\mathcal{U}_{\mathcal{P}}$, testing whether $w=a^{m} \in \underline{\text { UPOWER. }}$

## 8. Two-Way Deterministic PDA - Theorem 4

Here we can follow the same construction as in the proof of Theorem 2. (We might even use an easier version abandoning the reverse written parts.) Instead of guessing the kind of incorrectness, we have to check them one by one in an appropriate order, to make sure that we find the smallest occurring incorrectness first:

First, check the counter representation $c_{1}$. After checking $c_{1}, \ldots, c_{k-1}$, the machine checks $c_{k}$. Namely, we check that the smallest subcounter is $b_{0}$ and that the subcounters are correctly increasing, until we get a subcounter $b_{i}$ equal to the highest subcounter in $c_{k-1}$. Then check the highest subcounter in $c_{k}$ against $c_{k-1}$ and that the main counter is increasing correctly, by going back and forth between $c_{k-1}$ and $c_{k}$ for each bit with the related subcounter in the pushdown store.

To check that subcounters are correctly increasing, just load the current subcounter to the pushdown store and compare it with the next one. Even if such comparison fails, we can restore the current pushdown contents, by going back to the beginning of the tested counter and using the prefix which, so far, has been identical. To find a position related to the current subcounter within $c_{k-1}$, just load the current subcounter to the pushdown store and compare it with the subcounters in $c_{k-1}$, one after another, until the "proper" one is found. Since we compare without destroying the pushdown store, we can return back to the original position in $c_{k}$ in the same way.

## 9. Two-Way Quantum Counter Automata - Theorem $\underline{5}$

Recently, Yakarylmaz [32] introduced a new programming technique for 2QCCAs and it was shown that USQUARE $=\left\{a^{n^{2}} \mid n \geq 1\right\}$ can be recognized by 2QCCAs for any error bound by using $O(\sqrt{n})$ space on its counter for all accepted inputs. Based on this technique, we show that $O(\log n)$ space can also be useful.

2QCFAs can recognize POWER $=\left\{a^{n} b^{2^{n}} \mid n \geq 1\right\}$ such that each $w \in$ POWER is accepted with probability 1 and each $w \notin$ POWER rejected with a probability arbitrarily close to 1 [35]. Let $\mathcal{P}$ be such a 2QCFA, rejecting $w \notin$ POWER with a probability at least $\frac{8}{9}$ (Appendix). An important property of $\mathcal{P}$ is that it reads the input from left to right in an infinite loop and uses 3 quantum states.

We present a 2QCCA $\mathcal{U}_{\mathcal{P}}$ for UPOWER calling $\mathcal{P}$ as a subroutine such that each $a^{m}$ in UPOWER is accepted with probability 1 and with the counter value not exceeding $\log m$. Each $a^{m} \notin$ UPOWER is rejected with a probability above $1-\frac{1}{8^{m}+1}$. The pseudo-code for $\mathcal{U}_{\mathcal{P}}$ is given in Figure 4.

Let $w=a^{m}$ be the input. Using the counter, we implement a for-loop iterated for $i=1, \ldots, m$, in which we simulate $\mathcal{P}$ on the input $w^{\prime}=a^{i} b^{m}$. Clearly, if $a^{m}$ is in UPOWER, $\mathcal{P}$ always accepts $w^{\prime}=a^{\log m} b^{m}$ and so we exit the for-loop with the counter containing $i \leq \log m<m$. After the exit from the for-loop, $a^{m}$ is accepted with a probability $p \leq\left(\frac{1}{9}\right)^{m}$ but, since this process is nested in an outer infinite loop, $\mathcal{U}_{\mathcal{P}}$ accepts $a^{m}$ with probability $\sum_{j=0}^{\infty} p \cdot(1-p)^{j}=1$. Moreover, the counter value (hence, the space complexity) never exceeds $\log m$. Conversely, if $a^{m}$ is not in UPOWER, it is rejected with a probability $p^{\prime} \geq\left(\frac{8}{9}\right)^{m}$ by the for-loop (when $i=m$ ). Thus, it is accepted with probability $\left(1-p^{\prime}\right) \cdot p$ after the for-loop. Again, since this is nested in the outer infinite loop, $\mathcal{U}_{\mathcal{P}}$ rejects $a^{m}$ with probability

$$
\begin{aligned}
& \sum_{j=0}^{\infty} p^{\prime} \cdot\left(1-p^{\prime}-\left(1-p^{\prime}\right) \cdot p\right)^{j} \geq \sum_{j=0}^{\infty} p^{\prime} \cdot\left(1-p^{\prime}-\left(\frac{1}{9}\right)^{m}\right)^{j}=\frac{p^{\prime}}{p^{\prime}+(1 / 9)^{m}} \\
& \quad \geq \frac{(8 / 9)^{m}}{(8 / 9)^{m}+(1 / 9)^{m}}=1-\frac{1}{8^{m}+1} .
\end{aligned}
$$

This probability can raised closer to 1 , using $\left(\frac{1}{c}\right)^{m}$ with $c>9$ instead of $\left(\frac{1}{9}\right)^{m}$.

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## Appendix. The 2QCFA $\mathcal{P}$ for POWER

The description of $\mathcal{P}$ is as follows. Let $w \in\{a, b\}^{*}$ be an input. We can assume the input of the form $a^{m} b^{n}$, where $m>0$ and $n>0$. Otherwise, $\mathcal{P}$ rejects.

The quantum register has three states: $\left|q_{1}\right\rangle,\left|q_{2}\right\rangle,\left|q_{3}\right\rangle . \mathcal{P}$ encodes $2^{m}$ and $n$ into amplitudes of $\left|q_{2}\right\rangle$ and $\left|q_{3}\right\rangle$ and compares them by subtracting. The resulting amplitude is zero if and only if the amplitudes are equal. Based on this, the input is rejected. Since we use only rational numbers, we can bound a nonzero rejecting probability from below, with zero probability only for the members. With a carefully tuned accepting probability, the members are only accepted while the nonmembers are rejected with a probability that is sufficiently high. Since this gap is achieved only with a small probability, we run the procedure in an infinite loop.

In each iteration (round), the input is read from left to right in a realtime mood. At the beginning of each round, the quantum state is set to $\left|\psi_{0}\right\rangle=\left(\begin{array}{lll}1 & 0 & 0\end{array}\right)^{\mathrm{T}}$. We keep the unconditional quantum state until we read the left endmarker. Then $\mathcal{E}_{\Phi}=\left\{E_{\mathrm{q}, 1}, E_{\mathrm{C}, 2}\right\}$ is applied to the quantum register, i.e.,

$$
E_{\mathrm{C}, 1}=\frac{1}{2}\left(\begin{array}{lll}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 2
\end{array}\right) \quad \text { and } \quad E_{\mathrm{C}, 2}=\frac{1}{2}\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 2 & 0
\end{array}\right),
$$

where (i) the current round continues if the outcome " $\subset, 1$ " is observed, and (ii) the current round is terminated without any decision if the outcome " $\phi, 2$ " is observed. Before reading $a$ 's, the quantum state is

$$
\left|\widetilde{\psi_{0}}\right\rangle=\frac{1}{2} \cdot\left(\begin{array}{l}
1 \\
1 \\
0
\end{array}\right) .
$$

For each $a, \mathcal{E}_{a}=\left\{E_{a, 1}, E_{a, 2}\right\}$ is applied to the quantum register, i.e.,

$$
E_{a, 1}=\frac{1}{2}\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{array}\right) \quad \text { and } \quad E_{a, 2}=\frac{1}{2}\left(\begin{array}{ccc}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{array}\right),
$$

where (i) the current round continues if the outcome " $a, 1$ " is observed, and (ii) the current round is terminated without any decision if the outcome " $a, 2$ " is observed. Before reading $b$ 's, the quantum state is

$$
\left|\widetilde{\psi_{m}}\right\rangle=\left(\frac{1}{2}\right)^{m+1} \cdot\binom{2^{1}}{0}
$$

For each $b, \mathcal{E}_{b}=\left\{E_{b, 1}, E_{b, 2}, E_{b, 3}\right\}$ is applied to the quantum register, i.e.,

$$
E_{b, 1}=\frac{1}{2}\left(\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{array}\right), \quad E_{b, 2}=\frac{1}{2}\left(\begin{array}{ccc}
1 & 0 & -1 \\
1 & 0 & 0 \\
0 & 1 & 1
\end{array}\right), \quad \text { and } \quad E_{b, 3}=\frac{1}{2}\left(\begin{array}{ccc}
0 & 1 & -1 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

where (i) the current round continues if the outcome " $b, 1$ " is observed, and (ii) the current round is terminated without any decision if the outcome " $b, 2$ " or " $b, 3$ " is observed. Before reading the right endmarker, the quantum state is

$$
\left|\widetilde{\left.\psi_{|w|}\right\rangle}\right\rangle=\left(\frac{1}{2}\right)^{m+n+1} \cdot\binom{2^{1}}{n} .
$$

When reading the right endmarker, $\mathcal{E}_{\$}=\left\{E_{\S, 1}, E_{\S, 2}, E_{\S, 3}, E_{\S, 4}\right\}$ is applied to the quantum register, i.e.

$$
E_{\S, 1}=\frac{1}{4}\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right), E_{\S, 2}=\frac{1}{4}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 2 & -2 \\
0 & 2 & -2
\end{array}\right), E_{\S, 3}=\frac{1}{4}\left(\begin{array}{ccc}
0 & 2 & 2 \\
0 & 2 & 2 \\
3 & 0 & 0
\end{array}\right), \quad \text { and } \quad E_{\S, 4}=\frac{1}{4}\left(\begin{array}{ccc}
2 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right),
$$

where the actions based on the measurement outcomes are as follows:

- the input is accepted if the outcome " $\$, 1$ " is observed,
- the input is rejected if the outcome " $\$, 2$ " is observed, and
- the current round is terminated without any decision, otherwise.

Thus, if the outcome " $\$, 1$ " is observed, the quantum state is

$$
\left|\widetilde{\psi_{|w|+1}}\right\rangle=\left(\frac{1}{2}\right)^{m+n+3} \cdot\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right) .
$$

That is, in a single round, the input is always accepted with probability $\left(\frac{1}{4}\right)^{m+n+3}$. If the outcome " $\$, 2$ " is observed, then the quantum state is

$$
\left|\widetilde{\psi_{|w|+1}}\right\rangle=\left(\frac{1}{2}\right)^{m+n+3} \cdot\left(\begin{array}{c}
2\left(\begin{array}{c}
0 \\
2\left(2^{m}-n\right) \\
\left.22^{m}-n\right)
\end{array}\right) .
\end{array}\right.
$$

That is, in a single round, the input will always be rejected with a probability $\left(\frac{1}{4}\right)^{m+n+3} .8$. $\left(2^{m}-n\right)^{2}$, which is

- zero for any member and
- at least 8 times greater than the accepting probability for any nonmember.

Thus, we can conclude that $\mathcal{P}$ accepts any member with probability 1 and rejects any nonmember with a probability at least $\frac{8}{9}$.


[^0]:    A preliminary version of this work was presented at the $18^{\text {th }}$ International Conference on Developments in Language Theory (DLT 2014), August 5-8, 2014, Ekaterinburg, Russia [Lect. Notes Comput. Sci., 8633, pp. 315-26. Springer-Verlag, 2014].
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[^1]:    ${ }^{1}$ Throughout the paper, $\log x$ denotes the binary logarithm of $x$, unless otherwise specified.

[^2]:    ${ }^{2}$ The upper bound on $f(n)$ will be required not only to derive an upper bound for space, but also for tuning up some parameters, so that our algorithm will work correctly.

[^3]:    ${ }^{3}$ To present binary written numbers as usual, with the least significant bits on the right end, we use a worktape growing to the left, initially empty, with the right endmarker " -1 " and infinitely many blank symbols "\#" to the left of it. (See also Figure 2.) On the other hand, there is no endmarker at the end of the input: after reading the last input symbol, the machine stops immediately.
    ${ }^{4}$ Carefully implemented, we can increase and test $r_{1}, r_{2}$ in their respective tracks simultaneously, by a single double-sweep across the worktape (moving from the right worktape endmarker "-1" to the first blank symbol "\#", followed by going back). Since there are exactly $\ell$ nonblank symbols in between "\#" and "-1", we perform this with exactly $2 \cdot(\ell+1)$ steps, per each input tape position. More implementation details will be presented for a more advanced realtime version.

[^4]:    ${ }^{5}$ Formally, the worktape alphabet is $\Gamma=\{\#, \dashv\} \cup\left\{\left[b_{1}, \ldots, b_{6}\right] \mid b_{1}, \ldots, b_{6} \in\{0,1\}\right\}$.
    ${ }^{6}$ Throughout this section, the $t^{\text {th }}$ bit of a number $a$ is denoted by $[a]_{t}$. To avoid confusion with other notation, we enclose binary strings in quotation marks, e.g., " $b^{i}$ " represents $i$ replicated copies of the same bit $b \in\{0,1\}$ (a string), while $b^{i}$ denotes $b$ raised to the power of $i$ (a number).

[^5]:    ${ }^{7}$ From $\{16, \ldots, 1\}$, the set of candidates for $k$, we exclude $16,8,4,2$ (if $f(n)=2^{m}$, for some integer $m \geq 1$, then $1^{n} \notin \mathrm{AM}$ ) and $15,14,12,10,6,1$ (from [ $\underline{4}, \underline{5}$, we know that $f(n)=p^{m}$, for some prime $p$ and integer $m \geq 1$ ). Finally, we exclude 3 , since we do not need an extra cycle counting modulo $3 \cdot 2$, once we have a cycle counting modulo $9 \cdot 8$.

