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MASTER–SLAVE SYNCHRONIZATION OF LUR'E SYSTEMS WITH TIME-DELAY

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In this paper time-delay effects on the master–slave synchronization scheme are investigated. Sufficient conditions for master–slave synchronization of Lur'e systems are presented for a known time-delay in the master and slave systems. A delay-dependent synchronization criterion is given based upon a new Lyapunov–Krasovskii function. The derived criterion is a sufficient condition for global asymptotic stability of the error system, expressed by means of a matrix inequality. The feedback matrix follows from solving a nonlinear optimization problem. The method is illustrated for the synchronization of Chua's circuits, 5-scroll attractors and hyperchaotic attractors.

1. Introduction

Since the work of Pecora and Carroll [1991], the research on chaotic synchronization has received considerable attention. It basically deals with sufficient conditions for synchronization of identical or nonidentical systems [Wu & Chua, 1994]. Synchronization schemes have been investigated with local and global synchronization [Hasler, 1994], robust synchronization [Suykens *et al.*, 1999], partial synchronization [Hasler *et al.*, 1998] and generalized synchronization [Kocarev *et al.*, 1996]. An overview of synchronization methods has been recently presented in [Chen & Dong, 1998]. Synchronization has opened the way to investigate an engineering application of chaos, which is chaotic communications. Since 1992, a number of chaotic communication schemes have been proposed [Oppenheim *et al.*, 1992; Hasler, 1994; Wu & Chua, 1994]. In this paper, we deal with propagation delay in master–slave synchronization schemes. This problem has been recently reported in [Chen & Liu, 2000] which introduces the possibility of applying chaotic synchronization to optical communication. Chen and Liu [2000] called this problem a phase sensitivity due to the distance between two remote chaotic systems and has reported that the existence of a time-delay can destroy synchronization. Furthermore, the synchronization and bifurcation phenomena of two chaotic circuits which are coupled by a delay line have been investigated e.g. by Koike *et al.* [1997]. Experimental confirmation of synchronization of two chaotic circuits with the existence of delay has been studied by Kawate *et al.* [2000]. However, theoretical studies of this problem are still lacking.

On the other hand, in the area of control theory time-delay systems have been investigated and it is well known that delays often result in instabilities. Therefore, stability analysis of time-delay systems is an important subject in control theory [Hale, 1977; Mori *et al.*, 1983; Kamen, 1982, 1983; Tissir & Hmamed, 1996]. In the literature, stability criteria for time-delay systems are classified into two main categories: delay-independent criteria [Kamen, 1982, 1983; Chen & Latchman,

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1995] and delay-dependent criteria [Mori, 1985; Mori & Kamen, 1989; Chen, 1995]. Recently, time-delay systems have been intensively studied in [Mahmoud, 2000]. The Lyapunov–Krasovskii function [Krasovskii & Brenner, 1963] is a candidate function for asymptotical stability analysis of linear time-delay systems. Boyd *et al.* [1994] derived linear matrix inequalities (LMI's) for multiple delay [Boyd *et al.*, 1994] from a Lyapunov–Krasovskii function.

In this paper we consider master–slave schemes with identical Lur'e systems. Lur'e systems are a class of nonlinear systems which can be represented as a linear dynamical system, feedback interconnected to a nonlinearity that satisfies a sector condition. We suppose that the output of the master system is received at the slave system with delay (τ), which is assumed to be a known value. A delay-dependent criterion for global asymptotic stability of the error system is given which is expressed as a matrix inequality and is derived from an extended Lyapunov–Krasovskii function. The theoretical result is illustrated by a number of simulation examples for Chua's circuits, n-scroll attractors and hyperchaotic systems.

This paper is organized as follows. In Sec. 2 we present the master–slave synchronization scheme of Lur'e systems with time-delay. In Sec. 3 the error system and sufficient conditions for global asymptotic stability based on a Lyapunov–Krasovskii function for the delay-dependent and delay-independent cases are derived. In Sec. 4 we introduce a new Lyapunov function for the master–slave synchronization scheme and a sufficient condition for global asymptotic stability is given. Finally, in Sec. 5 examples are given for Lur'e systems: Chua's circuit, 5-scroll attractors and hyperchaotic attractors.

2. Time-Delay Synchronization Scheme

Consider the following master–slave synchronization scheme with static error feedback and time-delay τ

$$
\begin{aligned}
\mathcal{M} &: \begin{cases} \dot{x}(t) = Ax(t) + B\sigma(Cx(t)) \\ p(t) = Hx(t) \end{cases} \\
\mathcal{S} &: \begin{cases} \dot{y}(t) = Ay(t) + B\sigma(Cy(t)) + u(t) \\ q(t) = Hy(t) \end{cases} \\
\mathcal{C} &: \left\{ u(t) = G(p(t-\tau) - q(t-\tau)) \right.
\end{aligned}
\tag{1}
$$

with master system $\mathcal{M}$, slave system $\mathcal{S}$ and controller $\mathcal{C}$ (Fig. 1). The master and slave systems are Lur'e systems with state vectors $x,\, y \in \mathbb{R}^n$, output of subsystems $p,\, q \in \mathbb{R}^l$, respectively, and matrices $H \in \mathbb{R}^{l\times n}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n_h}$, $C \in \mathbb{R}^{n_h\times n}$. $\sigma(\cdot)$ satisfies a sector condition [Vidyasagar, 1993; Khalil, 1993] with $\sigma_i(\cdot)$ $i = 1, 2, \ldots, n_h$ belonging to sector $[0, k]$, i.e. $\sigma_i(\xi)[\sigma_i(\xi) - k\xi] \le 0$, $\forall\, \xi$ for $i = 1, 2, \ldots, n_h$. The scheme aims at synchronizing the master system to the slave system by applying full state error feedback to the slave system with control signal $u \in \mathbb{R}^n$ and feedback matrix $G \in \mathbb{R}^{n\times l}$. Master–slave synchronization has been studied for $\tau = 0$ in [Wu & Chua, 1994; Curran *et al.*, 1997]. The difference between this work and the present paper is the time-delay in the outputs, i.e. $p(t-\tau)$ and $q(t-\tau)$ instead of $p(t)$ and $q(t)$, respectively.

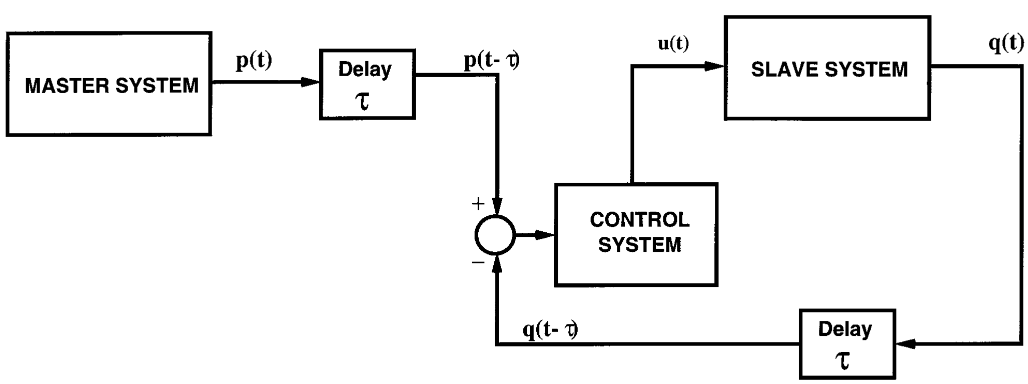


Fig. 1. Synchronization scheme.

3. Error System of Time-Delay Synchronization Scheme

Defining the signal $e(t) = x(t) - y(t)$ one obtains the error system $\mathcal{E}$

$$\mathcal{E} : \dot{e} = Ae + B\eta(Ce; y) + Fe(t - \tau) \qquad (2)$$

with $e = e(t)$, $F = -GH$ and $\eta(Ce; y) = \sigma(Ce + Cy) - \sigma(Cy)$. One assumes that the nonlinearity $\eta(Ce, y)$ belongs to sector $[0, k]$ [Curran & Chua, 1997; Suykens & Vandewalle, 1996]

$$0 \leq \frac{\eta_i(c_i^T e; y)}{c_i^T e} = \frac{\sigma_i(c_i^T e + c_i^T y) - \sigma_i(c_i^T y)}{c_i^T e} \leq k\,, \quad \forall\, e,\, y;\ i = 1, 2, \ldots, n_h \qquad (3)$$

The following inequality holds then [Boyd *et al.*, 1994; Khalil, 1992; Vidyasagar, 1993]

$$\eta_i(c_i^T e; y)[\eta(c_i^T e; y) - kc_i^T e] \leq 0\,, \quad \forall\, e,\, y;\ i = 1, 2 \ldots, n_h\,. \qquad (4)$$

Stability of the error system without time-delay ($\tau = 0$) in the feedback has been derived for quadratic Lyapunov functions [Wu & Chua, 1994; Curran & Chua, 1997] and Lur'e–Postnikov functions [Curran *et al.*, 1997]. The Lyapunov–Krasovskii function is a candidate Lyapunov function for time-delay systems [Krasovskii & Brennel, 1963; Hale, 1977; Mahmoud, 2000]:

$$V_1(e) = e^T Pe + \int_{-\tau}^{0} e(t+s)^T Qe(t+s)ds\,, \quad P = P^T > 0\,, \quad Q = Q^T > 0\,. \qquad (5)$$

By taking Lyapunov–Krasovskii function Eq. (5), it is straightforward to find a sufficient condition for global asymptotic stability of the error system $\mathcal{E}$.

Theorem 1. *Let $\Lambda = \text{diag}\{\lambda_i\}$ be a diagonal matrix with $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n_h$ then a sufficient condition for global asymptotic stability of the error system $\mathcal{E}$, based on the Lyapunov–Krasovskii function, is given by the matrix inequality*

$$Y = \begin{bmatrix} A^T P + PA + Q & PB + kC^T\Lambda & PF \\ B^T P + k\Lambda^T C & -2\Lambda & 0 \\ F^T P & 0 & -Q \end{bmatrix} < 0\,. \qquad (6)$$

Proof. By taking the time derivative of the Lyapunov–Krasovskii function Eq. (5) and applying the S-procedure [Boyd *et al.*, 1994], using the inequalities from the nonlinearities, one obtains

$$\begin{aligned} \dot{V}_1(e) &= \dot{e}^T Pe + e^T P\dot{e} + e(t)^T Qe(t) \\ &\quad - e(t-\tau)^T Qe(t-\tau) \\ &\leq e^T(A^T P + PA)e + \eta^T B^T Pe + e^T PB\eta \\ &\quad + e(t-\tau)^T F^T Pe + e^T PFe(t-\tau) \\ &\quad + e(t)^T Qe(t) - e(t-\tau)^T Qe(t-\tau) \\ &\quad - \sum_i 2\lambda_i\eta_i(\eta_i - kc_i^T e) \\ &= \xi^T Y\xi < 0 \end{aligned}$$

where $\xi = [e(t);\, \eta;\, e(t-\tau)]$. ■

The matrix inequality (6) does not include information on the delay. Therefore this result is a delay-independent stability criterion for synchronization. A necessary condition for $Y < 0$ is that the linear part of this system (A matrix) must be stable. An analysis for Chua's circuit, 5-scroll attractors and hyperchaotic attractors shows that a feasible point could not be found for these examples with Y negative definite. For Chua's circuit, a Lur'e representation is given by Güzeliş [1993], which has a stable A matrix, but even in this case one could not find a feasible point. In the literature it has been recommended that if delay-independent criteria fail, delay-dependent conditions should be applied [Tissir & Hmamed, 1996]. The following Lyapunov–Krasovskii function is a candidate function for a delay-dependent condition [Mahmoud, 2000]

$$\begin{aligned} V_2(e) &= e^T Pe + r_1 \int_{t-\tau}^{t} \int_{t+\theta}^{t} [e^T(s)A^T Ae(s)]dsd\theta \\ &\quad + r_2 \int_{t-\tau}^{t} \int_{t-\tau+\theta}^{t} [e^T(s)F^T Fe(s)]dsd\theta \end{aligned} \qquad (7)$$

where $P = P^T > 0$ and $r_1 > 0$, $r_2 > 0$ are weight factors. However, taking Eq. (7) as a candidate function and deriving an LMI from this would not be possible. For this reason we propose a new candidate Lyapunov function in the next section in order to find a sufficient condition for global asymptotic stability of error system $\mathcal{E}$.

4. Delay-Dependent Synchronization Criterion for Lur'e Systems

Now, we propose a new Lyapunov–Krasovskii function for delay-dependent stability criteria

$$\begin{aligned} V_3(e) &= e^T Pe + r_1 \int_{t-\tau}^{t} \int_{t+\theta}^{t} [e^T(s) A^T A e(s)] ds d\theta \\ &+ r_2 \int_{t-\tau}^{t} \int_{t+\theta-\tau}^{t} [e^T(s) F^T F e(s)] ds d\theta \\ &+ r_3 \int_{t-\tau}^{t} \int_{t+\theta}^{t} [\eta^T(Ce(s); \\ &y(s)) B^T B \eta(Ce(s); y(s))] ds d\theta \end{aligned} \tag{8}$$

where $P = P^T > 0$ and $r_1 > 0$, $r_2 > 0$, $r_3 > 0$. By taking this new Lyapunov function Eq. (8) we find a new sufficient condition for global asymptotic stability of error system $\mathcal{E}$.

Theorem 2. *Let $\Lambda = \mathrm{diag}\{\lambda_i\}$ be a diagonal matrix with $\lambda_i \geq 0$ for $i = 1, 2, \ldots, n_h$ and $\tau^* > 0$ be a scalar, then a sufficient condition for global asymptotic stability of error system ($\mathcal{E}$) from (8) for any constant time-delay τ satisfying $0 \leq \tau \leq \tau^*$ is given by the matrix inequality*

$$Y = \begin{bmatrix} Z & PB + kC\Lambda \\ B^T P + k\Lambda C^T & r_3 \tau B^T B - 2\Lambda \end{bmatrix} < 0 \tag{9}$$

where $Z = P(A+F) + (A+F)^T P + r_1 \tau A^T A + r_2 \tau F^T F + ((1/r_1) + (1/r_2) + (1/r_3)) \tau PFF^T P$.

Proof. We have

$$e(t-\tau) = e(t) - \int_{-\tau}^{0} \dot{e}(t+\theta) d\theta \tag{10}$$

Substituting $\dot{e}(t+\theta)$ into Eq. (10) one obtains

$$\begin{aligned} e(t-\tau) = e(t) - \int_{-\tau}^{0} \{&Ae(t+\theta) + B\eta(Ce(t+\theta); \\ &y(t+\theta)) + Fe(t+\theta-\tau)\} d\theta \end{aligned}$$

Substituting $e(t-\tau)$ back into Eq. (2)

$$\begin{aligned} \dot{e} &= (A+F)e + B\eta(Ce; y) \\ &- F\left\{ \int_{-\tau}^{0} \{Ae(t+\theta) + B\eta(Ce(t+\theta); \right. \\ &\left. y(t+\theta)) + Fe(t+\theta-\tau)\} d\theta \right\} \end{aligned} \tag{11}$$

Taking the time derivative of the Lyapunov function Eq. (8) along Eq. (11) we obtain

$$\begin{aligned} \dot{V}_3(e) = {} & e^T(P(A+F) + (A+F)^T P)e + e^T PB\eta(Ce; y) + \eta^T(Ce; y) B^T Pe \\ & - e^T PF \int_{-\tau}^{0} Ae(t+\theta) d\theta - e^T PF \int_{-\tau}^{0} B\eta(Ce(t+\theta); y(t+\theta)) d\theta \\ & - e^T PF \int_{-\tau}^{0} Fe(t+\theta-\tau) d\theta - \int_{-\tau}^{0} e^T(t+\theta) A^T F^T Pe d\theta \\ & - \int_{-\tau}^{0} \eta^T(Ce(t+\theta); y(t+\theta)) B^T F^T Pe d\theta \\ & - \int_{-\tau}^{0} e^T(t+\theta-\tau) F^T F^T Pe d\theta + r_1 \tau e^T A^T Ae + r_2 \tau e^T F^T Fe \\ & + r_3 \tau \eta^T(Ce; y) B^T B\eta(Ce; y) - \int_{-\tau}^{0} r_1 [e^T(t+\theta) A^T Ae(t+\theta)] d\theta \\ & - \int_{-\tau}^{0} r_2 [e^T(t+\theta-\tau) F^T Fe(t+\theta-\tau)] d\theta \\ & - \int_{-\tau}^{0} r_3 [\eta^T(Ce(t+\theta); y(t+\theta)) B^T B\eta(e(t+\theta); y(t+\theta))] d\theta \end{aligned}$$

We have the following inequalities from [Mahmoud, 2000, pp. 33–34, p. 401]

$$
\begin{aligned}
&-\int_{-\tau}^{0} e^T PFAe(t+\theta)d\theta \\
&\qquad -\int_{-\tau}^{0} e^T(t+\theta)A^T F^T Ped\theta \\
&\qquad \le r_1^{-1}\int_{-\tau}^{0} e^T PFF^T Ped\theta \\
&\qquad\quad + r_1\int_{-\tau}^{0} e^T(t+\theta)A^T Ae(t+\theta)d\theta
\end{aligned}
$$

and

$$
\begin{aligned}
&-\int_{-\tau}^{0} e^T PFFe(t+\theta-\tau)d\theta \\
&\qquad -\int_{-\tau}^{0} e^T(t+\theta-\tau)F^T F^T Ped\theta \\
&\le r_2^{-1}\int_{-\tau}^{0} e^T PFF^T Ped\theta \\
&\qquad + r_2\int_{-\tau}^{0} e^T(t+\theta-\tau)F^T Fe(t-\tau+\theta)d\theta
\end{aligned}
$$

and also

$$
\begin{aligned}
&-\int_{-\tau}^{0} e^T PFB\eta(e(t+\theta);\, y(t+\theta))d\theta \\
&\qquad -\int_{-\tau}^{0} \eta^T(e(t+\theta);\, y(t+\theta))B^T F^T Ped\theta \\
&\le r_3^{-1}\int_{-\tau}^{0} e^T PFF^T Ped\theta + r_3\int_{-\tau}^{0} \eta^T(e(t+\theta); \\
&\qquad y(t+\theta))B^T B\eta(e(t+\theta);\, y(t+\theta))d\theta
\end{aligned}
$$

Using the above inequalities, we get

$$
\begin{aligned}
\dot{V}_3(e) \le{}& e^T Ze + e^T PB\eta(Ce;\, y) + \eta^T(Ce;\, y)B^T Pe \\
&+ r_3\tau\eta^T(Ce;\, y)B^T B\eta(Ce;\, y)\,.
\end{aligned}
$$

Applying the S-procedure, by using the inequalities from the nonlinearities, gives

$$
\begin{aligned}
\dot{V}_3(e) \le{}& e^T Ze + e^T PB\eta(Ce;\, y) + \eta^T(Ce;\, y)B^T Pe \\
&+ r_3\tau\eta^T(Ce;\, y)B^T B\eta(Ce;\, y) \\
&+ \sum_i \lambda_i\eta_i(\eta_i - kc_{*_i}^T e) \\
\le{}& \xi^T Y\xi
\end{aligned}
$$

where $\xi = [e;\, \eta]$.

If $\dot{V}_3 < 0$ for τ^*, then the following inequality is satisfied for all $\tau \in [0,\, \tau^*]$

$$
\begin{aligned}
&\tau^*\left\{e^T\left(r_1A^TA + r_2F^TF + \left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)\Upsilon\right)e \right.\\
&\qquad \left. + r_3\eta^T(Ce;\, y)B^TB\eta(Ce;\, y)\right\} \\
&\quad \le -e^T(P(A+F) + (A+F)^TP)e \\
&\qquad - \eta^T(Ce;\, y)B^TPe - e^TPB\eta(Ce;\, y)
\end{aligned}
$$

where $\Upsilon = PFF^TP$. ■

The matrix inequality (9) includes information on the delay. Therefore, this result is a delay-dependent stability criterion for synchronization. Note that a necessary condition for synchronization here is that $A+F$ must be strictly Hurwitz.

5. Examples

We illustrate Theorem 2 for the examples of Chua's circuits, n-scroll attractors and hyperchaotic attractors.

5.1. *Chua's circuit*

Let us take the following representation of Chua's Circuit

$$
\begin{cases}
\dot{x} = \alpha(y - h(x)) \\
\dot{y} = x - y + z \\
\dot{z} = -\beta y
\end{cases}
$$

with nonlinear characteristic

$$
h(x) = m_1x + \frac{1}{2}(m_0 - m_1)(|x+c| - |x-c|)
$$

and parameters $m_0 = -(1/7)$, $m_1 = (2/7)$, $\alpha = 9$, $\beta = 14.28$, $c = 1$ in order to obtain the double scroll attractor [Chua *et al.*, 1986; Madan, 1993]. The system can be represented in Lur'e form by

$$
A = \begin{bmatrix} -\alpha m_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},
$$

$$
B = [-\alpha(m_0 - m_1);\, 0;\, 0]\,, \quad C = [1 \quad 0 \quad 0]
$$

and $\sigma(\xi) = (1/2)(|\xi+c| - |\xi-c|)$ belonging to sector $[0,\, k]$ with $k = 1$.

The matrix inequality (9) is employed as follows

$$\min_{\Lambda,P,F,r_1,r_2,r_3} \lambda_{\max}[Y(P,\, F,\, \Lambda,\, r_1,\, r_2,\, r_3)]$$

$$\text{such that} \begin{cases} P = P^T > 0 \\ \Lambda \geq 0 \\ r_1,\, r_2,\, r_3 > 0 \end{cases} \tag{12}$$

according to e.g. Suykens *et al.* [1997]. Sequential quadratic programming has been applied in Matlab's optimization toolbox for different values of τ. The matrix $G = [6.0229;\ 1.3367;\ -2.1264]$ stabilizes the error system for $\tau \in [0 \quad 0.039]$. No feasible points were found for $\tau > 0.039$. In the experiments $H = [1 \quad 0 \quad 0]$ was chosen which means that the master system is connected to the slave system with the first state variable only.

The synchronization scheme (Fig. 1) has been modeled in Matlab Simulink and the following simulation results have been obtained. The first observation is given in Fig. 2 for maximum delay $\tau^* = 0.039$. A second observation is given in Fig. 3 for a smaller delay $\tau = 0.01$ than the maximum delay. During the simulations it has been observed that two Chua's circuits synchronize until $\tau = 0.21$ for the same feedback matrix (Fig. 4). Synchronization could not be observed for τ bigger than 0.21. The simulation result for $\tau = 0.22$ is given in Fig. 5. As initial conditions of the master and slave systems were taken $x(0) = [-0.2;\ -0.33;\ 0.2]$, $y(0) = [0.5;\ -0.1;\ 0.66]$ in the simulations.

5.2. *5-scroll attractors*

A more complete family of n-scroll [Suykens & Vandewalle, 1993] instead of double and n-double scroll attractors has been obtained from a generalized Chua's circuit proposed in [Suykens *et al.*, 1997]. An experimental confirmation of 3- and 5-scroll attractors has been given by Yalçin *et al.* [2000]. The n-scroll circuit is given by

$$\begin{cases} \dot{x} = \alpha(y - h(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta y \end{cases} \tag{13}$$

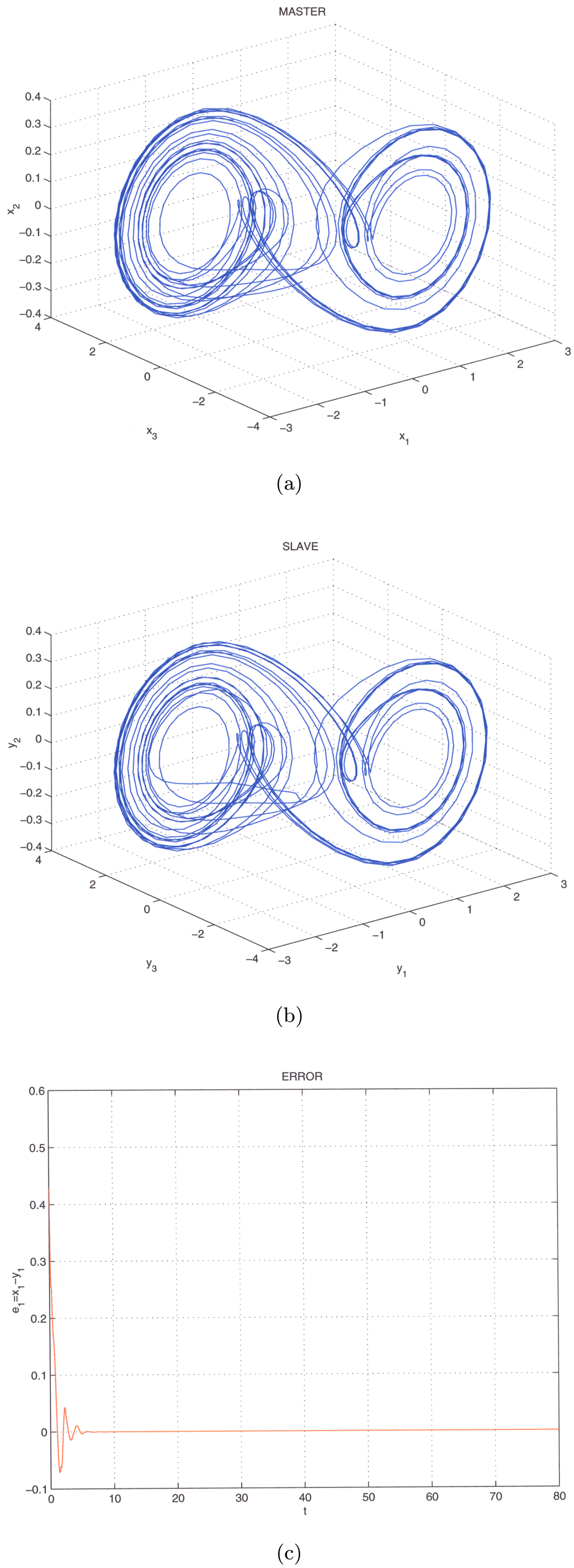
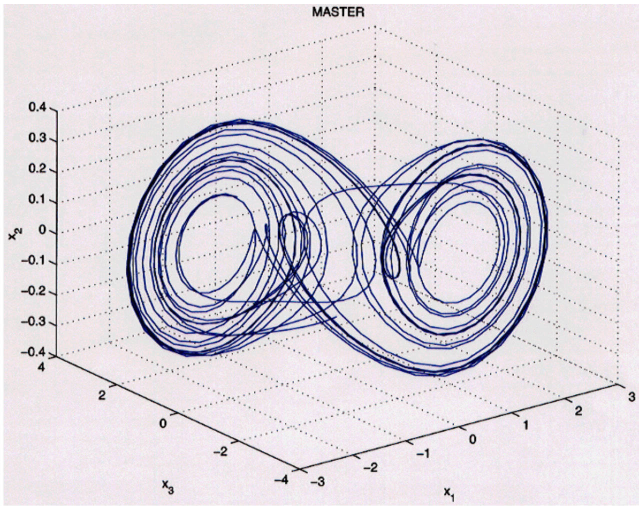
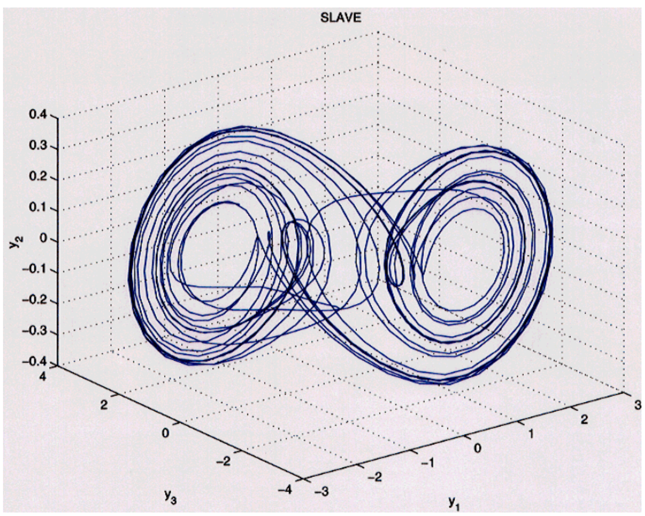


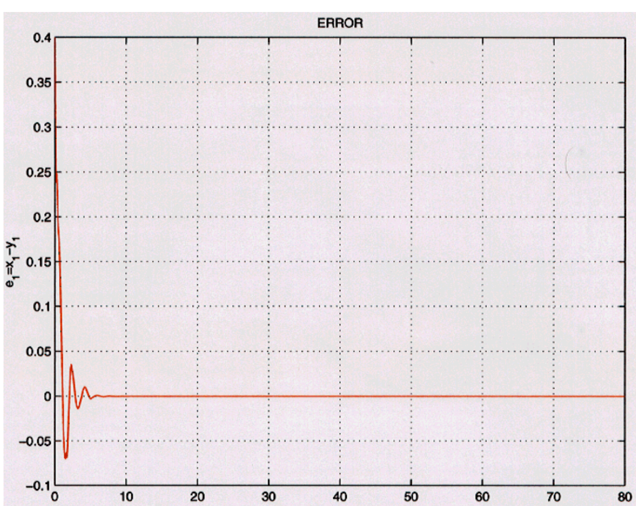
Fig. 2. Simulation result for master–slave synchronization of two identical Chua's circuits. The master system is coupled to the slave system with the first state variable and delay ($\tau = 0.039$): Three-dimensional view on the double scroll attractor generated for (a) master system and (b) slave system. (c) Error signal $(x_1(t) - y_1(t))$ with respect to time.



(a)

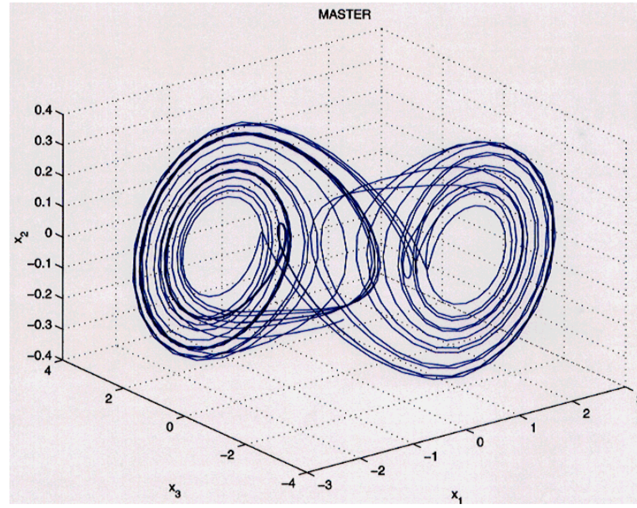


(b)

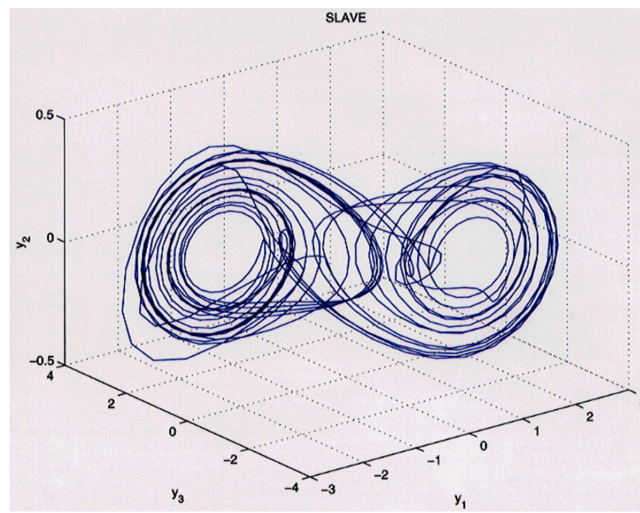


(c)

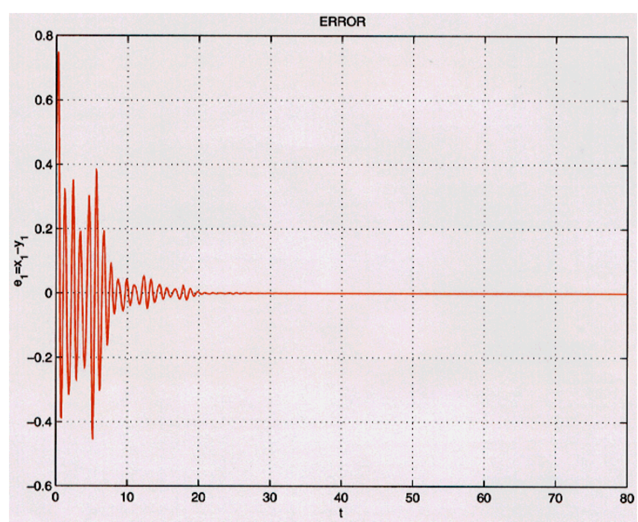
Fig. 3. Delay ($\tau = 0.01$).



(a)

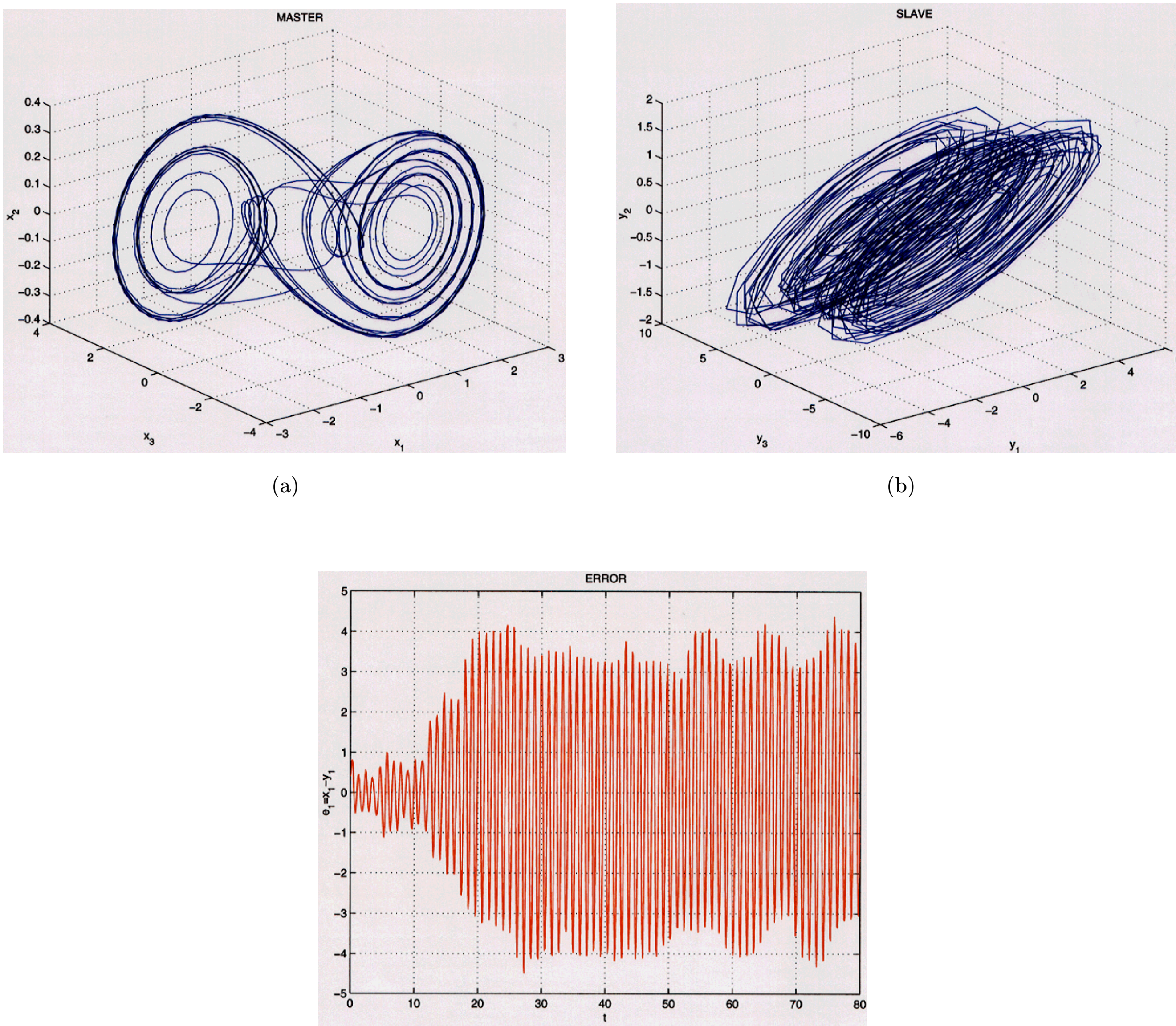


(b)



(c)

Fig. 4. Delay ($\tau = 0.21$).



(a) (b) (c)

Fig. 5. Delay ($\tau = 0.22$).

with a nonlinear characteristic having additional break points

$$h(x) = m_{2q-1}x + \frac{1}{2}\sum_{i=1}^{2q-1}(m_{i-1} - m_i)(|x + c_i| - |x - c_i|) \tag{14}$$

where q denotes a natural number [Suykens *et al.*, 1997]. Here, we will consider the 5-scroll attractor, which is obtained for $m = [0.9/7,\ -3/7,\ 3.5/7,\ -2.7/7,\ 4/7,\ -2.4/7]$, $c = [1,\ 2.15,\ 3.6,\ 6.2,\ 9]$, $\alpha = 9$, $\beta = 14.28$. The system is represented in another Lur'e form than given in [Suykens *et al.*, 1997], based upon [Güzeliş, 1993] with $-(1+\delta)x + f(x) = -h(x)$ and

$$A = \begin{bmatrix} -\alpha(1+\delta) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix},$$

$$B = [-\alpha;\ 0;\ 0], \quad C = [1 \quad 0 \quad 0]$$

and $\delta = 1$ where $f(x)$ belong to sector $[0,\ k]$ with $k = 2.5$. This representation results in $n_h = 1$.

The same optimization procedure has been applied as for Chua's circuit. For $\tau > 0.04$ no feasible points were found such that Y negative definite. The feedback vector $G = [0.6945;\ 0.7002;\ -0.1645]$ found for $\tau = 0.04$ has stabilized the error system between $\tau \in [0 \quad 0.04]$. In the experiments $H = [1 \quad 0 \quad 0]$ was chosen. Figure 6 shows simulation results for the maximum delay $\tau^* = 0.04$. Results for a smaller delay $\tau = 0.01$ are given in Fig. 7. In Matlab Simulink, synchronization has been observed until $\tau = 0.14$ (Fig. 8) for the same feedback matrix. Synchronization has not been observed when the delay was bigger than 0.14. Figure 9 shows results for $\tau = 0.15$. During the simulations the initial conditions of master and slave systems are taken as $x(0) = [-1.7;\ -0.4;\ -0.2]$, $y(0) = [0.2;\ -0.2;\ 0.33]$.

5.3. *Hyperchaotic system*

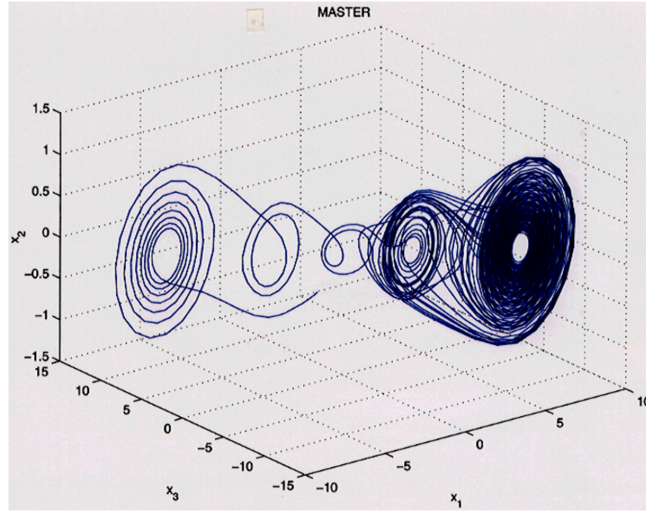
We consider the following system which consists of two unidirectionally coupled Chua circuits

$$\begin{cases} \dot{x}_1 = a(x_2 - h(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -bx_2 \\ \dot{x}_4 = a(x_5 - h(x_5)) \\ \dot{x}_5 = x_4 - x_5 + x_6 + K(x_5 - x_2) \\ \dot{x}_6 = -bx_5 \end{cases} \tag{15}$$

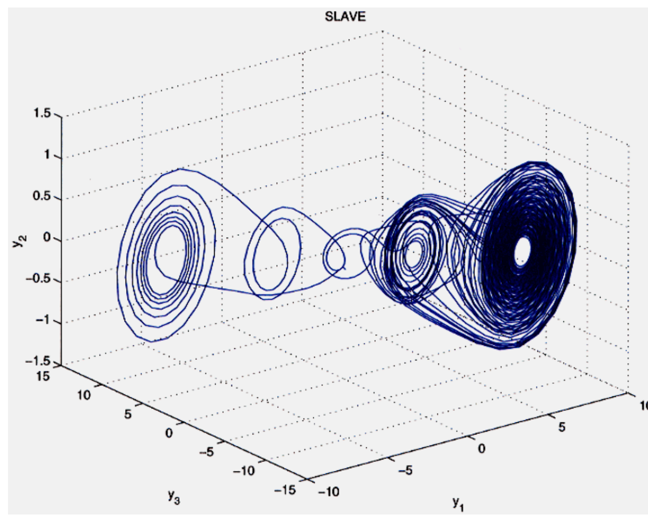
with nonlinear characteristic

$$h(x_i) = m_1 x_i + \frac{1}{2}(m_0 - m_1)(|x_i + c| - |x_i - c|),\ i = 1,\, 4 \tag{16}$$

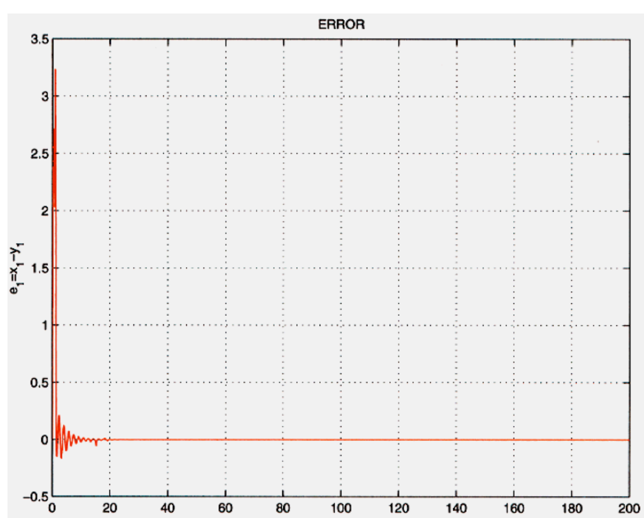
and parameters $m_0 = -(1/7)$, $m_1 = 2/7$, $a = 9$, $b = 14.28$, $c = 1$, $K = 0.01$. The system exhibits hyperchaotic behavior with a double–double scroll attractor [Kapitaniak & Chua, 1995]. This system was represented in Lur'e form by



(a)

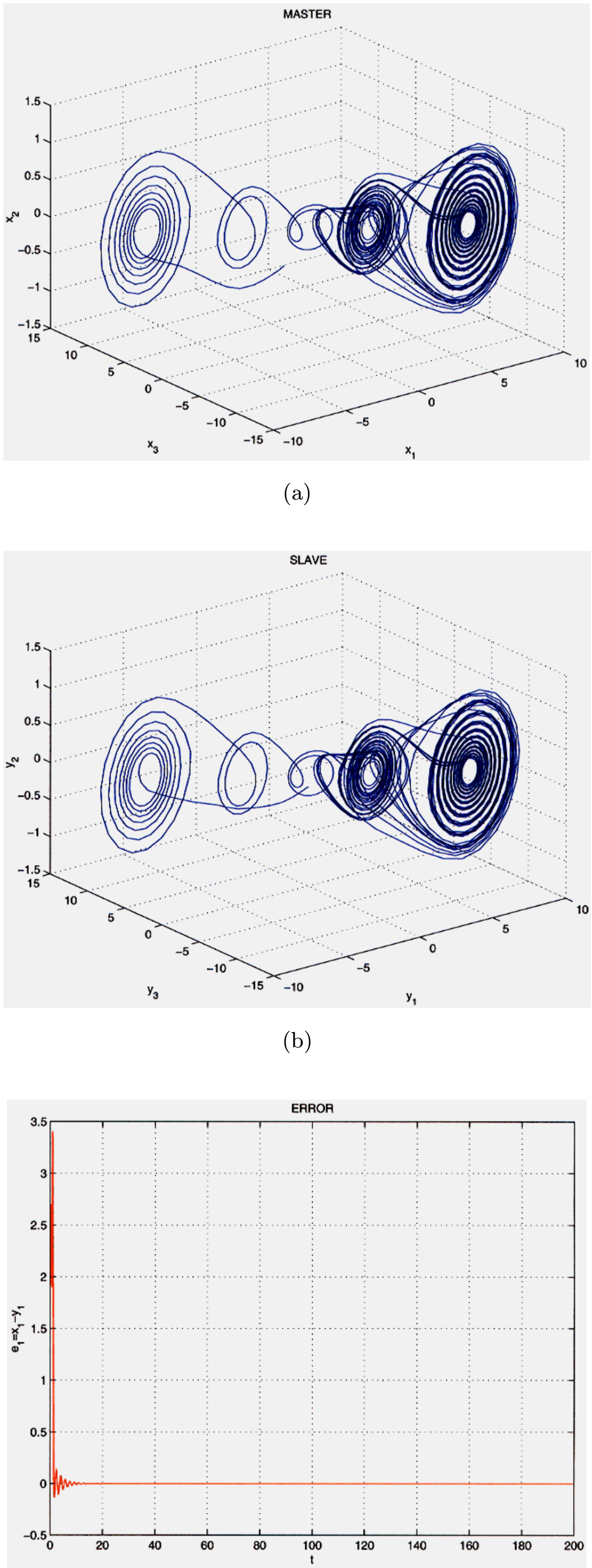


(b)



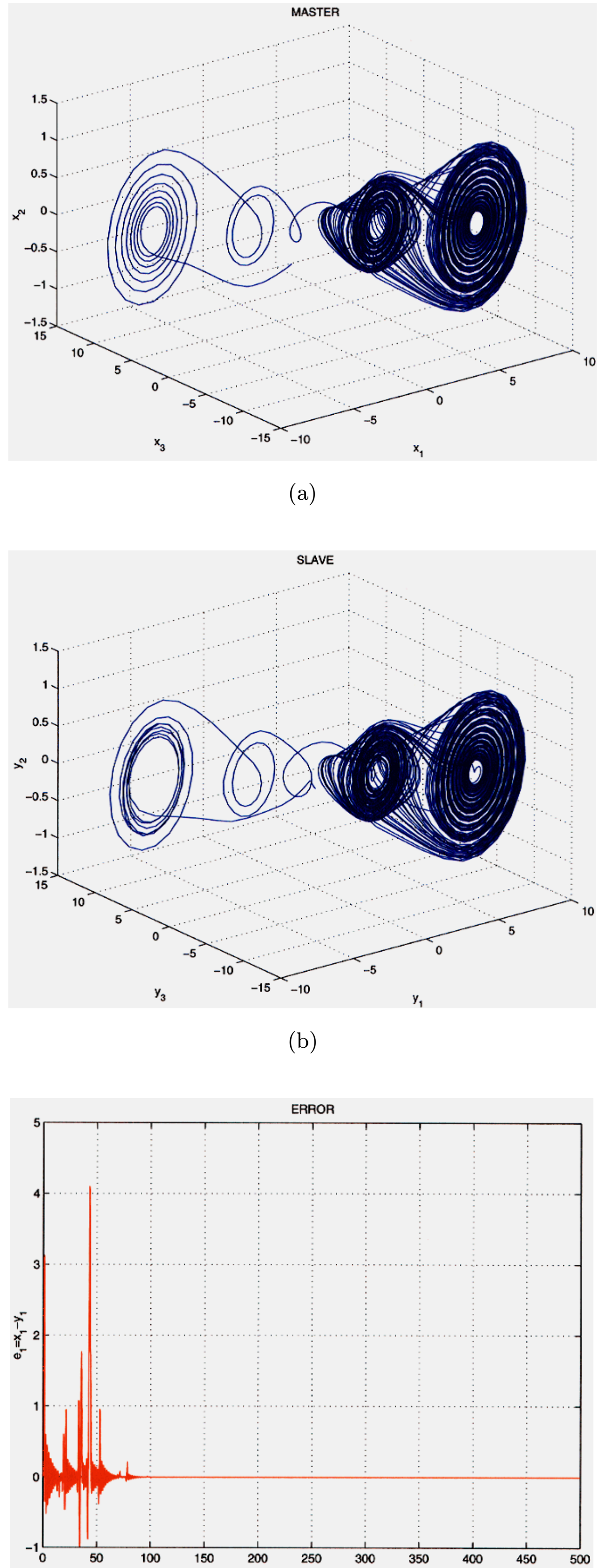
(c)

Fig. 6. Simulation result for master–slave synchronization of two identical 5-scroll attractors. The master system is coupled to the slave system with the first state variable and delay ($\tau = 0.04$): Three-dimensional view on the 5-scroll attractors generated at (a) master system and (b) slave system. (c) Error signal ($x_1(t) - y_1(t)$) with respect to time.



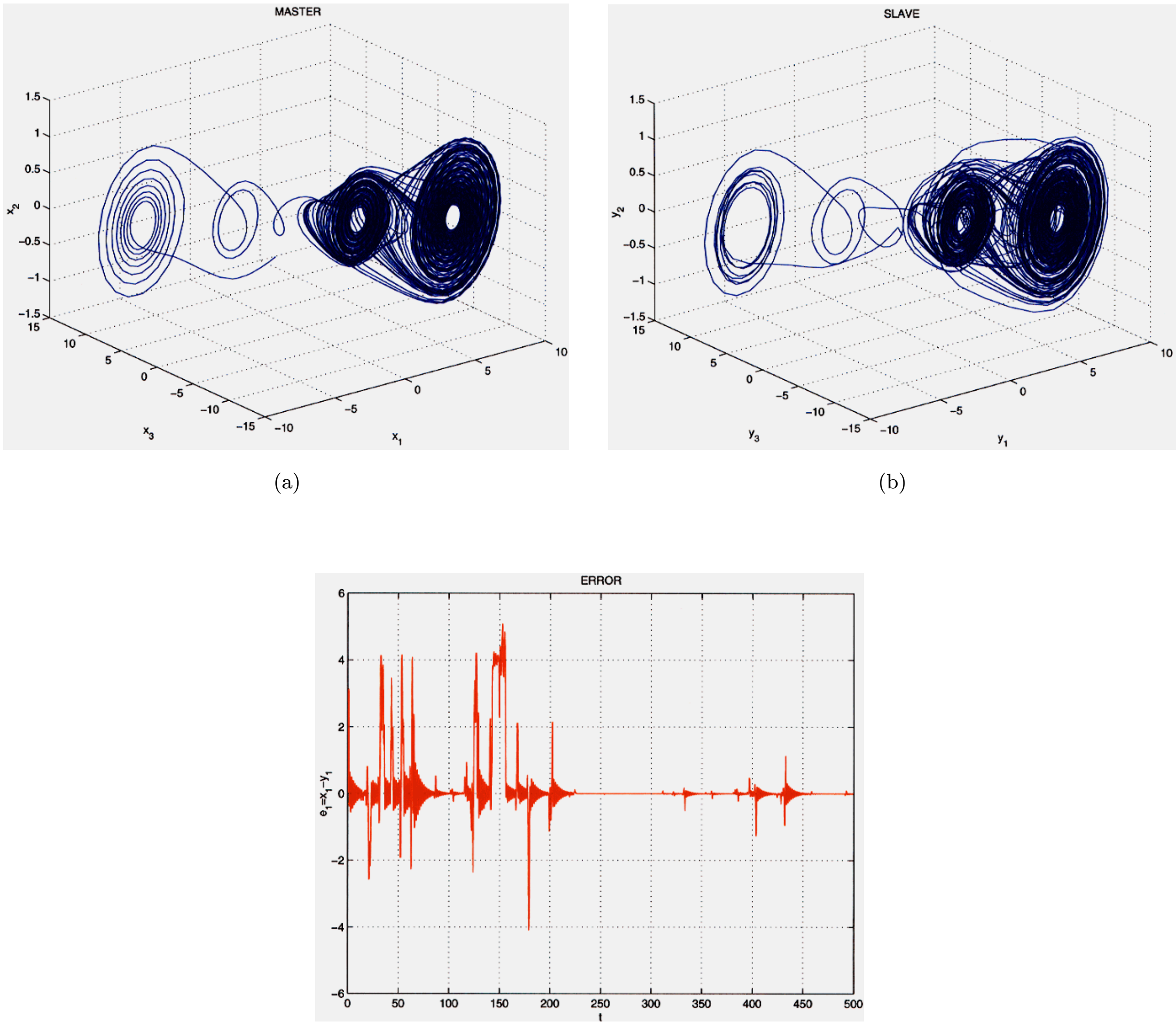
(a) (b) (c)

Fig. 7. Delay ($\tau = 0.01$).



(a) (b) (c)

Fig. 8. Delay ($\tau = 0.14$).



(a) (b)

(c)

Fig. 9. Delay ($\tau = 0.15$).

Suykens *et al.* [1998]. $h(x)$ belongs to sector $[0, 1]$ and

$$A = \begin{bmatrix} -am_1 & a & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & -b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -am_1 & a & 0 \\ 0 & -K & 0 & 1 & -1+K & 1 \\ 0 & 0 & 0 & 0 & -b & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -a(m_0 - m_1) & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -a(m_0 - m_1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Sequential quadratic programming has been applied

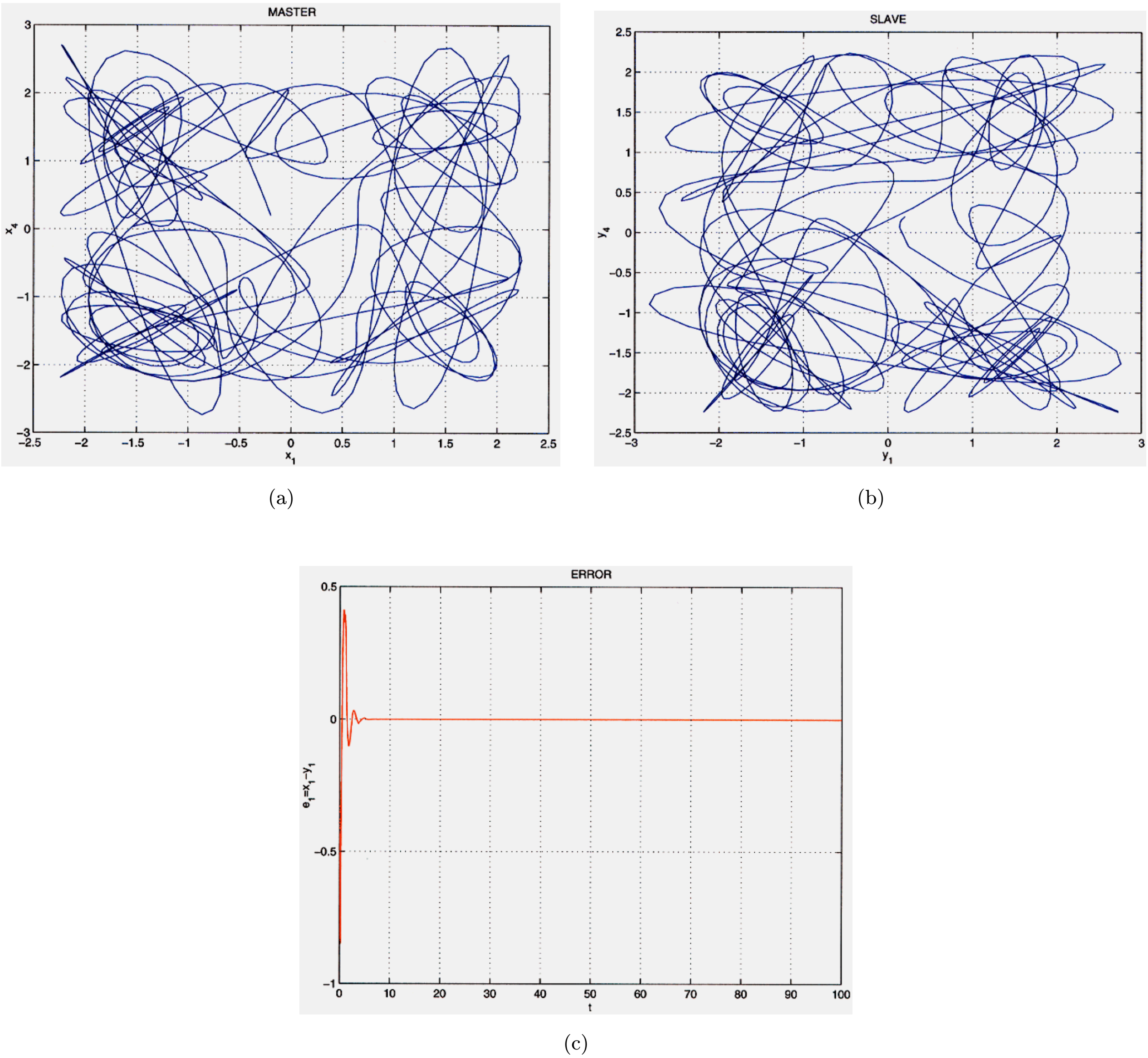
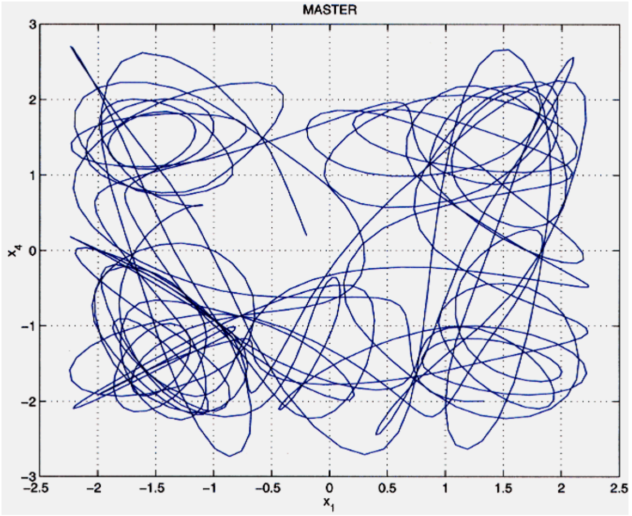


Fig. 10. Simulation result for master–slave synchronization of two identical hyperchaotic systems. The master system is coupled to the slave system with the first and fourth state variable and delay ($\tau = 0.038$): Double-double scroll attractor generated for (a) master system (projection onto the $x_1 - x_4$ plane) and (b) slave system (projection onto the $y_1 - y_4$ plane). (c) Error signal $(x_1(t) - y_1(t))$ with respect to time.

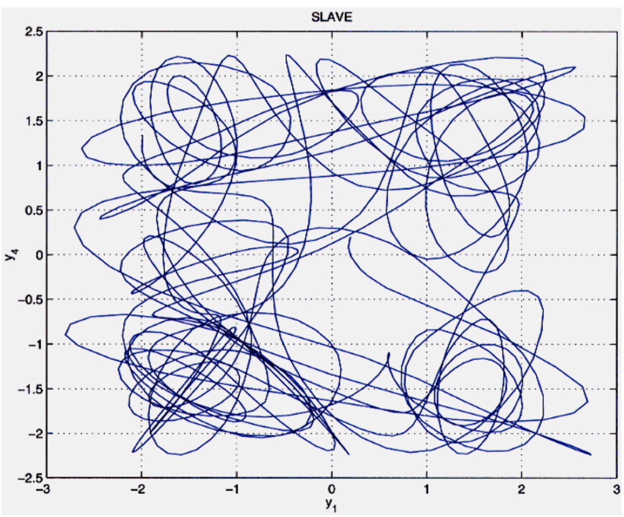
similar to the other examples. For $\tau > 0.038$ no feasible points were found such that Y is negative definite. The feedback matrix

$$G = \begin{bmatrix} 7.6909 & 2.1313 & -3.9865 & -0.3491 & 0.1811 & -0.5180 \\ -1.0520 & 0.0835 & 0.3455 & 8.0879 & 1.8021 & -4.8256 \end{bmatrix}$$

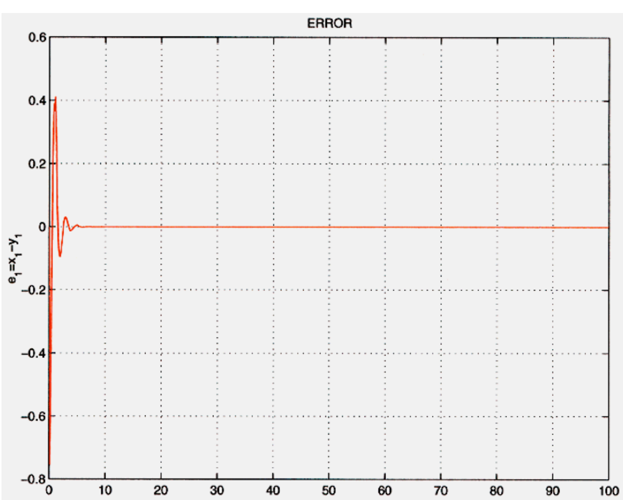
which is found for the maximum delay $\tau^* = 0.038$, has stabilized the error system for $\tau \in [0 \quad 0.038]$. In the experiments $H = [1\ 0\ 0\ 0\ 0\ 0;\ 0\ 0\ 0\ 1\ 0\ 0]$ has been taken. In Fig. 10 the result is given for the maximum obtained delay $\tau^* = 0.038$. Simulation results for a smaller delay $\tau = 0.01$ are shown in Fig. 11. In Matlab Simulink, synchronization has been observed until $\tau = 0.17$ (Fig. 12) for



(a)

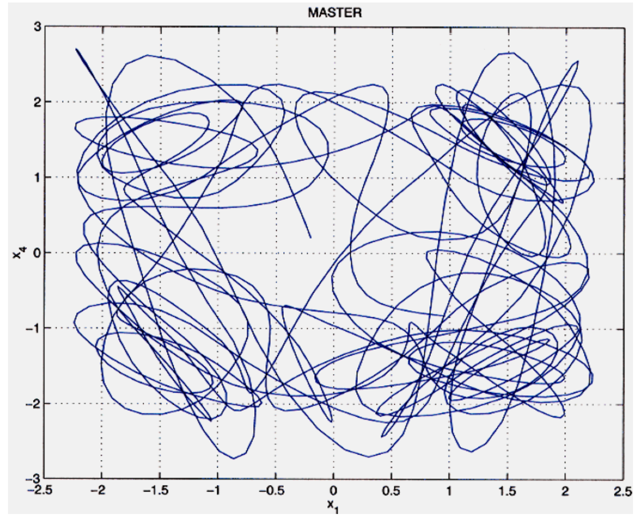


(b)

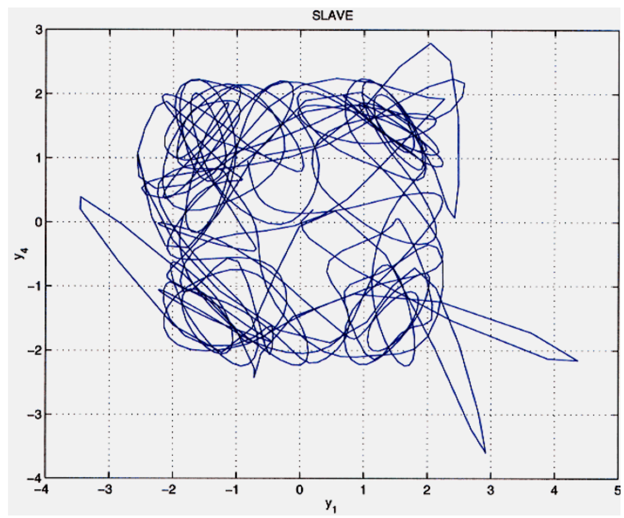


(c)

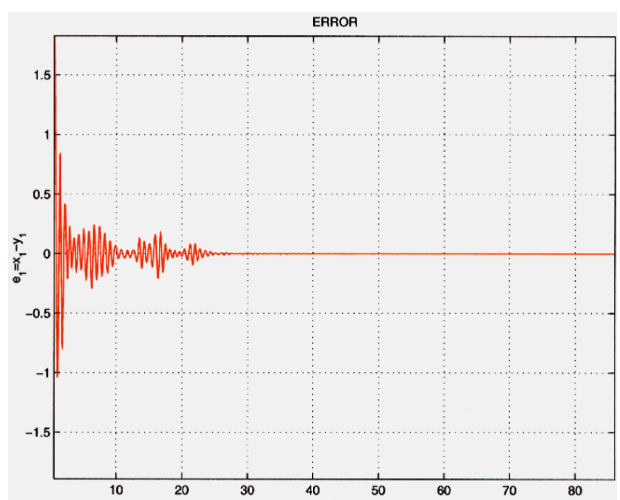
Fig. 11. Delay ($\tau = 0.01$).



(a)

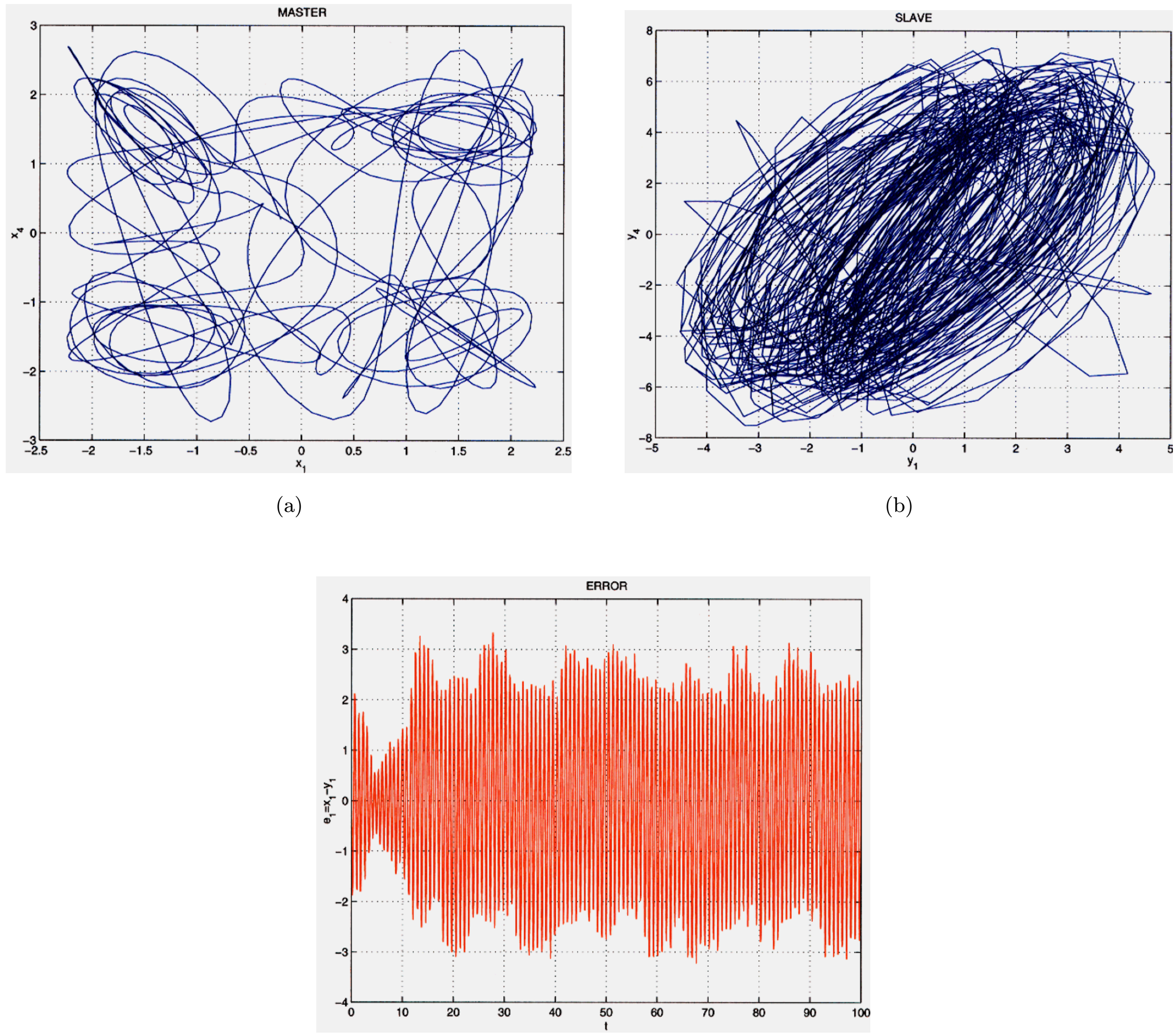


(b)



(c)

Fig. 12. Delay ($\tau = 0.17$).



(a) (b)

(c)

Fig. 13. Delay ($\tau = 0.18$).

the same feedback matrix. Synchronization has not been observed when the delay was larger than 0.17. Figure 13 shows the result for $\tau = 0.18$. During the simulations, the initial conditions of the master and slave systems are taken as $x(0) = [-0.2;\ -0.2;\ -0.33;\ 0.2;\ 0.9;\ 0.33]$, $y(0) = [0.2;\ -0.2;\ 0.33;\ 0.2;\ -0.2;\ 0.33]$.

6. Conclusion

In this paper a master–slave synchronization scheme for Lur'e systems has been investigated for a known delay existing between master and slave systems. Synchronization criteria have been classified into two categories: delay-independent and delay-dependent synchronization criteria. Sufficient conditions for global asymptotic stability of the error system have been given for these two categories. Delay-independent criteria have been applied to Chua's circuit, 5-scroll attractors and hyperchaotic attractors but feasible points could not be found. Therefore, a new Lyapunov–Krasovskii function has been introduced, which gives a delay-dependent synchronization criterion.

This condition has been successfully applied to the chaotic and hyperchaotic systems.

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