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Feng, Yi; Chen, Shaoxiang; Dai, Wenqiang

2015

Feng, Y., Chen, S., & Dai, W. (2015). The optimal order policy for a capacitated multiple product inventory system under symmetry. *Asia-Pacific Journal of Operational Research*, 32(02), 1550005-. doi:10.1142/S0217595915500050

<https://hdl.handle.net/10356/79449>

<https://doi.org/10.1142/S0217595915500050>

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Asia-Pacific Journal of Operational Research  
Vol. 32, No. 1 (2015) 1550005 (17 pages)  
© World Scientific Publishing Co. & Operational Research Society of Singapore  
DOI: 10.1142/S0217595915500050

## The Optimal Order Policy for a Capacitated Multiple Product Inventory System Under Symmetry

Yi Feng

*School of Management and Economics  
University of Electronic Science and Technology of China  
Chengdu, P. R. China, 610054  
fengyi@uestc.edu.cn*

Shaoxiang Chen

*Nanyang Business School  
Nanyang Technological University  
Singapore, 639798  
aschen@ntu.edu.sg*

Wenqiang Dai\*

*School of Management and Economics  
University of Electronic Science and Technology of China  
Chengdu, P. R. China, 610054  
wqdai@uestc.edu.cn*

Received 14 September 2011

Revised 4 February 2014

Accepted 6 March 2014

Published

Although many researches have conjectured the optimality of the hedging point policy for the multiple-product stochastic capacitated periodic review problem, it is still a challenge to prove the hypothesis. This paper considers a special case of the capacitated multiple-product periodic review problem where stochastic demand distribution, production rate, unit production cost and periodic expected inventory cost are the same for all different products. For this symmetric problem, we prove an optimal policy where ordering and non-ordering regions for every product are defined. Our research provides significant implications to the characterization of the optimal policy for the general problem.

*Keywords:* Stochastic programming; multiple-product; capacitated; periodic review; optimality.

\*Corresponding author

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## 1. Introduction

In the past 50 years, abundant research efforts aimed at finding the optimal ordering policies for stochastic dynamic inventory models. Many theories have been gradually developed for single-product systems (Iglehart, 1963; Veinott, 1966; Federgruen and Zipkin, 1986; Chen and Lambrecht, 1996; Chen, 2004a). However, only few theories have been established for capacitated multiple-product periodic review systems. Evans (1967) first investigates such a system without considering the setup cost. Furthermore, researchers (for example, Wein, 1992; Gershwin, 1994; de Vericourt *et al.*, 2000; Srivatsan and Dallery, 1998) have conjectured the optimality of the hedging point policy of production/ordering for the multiple-product model. Chen (2004b) characterizes the optimal hedging point policy for the two-product system.

However, extending Chen's (2004b) results to a system with more than two products is not straightforward, and still a great challenge in academia. This paper focuses on a symmetric multiple-product system in which parameters of the stochastic demand distribution, the production rate, the unit production cost, and the expected periodic inventory cost are the same for all different products. We establish an optimal policy which can be regarded as a special case of the hedging point policy. The policy is intuitive and can be described as follows. First, optimal ordering and non-ordering regions can be defined. Second, it is proved that the following ordering process can be repeated: Ordering the product with the least inventory until its inventory level reaches the second lowest of all inventory levels, or the capacity is used up, or the non-ordering region or the global minimum point is reached.

The symmetric model is rooted in many practical applications. For instance, in many mass customization cases (Lee *et al.*, 1993; Selladurai, 2004), manufacturing settings for different product variations are more or less the same. In some cases, even demand distributions are just slightly different (DeCroix and Arreola-Risa, 1998).

The remainder of this paper is organized as follows. The literature review is carried out in Sec. 2. Section 3 introduces the general multiple-product system. In Sec. 4, analytical results for the symmetric three-product model are presented. Section 5 extends the results in Sec. 4 to the symmetric  $m(m > 3)$ -product system and further to the infinite horizon case. Conclusions are drawn in Sec. 6.

## 2. Literature Review

Johnson (1967) and Kalin (1980) establish the optimality of a multi-dimensional (s, S) policy for the multiple-product model without considering capacity constraints. For the capacitated multiple-product model, the policy is far more complex. Evans (1967) may be the first one to consider the capacitated multiple-product, periodic review and stochastic demand system in a finite horizon. Under the assumption that the one-period expected cost function is strictly convex and second-order

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differentiable, he shows that an optimal policy can be partially characterized by the base-stock policy. When all product inventory levels are above the base-stock level, it is optimal to order nothing; when all product inventory levels are below the base-stock level, it is optimal to order up to the base-stock level if there is sufficient capacity. Evans (1967) only partially characterizes the order policy. Based on Evans's (1967) work, the model has been investigated intensively by the researchers. Peña-Perez and Zipkin (1997) provide a heuristic policy for the system. Ha (1997) proves that the optimality of the hedging point policy for the two-product system in which two products have the identical production time. Wein (1992) proposes an approximation for the multiclass queuing control problem. The solution conjectures the optimality of a hedging point policy when all products are backlogged. Srivatsan and Dallery (1998) and de Vericourt *et al.* (2000) provide a partial characterization of the optimal hedging point policy by different techniques. Gershwin (1994) provides a systematic review on research works in this field and also conjectures the optimality of the hedging point policy for the multiple product system.

For the two-product system, Chen (2004b) proves the optimality of the hedging point policy which can be described as follows. There are two curves intersect at one hedging point, and therefore divide the two-dimensional (2D) plane into three distinct regions, it is optimal to order none of the products 1 and 2 in different regions.

DeCroix and Arreola-Risa (1998) extend Evans's (1967) results to the infinite horizon case, and prove that the symmetric resource allocation policy is optimal to the model which is similar to the one in this paper. This paper further characterizes the policy, and defines optimal ordering and non-ordering regions for each product.

When setup costs (or times) and capacity constraints are simultaneously considered in the multiple-product model, the system becomes much more complex. Federgruen and Katalan (1998), Anupindi and Tayur (1998), and Stadtler (2003) propose some heuristic solutions. It is still a challenge to find an optimal policy for the system.

### 3. Model Description, Notation, and Dynamic Program Formulation

A capacitated multiple-product periodic review system considers a production system with a reliable and flexible machine that produces multiple distinct products in a make-to-stock mode. Machine's flexibility means that little time or cost is required for the changeover as described in Gershwin (1994). The stochastic periodic demand for each item is independently and identically distributed (i.i.d.). Unsatisfied demands are backlogged with the incurred penalty cost. The costs considered here include the production cost and the inventory cost. The production cost function is assumed to be linear and the inventory cost function is convex. The inventory cost is actually the holding cost when the inventory level is positive, or the penalty

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cost when there is a backlogging. Convex inventory cost can be justified by the case of the Just-In-Time manufacturing system in which inventory is not encouraged. The objective is to minimize the total expected discounted cost over the planning time horizon.

A general model for the  $m$ -product system can be expressed as the following dynamic program.

$$f_n(x_1, x_2, \dots, x_m) = \min_{\substack{\sum_{i=1}^m \frac{X_i - x_i}{u_i} \leq 1; \\ x_i \leq X_i, \forall i, 1 \leq i \leq m}} \left\{ \sum_{i=1}^m (L_i(X_i) + c_i(X_i - x_i)) + \alpha E(f_{n-1}(X_1 - d_1, X_2 - d_2, \dots, X_m - d_m)) \right\}, \quad (1)$$

where

$x_i(X_i)$ : The inventory level before (after) the order is placed for product  $i$  at the beginning of a period.

$L_i(X_i)$ : The one-period expected inventory cost function for product  $i$ . It is assumed to be convex.

$u_i$ : The production rate for product  $i$ .

$c_i$ : The unit production cost for product  $i$ .

$d_i$ : The one-period demand for product  $i$ , and it is an i.i.d. random variable.

$\alpha$ : The discount factor.

$f_n(x_1, x_2, \dots, x_m)$ : The  $n$ -period minimum expected discounted cost function.

Define  $g_i(X) = L_i(X) + c_iX$ , and

$$G_n(X_1, X_2, \dots, X_m) = \sum_{i=1}^m g_i(X_i) + \alpha E(f_{n-1}(X_1 - d_1, X_2 - d_2, \dots, X_m - d_m)). \quad (2)$$

Then Eq. (1) can be simplified as

$$f_n(x_1, x_2, \dots, x_m) = \min_{\substack{\sum_{i=1}^m \frac{X_i - x_i}{u_i} \leq 1; \\ x_i \leq X_i, \forall i, 1 \leq i \leq m}} \{G_n(X_1, X_2, \dots, X_m)\} - \sum_{i=1}^m c_i x_i. \quad (3)$$

The initial condition is  $f_0(x_1, x_2, \dots, x_m) = 0$ .

It can be easily shown by induction that  $G_n(X_1, X_2, \dots, X_m)$  and  $f_n(x_1, x_2, \dots, x_m)$  are convex (Chen, 2001).

One-sided directional partial derivatives of convex functions  $G_n(X_1, X_2, \dots, X_m)$  and  $f_n(x_1, x_2, \dots, x_m)$  exist by Theorem 23.1 in Rockafellar (1970)

and  $G'_{nX_i^+}(X_1, X_2, \dots, X_m)$  is defined as

$$G'_{nX_i^+}(X_1, X_2, \dots, X_m) = \lim_{t \rightarrow 0^+} \frac{G_n(X_1, \dots, X_i + t, \dots, X_m) - G_n(X_1, \dots, X_i, \dots, X_m)}{t}. \quad (4)$$

$t \rightarrow 0^+$  means that  $t$  approaches zero from the side which is greater than zero. The other partial derivatives can be defined similarly.

#### 4. Analytical Results for a Symmetric Three-Product System

First, we consider the following case: The periodic stochastic market demand is discrete and can be 2, 5, 8 respectively with probability 0.2, 0.5, 0.3. The unit holding cost is 1 and the shortage penalty cost is 6. The unit production cost is 3 and the periodic discount factor is 0.98. It is easy to compute the global minimum which is achieved at point (10, 10, 10). The problem with different production rates and different inventory levels will have different order quantities as demonstrated in the following Table 1.

From Table 1, we can observe some possible rules. For instance, if ordering is necessary and the production capacity allows, the product with the least inventory will be ordered till its inventory level reaches that of the product with the second least inventory; then both products will be ordered simultaneously till their inventory levels reach that of the product with the third least inventory. In this section, we will prove these rules are true for the model. Moreover, the larger production rate affects the order quantities with the same rule.

Here, model (3) is a simplified model, with its ordering policy being analyzed. The simplified model assumes that three products have the same i.i.d. stochastic demand, inventory cost, unit production cost, and production rate. Since cost parameters are the same, the subscripts in the notation are removed thereafter. Thus,

$$G_n(X_i, X_j, X_k) = g(X_i) + g(X_j) + g(X_k) + \alpha E(f_{n-1}(X_i - d, X_j - d, X_k - d)). \quad (5)$$

Table 1. Computational results for different inventory levels and production rates.

Starting inventory level	Inventory level after ordering ( $u = 3$ )	Inventory level after ordering ( $u = 5$ )
(2, 5, 1)	(3, 5, 3)	(4, 5, 4)
(2, 5, 5)	(5, 5, 5)	(5.66, 5.66, 5.66)
(2, 5, 9)	(5, 5, 9)	(6, 6, 9)
(2, 1, 5)	(3, 3, 5)	(4, 4, 5)
(2, 9, 5)	(5, 9, 5)	(6, 9, 6)

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$$f_n(x_i, x_j, x_k) = \min_{\substack{\frac{x_i - x_j}{u} + \frac{x_j - x_k}{u} + \frac{x_k - x_i}{u} \leq 1; \\ x_i \leq X_i, x_j \leq X_j, x_k \leq X_k}} \{G_n(X_i, X_j, X_k)\} - c(x_i + x_j + x_k). \quad (6)$$

Due to symmetry, it can be inferred that ordering and non-ordering regions for products  $i, j$ , and  $k$  might divide the three-dimensional (3D) space into three equal parts. The main theoretical result obtained in this section is that three planes of  $X_i = X_j$ ,  $X_j = X_k$ , and  $X_k = X_i$  intersect at one global minimum point with equal coordinate values and divide the 3D space into six equal parts. Propositions 4.1 and 4.2 characterize the ordering priority of these three products, and Proposition 4.3 presents the ordering policy.

Lemma 4.1 is straightforward.

**Lemma 4.1.** Suppose  $(r1, s1, t1)$  and  $(r2, s2, t2)$  are any sequence orders of  $(i, j, k)$ , then for all  $n$ ,  $G_n(x_{r1}, x_{s1}, x_{t1}) = G_n(x_{r2}, x_{s2}, x_{t2})$ , and  $f_n(x_{r1}, x_{s1}, x_{t1}) = f_n(x_{r2}, x_{s2}, x_{t2})$ .

**Lemma 4.2.** For both  $G_n(x_i, x_j, x_k)$  and  $f_n(x_i, x_j, x_k)$ , at least one global minimum point  $P(X_i^0, X_j^0, X_k^0)$  exists for each of them where their coordinate values are equal, i.e.,  $X_i^0 = X_j^0 = X_k^0$ .

**Proof.** Suppose one global optimum point is  $P_1(X_i^1, X_j^1, X_k^1)$ , without loss of generality, let  $X_i^1 \leq X_j^1 \leq X_k^1$ . If two equal signs are achieved, then the lemma has been proved. Suppose at least one equal sign of the two does not hold. Since  $P_1(X_i^1, X_j^1, X_k^1)$  is the global minimum point, by Lemma 4.1,  $P_2(X_k^1, X_i^1, X_j^1)$  and  $P_3(X_j^1, X_k^1, X_i^1)$  are also global minimum points. Due to the convexity of  $G_n(x_i, x_j, x_k)$  and  $f_n(x_i, x_j, x_k)$ , define  $P = \frac{1}{3}P_1 + \frac{1}{3}P_2 + \frac{1}{3}P_3 = (\frac{X_i^1 + X_j^1 + X_k^1}{3}, \frac{X_i^1 + X_j^1 + X_k^1}{3}, \frac{X_i^1 + X_j^1 + X_k^1}{3})$ , clearly  $P$  is a global optimum point with equal coordinate values. This completes the proof.  $\square$

**Proposition 4.1.** If  $x_i = x_j$ , then  $G'_{nx_i^+}(x_i, x_j, x_k) = G'_{nx_j^+}(x_i, x_j, x_k)$ ,  $f'_{nx_i^+}(x_i, x_j, x_k) = f'_{nx_j^+}(x_i, x_j, x_k)$ , if  $x_j = x_k$ , then  $G'_{nx_j^+}(x_i, x_j, x_k) = G'_{nx_k^+}(x_i, x_j, x_k)$ ,  $f'_{nx_j^+}(x_i, x_j, x_k) = f'_{nx_k^+}(x_i, x_j, x_k)$ , and if  $x_k = x_i$ , then  $G'_{nx_k^+}(x_i, x_j, x_k) = G'_{nx_i^+}(x_i, x_j, x_k)$ ,  $f'_{nx_k^+}(x_i, x_j, x_k) = f'_{nx_i^+}(x_i, x_j, x_k)$ .

**Proof.** The proof for the case of  $x_i = x_j$  will be given, and the results for the cases of  $x_j = x_k$  and  $x_k = x_i$  can be obtained similarly.

By (4), Lemma 4.1 and  $x_i = x_j$ ,

$$\begin{aligned} G'_{nx_i^+}(x_i, x_j, x_k) &= \lim_{t \rightarrow 0^+} \frac{G_n(x_i + t, x_j, x_k) - G_n(x_i, x_j, x_k)}{t}, \\ &= \lim_{t \rightarrow 0^+} \frac{G_n(x_j, x_i + t, x_k) - G_n(x_j, x_i, x_k)}{t}, \end{aligned}$$

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$$\begin{aligned}
 &= \lim_{t \rightarrow 0^+} \frac{G_n(x_i, x_j + t, x_k) - G_n(x_i, x_j, x_k)}{t}, \\
 &= G'_{nx_j^+}(x_i, x_j, x_k).
 \end{aligned}$$

Similar process is applicable to  $f'_{nx_i^+}(x_i, x_j, x_k)$ . This completes the proof.  $\square$

Proposition 4.1 states that when two items have equal inventory, their ordering priorities are also equal.

**Proposition 4.2.** *If  $x_i < x_j$ , then  $G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k)$ , and  $f'_{nx_i^+}(x_i, x_j, x_k) \leq f'_{nx_j^+}(x_i, x_j, x_k)$ , if  $x_i > x_j$ , then  $G'_{nx_i^+}(x_i, x_j, x_k) \geq G'_{nx_j^+}(x_i, x_j, x_k)$ , and  $f'_{nx_i^+}(x_i, x_j, x_k) \geq f'_{nx_j^+}(x_i, x_j, x_k)$ .*

**Proof.** The proof for  $x_i < x_j$  will be given. It is conducted by induction. First for  $n = 1$ , by Eq. (5),

$$G_1(x_i + t, x_j, x_k) = g(x_i + t) + g(x_j) + g(x_k), \quad \text{and}$$

$$G_1(x_i, x_j + t, x_k) = g(x_i) + g(x_j + t) + g(x_k).$$

Since  $g(\cdot)$  is convex, if  $x_i < x_j$ , then  $g(x_i + t) + g(x_j) \leq g(x_i) + g(x_j + t)$ , thus  $G_1(x_i + t, x_j, x_k) \leq G_1(x_i, x_j + t, x_k)$ , and

$$\begin{aligned}
 G'_{1x_i^+}(x_i, x_j, x_k) &= \lim_{t \rightarrow 0^+} \frac{G_1(x_i + t, x_j, x_k) - G_1(x_i, x_j, x_k)}{t} \\
 &\leq \lim_{t \rightarrow 0^+} \frac{G_1(x_i, x_j + t, x_k) - G_1(x_i, x_j, x_k)}{t} \\
 &= G'_{1x_j^+}(x_i, x_j, x_k).
 \end{aligned}$$

Assume that for  $n = N$ , if  $x_i < x_j$ , then  $G'_{Nx_i^+}(x_i, x_j, x_k) \leq G'_{Nx_j^+}(x_i, x_j, x_k)$ , and if  $x_i > x_j$ , then  $G'_{Nx_i^+}(x_i, x_j, x_k) \geq G'_{Nx_j^+}(x_i, x_j, x_k)$ . The assumption that if  $x_i < x_j$  then  $G'_{Nx_i^+}(x_i, x_j, x_k) \leq G'_{Nx_j^+}(x_i, x_j, x_k)$  implies:

$$\lim_{t \rightarrow 0^+} \frac{G_N(x_i + t, x_j, x_k) - G_N(x_i, x_j + t, x_k)}{t} \leq 0. \quad (7)$$

Furthermore, the following results are based on the assumptions and Proposition 4.1: If  $x_i < x_j$ , then  $G'_{Nx_i^+}(x_i, x_j, x_k) \leq G'_{Nx_j^+}(x_i, x_j, x_k)$ , if  $x_i > x_j$ , then  $G'_{Nx_i^+}(x_i, x_j, x_k) \geq G'_{Nx_j^+}(x_i, x_j, x_k)$ , and if  $x_i = x_j$ , then  $G'_{Nx_i^+}(x_i, x_j, x_k) = G'_{Nx_j^+}(x_i, x_j, x_k)$ .

Thus, under the condition of that ordering product  $i$  or  $j$  is necessary, we have two ordering policies:

Ordering policy 1: When  $x_i < x_j$ , ordering product  $i$  has the priority lower than ordering product  $j$ , and vice versa. Thus, it is optimal to order product  $i$  when  $x_i < x_j$  and product  $j$  when  $x_i > x_j$ .



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Ordering policy 2: When  $x_i = x_j$ , then it is always optimal to order products  $i$  and  $j$  simultaneously along with  $x_i = x_j$ . This is due to the following reasons. To the left-hand side of  $x_i = x_j$ , ordering product  $i$  is preferred to product  $j$ , and to the right-hand side of  $x_i = x_j$  ordering product  $j$  is preferred; thus, it is optimal to order products  $i$  and  $j$  simultaneously along with  $x_i = x_j$ . Although there is a possibility that Chattering Phenomena in Chen (2004b) exists,  $x_i = x_j$  is always included in the chattering area. Therefore, ordering along with  $x_i = x_j$  is always optimal.

According to the above policies, if  $x_i < x_j$ , then  $X_i \leq X_j$ ; if  $x_i = x_j$ , then  $X_i = X_j$ ; and if  $x_i > x_j$ , then  $X_i \geq X_j$ . Now it will be proved that if  $x_i < x_j$ ,  $f'_{N x_i^+}(x_i, x_j, x_k) \leq f'_{N x_j^+}(x_i, x_j, x_k)$ .

From Eq. (6),

$$f_N(x_i + t, x_j, x_k) = \min_{\substack{\frac{X_i - (x_i + t)}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i + t \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\} - c(x_i + x_j + x_k + t), \quad (8)$$

$$f_N(x_i, x_j + t, x_k) = \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - (x_j + t)}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j + t \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\} - c(x_i + x_j + x_k + t) \quad (9)$$

and

$$\begin{aligned} & f_N(x_i + t, x_j, x_k) - f_N(x_i, x_j + t, x_k) \\ &= \min_{\substack{\frac{X_i - (x_i + t)}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i + t \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\} \\ & - \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - (x_j + t)}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j + t \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\}. \end{aligned} \quad (10)$$

Assign  $X'_i = X_i - t$ , then  $X_i = X'_i + t$ , thus

$$\begin{aligned} & \min_{\substack{\frac{X_i - (x_i + t)}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i + t \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\} \\ &= \min_{\substack{\frac{X'_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X'_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X'_i + t, X_j, X_k)\} \\ &= \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i + t, X_j, X_k)\}. \end{aligned} \quad (11)$$

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Similarly, assign  $X'_j = X_j - t$ , then  $X_j = X'_j + t$  and

$$\begin{aligned} & \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - (x_j + t)}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j + t \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\} \\ &= \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j + t, X_k)\}. \end{aligned} \quad (12)$$

Clearly, both minimization functions at the right-hand sides of Eqs. (11) and (12) have the same constraint, thus have the same set of feasible solutions.

By Eqs. (11) and (12), Eq. (10) can be written as

$$\begin{aligned} & f_N(x_i + t, x_j, x_k) - f_N(x_i, x_j + t, x_k) \\ &= \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i + t, X_j, X_k)\} \\ &- \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j + t, X_k)\}. \end{aligned} \quad (13)$$

Since  $x_i < x_j$ , for a sufficiently small positive  $t$ , we have  $x_i + t \leq x_j$ . For the point  $(x_i + t, x_j, x_k)$  in Eq. (8), suppose the optimal point after ordering is  $(X_i^1, X_j^1, X_k^1)$ , it is clear that  $X_i^1 \leq X_j^1$ . For the point  $(x_i, x_j + t, x_k)$  in (9), suppose the optimal point after ordering is  $(X_i^2, X_j^2, X_k^2)$ , thus  $X_i^2 \leq X_j^2$  since  $x_i < x_j < x_j + t$ .

The next step here is to prove that  $G_N(X_i^1, X_j^1, X_k^1) \leq G_N(X_i^2, X_j^2, X_k^2)$ . Equation (7) and Proposition 4.1 state that for a sufficiently small positive  $t$ , and  $X_i \leq X_j$ , then  $G_N(X_i + t, X_j, X_k) - G_N(X_i, X_j + t, X_k) \leq 0$ . Since the minimization functions at the right-hand sides of Eqs. (11) and (12) have the same set of feasible solutions, therefore, for a sufficiently small positive  $t$ , and  $X_i \leq X_j$ , we have

$$\begin{aligned} & \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i + t, X_j, X_k)\} \\ &\leq \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j + t, X_k)\}. \end{aligned}$$

Since  $X_i^1 \leq X_j^1$ , thus by Eq. (11), for  $X_i \leq X_j$ , the following inequality holds:

$$G_N(X_i^1, X_j^1, X_k^1) \leq \min_{\substack{\frac{X_i - x_i}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j + t, X_k)\}. \quad (14)$$

For  $X_i^2 \leq X_j^2$ , consider the following two cases:

**Case 1:**  $X_i^2 < X_j^2$ . It means that there exist  $X_i$  and  $X_j$  such that following inequality holds.

$$X_i^2 = X_i < X_j + t = X_j^2.$$

When  $t \rightarrow 0^+$ ,  $X_i \leq X_j$ . This means that the minimum to the right-hand side of inequality (14) is attained at  $X_i \leq X_j$ . Thus, by Eqs. (12) and (14), the following

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result is true:

$$G_N(X_i^1, X_j^1, X_k^1) \leq G_N(X_i^2, X_j^2, X_k^2).$$

**Case 2:**  $X_i^2 = X_j^2$ . It means that  $(X_i^2, X_j^2, X_k^2)$  satisfies the following inequalities:

$$\frac{X_i^2 - x_i}{u} + \frac{X_j^2 - (x_j + t)}{u} + \frac{X_k^2 - x_k}{u} \leq 1, \quad x_i \leq X_i^2, \quad x_j + t \leq X_j^2, \quad x_k \leq X_k^2.$$

Since  $X_i^2 = X_j^2$ , and  $x_i < x_j$ , thus  $X_i^2 = X_j^2 \geq x_j + t > x_i + t$ , and  $X_j^2 \geq x_j + t > x_j$ ,  $x_k \leq X_k^2$ . This means that  $(X_i^2, X_j^2, X_k^2)$  is in the feasible region of  $\min_{\substack{\frac{X_i - (x_i + t)}{u} + \frac{X_j - x_j}{u} + \frac{X_k - x_k}{u} \leq 1; \\ x_i + t \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G_N(X_i, X_j, X_k)\}$ . Thus,

$$G_N(X_i^1, X_j^1, X_k^1) \leq G_N(X_i^2, X_j^2, X_k^2).$$

The above analysis yields the following inequality for  $t \rightarrow 0^+$ :

$$G_N(X_i^1, X_j^1, X_k^1) \leq G_N(X_i^2, X_j^2, X_k^2). \quad (15)$$

By Eq. (10), for  $t \rightarrow 0^+$ , we have  $f_N(x_i + t, x_j, x_k) - f_N(x_i, x_j + t, x_k) \leq 0$ . Thus,

$$\begin{aligned} & f'_{Nx_i^+}(x_i, x_j, x_k) - f'_{Nx_j^+}(x_i, x_j, x_k) \\ &= \lim_{t \rightarrow 0^+} \frac{f_N(x_i + t, x_j, x_k) - f_N(x_i, x_j + t, x_k)}{t} \leq 0. \end{aligned} \quad (16)$$

The next step is to prove that if  $X_i < X_j$ , then  $G'_{(N+1)X_i^+}(X_i, X_j, X_k) - G'_{(N+1)X_j^+}(X_i, X_j, X_k) \leq 0$ .

By Eq. (5),

$$\begin{aligned} & G'_{(N+1)X_i^+}(X_i, X_j, X_k) - G'_{(N+1)X_j^+}(X_i, X_j, X_k) \\ &= g'(X_i) - g'(X_j) + \alpha E(f'_{NX_i^+}(X_i - d, X_j - d, X_k - d) \\ &\quad - f'_{NX_j^+}(X_i - d, X_j - d, X_k - d)). \end{aligned} \quad (17)$$

Since  $X_i < X_j$ , thus

$$g'(X_i) - g'(X_j) \leq 0. \quad (18)$$

By Eq. (16), the following inequality can be obtained easily:

$$\alpha E\{f'_{NX_i^+}(X_i - d, X_j - d, X_k - d) - f'_{NX_j^+}(X_i - d, X_j - d, X_k - d)\} \leq 0. \quad (19)$$

Thus, by (18) and (19),  $G'_{(N+1)X_i^+}(X_i, X_j, X_k) - G'_{(N+1)X_j^+}(X_i, X_j, X_k) \leq 0$ .

This completes the first part of the proposition and also concludes the proof.  $\square$

Due to the symmetry of  $x_i(X_i)$ ,  $x_j(X_j)$ , and  $x_k(X_k)$ , the following results can be obtained:

- (1) if  $x_j < x_k$ , then  $G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k)$  and  $f'_{nx_j^+}(x_i, x_j, x_k) \leq f'_{nx_k^+}(x_i, x_j, x_k)$ ;

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- (2) if  $x_j > x_k$ , then  $G'_{nx_j^+}(x_i, x_j, x_k) \geq G'_{nx_k^+}(x_i, x_j, x_k)$  and  $f'_{nx_j^+}(x_i, x_j, x_k) \geq f'_{nx_k^+}(x_i, x_j, x_k)$ ;
- (3) if  $x_k < x_i$ , then  $G'_{nx_k^+}(x_i, x_j, x_k) \leq G'_{nx_i^+}(x_i, x_j, x_k)$  and  $f'_{nx_k^+}(x_i, x_j, x_k) \leq f'_{nx_i^+}(x_i, x_j, x_k)$ ;
- (4) if  $x_k > x_i$ , then  $G'_{nx_k^+}(x_i, x_j, x_k) \geq G'_{nx_i^+}(x_i, x_j, x_k)$  and  $f'_{nx_k^+}(x_i, x_j, x_k) \geq f'_{nx_i^+}(x_i, x_j, x_k)$ .

From Propositions 4.1 and 4.2, we can see that the plane  $x_i = x_j$  divides the whole 3D space into two parts of  $x_i < x_j$  with  $G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k)$  and  $x_i > x_j$  with  $G'_{nx_i^+}(x_i, x_j, x_k) \geq G'_{nx_j^+}(x_i, x_j, x_k)$ . Similar arguments are applicable to the planes of  $x_j = x_k$  and  $x_k = x_i$ . Thus, these three planes of  $x_i = x_j$ ,  $x_j = x_k$ , and  $x_k = x_i$  have an intersection curve which is the line of  $x_i = x_j = x_k$  as illustrated by line  $L_1$  in Fig. 1. By Lemma 4.2,  $L_1$  passes through a global minimum point  $S$ . In addition, these three planes also divide the space into six different sub-spaces which share the line  $L_1$  as the common boundary.

By the preceding analysis, the following results can be obtained as described in Fig. 1:

in sub-space ①:

$$x_i \leq x_j \leq x_k, \quad \text{and}$$

$$G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k), \quad (20)$$

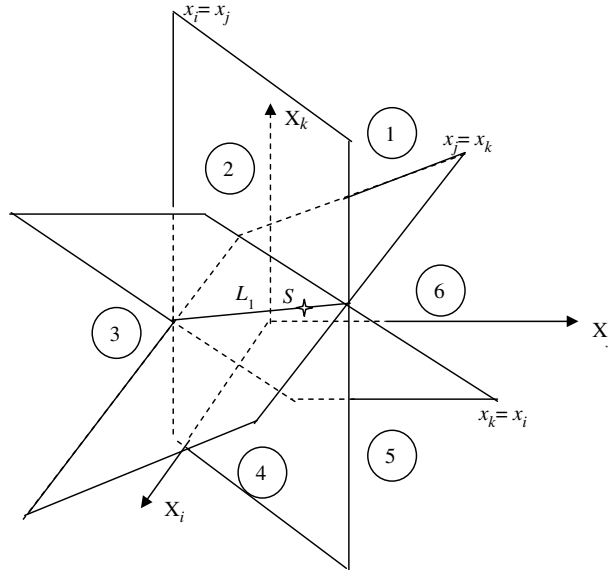


Fig. 1. Solution framework for the symmetric three-product system.

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in sub-space ②:

$$x_j \leq x_i \leq x_k, \quad \text{and} \\ G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k), \quad (21)$$

in sub-space ③:

$$x_j \leq x_k \leq x_i, \quad \text{and} \\ G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k) \leq G'_{nx_i^+}(x_i, x_j, x_k), \quad (22)$$

in sub-space ④:

$$x_k \leq x_j \leq x_i, \quad \text{and} \\ G'_{nx_k^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_i^+}(x_i, x_j, x_k), \quad (23)$$

in sub-space ⑤:

$$x_k \leq x_i \leq x_j, \quad \text{and} \\ G'_{nx_k^+}(x_i, x_j, x_k) \leq G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k), \quad (24)$$

in sub-space ⑥:

$$x_i \leq x_k \leq x_j, \quad \text{and} \\ G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k). \quad (25)$$

In sub-spaces ① and ⑥,  $G'_{nx_i^+}(x_i, x_j, x_k)$  has the smallest value compared with  $G'_{nx_j^+}(x_i, x_j, x_k)$  and  $G'_{nx_k^+}(x_i, x_j, x_k)$ . Because  $G_n(x_i, x_j, x_k)$  is convex,  $G'_{nx_i^+}(x_i, x_j, x_k)$  is a non-decreasing function of  $x_i$ . Therefore, the boundary between ordering something and nothing in sub-spaces ① and ⑥ is the surface  $X_i^*(x_j, x_k)$  which is defined as

$$G'_{nx_i^+}(x_i, x_j, x_k) \begin{cases} \geq 0, & \text{for } x_i \geq X_i^*(x_j, x_k) \\ < 0, & \text{for } x_i < X_i^*(x_j, x_k) \end{cases}. \quad (26)$$

Similar analysis is applicable to sub-spaces ② and ③, and sub-spaces ④ and ⑤.

The following proposition describes an optimal order policy for this system.

**Proposition 4.3.** (a) For any point  $P(x_i, x_j, x_k)$  in sub-space ① with  $x_i$  greater than or equal to  $X_i^*(x_j, x_k)$ , it is optimal to order nothing;

(b) For any point  $P(x_i, x_j, x_k)$  in sub-space ① with  $x_i$  less than  $X_i^*(x_j, x_k)$ , it is optimal to order product  $i$  until  $X_i^*(x_j, x_k)$  or a global minimum point is reached, or plane  $x_i = x_j$  is reached if sufficient capacity is available. If the plane  $x_i = x_j$  is reached, then order products  $i$  and  $j$  simultaneously along with the curve  $x_i = x_j$  with the given  $x_k$  value until  $X_i^*(x_j, x_k)$  or a global minimum point is reached, or line  $L_1$  is reached if the capacity allows. If line  $L_1$  is reached, then order products  $i$ ,

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$j$ , and  $k$  simultaneously along with the line until a global minimum point is reached provided any capacity is left.

Similar results hold for sub-spaces ②, ③, ④, ⑤, and ⑥ respectively. Sub-spaces ①, ②, ③, ④, ⑤, and ⑥ are defined by (20)–(25) respectively.

**Proof.** By the definition of  $X_i^*(x_j, x_k)$ , for a point  $P(x_i, x_j, x_k)$  in sub-space ①, if  $x_i \geq X_i^*(x_j, x_k)$ , then  $G'_{nx_i^+}(x_i, x_j, x_k) \geq 0$ , and  $0 \leq G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k)$ . It implies that ordering these three products will increase the cost. Hence, the part (a) is applicable.

If  $x_i < X_i^*(x_j, x_k)$ , then  $G'_{nx_i^+}(x_i, x_j, x_k) < 0$ . Furthermore, as  $G'_{nx_i^+}(x_i, x_j, x_k) \leq G'_{nx_j^+}(x_i, x_j, x_k) \leq G'_{nx_k^+}(x_i, x_j, x_k)$ , for any point in sub-space ①, ordering product  $i$  has more advantage. Thus, increasing  $x_i$  is optimal. The following scenarios will occur while increasing  $x_i$ .

**Scenario 1:** When  $X_i^*(x_j, x_k)$  is reached, by part (a), further ordering will no longer reduce the cost, so the optimum is reached.

**Scenario 2:** A global minimum point is reached. Obviously, further cost reduction is also impossible.

**Scenario 3:**  $x_i = x_j$  is reached. By the analysis at ordering policy 2 in the proof of Proposition 4.2, increasing products  $i$  and  $j$  simultaneously along with  $x_i = x_j$  is optimal if further ordering is necessary. While increasing  $x_i$  and  $x_j$ , scenario 1, 2, and 4 may occur and their respective analyses are applicable respectively.

**Scenario 4:** The line  $L_1$  is reached. Since  $L_1$  is the line with  $X_i = X_j = X_k$ , similarly, the optimal ordering is to increase the inventory of products  $i$ ,  $j$ , and  $k$  simultaneously along with line  $L_1$ , until the global minimum point is reached or the capacity is used up. This concludes part (b) of the proposition.

Similar analysis is applicable to sub-spaces ②, ③, ④, ⑤, and ⑥. This completes the proof.  $\square$

Thus, Proposition 4.3 proves the policy stated in Sec. 1. For a given  $x_k$ , by the definition of  $X_i^*(x_j, x_k)$ ,  $X_i^*(x_j, x_k)$  actually is a point on plane  $x_i = x_j$ . Therefore, on plane  $x_i = x_j$ ,  $X_i^*(x_j, x_k)$  is a function of  $x_k$  and can be rewritten as  $X'^*(x_k)$ . Clearly, the global minimum point is on the curve  $X'^*(x_k)$ .

Proposition 4.3 can be explained intuitively if we consider product 1 to be the product with the lowest inventory; product 2 to be the one with the second-lowest inventory; and product 3 to be the one with the highest inventory. By the optimal policy in Proposition 4.3, the firm first produces product 1 until its inventory reaches the same inventory level of product 2 at the beginning of the period. Then the firm produces products 1 and 2 at the same rate until both inventory levels reach the same inventory level of product 3 at the beginning of the period. Finally, the firm produces all three products at the same rate. This process will stop either

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when the product capacity is used up or when the marginal benefit of production becomes zero.

## 5. Extensions

### 5.1. Extension to the symmetric $m$ -product system

Similarly, the symmetric  $m$ -product model assumes that products in the system have the same i.i.d. stochastic demand, unit production cost and production rate respectively. Therefore, define

$$G_n(X_1, X_2, \dots, X_m) = g(X_1) + g(X_2) + \dots + g(X_m) \\ + \alpha E(f_{n-1}(X_1 - d, X_2 - d, \dots, X_m - d))$$

then the symmetric  $m$ -product system can be expressed as the following dynamic program.

$$f_n(x_1, x_2, \dots, x_m) = \min_{\substack{\frac{x_1-x_1}{u} + \frac{x_2-x_2}{u} + \dots + \frac{x_m-x_m}{u} \leq 1; \\ x_1 \leq X_1; x_2 \leq X_2; \dots; x_m \leq X_m}} \{G_n(X_1, X_2, \dots, X_m)\} - c \sum_{i=1}^n x_i.$$

With a little effort, Lemmas 4.1 and 4.2 and Propositions 4.1–4.3 can be extended for the symmetric  $m$ -product system, in which every  $x_i = x_j$  ( $i \neq j, 1 \leq i, j \leq m$ ) divides the  $m$ -dimensional space into two parts. All the parts have a common boundary of  $x_1 = x_2 = \dots = x_m$ , and construct the sub-spaces in the number of  $C_m^1 C_{m-1}^1 \dots C_2^1 C_1^1 = m!$ , where  $C_k^l = \frac{k!}{l!(k-l)!}$ .

By Katajainen and Pasanen (1999), the time complexity of sorting the inventory levels of  $(x_1, x_2, \dots, x_m)$  is  $O(m \log m)$ .

### 5.2. Extension to infinite horizon

DeCroix and Arreola-Risa (1998) prove that  $G_n(X_i, X_j, X_k)$  and  $f_n(x_i, x_j, x_k)$  converge monotonically and uniformly on any finite compact set. Hence, these limiting functions are convex and satisfy the following equation.

$$f(x_i, x_j, x_k) = \min_{\substack{\frac{x_i-x_i}{u} + \frac{x_j-x_j}{u} + \frac{x_k-x_k}{u} \leq 1; \\ x_i \leq X_i; x_j \leq X_j; x_k \leq X_k}} \{G(X_i, X_j, X_k)\} - c(x_i + x_j + x_k),$$

where  $f(x_i, x_j, x_k) = \lim_{n \rightarrow \infty} f_n(x_i, x_j, x_k)$  and  $G(x_i, x_j, x_k) = \lim_{n \rightarrow \infty} G_n(x_i, x_j, x_k)$ .

Clearly, let  $n \rightarrow \infty$ , Lemmas 4.1 and 4.2, Propositions 4.1 and 4.2 also hold for  $f(x_i, x_j, x_k)$  and  $G(x_i, x_j, x_k)$ . Therefore, for the infinite horizon case, Proposition 4.3 is true.

DeCroix and Arreola-Risa's (1998) result is also true for the  $m$ -product system. Therefore, the results in Lemmas 4.1 and 4.2 and Propositions 4.1–4.3 for the three-product infinite horizon case can be extended to the  $m$ -product infinite horizon case.

## 6. Discussions and Conclusions

This paper discusses a special case of the multiple-product flexible manufacturing system. An optimal policy is established for the problem. Our research provides a characterization for the structure of the optimal policy for a special case and also provides significant implications to the characterization of the optimal policy for the general problem.

Although this paper characterizes the production policy for the symmetric  $m$ -product system, it is still not clear on how production controls of two or three products affect each other in the optimal production policy. For the general two-product system, Chen (2004b) completely characterizes the optimal policy based on  $\mu$ -difference monotone which is a structural property to reflect the competitive advantage of producing two products. In this paper, the difficulty is avoided by Propositions 4.1 and 4.2 where production controls of two products are not affected by the production control of the third product. Future research can be directed toward characterizing the optimal policy for the general multiple-product model.

## Acknowledgments

The authors thank the Area Editor and two anonymous referees for their constructive comments and recommendations, and Dr. Arun Kumar and Dr. Lin Bing for proofreading the paper. The third author's research is supported in part by the NSF of China under Grant No. 70901012, the Specialized Research Foundation for the Doctoral Program of Higher Education of China, Grant No. 200806141084, and Fundamental Research Funds for the Central Universities (Grant No. ZYGX2013J134).

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**Yi Feng** is an associate professor in the Department of Management Science and Electronic Commerce at the University of Science and Technology of China. He has published papers in journals including *Operations Research Letters*, *International Journal of Production Economics*, *Computers and Operations Research* and *Asia-Pacific Journal of Operational Research*. His research includes logistics management, supply chain management, and operations management.

**Shaoxiang Chen** is a professor in Nanyang Business School of Nanyang Technological University. He has published papers in journals including *Operations Research*, *Management Science*, *IIE Transactions*, *Naval Research Logistics*, *Operations Research Letters*, and *European Journal of Operational Research*. His research includes logistics management, supply chain management, and operations management.

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**Wenqiang Dai** is an associate professor in the Department of Management Science and Electronic Commerce at the University of Science and Technology of China. He has published papers in journals including *Journal of Combinatorial Optimization* and *Theoretical Computer Science*. His research includes supply chain management, combinatorial optimization, and logistics network optimization.