

Continuation for Nonlinear Elliptic Partial Differential Equations Discretized by the Multiquadric Method

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Abstract

The Multiquadric Radial Basis Function (MQ) Method is a meshless collocation method with global basis functions. It is known to have exponential convergence for interpolation problems. We discretize nonlinear elliptic PDEs by the MQ method. This results in modest size systems of nonlinear algebraic equations which can be efficiently continued by standard continuation software such as AUTO and CONTENT. Examples are given of detection of bifurcations in 1D and 2D PDEs. These examples show high accuracy with small number of unknowns, as compared with known results from the literature.

Keywords: Continuation, elliptic PDEs, bifurcation analysis, multiquadric radial basis function method.

1 Introduction

Nonlinear multidimensional elliptic partial differential equations (PDEs) are the basis for many scientific and engineering problems, such as pattern formation in biology, viscous fluid flow phenomena, chemical reactions, crystal growth processes, etc. In these problems it is crucial to understand the qualitative dependence of the solution on the problem parameters.

During the past two decades the numerical continuation approach has become popular for qualitative study of solutions to nonlinear equations, see e.g. [33], [10], [34] and references therein. Several software packages, such as AUTO97 [9] and CONTENT [25], are currently available for bifurcation analysis of systems of nonlinear algebraic equations and ODEs, with only limited bifurcation analysis for 1D elliptic PDEs. For 2D PDEs, we mention the software package PLTMG [1] that allows to solve a class of boundary value problems on regions in the plane, to continue the solution with respect to a parameter and even to compute limit and branching points. This software combines a sophisticated finite element discretization with advanced linear algebra techniques. Numerical continuation for 1D and 2D elliptic PDEs is currently an active research area, see e.g. [31], [37], [35], [6], [7], [26], [5], and [18, Ch 10] for reaction diffusion equations; and [32], [29] for CFD. The typical approaches used are based on the finite element or finite difference discretization of the PDEs. They result in very large (thousands or tens of thousands for 2D problems) systems of nonlinear algebraic equations with sparse matrices. The continuation process is typically based on the predictor-corrector algorithms that require solving nonlinear systems by the Newton type method at each continuation step. For the bifurcation analysis during the continuation process, one usually needs to compute at least few eigenvalues of the Jacobian matrix at each continuation step. The methods currently used both for the continuation and the corresponding eigenvalue problems are variants of Krylov subspace methods and recursive projection (RPM). Solving the resulting linear system and the eigenvalue problem require sophisticated algorithms and considerable computer resources (CPU time, memory, disk space, etc.).

In this paper we report results of numerical experiments with continuation and detection of bifurcations for 1D and 2D elliptic PDEs discretized by the Multiquadric Radial Basis Function (MQ) method. The MQ

method was first introduced for solving PDEs in 1990 by Kansa [22], [23]. It is a meshless collocation method with global basis functions which leads to finite dimensional problems with full matrices. It was shown to give very high accuracy with a relatively small number of unknowns (tens or hundreds for 2D problems). The corresponding linear systems can be efficiently solved by direct methods. This opens a possibility for using standard continuation software, such as AUTO and CONTENT, designed for bifurcation analysis of modest size problems. We also note that the MQ method does not require predetermined location of the nodes as the spectral method does.

In Section 2 we summarize previous results on solving PDEs by the MQ method and our experiments with an eigenvalue problem.

In Section 3 we formulate an adaptation of the MQ method for the discretization of the parametrized elliptic PDEs.

In Section 4 we present results of our numerical experiments with continuation of solutions and detection of bifurcations for 1D and 2D elliptic PDEs.

In Section 5 we discuss our results.

2 Review of multiquadric method for elliptic PDEs

2.1 Summary of previous results

The concept of solving PDEs using the radial basis functions (RBF) was introduced by Kansa in 1990 [22], [23]. He implemented this approach for the solution of hyperbolic, parabolic, and elliptic PDEs using the MQ RBFs proposed by Hardy [19], [20] for interpolation of scattered data.

There exists an infinite class of RBFs. A radial basis function, $f(x)$, $x \in \mathbb{R}^n$, depends only upon the distances between the nodes. A MQ RBF is $g_j(x) = ((x - x_j)^2 + c_j^2)^{1/2}$, where x_j is a reference node and c_j is a shape parameter. In the comprehensive study by Franke [13], it is shown that MQ RBFs have the excellent properties for the interpolation of 2D scattered data. Among studied RBFs still only the MQ RBFs are proven to have the exponential convergence for the function interpolation [28], [39].

The numerical experiments by Kansa [22], [23], and Golberg and Chen [15] show high efficiency and very accurate solution with the MQ scheme. Kansa [23] showed that MQ method yields a high accuracy for parabolic and elliptic PDEs. Example for the transient convection-diffusion problem with steep initial front demonstrated highly accurate solution by the MQ method with a small number of nodes even for large cell Peclet number Pe . Test cases with 20 nodes for the MQ method ran for diffusion coefficient D in the range from 10^{-1} to 10^{-3} . The corresponding cell Pe number was from 0.5 to 50.0. Exact and MQ solution are indistinguishable graphically (apparent difference less than 10^{-4}) for $D = 10^{-1}$ and 10^{-2} , while small deviation (5%) was observed at $D = 10^{-3}$, $Pe = 50$. No instability or wiggles was seen. Finite difference simulation with $K = 200$ nodes and optimal combination of the central and upwind differences for the case $D = 10^{-1}$, $Pe = 5$ resulted in the error of 3%, which was still several orders less accurate than the MQ method solution.

In the numerical experiments with modeling the von Neumann blast wave Kansa [23] compared the exact solution and its derivatives with the MQ solution (35 nodes) and with finite difference ones (50, 500 and 5000 nodes). The error in value and gradients of pressure, density and energy was 10^{-6} or less for the MQ method, and in the range from 10^{-4} to 10^{-2} for the best finite difference result with 5000 nodes.

Golberg and Chen [16] showed that the solution of the 3D Poisson equation could be obtained with only 60 randomly distributed nodes to the same degree of accuracy as a FEM solution with 71,000 linear elements.

Sharan, Kansa, and Gupta [36] showed that the MQ method yields very accurate solutions for elliptic PDE problems, including the biharmonic equation, and that the MQ approach is simple to implement on domains with irregular boundaries. Dubal et al. [11] noted many benefits of using MQ RBFs to solve the initial value problem for a 3D nonlinear equation for the collision of two black holes. The resulting discrete system has 2000 unknowns and was solved directly.

Buhmann [3] showed that RBFs and, in particular, MQ RBFs are useful for constructing prewavelets and wavelets. Wavelets are most frequently used in time series analysis, but there are results for using wavelets to solve PDEs [12], [30]. As Buhmann points out, one can generate true wavelets by an orthonormalization process. The wavelets are an elegant way to achieve the same results as multi-grid schemes. The MQ

RBFs are attractive for prewavelet construction due to exceptional rates of convergence and their infinite differentiability.

Franke and Schaback paper [14] provides the first theoretical foundation for solving PDEs by collocation using the RBF methods.

Kansa and Hon [24] studied several methods to solve linear equations that arise from the MQ collocation problems. They studied the 2D Poisson equation, and showed that ill-conditioning of the system of equations could be circumvented by using the sub-structuring methods.

Kansa [23] introduced the concept of variable shape parameters c_j in the MQ scheme that appeared to work well in some cases. In the work by Kansa and Hon [24], a recipe based upon the local radius of curvature of solution surface was found to perform better than a constant shape parameter MQ scheme. A simple variable shape parameter formula is based the local radius of curvature. Kansa and Hon[24] tested the MQ method for the 2D Poisson equation with a set of exact solutions like $F = \exp(ax + by), \cos(ax + by), \sin(ax + by), \log(ax + by + c), \exp(-a(x - 1/2)2 - b(y - 1/2)2)$ and $\arctan(ax + by)$. They showed obtained a high accuracy (up to 10^{-5}) and a small residual norm (10^{-4}) on a modest node size set (121 nodes) while locally adapting the shape parameter c_j .

Franke [13] compared (global) RBF interpolation schemes against many popular compactly supported schemes such as finite difference method, and found that the global RBF schemes were superior on six criteria.

Madych [27] showed theoretically that the MQ interpolation scheme converges faster as the constant MQ shape parameter becomes progressively larger.

The multi-zone method of Wong et al. [38] is yet another alternative method for improving computational efficiency. This method is readily parallelizable, and the conditioning of the resulting matrices are much better.

Hon and Mao [21] showed that an adaptive algorithm that adjusted the nodes to follow the peak of the shock wave can produce extremely accurate results in 1D Burgers equation with only 10 nodes, even for extremely steep shocks with $Re = 10^4$.

2.2 A simple eigenvalue problem.

Accurate approximation of eigenvalue problems is essential for bifurcation analysis of PDEs. We have not found references in literature on the MQ-solution of eigenvalue problems. We therefore present here results for an eigenvalue problem for 1D Laplace operator. For details on the MQ discretization see Section 3. This is a scalar problem

$$\begin{aligned} -u'' &= \lambda u, \\ u(0) &= u(1) = 0, \end{aligned} \tag{1}$$

that has the exact solution:

$$(\lambda_m, U^m(x)) = ((\pi m)^2, \sin(\pi m x)), \quad m = 1, 2, \dots$$

where $(\lambda_m, U^m(x))$ is the m -th eigenpair of (1). Introduce the mesh $x_n = nh$, $n = 0, 1, \dots, N$, $h = 1/N$, and consider the standard second order finite difference (FDM) discretization of (1):

$$\begin{aligned} -\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} &= \lambda u_n, \quad n = 1, \dots, N-1, \\ u_0 &= u_N = 0. \end{aligned} \tag{2}$$

The corresponding approximate eigenpairs are given by

$$(\lambda_m^h, U_m^h) = \left(4N^2 \sin^2 \frac{\pi m}{2N}, \begin{bmatrix} \sin \frac{\pi m}{N} \\ \sin \frac{\pi 2m}{N} \\ \dots\dots\dots \\ \sin \frac{\pi (N-1)m}{N} \end{bmatrix} \right), \quad m = 1, \dots, N-1.$$

We also solved (1) using the MQ discretization for several values of the number K of internal nodes. Denote by $(\lambda_m^{MQ}, U_m^{MQ})$, $m = 1, \dots, K$ the corresponding approximate eigenpairs.

Table 1: Eigenvalue problem: comparison of eigenvalues and eigenfunctions

a) MQ method with uniform node distribution for $K = 5, 7$ and 9 .

m	λ_m (exact)	$\lambda_m^{MQ}, K = 5$	Rel. err. ε_λ^{MQ}	Rel. err. ε_U^{MQ}	Rel. err. $\varepsilon_\lambda^h, K = 47$
1	9.86961	9.86596	3.7×10^{-4}	3.7×10^{-4}	3.7×10^{-4}
2	39.4784	39.6492	4.3×10^{-3}	5.2×10^{-3}	1.5×10^{-3}

m	λ_m (exact)	$\lambda_m^{MQ}, K = 7$	Rel. err. ε_λ^{MQ}	Rel. err. ε_U^{MQ}	Rel. err. $\varepsilon_\lambda^h, K = 76$
1	9.86961	9.86821	1.4×10^{-4}	9.9×10^{-5}	1.4×10^{-4}
2	39.4784	39.4738	1.2×10^{-4}	1.8×10^{-4}	5.7×10^{-4}
3	88.8264	89.3648	6.0×10^{-3}	1.1×10^{-2}	1.3×10^{-3}

m	λ_m (exact)	$\lambda_m^{MQ}, K = 9$	Rel. err. ε_λ^{MQ}	Rel. err. ε_U^{MQ}	Rel. err. $\varepsilon_\lambda^h, K = 117$
1	9.86961	9.86901	6.0×10^{-5}	5.0×10^{-5}	6.0×10^{-5}
2	39.4784	39.4846	1.6×10^{-4}	2.1×10^{-4}	2.4×10^{-4}
3	88.8264	89.1667	3.8×10^{-3}	7.3×10^{-3}	5.4×10^{-4}
4	157.913	159.689	1.1×10^{-2}	2.5×10^{-2}	9.6×10^{-4}

b) MQ method with nonuniform node distribution for $K = 7$ and 9

m	λ_m (exact)	$\lambda_m^{MQ}, K = 7$	Rel. err. ε_λ^{MQ}	Rel. err. ε_U^{MQ}	Rel. err. $\varepsilon_\lambda^h, K = 3477$
1	9.86961	9.86961	6.8×10^{-8}	3.0×10^{-6}	6.8×10^{-8}
2	39.4784	39.4782	3.2×10^{-6}	3.0×10^{-4}	2.7×10^{-7}
3	88.8264	88.8139	1.4×10^{-4}	6.5×10^{-4}	5.4×10^{-4}

m	λ_m (exact)	$\lambda_m^{MQ}, K = 9$	Rel. err. ε_λ^{MQ}	Rel. err. ε_U^{MQ}	Rel. err. $\varepsilon_\lambda^h, K = 950$
1	9.86961	9.86960	9.1×10^{-7}	2.3×10^{-6}	9.1×10^{-7}
2	39.4784	39.4783	1.4×10^{-6}	2.0×10^{-5}	3.6×10^{-6}
3	88.8264	88.8241	2.6×10^{-5}	1.8×10^{-4}	8.2×10^{-6}
4	157.913	157.882	1.9×10^{-4}	1.8×10^{-3}	1.5×10^{-5}

The results of our computations are summarized in Table 1. We use the notation $\varepsilon_\lambda^{MQ}, \varepsilon_\lambda^h$ for the relative errors in $\lambda_m^{MQ}, \lambda_m^h$, respectively, and the notation ε_U^{MQ} for the L_∞ -norm error in U_m^{MQ} . For each MQ solution we provide a comparison with the FDM solution that has a sufficient number of nodes to give the same accuracy for λ_1 as the MQ method. In Part (a) of the table we use the uniform node distribution for the MQ method. Part (b) of the table shows that the accuracy of the MQ method can be significantly improved by adapting the node distribution: we moved only two nodes adjacent to boundary to reduce their distance from the boundary to $h_1 = h/4$ (while the remaining nodes are distributed uniformly).

One can see that the MQ method can give a highly accurate solution with a small number of unknowns, 10 – 100 times smaller than the number of unknowns in the FDM for the same accuracy.

3 Discretization of nonlinear elliptic PDEs by the MQ method

We consider the second order system of n parametrized nonlinear elliptic partial differential equations

$$D(\alpha)\Delta u - f(\nabla u, u, x, y, \alpha) = 0, \quad \alpha \in \mathbb{R}, u(\cdot), f(\cdot) \in \mathbb{R}^n, (x, y) \in \Omega \subset \mathbb{R}^2, \quad (3)$$

where $D(\alpha)$ is a positive diagonal $n \times n$ matrix, that depends smoothly on α , subject to boundary conditions

$$f^b\left(\frac{\partial u}{\partial n}, u, x, y, \alpha\right)\Big|_{\partial\Omega} = 0, f^b(\cdot) \in \mathbb{R}^n. \quad (4)$$

Here α is a control parameter, and we are interested in studying the dependence of the solutions to the boundary value problem (3), (4) on α .

We discretize the continuous problem by the multiquadric radial basis function (MQ) method [22], [23], [28] as follows. Introduce a set Θ_h of nodes (N internal and N_b on the boundary)

$$\Theta_h = \{(x_i, y_i) \mid i=1, N \subset \Omega, (x_i, y_i) \mid i=N+1, N+N_b \subset \partial\Omega\}$$

and look for the approximate solution to (3), (4) in the form

$$u_h(x, y) = a_0 + \sum_{j=1}^{j=N-1} a_j (g_j(c_j, x, y) - g_N(c_N, x, y)) + \sum_{j=N+1}^{j=N+N_b} a_j (g_j(c_j, x, y) - g_N(c_N, x, y)), \quad (5)$$

where $a_j \in \mathbb{R}^n$ are the unknown expansion coefficients and

$$g_j(c_j, x, y) = \sqrt{(x - x_j)^2 + (y - y_j)^2 + c_j^2}, \quad j = 1, \dots, N + N_b,$$

are the MQ basis functions, and $c_j > 0$ is called a *shape parameter* [23]. We then substitute $u_h(x, y)$ into (3), (4) and use collocation at the nodes Θ_h to obtain a finite dimensional system

$$\varphi_i(a, \alpha) \equiv D(\alpha) \Delta u_h(x_i, y_i) - f(\nabla u_h(x_i, y_i), u_h(x_i, y_i), x_i, y_i, \alpha) = 0, \quad i = 1, \dots, N, \quad (6)$$

$$\varphi_{i-N}^b(a, \alpha) \equiv f^b\left(\frac{\partial u_h(x_i, y_i)}{\partial n}, u_h(x_i, y_i), x_i, y_i, \alpha\right) = 0, \quad i = N + 1, \dots, N + N_b. \quad (7)$$

We next modify the discretized system to make it more suitable for continuation and bifurcation analysis. 1) We eliminate a_j , $j = N + 1, \dots, N + N_b$, associated with the boundary nodes, so as to minimize the number of unknowns. 2) We reformulate (5) in terms of nodal values u_i so that to have the correct eigenvalue problem (to avoid dealing with matrix stencils) for the Jacobian matrix of (6) for detecting bifurcations during the continuation process.

This is accomplished as follows. Denote $a^1 = (a_0, a_1, \dots, a_{N-1}) \in \mathbb{R}^{n \times N}$, $a^2 = (a_{N+1}, \dots, a_{N+N_b}) \in \mathbb{R}^{n \times N_b}$, $\varphi = (\varphi_1, \dots, \varphi_N)$, $\varphi^b = (\varphi_1^b, \dots, \varphi_{N_b}^b)$, and rewrite the system (6), (7) as

$$\varphi(a^1, a^2, \alpha) = 0, \quad \varphi(\cdot) \in \mathbb{R}^{n \times N}, \quad (8)$$

$$\varphi^b(a^1, a^2, \alpha) = 0, \quad \varphi^b(\cdot) \in \mathbb{R}^{n \times N_b}. \quad (9)$$

Assuming that the implicit function theorem is applicable here (which is usually the case), we solve (9) for a^2 to obtain

$$a^2 = \psi(a^1, \alpha), \text{ or, in components, } a_j = \psi_j(a^1, \alpha), \quad j = N + 1, \dots, N + N_b. \quad (10)$$

Substituting this into (8) yields

$$\varphi(a^1, \psi(a^1, \alpha), \alpha) = 0, \quad \varphi(\cdot) \in \mathbb{R}^{n \times N}. \quad (11)$$

We next want to reformulate (11) in terms of the nodal values $U = (u_1, u_2, \dots, u_N) \in \mathbb{R}^{n \times N}$ of the approximate solution at the internal nodes defined by $u_i = u_h(x_i, y_i)$. To this end we first eliminate $a^2 = (a_{N+1}, \dots, a_{N+N_b})$ from (5) by substituting (10) into (5) to obtain

$$u_h(x, y) = a_0 + \sum_{j=1}^{j=N-1} a_j (g_j(c_j, x, y) - g_N(c_N, x, y)) + \sum_{j=N+1}^{j=N+N_b} \psi_j(a^1, \alpha) (g_j(c_j, x, y) - g_N(c_N, x, y)) \quad (12)$$

We now define the map $\Gamma : a^1 \mapsto U = \Gamma(a^1)$. For $i = 0, \dots, N - 1$:

$$u_i = a_0 + \sum_{j=1}^{j=N-1} (g_j(c_j, x_i, y_i) - g_N(c_N, x_i, y_i)) a_j + \sum_{j=N+1}^{j=N+N_b} (g_j(c_j, x_i, y_i) - g_N(c_N, x_i, y_i)) \psi_j(a^1, \alpha). \quad (13)$$

Finally, substituting $a^1 = \Gamma^{-1}(U)$ into (11), we arrive at the finite dimensional continuation problem

$$G(U, \alpha) = 0, \quad U, G(\cdot) \in \mathbb{R}^{n \times N}, \alpha \in \mathbb{R}, \quad (14)$$

where

$$G(U, \alpha) = \varphi(\Gamma^{-1}(U), \psi(\Gamma^{-1}(U), \alpha), \alpha), \quad \Gamma : \mathbb{R}^N \rightarrow \mathbb{R}^N, \psi(\cdot) \in \mathbb{R}^{n \times N_b}.$$

Table 2: 1D Gelfand-Bratu equation: limit point comparison
a) results for MQ correspond to uniform node distribution

	[9], exact	[7], $K = 800$	MQ(u), $K = 5$	MQ(u), $K = 7$	MQ(u), $K = 9$
λ	3.513831	3.5137	3.512609	3.514224	3.514047
rel. error		3.7×10^{-5}	3.5×10^{-4}	-1.1×10^{-4}	-6.1×10^{-5}

b) results for MQ correspond to nonuniform node distribution

	[25], $K = 50$	[25], $K = 500$	MQ(nu), $K = 5$	MQ(nu), $K = 7$	MQ(nu), $K = 9$
λ	3.51145	3.51380	3.514010	3.513809	3.513828
rel. error	6.8×10^{-4}	8.8×10^{-6}	-5.1×10^{-5}	6.3×10^{-6}	8.5×10^{-7}

Remark 1 Note that in the case that the boundary condition (4) is linear, ψ_j are linear, and consequently Γ is an $N \times N$ matrix.

In Section 4 we consider examples of continuation of 1D PDEs with $\Omega = (0, 1)$ and 2D PDEs with $\Omega = (0, 1) \times (0, 1)$. In all 2D examples we have the same number N_s of nodes in x and y directions. We choose a constant shape parameter $c_j = s/(N_s - 1)$. Our typical choice for s is $4 \leq s \leq 12$.

We use two types of node distributions. In the case of uniform node distribution $(x_k, y_l) = (kh, lh)$, $k, l = 0, \dots, N_s$, $h = \frac{1}{N_s}$. In the case of nonuniform node distribution, the nodes adjacent to the boundary $\partial\Omega$ are placed at the distance $\tilde{h} = h_1 h$ from $\partial\Omega$, $0.1 \leq h_1 \leq 0.5$, while the remaining nodes are distributed uniformly. A criteria for the choice of h_1 was a minimum of L_2 -norm of the residual in Ω .

4 Numerical experiments for 1-D and 2-D elliptic PDEs

We present several examples of continuation of solutions to systems of nonlinear 1D and 2D elliptic PDEs. Each problem is discretized by the MQ method described in Section 3. We then perform continuation of the resulting system of algebraic equations (14) with AUTO97. The principal goal of our examples is to assess the accuracy of the detection of bifurcation points. We compare our results with some published results and, in one case, the results of our computations with an example in AUTO97 and CONTENT. We will use throughout the notation K for the number of unknowns in a particular method. For our MQ method $K = n \times N$, where n is the dimension of the system and N is the number of internal nodes. We denote by MQ(u) and MQ(nu) our MQ method with the uniform and nonuniform node distribution, respectively.

Example 1 1D Gelfand-Bratu equation. This is a scalar problem

$$\begin{aligned} u'' + \lambda e^u &= 0, \quad \text{in } \Omega = (0, 1), \\ u(0) &= u(1) = 0, \end{aligned} \tag{15}$$

that appears in combustion theory and is used as the demo example **exp** in AUTO97 [9] (fourth order adaptive orthogonal spline collocation method) and demo example in **brg** in CONTENT [25] (third order adaptive finite difference method). There is a limit (fold) point on the solution curve. We take the value of λ at the limit point found from demo **exp** ($K \geq 50$) as exact. The following table 2 shows comparison between numerical results in [7], our numerical results and our experiments with CONTENT.

Example 2 1D Brusselator problem. This is a reaction diffusion model for a trimolecular chemical reaction.

$$\begin{aligned} \frac{d_1}{l^2} u'' - (b+1)u + u^2 v + a &= 0, \quad \frac{d_2}{l^2} v'' + bu - u^2 v = 0, \quad \text{in } \Omega = (0, 1), \\ u(0) = u(1) &= a, \quad v(0) = v(1) = \frac{b}{a}. \end{aligned} \tag{16}$$

A stationary bifurcation occurs [6, Eq. (24)] at

$$b_n = 1 + \frac{d_1}{d_2} a^2 + \frac{\pi^2 n^2}{l^2} d_1 + \frac{l^2}{\pi^2 n^2} \frac{a^2}{d_2} > 0.$$

Table 3: 1D Brusselator equation: bifurcation points comparison

a) bifurcation point b_1

	exact	[6], $K = 80$	MQ(u), $K = 10$	MQ(u), $K = 14$	MQ(u), $K = 18$
b_1	19.680174	19.67547	19.67366	19.67786	19.67919
rel. error		2.4×10^{-4}	3.3×10^{-4}	1.2×10^{-4}	5.0×10^{-5}

b) bifurcation point b_2

	exact	[6], $K = 80$	MQ(u), $K = 10$	MQ(u), $K = 14$	MQ(u), $K = 18$
b_2	48.681060	48.6004	48.57476	48.63168	48.65605
rel. error		1.7×10^{-3}	2.2×10^{-3}	1.0×10^{-3}	5.1×10^{-4}

Table 4: 1D pattern formation problem, bifurcation points

[8, numerical]	0.047	0.080	0.093	0.159	0.140	0.238	0.186	0.317	0.232
[8, analytic]	0.0465	0.0793	0.093	0.159	0.140	0.238	0.186	0.317	0.233
MQ(nu)	0.0465	0.0793	0.093	0.159	0.140	0.238	0.186	0.317	0.233

For $l = d_1 = 1$, $d_2 = 2$, $a = 4$, $n = 1, 2$ this gives simple bifurcations: $b_1 = 9 + \pi^2 + \frac{8}{\pi^2} = 19.680174$, $b_2 = 9 + 4\pi^2 + \frac{2}{\pi^2} = 48.681060$, correspondingly. For the second order central difference method with uniform mesh of 41 mesh points ($K = 80$ unknowns), the corresponding approximate bifurcation points were found in [6, Section 6.1]. The following table 3 shows comparison between analytical, numerical results [6, Section 6.1] and our numerical results for values of b_1 and b_2 at simple bifurcation points.

Example 3 Pattern formation in a 1D system with mixed boundary conditions [8].

$$\begin{aligned} \frac{d_1}{\omega l^2} u'' + \beta - \kappa u - uv^2 &= 0, & \delta \frac{d_1}{\omega l^2} v'' + \kappa u + uv^2 - v &= 0, & \text{in } \Omega = (0, 1) \\ \theta_1 \frac{\partial u}{\partial n} &= \rho(1 - \theta_1)(\theta_3 u^s - u), & \delta \theta_2 \frac{\partial v}{\partial n} &= \delta \rho(1 - \theta_2)(\theta_3 v^s - v), & \text{on } \partial\Omega = \{0, 1\}. \end{aligned} \quad (17)$$

Here $\theta_i \in [0, 1]$, $i = 1, 2, 3$, are homotopy parameters. For $d_1 = 10^{-5}$, $\omega = 10^{-2}$, $\delta = 0.14$, $\beta = 1.0$, $\kappa = 0.001$, $(\theta_1, \theta_2, \theta_3) = (1, 1, 0)$ (Neumann problem). Eq. (17) was discretized by the second order central difference method with equidistant mesh of 41 mesh points ($K = 80$ unknowns). The following table [8, Table 1] shows a comparison between analytic and numerical results for values of l at simple bifurcation points.

Our numerical results (MQ(nu) method) with $K = 18$, coincide with the analytic results above. In addition, we found a bifurcation point at $l = 0.279$.

Example 4 2D Bratu problem

$$\begin{aligned} \Delta u + \lambda e^u, \Omega &= (0, 1) \times (0, 1), \\ u &|_{\partial\Omega} = 0. \end{aligned} \quad (18)$$

This problem was studied in [35]. It was discretized with the second order central difference method with uniform mesh and then continued using Implicit Block Elimination based on Recursive Projections. A limit point was detected for some value of λ (not reported in the paper), and spurious limit points were detected for $K = 961, 1521, 2209, 3025$ and λ sufficiently small. We reproduced the bifurcation diagram in [35], no spurious limit points were detected. The following table 5 gives the values of λ at the limit point computed by MQ method.

Example 5 2D Brusselator problem.

$$\begin{aligned} \frac{d_1}{l^2} \Delta u - (b + 1)u + u^2 v + a &= 0, & \frac{d_2}{l^2} \Delta v + bu - u^2 v &= 0, & \text{in } \Omega = (0, 1) \times (0, 1), \\ u &|_{\partial\Omega} = a, & v &|_{\partial\Omega} = \frac{b}{a}. \end{aligned} \quad (19)$$

Table 5: 2D Bratu equation, limit point

	[35], $225 \leq K \leq 3025$	MQ(u), $K = 25$	MQ(u), $K = 49$	MQ(u), $K = 81$
λ	not reported	6.873498	6.840836	6.827400

Table 6: 2D Brusselator equation: bifurcation points, uniform node distribution for MQ

a) bifurcation point b_1

	exact	[5], $K = 800$	MQ(u), $K = 50$	MQ(u), $K = 72$	MQ(u), $K = 98$
$b_{1,1}$	29.144494	29.104774	29.16280	29.17050	29.16062
rel. error		1.4×10^{-3}	-6.3×10^{-4}	-8.9×10^{-4}	-5.5×10^{-4}

b) bifurcation point b_2

	exact	[5], $K = 800$	MQ(u), $K = 50$	MQ(u), $K = 72$	MQ(u), $K = 98$
$b_{2,2}$	88.058156	87.47325	87.61578	87.86924	88.00143
rel. error		6.6×10^{-3}	5.0×10^{-3}	2.1×10^{-3}	6.4×10^{-4}

A stationary bifurcation occurs [5, Eq. (2.26)] for

$$b_{m,n} = 1 + \frac{d_1}{d_2}a^2 + d_1\pi^2 \left(\frac{m^2}{l^2} + n^2 \right) + \frac{a^2}{\pi^2 d_2} \left(\frac{l^2}{m^2 + l^2 n^2} \right) > 0.$$

For $l = d_1 = 1$, $d_2 = 2$, $a = 4$, $(m, n) = (1, 1)$, $(m, n) = (2, 2)$ this gives simple bifurcations: $b_{1,1} = 9 + 2\pi^2 + \frac{4}{\pi^2}$, $b_{2,2} = 9 + 8\pi^2 + \frac{1}{\pi^2}$, correspondingly. For the second order central difference method with equidistant mesh of 21 mesh points, the corresponding approximate bifurcation points are found in [5, Section 5]. The following tables 6, 7 show comparisons between analytical, numerical results [5, Eq. (2.26)] and our numerical results for values of $b_{1,1}$ and $b_{2,2}$ at simple bifurcation points.

A Hopf bifurcation occurs [5, Eq. (2.26)] for

$$b_{m,n} = 1 + a^2 + (d_1 + d_2) \left(\frac{m^2}{l^2} + n^2 \right) \pi^2$$

for some m , n , and l large enough. For $l = 10$, $d_1 = d_2 = 1$, $a = 10$, $(m, n) = (1, 2)$, this gives a Hopf bifurcation at $b_{1,2} = 101 + 2 \left(\frac{1}{100} + 2^2 \right) \pi^2 = 180.15$, see table 8.

5 Conclusions.

We presented the results of our experiments with the MQ method for continuation of solution of nonlinear 1D and 2D elliptic PDEs. We used small number of unknowns and obtained a high accuracy for detecting bifurcation points in our examples. Here are some sample results.

Table 7: 2D Brusselator equation: bifurcation points, nonuniform node distribution for MQ

a) bifurcation point b_1

	exact	MQ(nu), $K = 50$	MQ(nu), $K = 72$	MQ(nu), $K = 98$
$b_{1,1}$	29.144494	29.14621	29.14726	29.14431
rel. error		-5.9×10^{-5}	-9.5×10^{-5}	6.3×10^{-6}

b) bifurcation point b_2

	exact	MQ(nu), $K = 50$	MQ(nu), $K = 72$	MQ(nu), $K = 98$
$b_{2,2}$	88.058156	88.15470	87.93391	88.07288
rel. error		-1.1×10^{-3}	1.4×10^{-3}	-1.7×10^{-4}

Table 8: 2D Brusselator equation, Hopf bifurcation point

	exact	MQ(u), $K = 50$	MQ(nu), $K = 50$	MQ(u), $K = 72$	MQ(u), $K = 98$
$b_{1,2}$	180.15	181.8625	180.7880	181.0696	180.492
rel. error		-9.5×10^{-3}	-3.5×10^{-3}	-5.1×10^{-3}	-1.9×10^{-3}

(i) For the limit point in the 1D Gelfand-Bratu equation, the MQ method with 9 unknowns gives the relative errors 6.1×10^{-5} and 8.5×10^{-7} for the uniform and nonuniform node distributions, respectively. The relative error in the finite difference method with 500 nodes is 8.8×10^{-6} .

(ii) For the two bifurcation points in the 2D Brusselator problem, the MQ method with 98 unknowns gives the relative errors 5.5×10^{-4} , 6.4×10^{-4} for the uniform node distribution and 6.3×10^{-6} , 1.7×10^{-5} for the nonuniform node distribution. The corresponding relative errors in the finite difference method with 800 nodes are 1.4×10^{-3} , 6.6×10^{-3} .

(iii) for the first in the eigenvalue problem for the 1D Laplace operator with 9 unknowns gives the relative error 6×10^{-5} and 9×10^{-7} for the uniform and nonuniform node distributions, respectively. This is equivalent in accuracy to 117 and 950 node solution, respectively by the finite difference method.

The increase of the number of unknowns results in a better accuracy but also in a larger condition number of the operator Γ mapping solution nodal values to the expansion coefficients. This condition number is a limiting factor in our experiments. In the future, we plan to implement the ideas of Kansa et al. [24] to circumvent this ill conditioning problem.

In addition we found that even a simple adaptation of the nodes adjacent to the boundary can lead to a dramatic improvement of the accuracy in detecting bifurcation points. Adaptive choice of the shape parameter is another way to improve the accuracy that we plan to investigate.

Our results show that MQ method is an efficient method for continuation of solution nonlinear PDEs.

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References

- [1] R. E. Bank, *PLTMG : A software package for solving elliptic partial differential equations, Users' Guide 8.0*, SIAM Publications, Philadelphia 1998.
- [2] M.D. Buhmann and C.A. Micchelli, Multiquadric interpolation improved, *Computers Math. Applic.*, **24** (1992), 21-25.
- [3] M.D. Buhmann, Multiquadric prewavelets on nonequally spaced knots in one dimension, *Math. Comp.*, **64** (1995), 1611-1625.
- [4] R.E. Carlson, and T.A. Foley, The parameter R2 in multiquadric interpolation, *Compute. Math. Applic.*, **21**, No.9 (1991), 29-42.
- [5] C. S. Chien and M.-H. Chen, Multiple Bifurcations in a Reaction-Diffusion Problem *Computers Math. Applic.* **35**, No. 8 (1998), 15-39.
- [6] C. S. Chien, C.-L. Shen and Z. Mei, Numerical Continuation at Double Bifurcation Points of a Reaction-Diffusion Problem, *Int. J. Bifur. & Chaos* **8**, No. 1 (1997), 117-139.
- [7] B.D. Davidson, Large Scale Continuation and Numerical Bifurcation for Partial Differential Equations, *SIAM. J. Numer. Anal.* **24**, No. 5 (1997), 2008-2027.
- [8] R. Dillon, P. K. Maini, and H. G. Othmer, Pattern formation in generalized Turing systems. I. Steady-state patterns in systems with mixed boundary conditions, *J. Math. Biol.* **32** (1994), 345-393.
- [9] E.J. Doedel, A.R. Champneys, T.F. Fairgrieve, Yu.A. Kuznetsov, B. Sandstede, and X.J. Wang, *AUTO97: Continuation and bifurcation software for ordinary differential equations (with HomCont)* (1997).

- [10] E.J. Doedel, H.B. Keller and J.P. Kernévez, Numerical Analysis and Control of Bifurcation Problems, Part I: Bifurcation in Finite Dimensions, *Int. J. Bif. and Chaos*: **1**, No. 3 (1991), 493-520; Part II: Bifurcation in Infinite Dimensions, *Int. J. Bif. and Chaos*: **1**, No. 4 (1991), 745-772.
- [11] M.R. Dubal, S.R. Olivera and R.A. Matzner, In: Approaches to Numerical Relativity, ed. R. dInverno, Cambridge University Press, Cambridge, UK, 1993.
- [12] G.E. Fasshauer and J.W. Jerome, "Multi-step approximation algorithms: improved convergence rates though postconditioning and smoothing kernels, *Adv. Comp.. Math.* (to appear).
- [13] R. Franke, Scattered data interpolation: tests of some methods, *Math.Comp.*, 38 (1982), 181-199.
- [14] C. Franke and R. Schaback, Solving partial differential equations by collocation using radial basis functions, *Appl. Math. Comp.*, **93**, No. 1 (1998), 73-91.
- [15] M.A. Golberg and C.S. Chen, Improved multiquadric approximation for partial differential equations, *Engin. Anal with Bound. Elem.* 18 (1996), 9-17.
- [16] M.A. Golberg and C.S. Chen, *Discrete projection methods for integral equations*, Comput. Mech. Publ., Boston, MA (1997).
- [17] M.A. Golberg, and C.S. Chen, The theory of radial basis functions used in the dual reciprocity boundary element method, *Appl. Math. Comp.*, 60 (1994) 125-136.
- [18] W. Govaerts, *Numerical methods for bifurcations of dynamical equilibria*, Siam Publications, Philadelphia, to appear in 1999.
- [19] R.L. Hardy, Multiquadric equations of topography and other irregular surfaces, *J. Geophys. Res.*, 76 (1971), 1905-1915.
- [20] R.L. Hardy, Theory and applications of the multiquadric- biharmonic method: 20 years of discovery, *Comput. Math. Applic.*, **19**, No.8/9 (1990) 163-208.
- [21] Y.C. Hon, and X.Z. Mao, An efficient numerical scheme for Burgers equation, *Appl.Math. Comp.*, **95** (1998) 37-50.
- [22] E.J. Kansa, Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-I. Surface approximations and partial derivative estimates, *Comput. Math. Applic.*, **19**, No. 8/9 (1990) 127-145.
- [23] E.J. Kansa, Multiquadrics-a scattered data approximation scheme with applications to computational fluid dynamics-II. Solutions to hyperbolic, parabolic, and elliptic partial differential equations, *Comput. Math. Applic.*, **19**, No.8/9 (1990) 147-161.
- [24] E.J. Kansa, Y.C. Hon, Circumventing the ill-conditioning problem with Multiquadrics radial basis functions: Applications to elliptic partial differential equations, *Adv. Comp. Math.* (1998).
- [25] Yu.A. Kuznetsov, V.V. Levitin, *CONTENT, a multi-platform interactive environment to analyzing dynamical systems*. Dynamical Systems Laboratory, CWI, Amsterdam 1998, <ftp.cwi.nl/pub/content>.
- [26] Yu.A. Kuznetsov, A.R. Skovoroda, V.V. Levitin, W. Govaerts and B. Sijnave, Computation of Orbits and Steady States of 1D PDEs in CONTENT, *preprint*, 1998.
- [27] W.R. Madych, Miscellaneous error bounds for multiquadric and related interpolants, *Comput. Math. Applic.* **24**, No.12 (1992) 121-138.
- [28] W.R. Madych and S.A. Nelson, Multivariate interpolation and conditionally positive definite functions II, *Math. Comp.*, 54 (1990), 211-230.
- [29] C.K. Mamun and L.S. Tuckermann, Asymmetry and Hopf bifurcation in spherical Couette flow, *Phys. Fluids* **7** No. 1 (1995), 80-102.

- [30] F.J. Narcowich, R. Schaback, and J.D. Ward, Multi-level interpolation and approximation, preprint (1997).
- [31] R. Neubert, Predictor - Corrector techniques for detecting Hopf points, *Int. J. Bif. and Chaos*, **3** (1993) 1311 - 1318.
- [32] M. Poliashenko and C.K. Aidun, Computational Dynamics of Ordinary Differential Equations, *Int. J. Bif. and Chaos*, **5** No. 1 (1995), 159-174.
- [33] W.C. Rheinboldt, *Numerical Analysis of Parameterized Nonlinear Equations*. John Wiley & Sons, New York, 1986.
- [34] R. Seydel, *From Equilibrium to Chaos—Practical Bifurcation and Stability Analysis*. Elsevier Publishing, New York, Amsterdam, London, 1988.
- [35] H. Schwetlick, G. Timmermann and R. Lösche, Path Following for Large Nonlinear Equations by Implicit Block Elimination Based on Recursive Projections, *Lectures in Applied Mathematics*, **32**, (1996), 715-732.
- [36] M. Sharan, E.J. Kansa, and S. Gupta, Applications of the multiquadric method for the solution of elliptic partial differential equations, *Appl. Math. & Comput.*, **84** (1997) 275-302.
- [37] G. M. Shroff and H. B. Keller, Stabilization of unstable procedures : The recursive projection method, *SIAM J. Numer. Anal.* 30 (1993) 1099 - 1120.
- [38] Wong, S.M., Y.C. Hon, T.S. Li, S.L. Chung, and E.J. Kansa, Multi-zone decomposition of time-dependent problems using the multiquadric scheme, to appear in *Math. Comp. Applic.* (1998).
- [39] Z. Wu and R. Shaback, Local error estimates for radial basis function interpolation of scattered data, *IMA J. Num. Anal.* 13 (1993) 13-27.