Regularity Conditions for Iterated Shuffle on Commutative Regular Languages

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Abstract. We identify a subclass of the regular commutative languages that is closed under the iterated shuffle, or shuffle closure. In particular, it is regularity-preserving on this subclass. This subclass contains the commutative group languages and, for every alphabet Σ , the class $\mathbf{Com}^+(\Sigma^*)$ given by the ordered variety \mathbf{Com}^+ . Then, we state a simple characterization when the iterated shuffle on finite commutative languages gives a regular language again and state partial results for aperiodic commutative languages. We also show that the aperiodic, or starfree, commutative languages and the commutative group languages are closed under projection.

Keywords: finite automata \cdot commutative languages \cdot closure properties \cdot iterated shuffle \cdot shuffle closure \cdot regularity-preserving operations

1 Introduction

The shuffle and iterated shuffle have been introduced and studied to understand, or specify, the semantics of parallel programs. This was undertaken, as it appears to be, independently by Campbell and Habermann [3], by Mazurkiewicz [16] and by Shaw [27]. They introduced *flow expressions*, which allow for sequential operators (catenation and iterated catenation) as well as for parallel operators (shuffle and iterated shuffle) to specify sequential and parallel execution traces.

For illustration, let us reproduce the following very simple Reader-Writer Problem from [27], as an example involving the iterated shuffle. In this problem, a set of cyclic processes may be in read-mode, but only one process at a time is allowed to be in write-mode, and read and write operations may not proceed concurrently. Additionally, we impose that the processes have to come to an end, in [27] they are allowed to run indefinitely. This constraint could be specified, using our notation, by

 $((\text{StartRead} \cdot \text{Read} \cdot \text{EndRead})^{\sqcup,*} \cup \text{Write})^*,$

where " \sqcup , *" denotes the iterated shuffle and "*" the Kleene star.

Let us note that in [27] additional lock and signal instructions were allowed. Also in [24] similar expressions for process modeling were investigated, allowing the binary shuffle operation, but without inclusion of the iterated shuffle.

The shuffle operation as a binary operation, but not the iterated shuffle, is regularity-preserving on all regular languages. However, already the iterated shuffle of very simple languages can give non-regular languages. Hence, it is interesting to know, and to identify, quite rich classes for which this operation is regularity-preserving. Here, we give such a class which includes the commutative group languages and the languages described by the positive variety **Com**⁺. Additionally, we give a characterization for the regularity of the iterated shuffle when applied to finite commutative languages and state some partial results for aperiodic (or star-free) commutative languages.

We mention that subregular language classes closed under the binary shuffle operation were investigated previously [1, 2, 4, 9, 19, 23].

We also show that the commutative star-free languages and the commutative group languages are closed under projections. For further connections on regularity conditions and closure properties, in particular for the star-free languages, see the recent survey [22].

2 Preliminaries and Definitions

2.1 General Notions

Let Σ be a finite set of symbols called an *alphabet*. The set Σ^* denotes the set of all finite sequences, i.e., of all *words*. The finite sequence of length zero, or the *empty word*, is denoted by ε . For a given word we denote by |w| its length, and for $a \in \Sigma$ by $|w|_a$ the number of occurrences of the symbol a in w. A *language* is a subset of Σ^* . If $L \subseteq \Sigma^*$ and $u \in \Sigma^*$, then the *quotients* are the languages $u^{-1}L = \{v \in \Sigma^* \mid uv \in L\}$ and $Lu^{-1} = \{v \in \Sigma^* \mid vu \in L\}$.

We assume the reader to have some basic knowledge in formal language theory, as contained, e.g., in [12, 15]. For instance, we make use of regular expressions to describe languages.

Let $\Gamma \subseteq \Sigma$. Then, we define projection homomorphisms $\pi_{\Gamma} : \Sigma^* \to \Gamma^*$ onto Γ^* by $\pi_{\Gamma}(x) = x$ for $x \in \Gamma$ and $\pi_{\Gamma}(x) = \varepsilon$ for $x \notin \Gamma$.

By $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we denote the set of natural numbers, including zero. We will also consider the ordered set $\mathbb{N}_0 \cup \{\infty\}$ with \mathbb{N}_0 having the usual order and setting $n < \infty$ for any $n \in \mathbb{N}_0$.

A quintuple $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ is a finite *(incomplete)* deterministic automaton, where $\delta : Q \times \Sigma \to S$ is a partial transition function, Q a finite set of states, $q_0 \in S$ the start state and $F \subseteq Q$ the set of final states. The automaton \mathcal{A} is said to be *complete* if δ is a total function. The transition function $\delta : Q \times \Sigma \to S$ could be extended to a transition function on words $\delta^* : Q \times \Sigma^* \to S$ by setting $\delta^*(q, \varepsilon) = q$ and $\delta^*(q, wa) := \delta(\delta^*(q, w), a)$ for $q \in Q$, $a \in \Sigma$ and $w \in \Sigma^*$. In the remainder, we drop the distinction between both functions and will also denote this extension by δ . The language recognized by an automaton $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ is $L(\mathcal{A}) = \{w \in \Sigma^* \mid \delta(q_0, w) \in F\}$. A language $L \subseteq \Sigma^*$ is called regular if $L = L(\mathcal{A})$ for some finite automaton \mathcal{A} .

The following classic result will also be needed later.

Theorem 1 (Generalized Chinese Remainder Theorem [25]). The system of linear congruences

$$x \equiv r_i \pmod{m_i}$$
 $(i = 1, 2, \dots, k)$

has integral solutions x if and only if $gcd(m_i, m_j)$ divides $(r_i - r_j)$ for all pairs $i \neq j$ and all solutions are congruent modulo $\operatorname{lcm}(m_1, \ldots, m_k)$.

$\mathbf{2.2}$ Commutative Languages and the Shuffle Operation

For a given word $w \in \Sigma^*$, we define $\operatorname{perm}(w) := \{u \in \Sigma^* \mid \forall a \in \Sigma : |u|_a =$ $|w|_a$. If $L \subseteq \Sigma^*$, then we set $\operatorname{perm}(L) := \bigcup_{w \in L} \operatorname{perm}(w)$. A language is called commutative, if perm(L) = L. Let $\Sigma = \{a_1, \ldots, a_k\}$. The Parish mapping is $\psi: \Sigma^* \to \mathbb{N}_0^k$ given by $\psi(u) = (|u|_{a_1}, \dots, |u|_{a_k})$ for $u \in \Sigma^*$. We have perm(L) = $\psi^{-1}(\psi(L)).$

The *shuffle operation*, denoted by \sqcup , is defined by

$$u \sqcup v = \{ w \in \Sigma^* \mid w = x_1 y_1 x_2 y_2 \cdots x_n y_n \text{ for some words} \\ x_1, \ldots, x_n, y_1, \ldots, y_n \in \Sigma^* \text{ such that } u = x_1 x_2 \cdots x_n \text{ and } v = y_1 y_2 \cdots y_n \},$$

for $u, v \in \Sigma^*$ and $L_1 \sqcup L_2 := \bigcup_{x \in L_1, y \in L_2} (x \sqcup y)$ for $L_1, L_2 \subseteq \Sigma^*$. In writing formulas without brackets, we suppose that the shuffle operation binds stronger than the set operations, and the concatenation operator has the strongest binding.

If $L_1, \ldots, L_n \subseteq \Sigma^*$, we set $\coprod_{i=1}^n L_i = L_1 \sqcup \ldots \sqcup L_n$. The *iterated shuffle* of $L \subseteq \Sigma^*$ is $L^{\sqcup,*} = \bigcup_{n \ge 0} \coprod_{i=1}^n L$. We also set $L^{\sqcup,+} = \bigcup_{n \ge 1} \coprod_{i=1}^n L$.

Theorem 2 (Fernau et al. [6]). Let $U, V, W \subseteq \Sigma^*$. Then,

- 1. $U \sqcup U = V \sqcup U$ (commutative law);
- 2. $(U \sqcup V) \sqcup W = U \sqcup (V \sqcup W)$ (associative law);
- 3. $U \sqcup (V \cup W) = (U \sqcup V) \cup (U \sqcup W)$ (distributive over union);

- $\begin{array}{l} 4. \quad (U^{\sqcup,*})^{\sqcup,*} = U^{\sqcup,*}; \\ 5. \quad (U \cup V)^{\sqcup,*} = U^{\sqcup,*} \sqcup V^{\sqcup,*}; \\ 6. \quad (U \amalg V^{\sqcup,*})^{\sqcup,*} = (U \sqcup (U \cup V)^{\sqcup,*}) \cup \{\varepsilon\}. \end{array}$

The next result is taken from [6] and gives equations like perm(UV) = $\operatorname{perm}(U) \sqcup \operatorname{perm}(V)$ or $\operatorname{perm}(U^*) = \operatorname{perm}(U)^{\sqcup,*}$ for $U, V \subseteq \Sigma^*$. A semiring is an algebraic structure $(S, +, \cdot, 0, 1)$ such that (S, +, 0) forms a commutative monoid, $(S, \cdot, 1)$ is a monoid and we have $a \cdot (b+c) = a \cdot b + a \cdot c$, $(b+c) \cdot a = b \cdot a + c \cdot a$ and $0 \cdot a = a \cdot 0 = 0$.

Theorem 3 (Fernau et al. [6]). perm : $\mathcal{P}(\Sigma^*) \to \mathcal{P}(\Sigma^*)$ is a semiring morphism from the semiring $(\mathcal{P}(\Sigma^*), \cup, \cdot, \emptyset, \{\varepsilon\})$, that also respects the iterated catenation resp. iterated shuffle operation, to the semiring $(\mathcal{P}(\Sigma^*), \cup, \sqcup, \emptyset, \{\varepsilon\})$.

The class of commutative languages obeys the following closure properties.

Theorem 4 ([10, 11, 20, 21]). The class of commutative languages is closed under union, intersection, complement, projections, the shuffle operation and the iterated shuffle.

2.3 Aperiodic and Group Languages

The class of aperiodic languages was introduced in [18] and admits a wealth of other characterizations.

Definition 5. An automaton $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ is aperiodic, if there exists $n \ge 0$ such that, for all states $q \in Q$ and any word $w \in \Sigma^*$, we have $\delta(q, w^n) = \delta(q, w^{n+1})$.

We define the class of aperiodic languages.

Definition 6. A regular language is called aperiodic if there exists an aperiodic automaton recognizing it.

The class of *star-free regular languages* is the smallest class containing $\{\varepsilon\}$, Σ^* and $\{a\}$ for any $a \in \Sigma$ and closed under the boolean operations and concatenation. Let us state the following, due to [26].

Theorem 7 (Schützenberger [18, 26]). The class of star-free languages equals the class of aperiodic languages.

Next, we introduce the group languages.

Definition 8 (McNaughton [17]). A (pure-)group language¹ is a language recognized by an automaton $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ where every letter acts as a permutation on the state set², i.e., if $a \in \Sigma$, then the map $\delta_a : Q \to Q$ given by $\delta_a(q) = \delta(a, q)$ for $q \in Q$ is total and a permutation of Q. Such an automaton is called a permutation automaton.

Observe that a permutation automaton, as defined here, is always complete³.

Remark 1. Note some ambiguity here in the sense that if $\Sigma = \{a, b\}$, then $(aa)^*$ is not a group language over this alphabet, but it is over the unary alphabet $\{a\}$. Hence we mean the existence of an alphabet such that the language is recognized by a permutation automaton over this alphabet. By definition, $\{\varepsilon\}$ is considered to be a group language⁴. Also, group languages are closed under the boolean operations if viewed over a common alphabet, but not over different alphabets. For instance, $L = (aa)^* \cup (bbb)^*$ is not a group language.

¹ These were introduced in [17] under the name of pure-group events.

 $^{^2}$ Such automata are also called *permutation automata*, and the name stems from the fact that the transformation monoid of such an automaton forms a group.

³ Another way would be, to allow incomplete automata, to insist that every letter either gives a permutation or labels no transition.

⁴ It is not possible to give such an automaton for $|\Sigma| \ge 1$, but allowing $\Sigma = \emptyset$ the single-state automaton will do, or similarly as $\Sigma^* = \{\varepsilon\}$ in this case.

Commutative Aperiodic and Group Languages $\mathbf{2.4}$

The next definitions and results are taken from [20, 21]. For $a \in \Sigma$ and $n, r \ge 0$ set

$$F(a,r,n) = \{ u \in \Sigma^* \mid |u|_a \equiv r \pmod{n} \},\$$

and, for $a \in \Sigma$ and $t \ge 0$,

$$F(a,t) = \{ u \in \Sigma^+ \mid |u|_a \ge t \}.$$

Note that these sets are defined relative to an alphabet Σ .

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Example 1. Let Σ be a non-empty alphabet, $a \in \Sigma$ and $\Gamma \subseteq \Sigma$.

1. $F(a, 0, 1) = \Sigma^*$.

2.
$$F(a, 0, 2) \cap F(a, 3, 4) =$$

3.
$$F(a, 1) = \Sigma^* a \Sigma^*$$
.

4.
$$\Gamma^* = \Sigma^* \setminus \left(\bigcup_{b \in \Sigma \setminus \Gamma} F(b, 1) \right).$$

Theorem 9 ([20, 21]). Let Σ be an non-empty⁵ alphabet.

- 1. The class of commutative group languages over Σ is the boolean algebra generated by the languages of the form F(a, r, n), where $a \in \Sigma$ and $0 \leq r < n$.
- 2. The class of commutative aperiodic languages over Σ is the boolean algebra generated by the languages of the form F(a, t), where $a \in \Sigma$ and $t \ge 0$.
- 3. The class of all commutative regular languages over Σ is the boolean algebra generated by the languages of the form F(a,t) or F(a,r,n), where $t \ge 0$, $0 \leq r < n \text{ and } a \in \Sigma.$

A positive boolean algebra is a class of sets closed under union and intersection. In [21], the positive variety \mathbf{Com}^+ was introduced. A positive variety [20, 21] \mathcal{V} of languages maps any alphabet Σ to a subclass $\mathcal{V}(\Sigma^*)$ of languages over this alphabet that is closed under union, intersection, quotients and inverse homomorphisms. I only mention in passing that there is a rich theory between positive varieties of languages and so called pseudovarieties of finite ordered semigroups [20]. Originally, **Com**⁺ was defined in terms of certain ordered semigroups, but here, as we do not introduce these notions, we introduce it with an equivalent characterization from [21].

Definition 10 ([21]). For every alphabet Σ , the class $\operatorname{Com}^+(\Sigma^*)$ is the positive boolean algebra generated by the languages of the form F(a, t) and F(a, r, n), where $a \in \Sigma$ and $t \ge 0, 0 \le r < n$.

Lemma 11. Let Σ be a non-empty set⁶ and $\Gamma \subseteq \Sigma$ be a proper subset. Then, $\{\Gamma^*, \Gamma^+\} \cap \operatorname{Com}^+(\Sigma^*) = \emptyset.$

Note that the previous lemma, by choosing $\Gamma = \emptyset$, implies for $\Sigma \neq \emptyset$ that $\{\varepsilon\} \notin \mathbf{Com}^+(\Sigma^*)$. The sets F(a,t) were defined as subsets of Σ^+ [21], not Σ^* . However, this makes no difference as $\Sigma^+ = F(a,0) = \bigcup_{b \in \Sigma} F(a,1)$ and $F(a, 0, 1) = \Sigma^*$ and so $\{\Sigma^+, \Sigma^*\} \subseteq \operatorname{Com}^+(\Sigma^*)$.

⁵ For $\Sigma = \emptyset$, we set all these classes to equal $\{\emptyset, \{\varepsilon\}\}$. ⁶ For $\Sigma = \emptyset$, we set $\mathbf{Com}^+(\Sigma^*) = \{\emptyset, \{\varepsilon\}\}$.

3 Commutative Aperiodic and Group Languages under Projection

First, we strengthen Theorem 9 for commutative group languages.

Theorem 12. A commutative language $L \subseteq \Sigma^*$ is a group language if and only if it could be written as a finite union of languages of the form

$$\bigcap_{i=1}^{m} F(a_i, k_i, n_i)$$

where $a_i \in \Sigma$ and $0 \leq k_i < n_i$ for $i \in \{1, \ldots, m\}$ with $m \geq 0$.

A similar statement holds for the star-free languages. But we cannot use the languages F(a, t) introduced earlier. Set, for $a \in \Sigma$ and $k_1, k_2 \in \mathbb{N}_0 \cup \{\infty\}$,

$$I(a, k_1, k_2) = \{ u \in \Sigma^* \mid k_1 \le |u|_a < k_2 \}$$

Theorem 13. A commutative language $L \subseteq \Sigma^*$ is aperiodic if and only if it could be written as a finite union of sets of the form

$$\bigcap_{i=1}^{n} I(a_i, r_i, s_i),$$

where $0 \leq r_i < s_i$ and $a_i \in \Sigma$ for $i \in \{1, \ldots, n\}$ with $n \geq 0$.

Next, we state how these languages behave under projection.

Lemma 14. Let $\Gamma \subseteq \Sigma$, $n \ge 0$, $a_i \in \Sigma$ and $0 \le r_i < s_i$ for $i \in \{1, \ldots, n\}$. Then,

$$\pi_{\Gamma}\left(\bigcap_{i=1}^{n} I(a_i, r_i, s_i)\right) = \left(\bigcap_{\substack{i \in \{1, \dots, n\} \\ a_i \in \Gamma}} I(a_i, r_i, s_i)\right) \cap \Gamma^*.$$

With Lemma 14, we can prove that the star-free commutative languages are closed under projections.

Proposition 15. Let $L \subseteq \Sigma^*$ be commutative and star-free. Then, for any $\Gamma \subseteq \Sigma$, the language $\pi_{\Gamma}(L)$ is commutative star-free.

In general, for homomorphic mappings, this is not true, as a^* could be mapped homomorphically onto $(aa)^*$, and $(aa)^*$ is not star-free [18]. Also, more specifically, there exist non-commutative star-free languages with a non-star-free projection language. For example, the language $L = (aba)^*$ is star-free, as

$$L = \{\varepsilon\} \cup (aba\Sigma^* \cap \Sigma^* aba) \setminus (\Sigma^* \cdot \{aaa, bba, bab, abb\} \cdot \Sigma^*).$$

but $\pi_{\{a\}}(L) = (aa)^*$. Similarly, with Theorem 12, we can show the next result.

Proposition 16. Let $L \subseteq \Sigma^*$ be a commutative group language. Then, for any $\Gamma \subseteq \Sigma$, the language $\pi_{\Gamma}(L)$ is a commutative group language.

However, also here, this is false for general group languages. The language $(aa)^*$ could be mapped homomorphically onto $L = (abab)^*$, which is not a group language. Also, for projections, consider the group language given by the permutation automaton $\mathcal{A} = (\{a, b\}, \{0, 1, 2\}, \delta, 0, \{2\})$ with $\delta(0, a) = 1, \delta(1, a) = 0, \delta(2, a) = 2$ and $\delta(0, b) = 1, \delta(1, b) = 2, \delta(2, b) = 0$. Then, $\pi_{\{b\}}(L(\mathcal{A})) = bb^*$, which is not a group language. For example, b is the projection of $ab \in L(\mathcal{A})$, or bbb the projection of $abbab \in L(\mathcal{A})$.

4 A Class of Regular Languages Closed under Iterated Shuffle

Here, we introduce a subclass of commutative regular languages, which contains the commutative group languages, that is closed under iterated shuffle. In Definition 17, we introduce the *diagonal periodic* languages, and first establish that the iterated shuffle of such a language gives a language that is a union of diagonal periodic languages. We then use this result to show closure under this operation of our subclass, which either could be described as the positive boolean algebra generated by languages of the form F(a, n, k), F(a, k), Γ^* and Γ^+ for $\Gamma \subseteq \Sigma$, $a \in \Sigma$, $0 \leq k < n$, or as finite unions of diagonal periodic languages.

Note that, for already very simple languages, the iterated shuffle can give non-regular languages, for example $(a \sqcup b)^{\sqcup,*} = \{ab, ba\}^{\sqcup,*} = \{u \in \{a, b\}^* \mid |u|_a = |u|_b\}$, or $(a \sqcup \{b, bb\})^{\sqcup,*} = \{u \in \{a, b\}^* \mid |u|_b \leq |u|_a \leq 2|u|_b\}$.

Definition 17. A diagonal periodic language over $\Gamma \subseteq \Sigma$ is a language of the form

$$\coprod_{a\in\Gamma} a^{k_a} (a^{p_a})^*,$$

where $k_a \ge 0$ and $p_a > 0$ for $a \in \Gamma$ when $\Gamma \neq \emptyset$, or the language $\{\varepsilon\}$.

Remark 2. Let $\Sigma = \{a_1, \ldots, a_k\}$ In [5] a sequence of vectors $\rho = v_0, v_1, \ldots, v_k$ from \mathbb{N}_0^k was called a *base* if $v_i(j) = 0$ for $i, j \in \{1, \ldots, k\}$ such that $i \neq j$. The ρ set was defined as $\Theta(\rho) = \{v \in \mathbb{N}^k : v = v_0 + l_1v_1 + \ldots + l_kv_k \text{ for some } l_1, \ldots, l_k \in \mathbb{N}_0\}$. Then, in [5], a language $L \subseteq \Sigma^*$ was called *periodic* if, for some fixed order $\Sigma = \{a_1, \ldots, a_k\}$, there exists a base ρ such that $L = \psi^{-1}(\Theta(\rho))$. With this geometric view, the diagonal periodic languages are those periodic languages such that, for $i, j \in \{1, \ldots, k\}$, either

$$v_i(j) \neq 0$$
 or $v_i(j) = v_0(j) = 0$.

Intuitively, and very roughly, the vector $\sum_{a_i \in \Gamma} v_i$ points diagonally in the subspace corresponding to the letters in Γ , or more precisely, the dimension of the subspace spanned by v_1, \ldots, v_k is precisely $|\Gamma|$. Hence, the name diagonal periodic.

⁷ Note that the entries of $v \in \mathbb{N}_0^k$ are numbered by 1 to k, i.e., $v = (v(1), \ldots, v(k))$.

As the languages $a^{k_a}(a^{p_a})^*$, $a \in \Gamma$, are regular and the binary shuffle operation is regularity-preserving [14], we get the next result. But it was also established in [5, 10, 11] for the more general class of periodic languages.

Proposition 18. The diagonal periodic languages are regular and commutative.

Remark 3. Suppose, for each $a \in \Sigma$, we have a unary language $L_a \subseteq a^*$ and $\Gamma \subseteq \Sigma$. Then, $\pi_{\Gamma}(\bigsqcup_{a \in \Sigma} L_a) = \bigsqcup_{a \in \Gamma} L_a$ and $\pi_{\Sigma}^{-1}(\bigsqcup_{a \in \Gamma} L_a) = \bigsqcup_{a \in \Gamma} L_a \sqcup (\Sigma \setminus \Gamma)^*$. This could be worked out to give a different proofs for the results from Subsection 3.

Remark 4. The reason a subalphabet $\Gamma \subseteq \Sigma$ is included in Definition 17, and later in the statements, is due to Lemma 11, i.e., to have a larger class as given by \mathbf{Com}^+ .

Next, we investigate what languages we get if we apply the iterated shuffle to diagonal periodic languages.

Proposition 19. The iterated shuffle of a diagonal periodic language $L \subseteq \Sigma^*$ over $\Gamma \subseteq \Sigma^*$ is a finite union of diagonal periodic languages. In particular, it is regular.

The next lemma is the link between the languages F(a,t), $t \ge 0$, and F(a,r,n), $0 \le r < n$, and the diagonal periodic languages.

Lemma 20. Let $\Sigma_1, \Sigma_2 \subseteq \Sigma$. Suppose we have numbers t_a for $a \in \Sigma_1$ and $0 \leq r_a < n_a$ for $a \in \Sigma_2$. Then,

$$\bigcap_{a \in \Sigma_1} F(a, t_a) \cap \bigcap_{a \in \Sigma_2} F(a, r_a, n_a) = \coprod_{a \in \Sigma} a^{k_a} (a^{p_a})^*,$$

 $where^8$

$$k_a = \begin{cases} t_a + (n_a - ((t_a - r_a) \mod n_a)) & \text{if } a \in \Sigma_1 \cap \Sigma_2, t_a > r_a; \\ r_a & \text{if } a \in \Sigma_1 \cap \Sigma_2, t_a \leqslant r_a; \\ r_a & \text{if } a \in \Sigma_2 \backslash \Sigma_1; \\ t_a & \text{if } a \in \Sigma_1 \backslash \Sigma_2; \\ 0 & \text{if } a \notin \Sigma_1 \cup \Sigma_2. \end{cases}$$

and $p_a = \begin{cases} n_a \text{ if } a \in \Sigma_2; \\ 1 \text{ if } a \notin \Sigma_2. \end{cases}$

Now, we have everything together to prove our main theorem of this subsection.

Theorem 21. Let $L \subseteq \Sigma^*$ be in the positive boolean algebra generated by languages of the form F(a,k), F(a,k,n), Γ^+ and Γ^* for $\Gamma \subseteq \Sigma$. Then, the iterated shuffle of L is contained in this positive boolean algebra. In particular, the iterated shuffle is regular.

⁸ For $x, n \in \mathbb{N}$, by $x \mod n$ we denote the unique number $r \in \{0, \ldots, n-1\}$ such that $r \equiv x \pmod{n}$.

Proof (*sketch*). As intersection distributes over union, L could be written as an intersection over the generating languages. Now,

$$F(a, k_1) \cap F(a, k_2) = F(a, \max\{k_1, k_2\})$$

and, by the generalized Chinese Remainder Theorem, Theorem 1, every intersection $\bigcap_{i=1}^{m} F(a, r_i, n_i)$ is either the empty set, or also a set of the form F(a, r, n). So, every such intersection could be written in the form

$$\left(\bigcap_{a\in\Sigma_1}F(a,t_a)\right)\cap\left(\bigcap_{a\in\Sigma_2}F(a,r_a,n_a)\right)\cap L$$

where $L \in \{\Gamma^+, \Gamma^*\}$ for some $\Gamma \subseteq \Sigma$ and $\Sigma_1, \Sigma_2 \subseteq \Sigma$. By Lemma 20, these language are diagonal periodic over Γ . By Theorem 2, the iterated shuffle of L is a finite shuffle product of iterated shuffles of these languages, which are regular by Proposition 19. Hence, they are a finite shuffle product of regular languages and as the binary shuffle product is a regularity-preserving operation [14], the language L is regular. More precisely, as the iterated shuffles are finite unions of diagonal periodic languages, the result could be written as a finite union of diagonal periodic languages, which, by Lemma 20, are contained in this class. \Box

The method of proof of Theorem 21 also gives the next result.

Proposition 22. The positive boolean algebra generated by languages of the form F(a,k), F(a,k,n), $0 \leq k < n$, Γ^+ and Γ^* , $\Gamma \subseteq \Sigma$, is precisely the language class of finite unions of the diagonal periodic languages.

Corollary 23. The iterated shuffle of a commutative group language is regular.

Proof. By Theorem 12, the class introduced in Theorem 21 contains the group languages. \Box

Corollary 24. The variety Com^+ is closed under iterated shuffle.

Also, as, for $U_a, V_a \subseteq \{a\}^*$, $a \in \Sigma$, we have $(\coprod_{a \in \Sigma} U_a) \sqcup (\coprod_{a \in \Sigma} V_a) = (\coprod_{a \in \Sigma} (U_a \cdot V_a))$, and with Theorem 2, we can deduce, by Proposition 22, the next result. This extends an old result by J.F. Perrot [19] stating that the star-free commutative language are closed under binary shuffle.

Proposition 25. The positive boolean algebra generated by the languages F(a, k), F(a, k, n), $0 \leq k < n$, Γ^+ and Γ^* for $\Gamma \subseteq \Sigma$ is closed under binary shuffle.

5 Characterizing Regularity of the Iterated Shuffle

First, in Subsection 5.1, we will give a necessary and sufficient condition when the iterated shuffle of a commutative finite language is regular. Then, in Subsection 2.3, we will present partial results for aperiodic commutative language. Lastly, in Subsection 5.3, we discuss decision procedures related to regularity, the commutative closure and the iterated shuffle.

5.1 Finite Commutative Languages

Here, we investigate finite commutative languages.

Theorem 26. Let $L \subseteq \Sigma^*$ be a finite language. Then, $\operatorname{perm}(L)^{\sqcup,*}$ is regular if and only if for any $a \in \Sigma$ with $\Sigma^* a \Sigma^* \cap L \neq \emptyset$ we have $a^+ \cap L \neq \emptyset$.

By the next corollary, we find that we can characterize regularity of expressions, for instance, of the form

$$\operatorname{perm}(u_1^+) \sqcup \ldots \sqcup \operatorname{perm}(u_n^+) = \operatorname{perm}(u_1 \cdots u_n) \sqcup \operatorname{perm}(u_1^*) \sqcup \operatorname{perm}(u_n^*)$$
$$= \operatorname{perm}(u_1 \cdots u_n) \sqcup \operatorname{perm}(\{u_1, \ldots, u_n\})^{\sqcup, *}$$

with Theorem 26, where the above equalities are implied by Theorem 2 and Theorem 3.

Corollary 27. Let $u \in \Sigma$ and $L \subseteq \Sigma^*$ be a finite language. Then, $\operatorname{perm}(u) \sqcup \operatorname{perm}(L)^{\sqcup,*}$ is regular if and only if for any $a \in \Sigma$ with $\Sigma^* a \Sigma^* \cap L \neq \emptyset$, we have $a^+ \cap L \neq \emptyset$.

5.2 Aperiodic Commutative Languages

Here, we investigate aperiodic commutative languages.

Proposition 28. Every aperiodic commutative language could be written as a finite union of languages of the form $perm(u) \sqcup \Gamma^*$ for $u \in \Sigma^*$ and $\Gamma \subseteq \Sigma$.

Remark 5. By a result from [14, Page 9], it follows that a letter which permutes with every other letter has to permute the states of every strongly connected component. This could be used to prove that the minimal automaton of an aperiodic commutative language cannot have non-trivial loops, i.e., every loop must be a self-loop, which could also be used to give a proof of Proposition 28.

With Theorem 26 we get the next result.

Proposition 29. Let $u \in \Sigma^*$ and $\Gamma \subseteq \Sigma$. The iterated shuffle of perm $(u) \sqcup \Gamma^*$ is regular if and only if there exists $a \in \Sigma$ such that $u \subseteq a^+$ or when $u \in \Gamma^*$.

Next, we give a simple sufficient criterion of regularity for a binary alphabet.

Lemma 30. Let $\Sigma = \{a, b\}$ and $L \subseteq \Sigma^*$ be regular. Then, if there exists $u \in \Sigma^*$ such that $\operatorname{perm}(u) \sqcup \Sigma^* \subseteq \operatorname{perm}(L)$, then $\operatorname{perm}(L)$ is regular.

Lastly, a few examples of aperiodic commutative languages, some of them yielding non-regular languages and some of them regular languages when applying the iterated shuffle.

Example 2. Let $\Sigma = \{a, b, c\}$.

- 1. The iterated shuffle of $\{ab, ba\} \cup \{c\} \sqcup \{a, b\}^*$ is not regular.
- 2. The iterated shuffle of $\{ab, ba\} \sqcup \{c\}^* \cup \{ac\} \sqcup \{a, b\}^*$ is not regular.
- 3. The iterated shuffle of $\{ab, ba\} \cup \{c\} \sqcup \{a, b\}^* \cup \operatorname{perm}(abb) \sqcup \{a, b\}^*$ is regular.
- 4. The iterated shuffle of $\{ab, ba\} \cup \{c\} \sqcup \{a, b\}^* \cup \operatorname{perm}(abb) \sqcup \{a\}^* \cup \{bb\}$ is regular.

5.3 Decision Procedures

In [7, 8] it was shown that for regular $L \subseteq \Sigma^*$, it is decidable if perm(L) is regular. As perm(L)^{\square ,*} = perm(L^*), also the regularity of the iterated shuffle on commutative regular languages is decidable. This result was also shown directly, without citing [7, 8], in [13, 14]. However, the precise computational complexity was not clear, and by a statement given in [6, Theorem 45] it follows that for a regular language given by a regular expression it is NP-hard to decide if the commutative closure is regular. On the contrary, the conditions stated in Theorem 26 could be tested in polynomial time for a finite commutative language given by a deterministic, a non-deterministic or a regular expression as input. This follows as non-emptiness of intersection with the fixed languages $\Sigma^* a \Sigma^*$ and a^+ , $a \in \Sigma$, could be done in polynomial time by the product automaton construction.

6 Conclusion

A general criterion as given for finite (commutative) languages in Theorem 26, which gives a polynomial time decision procedure, for general commutative regular languages is an open problem. For the subclass closed under iterated shuffle identified in Subsection 4, a sharp bound for the size of a recognizing automaton of the iterated shuffle is unknown.

Acknowledgement. I thank the anonymous reviewers for careful reading, pointing out typos and unclear formulations and providing additional references.

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A Proofs for Section 2 (Preliminaries and Definitions)

The Nerode right-congruence with respect to $L \subseteq \Sigma^*$ is defined, for $u, v \in \Sigma^*$, by $u \equiv_L v$ if and only if

$$\forall x \in \Sigma^* : ux \in L \leftrightarrow vx \in L.$$

The equivalence class, for $w \in \Sigma^*$, is denoted by $[w]_{\equiv_L} = \{x \in \Sigma^* \mid x \equiv_L w\}$. A language is regular if and only if the above right-congruence has finite index.

Theorem 4 ([10, 11, 20, 21]). The class of commutative languages is closed under union, intersection, complement, projections, the shuffle operation and the iterated shuffle.

Proof. In [20], closure under the boolean operations and shuffle is shown. For closure under projections, note that for any $L \subseteq \Sigma^*$ and $\Gamma \subseteq \Sigma$, we have $\operatorname{perm}(\pi_{\Gamma}(L)) = \pi_{\Gamma}(\operatorname{perm}(L))$. For iterated shuffle, if $u \in L^{\sqcup,*}$, then $u \in L^{\sqcup,n}$ for some $n \ge 0$. Hence, as $L^{\sqcup,n}$ is commutative, $\operatorname{perm}(u) \subseteq L^{\sqcup,n}$. So, $L^{\sqcup,*}$ is also commutative.

Lemma 11. Let Σ be a non-empty set⁹ and $\Gamma \subseteq \Sigma$ be a proper subset. Then, $\{\Gamma^*, \Gamma^+\} \cap \operatorname{Com}^+(\Sigma^*) = \emptyset$.

Proof. Every language in $\mathbf{Com}^+(\Sigma^*)$ could be written as a union over intersections of languages of the form F(a,t) and F(a,r,n), where $a \in \Sigma$ and $t \ge 0$, $0 \le r < n$. Let $L \subseteq \Sigma^*$ be such an intersection of these languages. If $L \ne \emptyset$, by Theorem 1, we can suppose for each $a \in \Sigma^*$ at most one set of the form F(a,r,n) for $0 \le r < n$ appears in an expression for L as an intersection. Also, as $F(a,t_1) \cap F(a,t_2) = F(a,\max\{k_1,k_2\})$ for $t_1, t_2 \ge 0$ we can also suppose for each letter at most one set of the form F(a,t) for $t \ge 0$ appears in an expression for L for any $a \in \Sigma$.

Fix $a \in \Sigma$. As, for $b \in \Sigma$, $F(b,t) = (F(b,t) \cap b^*) \sqcup (\Sigma \setminus \{b\})^*$ and $F(b,r,n) = (F(b,r,n) \cap b^*) \sqcup (\Sigma \setminus \{b\})^*$, we can then deduce that $a^* \cap L$ is non-empty, actually infinite. Hence, every union of such languages has this property, which gives the claim. In particular, no non-empty language $L \subseteq \Gamma^*$ is in $\mathbf{Com}^+(\Sigma^*)$.

B Proofs for Section 3 (Commutative Aperiodic and Group Languages under Projection)

Theorem 12. A commutative language $L \subseteq \Sigma^*$ is a group language if and only if it could be written as a finite union of languages of the form

$$\bigcap_{i=1}^{m} F(a_i, k_i, n_i),$$

where $a_i \in \Sigma$ and $0 \leq k_i < n_i$ for $i \in \{1, \ldots, m\}$ with $m \geq 0$.

Proof. Let $L \subseteq \Sigma^*$ be a commutative group language. Then, by Theorem 9, L is in the boolean algebra generated by languages of the form F(a, n, k). First, by using DeMorgan's laws, L is in the positive boolean closure of languages of the form F(a, k, n) or $\overline{F(a, k, n)}$. Now,

$$\overline{F(a,k,n)} = \bigcup_{i \in \{0,\dots,n-1\} \setminus \{k\}} F(a,i,n).$$

⁹ For $\Sigma = \emptyset$, we set $\mathbf{Com}^+(\Sigma^*) = \{\emptyset, \{\varepsilon\}\}.$

Hence, we can suppose L is in the positive boolean closure of languages of the form F(a, k, n). As intersection distributes over union, we can then write L as a union of intersection of languages of the form F(a, k, n), i.e., L is a union of languages of the form

$$\bigcap_{i=1}^{m} F(a_i, k_i, n_i).$$

Hence, we have shown the claim

Conversely, if L is written as a union over languages of the form as written, then, by Theorem 9, it is a commutative group language.

Theorem 13. A commutative language $L \subseteq \Sigma^*$ is aperiodic if and only if it could be written as a finite union of sets of the form

$$\bigcap_{i=1}^n I(a_i, r_i, s_i),$$

where $0 \leq r_i < s_i$ and $a_i \in \Sigma$ for $i \in \{1, \ldots, n\}$ with $n \geq 0$.

Proof. We use the characterization stated in Theorem 9. First, let $L \subseteq \Sigma^*$ be a commutative and star-free language. We have

$$\overline{F(a,k)} = \{\varepsilon\} \cup \{u \in \Sigma^* \mid |u|_a < k\} = I(a,0,k),$$
$$F(a,k) = \begin{cases} I(a,k,\infty) & \text{if } k > 0; \\ \bigcup_{b \in \Sigma} I(b,1,\infty) & \text{if } k = 0. \end{cases}$$

Hence, L is in the positive boolean algebra generated by sets of the form I(a, r, s). As intersection distributes over union, we can then write L as a union of intersections of languages of the form I(a, r, s), i.e., L is a union of languages of the form

$$\bigcap_{i=1}^n I(a_i, r_i, s_i).$$

Conversely, suppose L has the form as written in the statement. Then,

$$I(a,r,s) = \begin{cases} F(a,r) \cap \overline{F(a,s)} & \text{if } r > 0, s \neq \infty; \\ F(a,r) & \text{if } r > 0, s = \infty; \\ \overline{F(a,s)} & \text{if } r = 0, s \neq \infty; \\ \Sigma^* & \text{if } r = 0, s = \infty. \end{cases}$$

As $\Sigma^* = \overline{(F(a,0) \cap \overline{F(a,0)})}$, we find that *L* is in the boolean closure of languages of the form F(a,k). Hence, by Theorem 9, *L* is a commutative star-free language.

Lemma 14. Let $\Gamma \subseteq \Sigma$, $n \ge 0$, $a_i \in \Sigma$ and $0 \le r_i < s_i$ for $i \in \{1, \ldots, n\}$. Then,

$$\pi_{\Gamma}\left(\bigcap_{i=1}^{n} I(a_i, r_i, s_i)\right) = \left(\bigcap_{\substack{i \in \{1, \dots, n\}\\a_i \in \Gamma}} I(a_i, r_i, s_i)\right) \cap \Gamma^*.$$

Proof. Let $a \in \Sigma$ and $0 \leq k_1 < k_2$. Then, for $a \in \Gamma$ we have $k_1 \leq |\pi_{\Gamma}(u)|_a < k_2$ if and only if $k_1 \leq |u|_a < k_2$. The letters not in Γ are deleted and do not appear in the image words, hence do not contribute to the result.

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Proposition 15. Let $L \subseteq \Sigma^*$ be commutative and star-free. Then, for any $\Gamma \subseteq \Sigma$, the language $\pi_{\Gamma}(L)$ is commutative star-free.

Proof. By Theorem 4, the projected language is commutative. By Theorem 9 and as, for $U, V \subseteq \Sigma^*$,

$$\pi_{\Gamma}(U \cup V) = \pi_{\Gamma}(V) \cup \pi_{\Gamma}(V)$$

and the star-free languages are closed under union and intersection, we only need to show, by Theorem 13 and Lemma 14, that Γ^* is star-free. But this is shown in Example 1.

Proposition 16. Let $L \subseteq \Sigma^*$ be a commutative group language. Then, for any $\Gamma \subseteq \Sigma$, the language $\pi_{\Gamma}(L)$ is a commutative group language.

Proof. The proof is similar to the proof of Proposition 15, but using Theorem 12, a similar property for the intersection of sets of the form F(a, r, n) and the fact that Γ^* is a group language when considering Γ as the whole alphabet in the image of π_{Γ} (but not when it is a proper subalphabet of Σ , compare Remark 1).

C Proofs for Section 4 (A Class of Regular Languages Closed under Iterated Shuffle)

Proposition 19. The iterated shuffle of a diagonal periodic language $L \subseteq \Sigma^*$ over $\Gamma \subseteq \Sigma^*$ is a finite union of diagonal periodic languages. In particular, it is regular.

Proof. Let $L \subseteq \Sigma^*$ be a diagonal periodic language over $\Gamma \subseteq \Sigma$. Write

$$L = \coprod_{a \in \Gamma} a^{k_a} (a^{p_a})^*$$

for numbers $k_a \ge 0$, $p_a > 0$ with $a \in \Gamma$. Now, by Theorem 2 and as for unary language concatenation and shuffle coincide, we find $\coprod_{i=1}^m L = \coprod_{a \in \Gamma} a^{m \cdot k_a} (a^{p_a})^*$. So,

$$L^{\coprod, *} = \{\varepsilon\} \cup \bigcup_{m>0} \bigsqcup_{a \in \Gamma} a^{m \cdot k_a} (a^{p_a})^*.$$

Hence, $u \in L^{\sqcup,*}$ if and only if there exists $r_a \ge 0$, $a \in \Gamma$, such that $u \in \coprod_{a \in \Gamma} a^{m \cdot k_a + r_a \cdot p_a}$ for some m > 0. Now, fix $a \in \Gamma$ and consider the sum $m \cdot k_a + r_a \cdot p_a$. We have, for any $t \in \mathbb{Z}$,

$$m \cdot k_a + r_a \cdot p_a = (m - tp_a) \cdot k_a + (r_a + tk_a) \cdot p_a$$

In particular, we can choose $t \in \mathbb{Z}$ such that $1 \leq m - tp_a \leq p_a$ and $r_a + tk_a \geq 0$. Hence,

$$\bigcup_{m>0} a^{m \cdot k_a} (a^{p_a})^* = \bigcup_{i=1}^{p_a} a^{i \cdot k_a} (a^{p_a})^*.$$

Let N be the least common multiple of the numbers $p_a, a \in \Gamma$. Suppose

$$u \in \bigcup_{m > 0} \bigsqcup_{a \in \Gamma} a^{m \cdot k_a} (a^{p_a})^*$$

Then, there exist numbers $r_a \ge 0$, $a \in \Gamma$, and m > 0 such that

$$u = \coprod_{a \in \Gamma} a^{m \cdot k_a + r_a \cdot p_a}$$

Similarly as above, for any $a \in \Gamma$ and $t \ge 0$, we have

$$m \cdot k_a + r_a \cdot p_a = (m - tN) \cdot k_a + \left(r_a + t\frac{N}{p_a}k_a\right) \cdot p_a.$$

So, we can choose $t \ge 0$ such that $1 \le m - tN \le N$ and

$$u = \bigsqcup_{a \in \Gamma} a^{(m-tN) \cdot k_a + \left(r_a + t\frac{N}{p_a}k_a\right) \cdot p_a} \in \bigcup_{i=1}^N \bigsqcup_{a \in \Gamma} a^{i \cdot k_a} (a^{p_a})^*.$$

Hence, we have shown $L^{\sqcup,*} \subseteq \{\varepsilon\} \cup \bigcup_{i=1}^N \bigsqcup_{a \in \Gamma} a^{i \cdot k_a} (a^{p_a})^*$. The other inclusion is obvious, and we find

$$L^{\amalg,\ast} = \{\varepsilon\} \cup \bigcup_{i=1}^{N} \bigsqcup_{a \in \Gamma} a^{i \cdot k_a} (a^{p_a})^{\ast}.$$

So, as a finite union of diagonal periodic languages, hence regular languages by Proposition 18, the language $L^{\sqcup,*}$ is itself regular.

Lemma 20. Let $\Sigma_1, \Sigma_2 \subseteq \Sigma$. Suppose we have numbers t_a for $a \in \Sigma_1$ and $0 \leq r_a < n_a$ for $a \in \Sigma_2$. Then,

$$\bigcap_{a \in \Sigma_1} F(a, t_a) \cap \bigcap_{a \in \Sigma_2} F(a, r_a, n_a) = \bigsqcup_{a \in \Sigma} a^{k_a} (a^{p_a})^*,$$

 $where^{10}$

$$k_a = \begin{cases} t_a + (n_a - ((t_a - r_a) \mod n_a)) & \text{if } a \in \Sigma_1 \cap \Sigma_2, t_a > r_a; \\ r_a & \text{if } a \in \Sigma_1 \cap \Sigma_2, t_a \leqslant r_a; \\ r_a & \text{if } a \in \Sigma_2 \backslash \Sigma_1; \\ t_a & \text{if } a \in \Sigma_1 \backslash \Sigma_2; \\ 0 & \text{if } a \notin \Sigma_1 \cup \Sigma_2. \end{cases}$$

Proof. For the first case, let us first state an auxiliary claim. Let $t, r, n \ge 0$ with t > r and $0 \le r < n$.

<u>Claim</u>: For the unique number $m \ge 0$ with $r + m \cdot n < t \le r + (m+1) \cdot n$ we have $(m+1)n = t + (n - ((t-r) \mod n)).$

Proof of the Claim: As $mn < t - r \leq (m+1)n$, we have $mn + ((t-r) \mod n) = t - r$. Hence,

$$(m+1)n = mn + ((t-r) \mod n)) + (n - ((t-r) \mod n))$$

= t - r + (n - ((t - r) mod n)),

which gives the claim. [End, Proof of the Claim]

and $p_a = \begin{cases} n_a & \text{if } a \in \Sigma_2; \\ 1 & \text{if } a \notin \Sigma_2. \end{cases}$

¹⁰ For $x, n \in \mathbb{N}$, by $x \mod n$ we denote the unique number $r \in \{0, \ldots, n-1\}$ such that $r \equiv x \pmod{n}$.

We have, for any $a \in \Sigma$,

$$F(a,t) \cap F(a,r,n) = (a^t a^* \cap a^r (a^n)^*) \sqcup (\Sigma \setminus \{a\})^*.$$

And, by the above claim,

$$a^{t}a^{*} \cap a^{r}(a^{n})^{*} = \begin{cases} a^{r}(a^{n})^{*} & \text{if } t \leq r; \\ a^{t+(n-((t-r) \bmod n)}(a^{n})^{*} & \text{if } t > r. \end{cases}$$

Furthermore,

$$F(a, r, n) = a^{r}(a^{n})^{*} \sqcup (\Sigma \setminus \{a\})^{*};$$

$$F(a, t) = a^{t}a^{*}.$$

And these formulas give the claim.

Corollary 24. The variety Com^+ is closed under iterated shuffle.

Proof. By Definition 10, the class introduced in Theorem 21 contains $\mathbf{Com}^+(\Sigma^*)$ for any alphabet Σ . Furthermore, by the method of proof of Theorem 21 and as the iterated shuffle does not introduce new letters, and does not remove old letters, we do not leave the class $\mathbf{Com}^+(\Sigma^*)$.

D Proofs for Section 5 (Characterizing Regularity of the Iterated Shuffle)

Theorem 26. Let $L \subseteq \Sigma^*$ be a finite language. Then, $\operatorname{perm}(L)^{\sqcup,*}$ is regular if and only if for any $a \in \Sigma$ with $\Sigma^* a \Sigma^* \cap L \neq \emptyset$ we have $a^+ \cap L \neq \emptyset$.

Proof. Let $L = \{u_1, \ldots, u_n\}$. Then, by Theorem 2 and Theorem 3,

$$\operatorname{perm}(L)^{\amalg,*} = \operatorname{perm}(u_1^*) \sqcup \ldots \sqcup \operatorname{perm}(u_n^*).$$
(1)

Suppose that for $a \in \Sigma$, we find $u \in L$ with $|u|_a > 0$ but $a^+ \cap L = \emptyset$. Let $b \in \Sigma \setminus \{a\}$ be such that $|u|_b > 0$. Set $P = \pi_{\{a,b\}}(L)$, $m = \max\left\{\frac{|u|_a}{|u|_b} \mid v \in P\right\} \in \mathbb{Q}$ and choose $w \in P$ with $|w|_b m = |w|_a$. Let 0 < s < t, then, as $\frac{tm}{s} > m$ and by the maximal choice of m,

$$a^{s|w|_a}b^{|w|_bs} \in P^{\sqcup,*}$$
 and $a^{t|w|_a}b^{|w|_bs} \notin P^{\sqcup,*}$.

So, the words $a^{s|w|_a}$ and $a^{t|w|_a}$ are not equivalent for the Nerode right-congruence. Hence, the infinitely many words $\{a^{|w|_a r} \mid r > 0\}$ are pairwise non-equivalent for the Nerode right-congruence of $P^{\sqcup,*}$ and so $P^{\sqcup,*}$ has infinitely many distinct Nerode right-congruence classes and so is not regular. As $P^{\sqcup,*} = \pi_{\{a,b\}}(L)^{\sqcup,*} = \pi_{\{a,b\}}(L^{\sqcup,*})$, we find that $L^{\sqcup,*}$ is not regular.

Now, suppose the condition is true. By Equation (1), we have

$$v \in \psi(\operatorname{perm}(L)^{\boxtimes,*}) \Leftrightarrow \exists c_1, \dots, c_n \in \mathbb{N}_0 : v = c_1 \psi(u_1) + \dots + c_n \psi(u_n).$$
(2)

Next, we select k unary words from $\{u_1, \ldots, u_n\}$ such that for every letter used in L we have exactly one such non-empty word over this letter in this set of selected words. We assume these to be the first k words among u_1, \ldots, u_n . More precisely, without loss of generality, let $1 \leq k \leq n$ be such that for any $u_i, i \in \{1, \ldots, k\}$, we have $u_i \in a^+$ for

some $a \in \Sigma$ and for any $a \in \Sigma$ with $\Sigma^* a \Sigma^* \cap L \neq \emptyset$ we have $|\{u_1, \ldots, u_k\} \cap a^+| = 1$. Then, for any $i \in \{1, \ldots, k\}$, we can write $\psi(u_i) = m_i \cdot \psi(a)$, where $u_i \in a^+$ and $m_i > 0$. Also, denote by $a_i \in \Sigma$ the letter such that $\psi(u_i) = m_i \cdot \psi(a_i)$. Then, for $i, j \in \{1, \ldots, k\}$, by the assumptions, $u_i \neq u_j$ implies $a_i \neq a_j$ and $L \subseteq \{a_1, \ldots, a_k\}^*$. If, for $i \in \{k+1, \ldots, n\}$, we have $c_i \ge m_1 \cdots m_k$ in Equation (2), then, if we select number $x_a \ge 0, a \in \Sigma$, such that $\psi(u_i) = \sum_{a \in \Sigma} x_a \psi(a) = \sum_{j=1}^k x_{a_j} \psi(a_j)$, as

$$x_{a_1} \frac{m_1 \cdots m_k}{m_1} \psi(u_1) + \ldots + x_{a_k} \frac{m_1 \cdots m_k}{m_k} \psi(u_k)$$

= $x_{a_1} \frac{m_1 \cdots m_k}{m_1} m_1 \psi(a_1) + \ldots + x_{a_k} \frac{m_1 \cdots m_k}{m_k} m_k \psi(a_k)$
= $m_1 \cdots m_k \psi(u_i)$

we have

Hence, we can choose the coefficients in Equation (2) such that, for any $i \in \{k + 1, ..., n\}$,

$$c_i < m_1 \cdots m_k.$$

As, by Theorem 4, perm $(L)^{\sqcup,*}$ is commutative, we have, for $w \in \Sigma^*$,

$$w \in \operatorname{perm}(L)^{\sqcup,*} \Leftrightarrow \psi(w) \in \psi(\operatorname{perm}(L)^{\sqcup,*})$$

and, for $c_1, \ldots, c_n \ge 0$,

$$\psi(w) = c_1 \psi(u_1) + \ldots + c_n \psi(u_n) \Leftrightarrow w \in \operatorname{perm}(u_1^{c_1}) \sqcup \ldots \sqcup \operatorname{perm}(u_n^{c_n}).$$

By the previous reasoning, we conclude

$$\operatorname{perm}(u_1^*) \sqcup \ldots \sqcup \operatorname{perm}(u_n^*) = \bigcup_{\substack{(c_{k+1},\ldots,c_n)\\ 0 \leqslant c_i < m_1 \cdots m_k}} \operatorname{perm}(u_1^*) \sqcup \ldots \sqcup \operatorname{perm}(u_k^*) \sqcup \operatorname{perm}(u_{k+1}^{c_{k+1}}) \sqcup \ldots \sqcup \operatorname{perm}(u_n^{c_n}).$$

As, for $i \in \{1, \ldots, k\}$, we have $u_i \in a_i^+$, $\operatorname{perm}(u_i^*) = u_i^*$. So, as the binary shuffle operation is regularity-preserving [14], every part of the union is regular and as the union is finite we find that $\operatorname{perm}(L)^{\sqcup,*}$ is regular.

Corollary 27. Let $u \in \Sigma$ and $L \subseteq \Sigma^*$ be a finite language. Then, $\operatorname{perm}(u) \sqcup \operatorname{perm}(L)^{\sqcup,*}$ is regular if and only if for any $a \in \Sigma$ with $\Sigma^* a \Sigma^* \cap L \neq \emptyset$, we have $a^+ \cap L \neq \emptyset$.

Proof. If $U \subseteq \Sigma^*$ is any commutative language and $u \in \Sigma^*$, then perm $(u) \sqcup U$ is regular if and only if U is regular. One implication is clear as the binary shuffle operation is regularity-preserving [14].

For the other implication, first note that $U \subseteq u^{-1}(\operatorname{perm}(u) \sqcup U)$. Now we argue that $U \supseteq u^{-1}(\operatorname{perm}(u) \sqcup U)$ holds true. If $x \in \Sigma^*$ is such that $ux \in \operatorname{perm}(u) \sqcup U$, then

there exists $y \in U$ such that $ux \in perm(uy)$. This implies $x \in perm(y)$ and so, as U is commutative, $x \in U$. Hence, we find

$$U = u^{-1}(\operatorname{perm}(u) \sqcup U)$$

and as the quotient by a word is a regularity-preserving operation, the other implication follows. $\hfill \square$

Proposition 28. Every aperiodic commutative language could be written as a finite union of languages of the form $perm(u) \sqcup \Gamma^*$ for $u \in \Sigma^*$ and $\Gamma \subseteq \Sigma$.

Proof. For the sets from Section 3 we have, with $a \in \Sigma$ and $r_1, s_1, r_2, s_2 \ge 0$,

$$I(a, r_1, s_1) \cap I(a, r_2, s_2) = I(a, \max\{r_1, r_2\}, \min\{s_1, s_2\}).$$

Hence, we can suppose in the intersections from Theorem 13 that all letters are different. Then, with $a_i \neq a_j$ for $i, j \in \{1, \ldots, n\}$ distinct, we have

$$\bigcap_{i=1} I(a_i, r_i, s_i) = \bigsqcup_{\substack{i \in \{1, \dots, n\}\\ s_i = \infty}} a_i^{r_i} \sqcup \sqcup \bigsqcup_{\substack{i \in \{1, \dots, n\}\\ s_i < \infty}} \{a_i^{r_i}, a_i^{r_i+1}, \dots, a_i^{s_i}\} \sqcup \Gamma^*$$

with $\Gamma = \Sigma \setminus \{a_1, \ldots, a_n\} \cup \{a_i : \exists i \in \{1, \ldots, n\} : s_i = \infty\}$. As, for $u \in \Sigma^*$, we have $\operatorname{perm}(u) = \bigsqcup_{a \in \Sigma} a^{|u|_a}$ and, by Theorem 2, the shuffle operation distributes over union, we can write the above set as a finite union of sets of the form $\operatorname{perm}(u) \sqcup \Gamma^*$. Then, Theorem 13 gives the claim.

Proposition 29. Let $u \in \Sigma^*$ and $\Gamma \subseteq \Sigma$. The iterated shuffle of perm $(u) \sqcup \Gamma^*$ is regular if and only if there exists $a \in \Sigma$ such that $u \subseteq a^+$ or when $u \in \Gamma^*$.

Proof. By Theorem 2 and Theorem 3,

 $(\operatorname{perm}(u) \sqcup \Gamma^*)^{\sqcup,*}$

n

$$= \{\varepsilon\} \cup \operatorname{perm}(u^+) \sqcup \bigsqcup_{a \in \Gamma} a^* = \{\varepsilon\} \cup \operatorname{perm}(u) \sqcup \operatorname{perm}(\{u\} \cup \Gamma)^{\sqcup,*}.$$

Then, apply Corollary 27 and the simple fact that a language $L \subseteq \Sigma^*$ is regular if and only if $L \cup \{\varepsilon\}$ is regular.

Lemma 30. Let $\Sigma = \{a, b\}$ and $L \subseteq \Sigma^*$ be regular. Then, if there exists $u \in \Sigma^*$ such that $\operatorname{perm}(u) \sqcup \Sigma^* \subseteq \operatorname{perm}(L)$, then $\operatorname{perm}(L)$ is regular.

Proof. Let $u \in \Sigma^*$ such that $\operatorname{perm}(u) \sqcup \Sigma^* \subseteq \operatorname{perm}(L)$. Note that $\operatorname{perm}(u) \sqcup \Sigma^*$ is regular. It is well-known [6] that, if $L \subseteq \Sigma^*$ is regular, then $\psi(\operatorname{perm}(L)) = \psi(L)$ is a finite union of linear sets, i.e., sets of the form

$$\left\{v_0 + \sum_{i=1}^n k_i v_i \mid \{k_1, \dots, k_n\} \subseteq \mathbb{N}_0\right\}$$

for vectors $\{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{N}_0^{|\Sigma|}$. Hence, we show that if $A \subseteq \mathbb{N}_0^{|\Sigma|}$ is a linear subset, then $\psi^{-1}(A) \cup \operatorname{perm}(u) \sqcup \Sigma^*$ is regular, which, inductively, gives our claim. Write

$$A = \left\{ v_0 + \sum_{i=1}^n k_i v_i \mid \{k_1, \dots, k_n\} \subseteq \mathbb{N}_0 \right\}$$

for vectors $\{v_0, v_1, \ldots, v_n\} \subseteq \mathbb{N}_0^{|\Sigma|}$. Without loss of generality, we suppose none of the vectors v_1, \ldots, v_n is zero, the vectors v_1, \ldots, v_m are axis-parallel, i.e., exactly one entry is non-zero, and the vectors v_{m+1}, \ldots, v_n are sloped, i.e., we have at least two non-zero entries. Choosing words $u_i \in \Sigma^*$, $i \in \{0, \ldots, n\}$, with $\psi(u_i) = v_i$, we have

$$\psi^{-1}(A) = \operatorname{perm}(u_0) \sqcup \bigsqcup_{i=1}^{n} \operatorname{perm}(u_i^*).$$
(3)

If m = n, then all the vectors are axis-parallel. Then, the words u_1, \ldots, u_n are unary and if we write $u_i \in a_i^*$ in this case, we find $\operatorname{perm}(u_i^*) = u_i^*$ and $\psi^{-1}(A)$ is regular. Hence, $\psi^{-1}(A) \cup \operatorname{perm}(u) \sqcup \Sigma^*$ is regular. So, suppose m < n. As $\Sigma = \{a, b\}$, for any $i \in \{m + 1, \ldots, n\}$, in the the vector v_i all entries are non-zero. Hence, for the fixed $u \in \Sigma^*$ chosen above, we can choose numbers $N_i \ge 0$ such that

$$\psi(u) \leqslant v_0 + N_i v_i.$$

So, if $v = v_0 + k_1 v_1 + \ldots + k_n v_n$ with $k_i \ge N_i$ for $i \in \{1, \ldots, n\}$, then $\psi^{-1}(v) \subseteq \operatorname{perm}(u) \sqcup \Sigma^*$ and $\operatorname{perm}(u) \sqcup \Sigma^* \cup \psi^{-1}(A)$ equals

perm(u)
$$\sqcup \varSigma^* \cup$$

 $\psi^{-1}\left(\left\{v_0 + \sum_{i=1}^n k_i v_i \mid \{k_1, \dots, k_n\} \subseteq \{0, \dots, \max\{N_1, \dots, N_n\} - 1\}\right\}\right).$

Hence, it is a regular language.

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