On Customers Acting as Servers

Efrat Perel and Uri Yechiali

Department of Statistics and Operations Research,

School of Mathematical Sciences,

Tel-Aviv University, Tel-Aviv, Israel.

[naamatie@post.tau.ac.il] [uriy@post.tau.ac.il]

Abstract

We consider systems comprised of two interlacing $M/M/\bullet/\bullet$ type queues, where customers of each queue are the servers of the other queue. Such systems can be found for example in file sharing programs, SETI@home project, and other applications (see e.g. Arazi, Ben-Jacob and Yechiali (2005)). Denoting by L_i the number of customers in queue i (Q_i), $i = 1, 2$, we assume that Q_1 is a multi-server finite-buffer system with an overall capacity of size N, where the customers there are served by the L_2 customers present in Q_2. Regarding Q_2, we study two different scenarios described as follows: (i) All customers present in Q_1 join hands together to form a single server for the customers in Q_2, with service time Exponentially distributed with an overall intensity $\mu_2 L_1$. That is, the *service rate* of the customers in Q_2 changes dynamically, following the state of Q_1. (ii) Each of the customers present in Q_1 *individually* acts as a server for the customers in Q_2, with service time Exponentially distributed with mean $1/\mu_2$. In other words, the *number of servers* at Q_2 changes according to the queue size fluctuations of Q_1.

We present a probabilistic analysis of such systems, applying both Matrix Geometric method and Probability Generating Functions (PGFs) approach, and derive the stability condition for each model, along with its 2-dimensional stationary distribution function. We reveal a relationship between the roots of a given matrix, related to the PGFs, and the stability condition of the systems. In addition, we calculate the means of L_i, $i = 1, 2$, along with their correlation coefficient, and obtain the probability of blocking at Q_1. Finally, we present numerical examples and compare between the two models.

Keywords Customers Act as Servers · QBD · $M/M/1/\infty$ · $M/M/c/\infty$ · $M/M/\bullet/N$

Mathematic Subject Classification (2000) Primary: 60K25

1 Introduction

Scenarios in which customers in a queue render service elsewhere, while waiting for their own service to start or to be completed, are common in networks comprised of nodes that can receive and provide service at the same time. An example related to computer networks is presented in Arazi, Ben-Jacob and Yechiali (2005). Another application arises from distributed computer architectures labeled "peer-to-peer", designed for sharing computer resources (such as Seti@Home and others, see e.g. Androutsellis-Theotokis and Spinellis (2004) and references there). When activating such programs users connect into a peer-to-peer network to search for files on the computers of other users (i.e. peers) connected to the network. Files of interest can then be downloaded directly from those users. Typically, large files are broken down into smaller portions, which may be obtained from multiple peers and then reassembled by the downloader. This is done while the peer is simultaneously uploading the portions it already has to other peers. Hence, once a user activates a file sharing program, he/she operates as a server for the other connected users, and also as a customer downloading a file.

First steps in the analysis of queues where customers act as servers was presented in Perel and Yechiali (2008) and in Sendfeld (2009), where only the customers of one queue act as servers for the customers of the other queue. In the present work we extend the scope of the analysis to the case where the customers of both queues act as servers , namely, customers of each queue are the servers of the other queue.

Specifically, consider a system comprised of two connected and dependent queues, where customers of each queue render service to the customers of the other queue. We study two models as follows:

In Model 1 (Section 2) we assume that one queue, Q_1, operates as a multi-server finite-buffer $M(\lambda_1)/M(\mu_1)/L_2/N$ system with Poisson arrival rate λ_1 and Exponential service time with mean $1/\mu_1$ for each individual customer, where the potential servers at Q_1 are the L_2 customers present in Q_2. That is, each customer present in Q_2 *individually* acts as a server for the customers in Q_1, such that, at any given moment, the actual number of active servers in Q_1 is $Min(N, L_2)$, since Q_1 has a limited overall capacity of size N . The other queue, Q_2, operates as an unlimited-buffer *single-server* $M(\lambda_2)/M(\mu_2 L_1)/1/\infty$ system with Poisson arrival rate λ_2, but with dynamically changing *service rate* $\mu_2 L_1$ for each individually served customer. That is, the L_1 customers present in Q_1 join hands together and form a *single* server having a combined service rate of $\mu_2 L_1$ for the customers in Q_2.

In Model 2 (Section 3) we assume that Q_1 operates as in Model 1, namely as an $M(\lambda_1)/M(\mu_1)/L_2/N$ system, but Q_2 operates as a *multi-server* (rather than a single-server) $M(\lambda_2)/M(\mu_2)/L_1/\infty$ system, where each of the L_1 customers present in Q_1 act as an individul server for a customer in Q_2.

We formulate each of the two models as a two-dimensional continuous-time Markov chain and study its steady-state behavior. We apply both Matrix Geometric approach, as well as Probability Generating Functions (PGFs) to analyze those systems. We show that the stability condition for Q_2 is the same in the two models and is given by $\lambda_2 < \mu_2 \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]$, where $\mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]$ is the mean queue size in Erlang's loss system (see Cooper (1981)). We discover a relationship between the roots of a given matrix, related to the PGFs, and this stability condition. Arguing that $Cov\,(L_1, L_2)$ is non positive we establish an *analytic* lower bound for $\mathbb{E}[L_2]$ as a function of $\mathbb{E}[L_1]$. In addition, we calculate the probability of blocking at Q_1. Finally, numerical examples are presented and the models are compared.

2 Model 1

Our first model analyzes the case where Q_1 is a multi-server finite-buffer $M(\lambda_1)/M(\mu_1)/L_2/N$ system, while Q_2 is an unlimited-buffer single-server $M(\lambda_2)/M(\mu_2 L_1)/1/\infty$ queue. All arrival and service processes are mutually independent.

2.1 Balance equations

The pair (L_1, L_2) defines an irreducible continuous-time Markov chain for which the transition-rate diagram is depicted in Figure 2.1. Let $P_{nm} = \mathbb{P}(L_1 = n, L_2 = m)$, $0 \leq n \leq N$ and $0 \leq m$, denote the system's stationary probabilities (we will derive the stability condition in the sequel). Then, the set of balance equations is given as follows:

$\underline{n = 0}:$

$$m = 1: \quad (\lambda_1 + \lambda_2)P_{01} = \mu_1 P_{11}$$

$$2 \leq m: \quad (\lambda_1 + \lambda_2)P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m} \tag{2.1}$$

$\underline{n = 1}:$

$$m = 0: \quad (\lambda_1 + \lambda_2)P_{10} = \mu_2 P_{11}$$

$$m = 1: \quad (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)P_{11} = \lambda_1 P_{01} + \lambda_2 P_{10} + \mu_1 P_{21} + \mu_2 P_{12}$$

$$2 \leq m: \quad (\lambda_1 + \lambda_2 + \mu_1 + \mu_2)P_{1m} = \lambda_1 P_{0m} + \lambda_2 P_{1,m-1} + 2\mu_1 P_{2m} + \mu_2 P_{1,m+1} \tag{2.2}$$

$\underline{2 \leq n \leq N - 1}:$

$$m = 0: \quad (\lambda_1 + \lambda_2)P_{n0} = \lambda_1 P_{n-1,0} + n\mu_2 P_{n1}$$

$$1 \leq m \leq n: \quad (\lambda_1 + \lambda_2 + m\mu_1 + n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1}$$

$$n < m: \quad (\lambda_1 + \lambda_2 + n\mu_1 + n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + (n+1)\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \tag{2.3}$$

$\underline{n = N}:$

$$m = 0: \quad \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + N\mu_2 P_{n1}$$

$$1 \leq m \leq N: \quad (\lambda_2 + m\mu_1 + N\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1}$$

$$N < m: \quad (\lambda_2 + N\mu_1 + N\mu_2)P_{Nm} = \lambda_1 PN - 1, m + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \tag{2.4}$$

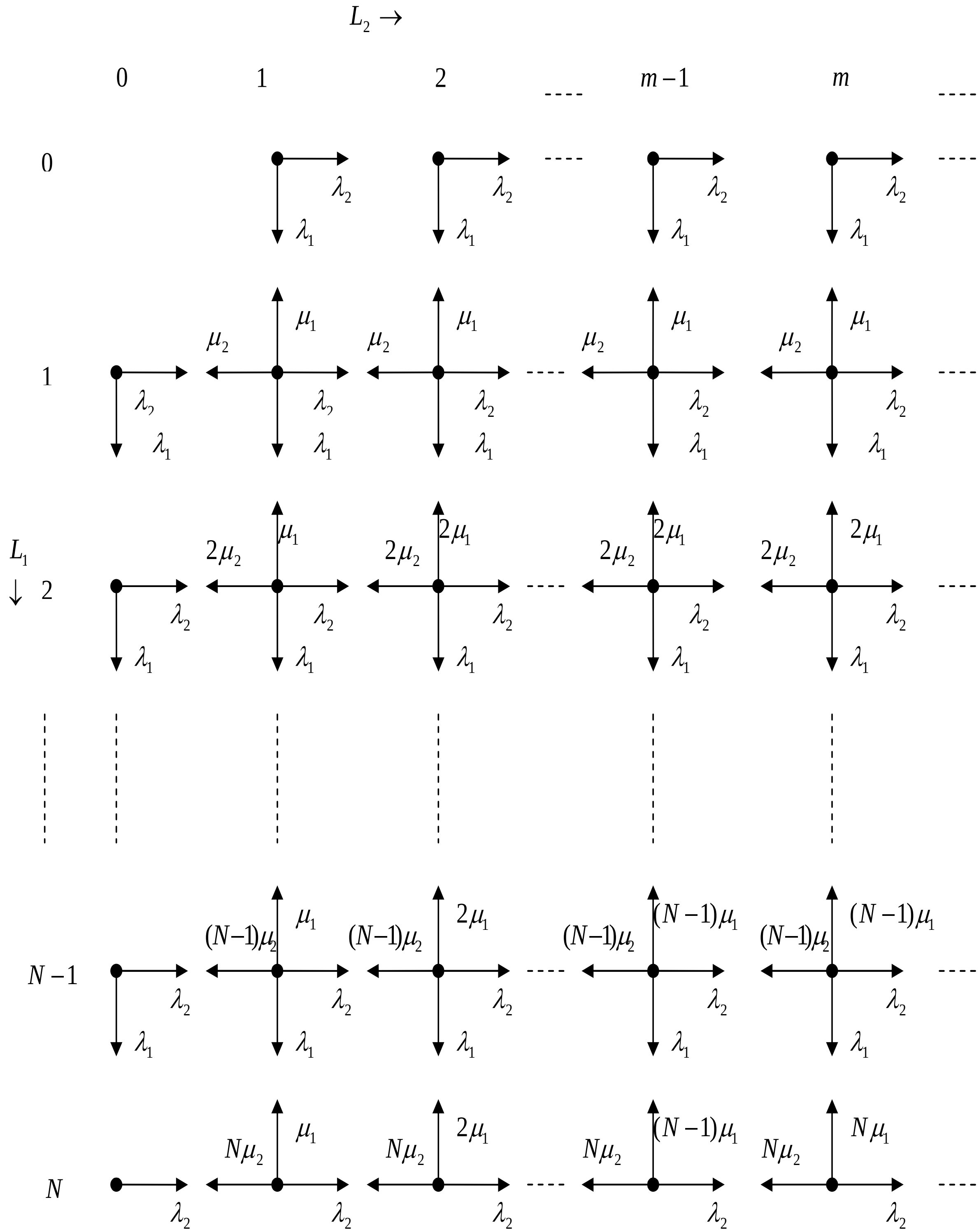

Figure 2.1: Transition rate diagram of (L_1, L_2) for Model 1.

Define (where $P_{00} = 0$) the marginal probabilities

$$\mathbb{P}(L_1 = n) \equiv P_{n\bullet} = \sum_{m=0}^{K} P_{nm} \quad \text{for} \quad 0 \leq n \leq N,$$
$$\mathbb{P}(L_2 = m) \equiv P_{\bullet m} = \sum_{n=0}^{N} P_{nm} \quad \text{for} \quad 0 \leq m.$$

Then for every $0 \leq m$, summing equations (2.1)-(2.4) over n yields

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} \mathbb{E}[L_1 | L_2 = m+1]. \tag{2.5}$$

By summing (2.5) over m we get

$$\lambda_2 \sum_{m=0}^{\infty} P_{\bullet m} = \mu_2 \sum_{m=0}^{\infty} P_{\bullet m+1} \mathbb{E}[L_1 | L_2 = m+1]. \tag{2.6}$$

Therefore, $\lambda_2 = \mu_2(\mathbb{E}[L_1] - P_{\bullet 0}\mathbb{E}[L_1|L_2 = 0]) = \mu_2 \left(\mathbb{E}[L_1] - \sum_{n=1}^{N} nP_{n0}\right)$.

That is,

$$\mathbb{E}[L_1] = \lambda_2/\mu_2 + \sum_{n=1}^{N} nP_{n0}. \tag{2.7}$$

The second term in the RHS of (2.7) represents the mean number of customers in Q_1 that stay idle when there are no customers to be served in Q_2.

Furthermore, by Summing equations (2.1)-(2.4) over m we get, for every $0 \leq n \leq N-1$,

$$\lambda_1 P_{n\bullet} = (n+1)\mu_1 P_{n+1\bullet} - \mu_1 \sum_{m=0}^{n} (n+1-m) P_{n+1,m}. \tag{2.8}$$

Summing equation (2.8) over n yields

$$\sum_{n=0}^{N-1} \lambda_1 P_{n\bullet} = \mu_1 \sum_{n=0}^{N-1} (n+1) P_{n+1\bullet} - \mu_1 \sum_{n=0}^{N-1} \sum_{m=0}^{n} (n+1-m) P_{n+1,m}.$$

Hence,

$$\lambda_1(1 - P_{N\bullet}) = \mu_1(\mathbb{E}[L_1] - \mu_1 \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{nm},$$

or

$$\mathbb{E}[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{nm}. \tag{2.9}$$

The first term in the RHS of (2.9) is the mean number of customers being served in Q_1, while the second term is the mean number of customers waiting to be served there.

Equating (2.7) and (2.9) we get

$$\mathbb{E}[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{nm} = \lambda_2/\mu_2 + \sum_{n=1}^{N} nP_{n0}. \tag{2.10}$$

Therefore, the probability of blocking at Q_1 is given by

$$\mathbb{P}(\text{Blocking at } Q_1) \equiv P_{N\bullet} = 1 - \frac{\lambda_2/\mu_2 - \sum_{n=2}^{N}\sum_{m=1}^{n-1}(n-m)P_{nm}}{\lambda_1/\mu_1}. \tag{2.11}$$

In Subsections 2.3 and 2.4 in the sequel we will show how to calculate the (yet unknown) probabilities $(P_{nm})_{0\le n\le N,\ 0\le m}$.

2.2 Generating Functions

Define, for each $0 \le n \le N$, the probability generating function, $G_n(z) = \sum_{m=0}^{\infty} P_{nm} z^m$. Multiplying by z^m each equation for m in the sets (2.1)-(2.4), summing over m and rearranging terms we get

$\underline{n = 0}$:

$$(\lambda_1 + \lambda_2(1-z))\, G_0(z) = \mu_1 G_1(z) - \mu_1 P_{10} \tag{2.12}$$

$\underline{1 \le n \le N-1}$:

$$\begin{aligned}((\lambda_1 + n\mu_1)\, z + (\lambda_2 z + n\mu_2)(1-z))\, G_n(z) &= \lambda_1 z G_{n-1}(z) + (n+1)\mu_1 z G_{n+1}(z) - n\mu_2 P_{n0}(1-z) \\ &\quad + \mu_1 z \left(\sum_{m=0}^{n-1}(n-m)z^m P_{nm} - \sum_{m=0}^{n}(n+1-m)z^m P_{n+1,m}\right)\end{aligned} \tag{2.13}$$

$\underline{n = N}$:

$$(N\mu_1 z + (\lambda_2 z + N\mu_2)(1-z))\, G_N(z) = \lambda_1 z G_{N-1}(z) - N\mu_2 P_{N0}(1-z) + \mu_1 z \sum_{m=0}^{N-1}(N-m)z^m P_{Nm} \tag{2.14}$$

The sets (2.12)-(2.14) comprise a system of linear equations of the form

$$A(z)\vec{G}(z) = \vec{P}(z),$$

where, the vectors $\vec{G}(z)$ and $\vec{P}(z)$ and the matrix $A(z)$ are defined as follows:

$$\vec{G}(z) = (G_0(z), G_1(z), ..., G_N(z))^t$$

$$\vec{P}(z) = (P_0(z), P_1(z), ..., P_N(z))^t$$

with

$$P_n(z) = \begin{cases} -\mu_1 P_{10} & , n = 0 \\ -n\mu_2 P_{n0}(1-z) + \mu_1 z \left(\sum_{m=0}^{n-1} (n-m) z^m P_{nm} - \sum_{m=0}^{n} (n+1-m) z^m P_{n+1,m} \right) & , 1 \le n \le N \\ -N\mu_2 P_{N0}(1-z) + \mu_1 z \sum_{m=0}^{N-1} (N-m) z^m P_{Nm} & , n = N \end{cases}$$

$$A(z) = \begin{pmatrix} \alpha_0^{(N)}(z) & -\mu_1 & 0 & \cdots & \cdots & 0 \\ -\lambda_1 z & \alpha_1^{(N)}(z) & -2\mu_1 z & 0 & \cdots & 0 \\ 0 & -\lambda_1 z & \alpha_2^{(N)}(z) & -3\mu_1 z & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & -N\mu_1 z \\ 0 & \cdots & \cdots & 0 & -\lambda_1 z & \alpha_N^{(N)}(z) \end{pmatrix},$$

where

$$\alpha_0^{(N)}(z) = \lambda_1 + \lambda_2(1-z),$$

$$\alpha_n^{(N)}(z) = (\lambda_1 + n\mu_1) z + (\lambda_2 z - n\mu_2)(1-z), \quad \text{for} \quad 1 \le n < N,$$

$$\alpha_N^{(N)}(z) = N\mu_1 z + (\lambda_2 z - N\mu_2)(1-z).$$

To obtain $G_n(z)$ we use Cramer's rule. I.e., for every $0 \le n \le N$, $G_n(z) = \frac{|A_n(z)|}{|A(z)|}$, where $|A|$ is the determinant of the matrix A and $A_n(z)$ is a matrix obtained from $A(z)$ by replacing its nth column by $\vec{P}(z)$. This leads to an expression of $G_n(z)$ in terms of the $N(N+1)/2$ unknown probabilities, P_{10}; P_{20}, P_{21}; P_{N0}, P_{N1}, ..., $P_{N,N-1}$, appearing in $\vec{P}(z)$. In order to find $\vec{P}(z)$ we need to find $N(N+1)/2$ equations relating those $N(N+1)/2$ variables. We do that in the next section by characterizing and using the roots of $|A(z)|$. Since $G_n(z)$ is a probability generating function defined for all $0 \le z \le 1$, each root of $|A(z)|$ in that interval is a root of $|A_n(z)|$, for every $0 \le n \le N$.

2.3 Derivation of P_{10}; P_{20}, P_{21}; P_{N0}, P_{N1}, ..., $P_{N,N-1}$ and $\mathbb{E}[L_2]$

Theorem 2.1. *For any $\lambda_1 > 0$, μ_1, $\lambda_2 \geq 0$, $\mu_2 > 0$ and $N \geq 1$, $|A(z)|$ is a polynomial of degree $2N+1$ possessing $N-1$ distinct roots in the open interval $(0,1)$, a single root at $z=1$, and N roots in the open interval $(1,\infty)$. Another root exists in the open interval $(0,1)$ if the condition $\lambda_2 > \mu_2\mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]$ holds.*

Proof. Let $q_0^{(N)} = 1$. Define the minors of the diagonal of $A(z)$, starting from the higher left side corner, as follows:

$$q_1^{(N)}(z) = \alpha_0^{(N)}(z), \quad q_2^{(N)}(z) = \begin{vmatrix} \alpha_0^{(N)}(z) & -\mu_1 \\ -\lambda_1 & \alpha_1^{(N)}(z) \end{vmatrix}, ..., \quad q_{N+1}^{(N)}(z) = |A(z)| \tag{2.15}$$

The polynomials $q_n^{(N)}$, $1 \leq n \leq N+1$, satisfy the following equations:

$$\begin{aligned} q_1^{(N)}(z) &= \alpha_0^{(N)}(z)q_0^{(N)}(z), \\ q_2^{(N)}(z) &= \alpha_1^{(N)}(z)q_1^{(N)}(z) - \lambda_1\mu_1 z q_0^{(N)}(z), \\ q_n^{(N)}(z) &= \alpha_{n-1}^{(N)}(z)q_{n-1}^{(N)}(z) - (n-1)\lambda_1\mu_1 z^2 q_{n-2}^{(N)}(z) \quad \text{for } \ 3 \leq n \leq N+1. \end{aligned} \tag{2.16}$$

From (2.15) and (2.16) we conclude that

1. $q_0^{(N)}(z) = 1$ and therefore has no roots.

2. $q_n^{(N)}(z)$ and $q_{n+1}^{(N)}(z)$ have no joint roots in $(0,\infty)$. Otherwise, suppose they have a joint root, then it would also be a root for $q_{n-1}^{(N)}(z)$, $q_{n-2}^{(N)}(z)$, ..., $q_0^{(N)}(z)$ which contradicts 1.

3. $\mathrm{Sign}\Big(q_n^{(N)}(0)\Big) = (-1)^{n+1}$, for all $1 \leq n \leq N+1$.

4. $\mathrm{Sign}\Big(q_n^{(N)}(\infty)\Big) = (-1)^n$, for all n.

5. $q_n^{(N)}(1) = \lambda_1^n$, for all $0 \leq n \leq N$ and $q_{N+1}^{(N)}(1) = 0$.

6. $\mathrm{Sign}\Big(\alpha_n^{(N)}(0)\Big) = -1$, for all $1 \leq n \leq N$.

7. Given $\tilde{z}$ a root of $q_n^{(N)}(z)$, then sign $\Big(q_{n-1}^{(N)}(\tilde{z})q_{n+1}^{(N)}(\tilde{z})\Big) = -1$

8. $q_n^{(N)}(z)$ is a polynomial of degree $2n-1$ for $1 \leq n \leq N+1$.

9. For $1 \leq n \leq N$ the polynomial $q_n^{(N)}(z)$ has $2n-1$ distinct roots, where $n-1$ of them are in the open interval $(0,1)$ and the other n are in the open interval $(1,\infty)$

From the above we conclude that $q_1^{(N)}(z)$ has only one root, $z_{1,1} = 1+\frac{\lambda_1}{\lambda_2} > 1$. $q_2^{(N)}(0) < 0$, $q_2^{(N)}(1) = \lambda_1^2 > 0$, $q_2^{(N)}(z_{1,1}) < 0$, $q_2^{(N)}(\infty) > 0$. Therefore, the 3 roots of $q_2^{(N)}(z)$ satisfy: $z_{2,1} \in (0,1)$, $z_{2,2} \in (1, z_{1,1})$, $z_{2,3} \in (z_{1,1}, \infty)$. Similarly, $q_3^{(N)}(z)$ is of degree 5 and therefore can have no more than 5 roots. Also $q_3^{(N)}(0) > 0$, $q_3^{(N)}(z_{2,1}) < 0$, $q_3^{(N)}(1) = \lambda_1^3 > 0$, $q_3^{(N)}(z_{2,2}) < 0$, $q_3^{(N)}(z_{2,3}) > 0$, $q_3^{(N)}(\infty) < 0$. This implies that $q_3^{(N)}(z)$ has exactly 5 distinct roots satisfying: $z_{3,1} \in (0, z_{2,1})$, $z_{3,2} \in (z_{2,1}, 1)$, $z_{3,3} \in (1, z_{2,2})$, $z_{3,4} \in (z_{2,2}, z_{2,3})$, $z_{3,5} \in (z_{2,3}, \infty)$.

In general, for $2 \leq n \leq N$, given $2n-3$ distinct roots of $q_{n-1}^{(N)}(z)$, the roots of $q_n^{(N)}(z)$ satisfy: $z_{n,1} \in (0, z_{n-1,1})$, $z_{n,2} \in (z_{n-1,1}, z_{n-1,2})$, ..., $z_{n,n-1} \in (z_{n-1,n-2}, 1)$, $z_{n,n} \in (1, z_{n-1,n-1})$, ..., $z_{n,2n-1} \in (z_{n-1,2n-3}, \infty)$.

$q_{N+1}^{(N)}(z)$ has $2N+1$ roots where the first $N-1$ are within the interval $(0,1)$ satisfying $z_{N+1,1} \in (0, z_{N,1})$, $z_{N+1,2} \in (z_{N,1}, z_{N,2})$, ..., $z_{N+1,N-1} \in (z_{N,N-2}, z_{N,N-1})$. As for the N-th root, $z_{N+1,N}$, we observe that, since $z_{N,N-1} \in (z_{N-1,N-2}, 1)$, we have that $q_{N-1}^{(N)}(z_{N,N-1}) > 0$ and therefore, $q_{N+1}^{(N)}(z_{N,N-1}) = -\lambda_1\mu_1(z_{N,N-1})^2 q_{N-1}^{(N)}(z_{N,N-1}) < 0$. $q_{N+1}^{(N)}(1) = 0$, and we need to check whether another root (besides the $N-1$ already mentioned) exists in $(z_{N,N-1}, 1)$. We will show that under a stationary condition, such a root does not exist. In such a case, the $N-1$ distinct roots of $q_{N+1}^{(N)}(z)$ in $(0,1)$ will provide $N-1$ equations relating the $N(N+1)/2$ unknown probabilities. By induction over n we obtain (see Proposition A.1)

$$q_n^{(N)}(z) = \lambda_1^n z^{n-1} + (1-z)h_n^{(N)}(z), \quad \text{for } \; 1 \leq n \leq N, \tag{2.17}$$

$$q_{N+1}^{(N)}(z) = (1-z)h_{N+1}^{(N)}(z). \tag{2.18}$$

Another root exists in $(z_{N,N-1}, 1)$ if and only if $h_{N+1}^{(N)}(1) > 0$.

Substituting (2.17) in (2.16) yields the following (see Proposition A.1):

$$h_1^{(N)}(z) = \lambda_2$$

$$h_2^{(N)}(z) = (\lambda_2 z - \mu_2)(\lambda_1 + \lambda_2(1-z)) + \lambda_2 z(\lambda_1 + \mu_1)$$

$$h_n^{(N)}(z) = \alpha_{n-1}^{(N)}(z) h_{n-1}^{(N)}(z) - (n-1)\lambda_1\mu_1 z^2 h_{n-2}^{(N)}(z) + \lambda_1^{n-1} z^{n-2}(\lambda_2 z - (n-1)\mu_2), \quad 3 \leq n \leq N \tag{2.19}$$

$$h_{N+1}^{(N)}(z) = \alpha_N^{(N)}(z) h_N^{(N)}(z) - N\lambda_1\mu_1 z^2 h_{N-1}^{(N)}(z) + \lambda_1^N z^{N-1}(\lambda_2 z - N\mu_2). \tag{2.20}$$

Substituting $z = 1$ in the above gives

$$h_1^{(N)}(1) = \lambda_2$$

$$h_2^{(N)}(1) = (\lambda_2 - \mu_2)\lambda_1 + \lambda_2(\lambda_1 + \mu_1)$$

$$h_n^{(N)}(1) = (\lambda_1 + (n-1)\mu_1) h_{n-1}^{(N)}(1) - (n-1)\lambda_1\mu_1 z^2 h_{n-2}^{(N)}(1) + \lambda_1^{n-1}(\lambda_2 - (n-1)\mu_2), \quad 3 \leq n \leq N \tag{2.21}$$

$$h_{N+1}^{(N)}(1) = N\mu_1 h_N^{(N)}(1) - N\lambda_1\mu_1 h_{N-1}^{(N)}(z) + \lambda_1^N(\lambda_2 - N\mu_2). \tag{2.22}$$

Therefore (see Proposition A.2), for every $N \geq 2$,

$$h_{N+1}^{(N)}(1) = \lambda_2 N! \mu_1^N \sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 N! \mu_1^N \sum_{n=1}^{N} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}. \tag{2.23}$$

Since another root for $|A(z)|$ exists if and only if $h_{N+1}^{(N)}(1) > 0$,

$$h_{N+1}^{(N)}(1) = \lambda_2 N! \mu_1^N \sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 N! \mu_1^N \sum_{n=1}^{N} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} > 0.$$

This implies that another root exists if and only if

$$\frac{\lambda_2}{\mu_2} > \frac{\sum_{n=1}^{N} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}}{\sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}} = \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]. \tag{2.24}$$

This completes the proof of Theorem 2.1. □

Remark: We will show in the next Subsection 2.4 that if condition (2.24) holds (namely, there exists an extra root in $(0, 1)$) the system is unstable.

To find the $N(N + 1)/2$ unknown probabilities appearing in $\vec{P}(z)$, when (2.24) does not hold, we use the $N - 1$ distinct roots in the open interval $(0, 1)$, which provide us with $N - 1$ equations for those probabilities.

Another $N(N-1)/2$ equations are taken from the balance equations for states (n,m), $2 \leq n \leq N$, $0 \leq m \leq n-2$. Together with equation (2.11), and since $P_{N\bullet} = G_N(1)$, we have a linear set of $N(N+1)/2$ distinct equations in the $N(N+1)/2$ unknown probabilities. Once the above $N(N+1)/2$ probabilities are calculated, the PGFs are completely determined and $\mathbb{E}[L_1]$ is calculated by (2.7).

The mean total number of customers in Q_2, $\mathbb{E}[L_2]$, is obtained by summing the derivatives of $G_n(z)$ over n at $z=1$. That is,

$$\mathbb{E}[L_2] = \sum_{n=0}^{N} G_n'(1) = \sum_{n=0}^{N} \mathbb{E}[L_2|L_1=n]\mathbb{P}\left(L_1=n\right). \tag{2.25}$$

Also, by multiplying equation (2.12) by z, summing it with (2.13) over $1 \leq n \leq N$, and adding (2.14) we get

$$\begin{aligned}
&\lambda_1 z \sum_{n=0}^{N-1} G_n(z) + \mu_1 z \sum_{n=1}^{N} nG_n(z) + (1-z)\sum_{n=0}^{N}\left(\lambda_2 z - n\mu_2\right)G_n(z) \\
&= \lambda_1 z \sum_{n=1}^{N} G_(n-1)(z) + \mu_1 z \sum_{n=0}^{N-1}(n+1)G_(n+1)(z) - (1-z)\mu_2 \sum_{n=0}^{N} nP_{n0} \\
&+ \mu_1\left(\sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)z^m P_{nm} - \sum_{n=0}^{N-1}\sum_{m=0}^{n}(n+1-m)z^m P_{n+1,m}\right),
\end{aligned} \tag{2.26}$$

implying that

$$\sum_{n=0}^{N}\left(\lambda_2 z - n\mu_2\right)G_n(z) = -\mu_2\sum_{n=0}^{N} nP_{n0} \tag{2.27}$$

Differentiating both sides of (2.27) and setting $z=1$ yields

$$\sum_{n=0}^{N}\lambda_2 G_n(1) + \sum_{n=0}^{N}\left(\lambda_2 - n\mu_2\right)G_n'(1) = \lambda_2 + \lambda_2\mathbb{E}[L_2] - \mu_2\mathbb{E}[L_1 \cdot L_2] = 0. \tag{2.28}$$

Therefore,

$$\mathbb{E}[L_1 \cdot L_2] = \frac{\lambda_2 + \lambda_2\mathbb{E}[L_2]}{\mu_2} = \frac{\lambda_2}{\mu_2} + \frac{\lambda_2\mathbb{E}[L_2]}{\mu_2} = \frac{\lambda_2}{\mu_2}\left(1 + \mathbb{E}[L_2]\right). \tag{2.29}$$

Thus,

$$\begin{aligned}
Cov\left(L_1, L_2\right) &= \mathbb{E}[L_1 \cdot L_2] - \mathbb{E}[L_1]\cdot\mathbb{E}[L_2] = \frac{\lambda_2}{\mu_2} + \frac{\lambda_2\mathbb{E}[L_2]}{\mu_2} - \mathbb{E}[L_1]\cdot\mathbb{E}[L_2] \\
&= \frac{\lambda_2}{\mu_2} - \mathbb{E}[L_2]\left(\mathbb{E}[L_1] - \frac{\lambda_2}{\mu_2}\right).
\end{aligned} \tag{2.30}$$

We argue that $Cov\left(L_1, L_2\right) \leq 0$, since increasing values of L_1 reduce the magnitude of L_2. Hence, we can derive a closed-form lower bound for $\mathbb{E}[L_2]$ (in terms of $\mathbb{E}[L_1]$):

$$\mathbb{E}[L_2] \geq \frac{\frac{\lambda_2}{\mu_2}}{\mathbb{E}[L_1] - \frac{\lambda_2}{\mu_2}} = \frac{\lambda_2}{\mu_2 \mathbb{E}[L_1] - \lambda_2}. \tag{2.31}$$

Clearly, by Little's Law, $\mathbb{E}[W_2] = \frac{\mathbb{E}[L_2]}{\lambda_2} \geq \frac{1}{\mu_2 \mathbb{E}[L_1] - \lambda_2}$.

2.4 Matrix Geometric Method for Deriving $(P_{nm})_{0 \leq n \leq N,\ 0 \leq m}$

We now use Matrix Geometric approach for deriving $(P_{nm})_{0 \leq n \leq N,\ 0 \leq m}$ and for further analysis of the system. Our queueing system can be described as having $N+1$ 'phases', where phase n indicates that the service rate in Q_2 is $n\mu_2$. State (n, m) denotes that there are m jobs in Q_2, $0 \leq m$, and the system is in phase n, $0 \leq n \leq N$. We construct a quasi birth-and-death (QBD) process (Nuets (1981), Latouche and Ramaswami (1999)) with generator Q given by

$$Q = \begin{pmatrix}
A_1^0 & A_0^0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
A_2^1 & A_1^1 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & A_2 & A_1^2 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots \\
\vdots & \vdots & \mathbf{0} & A_2 & A_1^N & A_0 & \mathbf{0} & \cdots & \cdots \\
\vdots & \vdots & \vdots & \mathbf{0} & A_2 & A_1^N & A_0 & \mathbf{0} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix},$$

where $\mathbf{0}$ is a matrix of zeros, and starting from the upper diagonal, A_0^0, A_0; A_1^0, A_1^1, A_1^2, …, A_1^N; A_2^1, A_2 are the following matrices: A_0^0 is of size $N \times (N+1)$; A_0 is of size $(N+1) \times (N+1)$; A_1^0 is of size $N \times N$; A_1^1, …, A_1^N are each of size $(N+1) \times (N+1)$; A_2^1 is of size $(N+1) \times N$; and A_2 is of size $(N+1) \times (N+1)$. They are given by

$$A_0^0 = \begin{pmatrix}
0 & \lambda_2 & 0 & \cdots & \cdots & 0 \\
\vdots & 0 & \lambda_2 & 0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \lambda_2
\end{pmatrix}, \qquad A_0 = diag(\lambda_2),$$

$$A_1^0 = \begin{pmatrix} -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \cdots & 0 \\ 0 & -(\lambda_1+\lambda_2) & \lambda_1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \lambda_1 \\ \vdots & \ddots & \ddots & 0 & -\lambda_2 \end{pmatrix},$$

For all $1 \le n \le N$,

$$(A_1^n)_{ij} = \begin{cases} -(\lambda_1+\lambda_2+i\mu_1+i\mu_2) & j=i=0,1,...,n \\ -(\lambda_1+\lambda_2+n\mu_1+i\mu_2) & j=i=n+1,...,N-1 \\ -(\lambda_2+n\mu_1+N\mu_2) & j=i=N \\ \lambda_1 & j=i+1\ ,\ i=0,1,...,N-1 \\ i\mu_1 & j=i-1\ ,\ i=1,...,m \\ n\mu_1 & j=i-1\ ,\ i=m+1,...,N \\ 0 & otherwise \end{cases}$$

$$A_2^1 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \mu_2 & 0 & \cdots & \cdots & \vdots \\ 0 & 2\mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & 3\mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & N\mu_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \mu_2 & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & 2\mu_2 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & N\mu_2 \end{pmatrix}.$$

Let $A_1 = A_1^N$, then the matrix $\tilde{Q} = A_0 + A_1 + A_2$ is the infinitesimal generator of Erlang's classical loss system $M(\lambda_1)/M(\mu_1)/N/N$ (see (1981)). Let $\vec{\pi} = (\pi_0, \pi_1, ..., \pi_N)$ be the stationary probability vector of the matrix $\tilde{Q}$, i.e. $\vec{\pi}\tilde{Q} = \vec{0}$ and $\vec{\pi}\vec{e} = 1$, where $\vec{e}$ is a vector of 1's. Then, $\pi_n = \frac{\frac{1}{n!}\left(\frac{\lambda_1}{\mu_1}\right)^n}{\sum_{k=0}^N \frac{1}{k!}\left(\frac{\lambda_1}{\mu_1}\right)^k}$ for all $0 \le n \le N$. Substituting $\vec{\pi}$ in the stability condition $\vec{\pi}A_2\vec{e} > \vec{\pi}A_0\vec{e}$ (see Neuts (1981), p. 83), we arrive at

$$\vec{\pi}A_0\vec{e} = \lambda_2 \sum_{n=0}^N \pi_n = \lambda_2 < \vec{\pi}A_2\vec{e} = \mu_2 \sum_{n=0}^N n\pi_n = \mu_2 \frac{\sum_{n=0}^N \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{(n-1)!}}{\sum_{k=0}^N \left(\frac{\lambda_1}{\mu_1}\right)^k \frac{1}{k!}} = \mu_2 \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}] \quad (2.32)$$

That is, the stability condition is

$$\frac{\lambda_2}{\mu_2} < \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]. \tag{2.33}$$

Indeed, the stability condition (2.33) contradicts condition (2.24), as stated.

Define the steady state probability vectors $\vec{P}_0 = (P_{10}, ..., P_{N0})$ and $\vec{P}_m = (P_{0m}, P_{1m}, ..., P_{Nm})$ for all $1 \leq m$. Then,

$$\vec{P}_m = \vec{P}_{N-1} R^{m-(N-1)}, \quad m \geq N-1,$$

where R is the minimal non negative solution of the matrix quadratic equation $A_0 + RA_1 + R^2 A_2 = 0$ (see Neuts (1981), Section 1.9 and Latouche and Ramaswami (1999), where computational procedures for finding the matrix R are discussed). The vectors $\vec{P}_0$, $\vec{P}_1$, ..., $\vec{P}_{N-1}$, can be found by solving the following linear system of equations:

$$\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\
&\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2 = \vec{0} \\
&\vec{P}_{m-1} A_0 + \vec{P}_m A_1^m + \vec{P}_{m+1} A_2 = \vec{0}\ , \quad 2 \leq m \leq N-2 \\
&\vec{P}_{N-2} A_0 + \vec{P}_{N-1}\left(A_1^{N-1} + RA_2\right) = \vec{0} \\
&\sum_{m=0}^{N-2} \vec{P}_m \vec{e} + \vec{P}_{N-1}[I-R]^{-1}\vec{e} = 1,
\end{aligned} \tag{2.34}$$

where I is the identity matrix. The mean total number of customers in Q_2, $\mathbb{E}[L_2]$, is given by

$$\begin{aligned}
\mathbb{E}[L_2] &= \sum_{m=0}^{\infty} m\vec{P}_m\vec{e} = \sum_{m=0}^{N-2} m\vec{P}_m\vec{e} + \sum_{N-1}^{\infty} m\vec{P}_{N-1}R^{m-N+1}\vec{e} \\
&= \sum_{m=0}^{N-2} m\vec{P}_m\vec{e} + (N-2)\vec{P}_{N-1}[I-R]^{-1}\vec{e} + \vec{P}_{N-1}[I-R]^{-2}\vec{e}.
\end{aligned} \tag{2.35}$$

3 Model 2

In this model Q_1 is an $M(\lambda_1)/M(\mu_1)/L_2/N$ system, as in Model 1, but, Q_2 is an $M(\lambda_2)/M(\mu_2)/L_1/\infty$ system, as described in the Introduction.

3.1 Balance equations

The transition-rate diagram depicting the states of the system (L_1, L_2) is shown in Figure 3.1. The set of balance equations for the system's stationary probabilities is given below, where for $\underline{n=0}$ and $\underline{n=1}$ the equations are the same as (2.1) and (2.2), respectively.

$\underline{2 \le n \le N-1}:$

$$m = 0: \quad (\lambda_1 + \lambda_2)P_{n0} = \lambda_1 P_{n-1,0} + \mu_2 P_{n1}$$

$$1 \le m < n: \quad (\lambda_1 + \lambda_2 + m\mu_1 + m\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + m\mu_1 P_{n+1,m} + (m+1)\mu_2 P_{n,m+1}$$

$$m = n: \quad (\lambda_1 + \lambda_2 + n\mu_1 + n\mu_2)P_{nn} = \lambda_1 P_{n-1,n} + \lambda_2 P_{n,n-1} + n\mu_1 P_{n+1,n} + n\mu_2 P_{n,n+1}$$

$$n < m: \quad (\lambda_1 + \lambda_2 + n\mu_1 + n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + (n+1)\mu_1 P_{n+1,m} + n\mu_2 P_{n,m+1} \tag{3.1}$$

$\underline{n=N}:$

$$m = 0: \quad \lambda_2 P_{N0} = \lambda_1 P_{N-1,0} + \mu_2 P_{N1}$$

$$1 \le m < N: \quad (\lambda_2 + m\mu_1 + m\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + (m+1)\mu_2 P_{N,m+1}$$

$$N \le m: \quad (\lambda_2 + N\mu_1 + N\mu_2)P_{Nm} = \lambda_1 PN-1,m + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1} \tag{3.2}$$

Similarly to Subsection 2.1, by algebraic manipulations we arrive at

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} \mathbb{E}[L_1 | L_2 = m+1] - \mu_2 \sum_{n=m+1}^{N} (n-m-1)P_{n,m+1}, \quad 0 \le m \le N-1$$

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} \mathbb{E}[L_1 | L_2 = m+1], \quad N \le m. \tag{3.3}$$

By summing (3.3) over m we get

$$\lambda_2 = \mu_2 \left(\mathbb{E}[L_1] - \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m)P_{nm} \right). \tag{3.4}$$

That is,

$$\mathbb{E}[L_1] = \lambda_2/\mu_2 + \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m)P_{nm}. \tag{3.5}$$

Comparing (3.5) to (2.7) one observes the difference of the second term in the RHS. Here, as in (2.7), this term represents the mean number of idle customers in Q_1. This difference is a consequence of the service

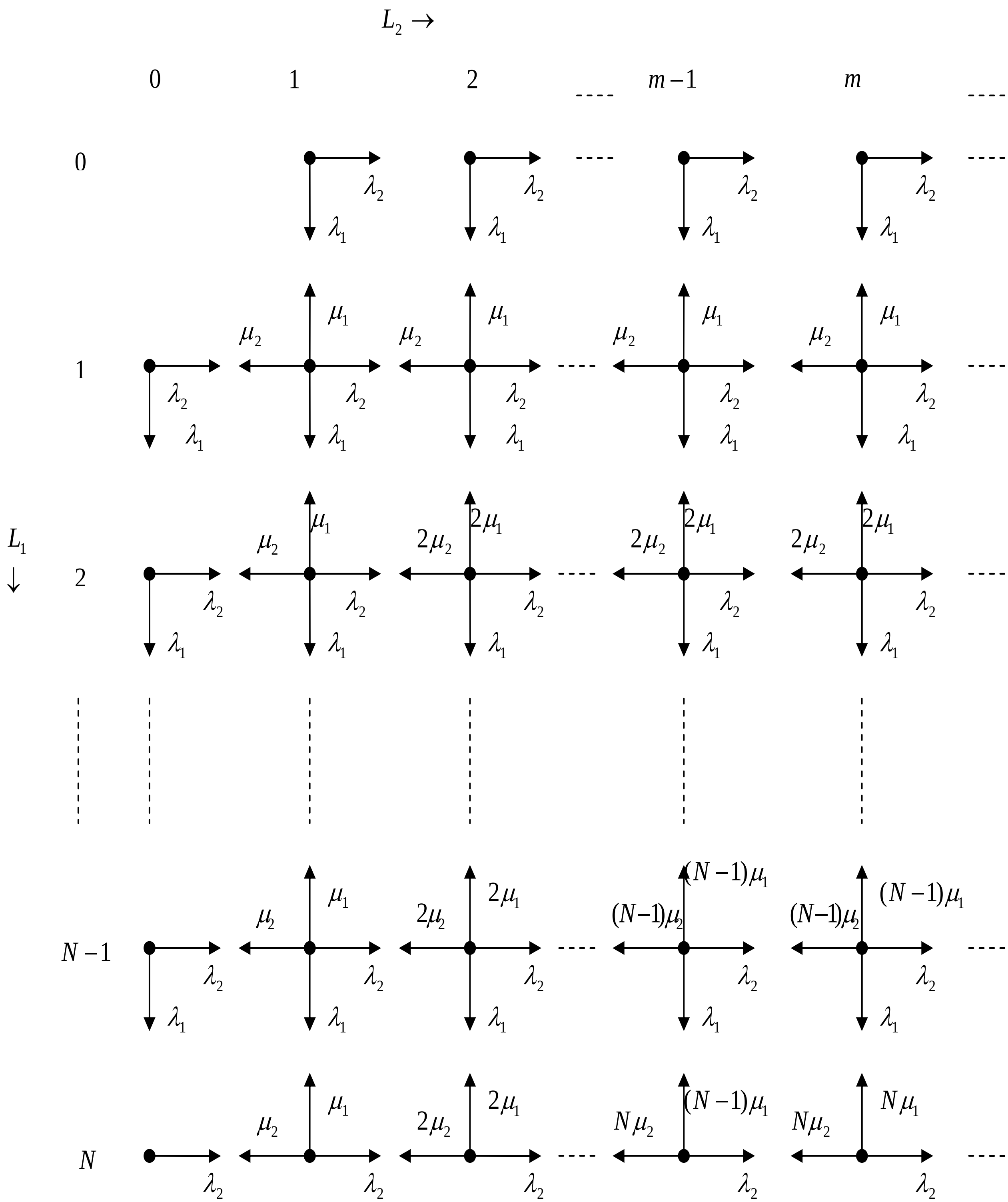

Figure 3.1: Transition rate diagram of (L_1, L_2) for Model 2.

regime in Q_2.

Now, since the service regime in Q_1 is the same as in Model 1, equation (2.9) holds here as well. Equating equations (3.5) and (2.9) results in

$$\mathbb{E}[L_1] = (1 - P_{N\bullet})\lambda_1/\mu_1 + \sum_{n=1}^{N}\sum_{m=0}^{n-1}(n-m)P_{nm} = \lambda_2/\mu_2 + \sum_{n=1}^{N}\sum_{m=0}^{n}(n-m)P_{nm}, \tag{3.6}$$

implying that the probability of blocking at Q_1 is

$$\mathbb{P}(\text{Blocking at } Q_1) \equiv P_{N\bullet} = 1 - \frac{\lambda_2/\mu_2}{\lambda_1/\mu_1}. \tag{3.7}$$

Note that the stability condition (2.33)implies that $P_{N\bullet} > 0$, since $\frac{\lambda_2}{\mu_2} < \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}] < \frac{\lambda_1}{\mu_1}$, for any $N < \infty$.

Rewriting equation(3.7) as

$$(1 - P_{N\bullet})\lambda_1/\mu_1 = \lambda_2/\mu_2, \tag{3.8}$$

reveals an interesting result: the carried load of Q_1, namely $(1 - P_{N\bullet})\lambda_1/\mu_1$, is equal to the carried load of Q_2, λ_2/μ_2, *independent of the capacities of the queues.*

3.2 Generating Functions

Repeating the derivations presented in Subsection 2.2, now with respect to the transition-rate diagram figure 3.1, we obtain, in a similar manner,

$$A(z)\vec{G}(z) = \vec{P}(z),$$

where, $A(z)$ and $\vec{G}(z)$ are the same as in Model 1, but $\vec{P}(z)$ is given by

$$P_n(z) = \begin{cases} -\mu_1 P_{10}, & n = 0 \\ (\mu_1 z - \mu_2(1-z))\sum_{m=0}^{n-1}(n-m)z^m P_{nm} - \mu_1 z \sum_{m=0}^{n}(n+1-m)z^m P_{n+1,m}, & 1 \le n \le N \\ (\mu_1 z - \mu_2(1-z))\sum_{m=0}^{N-1}(N-m)z^m P_{Nm}, & n = N \end{cases}$$

As before, to obtain $G_n(z)$ we use Cramer's rule. This leads to an expression of $G_n(z)$ in terms of $N(N+1)/2$ unknown probabilities appearing in $\vec{P}(z)$.

3.3 Derivation of P_{10}; P_{20}, P_{21}; ...; P_{N0}, P_{N1}, ..., $P_{N,N-1}$ and $\mathbb{E}[L_2]$

In order to find P_{10}; P_{20}, P_{21};... ; P_{N0}, P_{N1}, ..., $P_{N,N-1}$ we need to find $N(N+1)/2$ equations relating those $N(N+1)/2$ variables. We do that by using the roots of $|A(z)|$. From Theorem 2.1, $|A(z)|$ has $N-1$ distinct roots in the open interval $(0,1)$ (for $\lambda_1 > 0$, μ_1, $\lambda_2 \geq 0$, $\mu_2 > 0$ and provided that $\lambda_2 < \mu_2\mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]$). Another $N(N-1)/2$ equations are taken from the balance equations for states (n,m), $2 \leq n \leq N$, $0 \leq m \leq n-2$. The last equation we use is (3.7). Thus, we have $N(N+1)/2$ equations relating those $N(N+1)/2$ variables as requested.

In a similar manner as in the end of Subsection 2.3 we get

$$\begin{aligned} Cov\,(L_1, L_2) &= \mathbb{E}[L_1 \cdot L_2] - \mathbb{E}[L_1] \cdot \mathbb{E}[L_2] \\ &= \frac{\lambda_2}{\mu_2} + \sum_{n=2}^{N}\sum_{m=1}^{n-1} m(n-m)P_{nm} - \mathbb{E}[L_2]\left(\mathbb{E}[L_1] - \frac{\lambda_2}{\mu_2}\right). \end{aligned} \tag{3.9}$$

Arguing that $Cov\,(L_1, L_2) \leq 0$, we obtain

$$\mathbb{E}[L_2] \geq \frac{\frac{\lambda_2}{\mu_2} + \sum_{m=1}^{n-1} m(n-m)P_{nm}}{\mathbb{E}[L_1] - \frac{\lambda_2}{\mu_2}} = \frac{\lambda_2 + \mu_2 \sum_{m=1}^{n-1} m(n-m)P_{nm}}{\mu_2\mathbb{E}[L_1] - \lambda_2}. \tag{3.10}$$

Clearly, by Little's Law, $\mathbb{E}[W_2] = \frac{\mathbb{E}[L_2]}{\lambda_2}$.

3.4 Matrix Geometric Method for Deriving $(P_{nm})_{0 \leq n \leq N,\ 0 \leq m}$

Applying the Matrix Geometric approach, the system of balance equations can be described as a queueing system with $N+1$ phases, where phase n indicates that there are n servers available at Q_2. We construct a

QBD process with generator Q, given by

$$Q = \begin{pmatrix} A_1^0 & A_0^0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ A_2^1 & A_1^1 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & A_2^2 & A_1^2 & A_0 & \mathbf{0} & \cdots & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \mathbf{0} & A_2^N & A_1^N & A_0 & \mathbf{0} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \mathbf{0} & A_2^N & A_1^N & A_0 & \mathbf{0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix},$$

where, A_0^0, A_0, A_1^0, A_1^N and A_2^N are the same as in Subsection 2.4, but A_1^1, A_1^2, ..., A_1^{N-1}; A_2^1, A_2^2, ..., A_2^{N-1} are slightly different and are given by

For all $1 \leq n \leq N-1$,

$$(A_1^n)_{ij} = \begin{cases} -(\lambda_1 + \lambda_2 + i\mu_1 + i\mu_2) & j = i = 0, 1, ..., n \\ -(\lambda_1 + \lambda_2 + n\mu_1 + n\mu_2) & j = i = n+1, ..., N-1 \\ -(\lambda_2 + n\mu_1 + n\mu_2) & j = i = N \\ \lambda_1 & j = i+1 \ , \ i = 0, 1, ..., N-1 \\ i\mu_1 & j = i-1 \ , \ i = 1, ..., n \\ n\mu_1 & j = i-1 \ , \ i = n+1, ..., N \\ 0 & otherwise \end{cases}$$

$$A_2^1 = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \mu_2 & 0 & \cdots & \cdots & \vdots \\ 0 & \mu_2 & 0 & \cdots & \vdots \\ \vdots & \ddots & \mu_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \mu_2 \end{pmatrix}, \quad \text{For all } 2 \leq n \leq N-1, \quad (A_2^n)_{ij} = \begin{cases} i\mu_2 & j = i-1 \ , \ i = 1, ..., n-1 \\ n\mu_2 & j = i-1 \ , \ i = n, ..., N \\ 0 & otherwise \end{cases}$$

Again, letting $A_1 = A_1^N$, $A_2 = A_2^N$ then the matrix $\tilde{Q} = A_0 + A_1 + A_2$ is the infinitesimal generator of Erlang's classical loss system $M(\lambda_1)/M(\mu_1)/N/N$. Let $\vec{\pi} = (\pi_0, \pi_1, ..., \pi_N)$ be as in Subsection 2.4 then, substituting $\vec{\pi}$ in the stability condition $\vec{\pi}A_2\vec{e} > \vec{\pi}A_0\vec{e}$ we arrive at

$$\vec{\pi}A_0\vec{e} = \lambda_2 \sum_{n=0}^{N} \pi_n = \lambda_2 < \vec{\pi}A_2\vec{e} = \mu_2 \sum_{n=0}^{N} n\pi_n = \mu_2 \frac{\sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{(n-1)!}}{\sum_{k=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^k \frac{1}{k!}} = \mu_2 \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}] \tag{3.11}$$

That is, the stability condition is $\frac{\lambda_2}{\mu_2} < \mathbb{E}[L_{M(\lambda_1)/M(\mu_1)/N/N}]$, *exactly as in Model 1.* Define the steady state probability vectors $\vec{P}_0 = (P_{10}, ..., P_{N0})$ and $\vec{P}_m = (P_{0m}, P_{1m}, ..., P_{Nm})$ for all $1 \leq m$. Then,

$$\vec{P}_m = \vec{P}_{N-1} R^{m-(N-1)}, \quad m \geq N-1,$$

where R is the minimal non negative solution of the matrix quadratic equation $A_0 + RA_1 + R^2 A_2 = 0$. The vectors $\vec{P}_0$, $\vec{P}_1$, ..., $\vec{P}_{N-1}$, can be found by solving the following linear system of equations:

$$\begin{aligned}
&\vec{P}_0 A_1^0 + \vec{P}_1 A_2^1 = \vec{0} \\
&\vec{P}_0 A_0^0 + \vec{P}_1 A_1^1 + \vec{P}_2 A_2^2 = \vec{0} \\
&\vec{P}_{m-1} A_0 + \vec{P}_m A_1^m + \vec{P}_{m+1} A_2^{m+1} = \vec{0}\ , \quad 2 \leq m \leq N-2 \\
&\vec{P}_{N-2} A_0 + \vec{P}_{N-1} \left(A_1^{N-1} + RA_2^N\right) = \vec{0} \\
&\sum_{m=0}^{N-2} \vec{P}_m \vec{e} + \vec{P}_{N-1}[I-R]^{-1}\vec{e} = 1
\end{aligned} \tag{3.12}$$

where I is the identity matrix. The mean total number of customers in Q_2, $\mathbb{E}[L_2]$, is given by

$$\mathbb{E}[L_2] = \sum_{m=0}^{\infty} m\vec{P}_m\vec{e} = \sum_{m=0}^{N-2} m\vec{P}_m\vec{e} + (N-2)\vec{P}_{N-1}[I-R]^{-1}\vec{e} + \vec{P}_{N-1}[I-R]^{-2}\vec{e} \tag{3.13}$$

4 Numerical Examples

We present some numerical results for both models. In Table 4.1 we show results for Model 1, using the set of parameters $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$ and $\mu_2 = 2$ for $N = 2$ and $N = 3$.

In Table 4.2 we show results for Model 2, using the same set of parameters $\lambda_1 = 2$, $\mu_1 = 1$, $\lambda_2 = 1$ and $\mu_2 = 2$ for $N = 2$ and $N = 3$.

It is seen that, for same parameter values, $\mathbb{E}[L_1]$ in Model 1 is larger than in Model 2. This follows since in Model 1 all customers of Q_1 join hands together in serving Q_2, reducing its size. This affects the size of Q_1 since less customers are present in Q_2 to serve Q_1. The opposite holds for $\mathbb{E}[L_2]$.

Table 4.1: Numerical Results for Model 1 with $N = 2$ and $N = 3$

	P_{10}	P_{20}	P_{21}	P_{30}	P_{31}	P_{32}	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$Cov\,(L_1, L_2)$
$N = 2$	0.0282	0.6209	0.1411	—	—	—	1.7702	0.6642	-0.6642
$N = 3$	0.0056	0.0379	0.0256	0.751	0.1125	0.0164	2.8342	0.2999	-0.12

Table 4.2: Numerical Results for Model 2 with $N = 2$ and $N = 3$

	P_{10}	P_{20}	P_{21}	P_{30}	P_{31}	P_{32}	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$Cov\,(L_1, L_2)$
$N = 2$	0.0394	0.4719	0.1966	—	—	—	1.6798	0.9254	-0.3951
$N = 3$	0.0141	0.0524	0.0645	0.5017	0.1984	0.0407	2.48	0.5631	-0.0721

5 Summary

In this paper we extend the scope of analytic investigation of 2-queue models where customers of only one queue act as servers, to the case where *both* customers in *both* queues act as servers, each group serving the opposite queue. We derive the stability conditions of such queues, revealing their connection to the roots of a given matrix related to the PGFs. We obtain the system's 2-dimensional stationary probabilities, and calculate the mean queue size of each queue, as well as the correlation between them. Numerical results further exhibit the inter-relationship between the two queues.

References

Androutsellis-Theotokis S and D Spinellis (2004). A survey of peer-to-peer content distribution technologies. *ACM Computing Surveys*, 36(4), 335–371.

Arazi, A, E Ben-Jacob and U Yechiali (2005). Controlling an oscillating Jackson-type network having state-dependant service rates. *Mathematical Methods of Operations Research*, 62, 453–466.

Cooper, R B (1981). *Introduction to Queueing Theory*. North Holland (second Edition).

Latouche, G and V Ramaswami (1999). *Introduction to matrix analytic methods in stochastic mod-*

eling, SIAM and ASA, Philadelphia.

Neuts, M F (1981). *Matrix geometric solutions in stochastic models - an algorithmic approach.* Johns Hopkins, Baltimore.

Perel, E and U Yechiali (2008). Queues where customers of one queue act as servers the of the other queue. *Qeueueing Systems*, 60, 271–288.

Sendfeld, W P (2009). *Two-Dimensional Overflow Queueing Systems.* Doctoral Thesis, Institute for Mathematics, University of Osnabruk.

A Appendix

Proposition A.1. *$q_n^{(N)}(z)$ is of the form $q_n^{(N)}(z) = \lambda_1^n z^{n-1} + (1-z)h_n^{(N)}(z)$, for $1 \le n \le N$, and $q_{N+1}^{(N)}(z)$ is of the form $q_{N+1}^{(N)}(z) = (1-z)h_{N+1}^{(N)}(z)$, where $h_n^{(N)}(z) = \alpha_{n-1}^{(N)}(z)h_{n-1}^{(N)}(z) - (n-1)\lambda_1\mu_1 z^2 h_{n-2}^{(N)}(z) + \lambda_1^{n-1}z^{n-2}(\lambda_2 z - (n-1)\mu_2)$, for all $3 \le n \le N+1$.*

Proof. We will prove the proposition using induction over n.

For $n = 1$ we have $q_1^{(N)}(z) = \lambda_1 + \lambda_2(1-z)$.

For $n = 2$, $q_2^{(N)}(z) = \lambda_1^2 z + (1-z)\left(\lambda_2\left((\lambda_1+\mu_1)z + (\lambda_2 z - n\mu_2)(1-z)\right) + \lambda_1(\lambda_2 - \mu_2)\right)$.

Suppose the proposition is valid for some n, $2 \le n \le N-1$. We will show that it is valid for $n+1$.

$$
\begin{aligned}
q_{n+1}^{(N)}(z) =& \alpha_n^{(N)}(z)q_n^{(N)}(z) - n\lambda_1\mu_1 z^2 q_{n-1}^{(N)}(z) \\
=& \left((\lambda_1 + n\mu_1)z + (\lambda_2 z - n\mu_2)(1-z)\right)\left(\lambda_1^n z^{n-1} + (1-z)h_n^{(N)}(z)\right) \\
& - n\lambda_1\mu_1 z^2\left(\lambda_1^{n-1}z^{n-2} + (1-z)h_{n-1}^{(N)}(z)\right) \\
=& \lambda_1^{n+1}z^n + n\lambda_1^n\mu_1 z^n + \lambda_1^n z^{n-1}(\lambda_2 z - n\mu_2)(1-z) + \alpha_n^{(N)}(z)h_n^{(N)}(z) \\
& - n\lambda_1^n\mu_1 z^n - n\lambda_1\mu_1 z^2(1-z)h_{n-1}^{(N)}(z) \\
=& \lambda_1^{n+1}z^n + (1-z)\left(\alpha_n^{(N)}(z)h_n^{(N)}(z) - n\lambda_1\mu_1 z^2 h_{n-1}^{(N)}(z) + \lambda_1^n z^{n-1}(\lambda_2 z - n\mu_2)\right).
\end{aligned}
$$

Therefore,

$$
h_{n+1}^{(N)}(z) = \alpha_n^{(N)}(z)h_n^{(N)}(z) - n\lambda_1\mu_1 z^2 h_{n-1}^{(N)}(z) + \lambda_1^n z^{n-1}(\lambda_2 z - n\mu_2). \tag{A.1}
$$

As for $q_{N+1}^{(N)}(z)$, we have

$$\begin{aligned}
q_{N+1}^{(N)}(z) =& \alpha_N^{(N)}(z) q_N^{(N)}(z) - N\lambda_1\mu_1 z^2 q_{N-1}^{(N)}(z) \\
=& \left(N\mu_1 z + (\lambda_2 z - N\mu_2)(1-z)\right) \left(\lambda_1^N z^{N-1} + (1-z) h_N^{(N)}(z)\right) \\
& - N\lambda_1\mu_1 z^2 \left(\lambda_1^{N-1} z^{N-2} + (1-z) h_{N-1}^{(N)}(z)\right) \\
=& N\lambda_1^N \mu_1 z^N + \lambda_1^N z^{N-1}(\lambda_2 z - N\mu_2)(1-z) \\
& + \alpha_N^{(N)}(z)(1-z) h_N^{(N)}(z) - N\lambda_1^N \mu_1 z^N - N\lambda_1\mu_1 z^2 (1-z) h_{N-1}^{(N)}(z) \\
=& (1-z)\left(\alpha_N^{(N)}(z) h_N^{(N)}(z) - N\lambda_1\mu_1 h_{N-1}^{(N)}(z) + \lambda_1^N z^{N-1}(\lambda_2 z - N\mu_2)\right).
\end{aligned}$$

Hence,

$$h_{N+1}^{(N)}(z) = \alpha_N^{(N)}(z) h_N^{(N)}(z) - N\lambda_1\mu_1 h_{N-1}^{(N)}(z) + \lambda_1^N z^{N-1}(\lambda_2 z - N\mu_2). \tag{A.2}$$

This completes the proof of Proposition A.1. □

Proposition A.2. *For all $N \geq 2$, $h_{N+1}^{(N)}(1) = \lambda_2 N! \mu_1^N \sum_{n=0}^N \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 N! \mu_1^N \sum_{n=1}^N n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}$.*

Proof. We will show that the proposition holds by induction over N.

For $N = 2$, we have

$$h_1^{(2)}(1) = \lambda_2 \tag{A.3}$$

$$h_2^{(2)}(1) = \lambda_2(2\lambda_1 + \mu_1) - \mu_2\lambda_1 \tag{A.4}$$

$$h_3^{(2)}(1) = 2\mu_1 h_2^{(2)}(1) - 2\lambda_1\mu_1 h_1^{(2)}(1) + \lambda_1^2(\lambda_2 - 2\mu_2) \tag{A.5}$$

Substituting (A.3) and (A.4) in (A.5) we get

$$\begin{aligned}
h_3^{(2)}(1) =& 2\mu_1 h_2^{(2)}(1) - 2\lambda_1\mu_1 h_1^{(2)}(1) + \lambda_1^2(\lambda_2 - 2\mu_2) \\
=& 2\mu_1 \left(\lambda_2(2\lambda_1 + \mu_1) - \mu_2\lambda_1\right) - 2\lambda_1\mu_1\lambda_2 + \lambda_1^2(\lambda_2 - 2\mu_2) \\
=& \lambda_2 \left(2\mu_1^2 + 2\lambda_1\mu_1 + \lambda_1^2\right) - \mu_2 \left(2\lambda_1\mu_1 + 2\lambda_1^2\right) \\
=& \lambda_2 2! \mu_1^2 \sum_{n=0}^{2} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 2! \mu_1^2 \sum_{n=1}^{2} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}.
\end{aligned}$$

For $N = 3$, we see that

$$h_1^{(3)}(1) = \lambda_2 \tag{A.6}$$

$$h_2^{(3)}(1) = \lambda_2(2\lambda_1 + \mu_1) - \mu_2\lambda_1 \tag{A.7}$$

$$h_3^{(3)}(1) = (\lambda_1 + 2\mu_1)h_2^{(3)}(1) - 2\lambda_1\mu_1 h_1^{(3)}(1) + \lambda_1^2(\lambda_2 - 2\mu_2) \tag{A.8}$$

$$h_4^{(3)}(1) = 3\mu_1 h_3^{(3)}(1) - 3\lambda_1\mu_1 h_2^{(3)}(1) + \lambda_1^3(\lambda_2 - 3\mu_2) \tag{A.9}$$

Substituting (A.6) and (A.7) in (A.8) we get

$$\begin{aligned}
h_3^{(3)}(1) =& (\lambda_1 + 2\mu_1)h_2^{(2)}(1) - 2\lambda_1\mu_1 h_1^{(2)}(1) + \lambda_1^2(\lambda_2 - 2\mu_2) \\
=& (\lambda_1 + 2\mu_1)\left(\lambda_2(2\lambda_1 + \mu_2) - \mu_2\lambda_1\right) - 2\lambda_1\mu_1\lambda_2 + \lambda_1^2(\lambda_2 - 2\mu_2) \\
=& \lambda_2\left((\lambda_1 + 2\mu_1)(2\lambda_1 + \mu_1) - 2\lambda_1\mu_1 + \lambda_1^2\right) - \mu_2\left(\lambda_1(\lambda_1 + 2\mu_1) + 2\lambda_1^2\right) \\
=& \lambda_2\left(2\mu_1^2 + 3\lambda_1\mu_1 + 3\lambda_1^2\right) - \mu_2\left(2\lambda_1\mu_1 + 3\lambda_1^2\right).
\end{aligned} \tag{A.10}$$

Again, substituting (A.7) and (A.10) in (A.9), we get

$$\begin{aligned}
h_4^{(3)}(1) =& 3\mu_1 h_3^{(3)}(1) - 3\lambda_1\mu_1 h_2^{(3)}(1) + \lambda_1^3(\lambda_2 - 3\mu_2) \\
=& 3\mu_1\left(\lambda_2\left(2\mu_1^2 + 3\lambda_1\mu_1 + 3\lambda_1^2\right) - \mu_2\left(2\lambda_1\mu_1 + 3\lambda_1^2\right)\right) - 3\lambda_1\mu_1\left(\lambda_2(2\lambda_1 + \mu_1) - \mu_2\lambda_1\right) + \lambda_1^3(\lambda_2 - 3\mu_2) \\
=& \lambda_2\left(6\mu_1^3 + 6\lambda_1\mu_1^2 + 3\lambda_1^2\mu_1 + \lambda_1^3\right) - \mu_2\left(6\lambda_1\mu_1^2 + 6\lambda_1^2\mu_1 + 3\lambda_1^3\right) \\
=& \lambda_2 3!\mu_1^3 \sum_{n=0}^{3}\left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 3!\mu_1^3 \sum_{n=1}^{3} n\left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}.
\end{aligned}$$

Assume that the proposition holds for every $k \le N$, we now show that it holds for N. Notice that for all $k \le N$, $h_n^{(k)}(1) = h_n^{(N)}(1)$, for every $n \le k$. In particular, for $k = N-1$, $h_{N-1}^{(N-1)}(1) = h_{N-1}^{(N)}(1)$.
In addition, $h_N^{(N-1)}(1) = h_N^{(N)}(1) - \lambda_1 h_{N-1}^{(N-1)}(1) = h_N^{(N)}(1) - \lambda_1 h_{N-1}^{(N)}(1)$, meaning that

$$h_N^{(N)}(1) = h_N^{(N-1)}(1) + \lambda_1 h_{N-1}^{(N)}(1).$$

Therefore, by the definition of $h_{N+1}^{(N)}(1)$ we have

$$
\begin{aligned}
h_{N+1}^{(N)}(1) =& N\mu_1 h_N^{(N)}(1) - N\lambda_1\mu_1 h_{N-1}^{(N)}(1) + \lambda_1^N(\lambda_2 - N\mu_2) \\
=& N\mu_1 \left(h_N^{(N-1)}(1) + \lambda_1 h_{N-1}^{(N)}(1) \right) - N\lambda_1\mu_1 h_{N-1}^{(N)}(1) + \lambda_1^N(\lambda_2 - N\mu_2) \\
=& N\mu_1 h_N^{(N-1)}(1) + \lambda_1^N(\lambda_2 - N\mu_2) \\
=& N\mu_1 \left(\lambda_2 (N-1)!\mu_1^{N-1} \sum_{n=0}^{N-1} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 (N-1)!\mu_1^{N-1} \sum_{n=1}^{N-1} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} \right) + \lambda_1^N(\lambda_2 - N\mu_2) \\
=& \lambda_2 \left(N(N-1)!\mu_1^N \sum_{n=0}^{N-1} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} + \lambda_1^N \right) - \mu_2 \left(N(N-1)!\mu_1^N \sum_{n=1}^{N-1} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} + N\lambda_1^N \right) \\
=& \lambda_2 N!\mu_1^N \sum_{n=0}^{N} \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!} - \mu_2 N!\mu_1^N \sum_{n=1}^{N} n \left(\frac{\lambda_1}{\mu_1}\right)^n \frac{1}{n!}.
\end{aligned}
$$

This completes the proof of Proposition A.2 $\square$