International Journal of Bifurcation and Chaos, Vol. 12, No. 1 (2002) 129–134
© World Scientific Publishing Company

FURTHER INVESTIGATION OF HYSTERESIS IN CHUA'S CIRCUIT

J. BORRESEN and S. LYNCH
Department of Computing and Mathematics,
The Manchester Metropolitan University,
Manchester, M1 5GD, UK

Received February 1, 2001; Revised April 1, 2001

For a system to display bistable behavior (or hysteresis), it is well known that there needs to be a nonlinear component and a feedback mechanism. In the Chua circuit, nonlinearity is supplied by the Chua diode (nonlinear resistor) and in the physical medium, feedback would be inherently present, however, with standard computer models this feedback is omitted. Using Poincaré first return maps, bifurcations for a varying parameter in the Chua circuit equations are investigated for both increasing and decreasing parameter values. Evidence for the existence of a small bistable region is shown and numerical methods are applied to determine the behavior of the solutions within this bistable region.

1. Introduction

One important area of study in the field of nonlinear dynamics that attracts much interest from both mathematicians and scientists of varying disciplines is the study of bistability. It is possible for certain physical and mathematical systems to display multistable behavior, where one of a possible number of steady-states can be attained depending upon the initial conditions chosen. Of special interest are bistable regions, where for a given set of parameter values there exist two distinct stable solution outcomes. In order for a system to show bistable phenomena it must contain both nonlinear and feedback processes; the steady-state reached in the bistable region is then dependent upon the history of the system. The maximum number of steady-states possible for certain continuous systems is considered in [Christopher & Lynch, 1999], and multistability and bistability are discussed in some detail in [Lynch, 2000, 2001].

Bistable behavior, though not often easy to locate, has been proven to exist in numerous physical forms. It is present in mechanical systems [Lynch & Christopher, 1999], electromagnetism [Szczyglowski, 2001], chemical kinetics [Scott, 1994], astrochemical cloud models [Nejad, 1999] and nonlinear optics [Lynch & Steele, 2000; Lynch *et al.*, 1998; Steele *et al.*, 1997], where optical bistability has potential applications in high speed all-optical signal processing and all-optical computing.

A general introduction to Chua's circuit is discussed in some detail in [Madan, 1993]. Recently, Kal'yanov [1998] proposed the existence of hysteresis in a system similar to the original Chua's equations by identifying the characteristic "step" in bifurcation maps using maximal radius analysis for a varying parameter. In this paper, a feedback mechanism is applied to demonstrate hysteresis in computer models of the original Chua's circuit equations. Poincaré first-return maps are used along with a feedback process where a parameter is ramped up and then ramped down. Bifurcation diagrams are plotted which clearly show the bistable behavior. Chua's circuit and the governing coupled differential equations are well documented and can be described in various forms. In this paper, only the simplest of scaling transformations will be used to allow for greater transparency and ease of calculations.

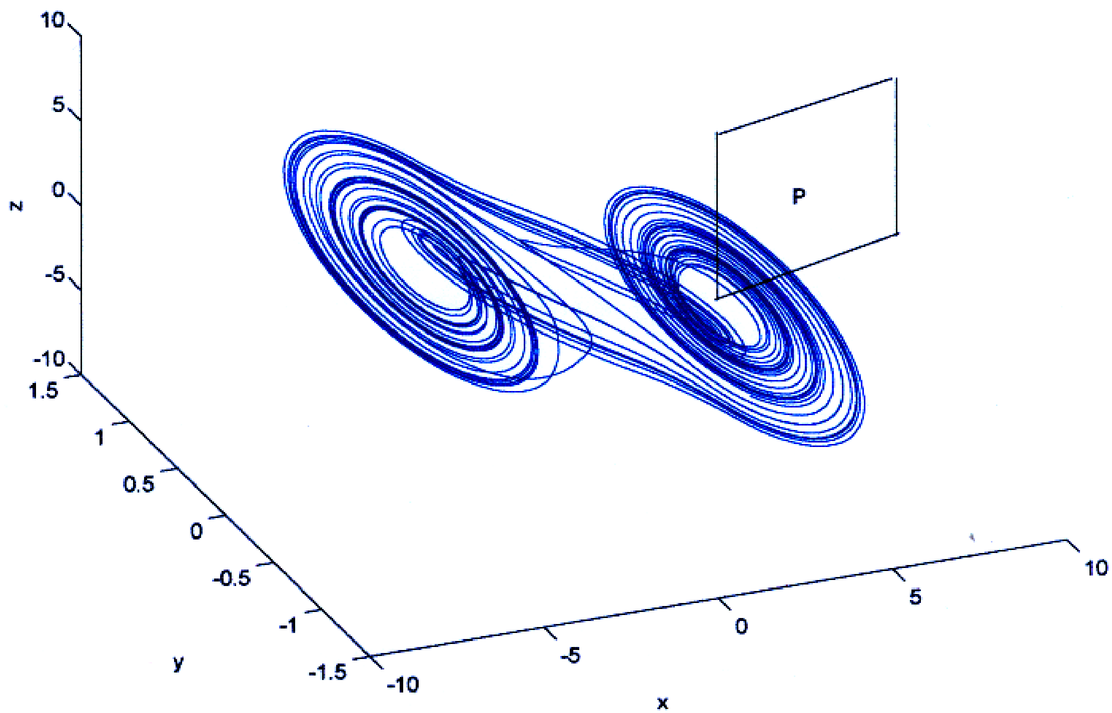


Fig. 1. Chua circuit simulation for $a = 0.57$, $b = 0.1$, $c = 0.5$, $d = -0.5$, $e = -0.8$ showing the double scroll attractor, generated using MATLAB. A Poincaré section (P) is also shown.

It is a characteristic of Chua's circuit that wildly varying qualitative behavior can be obtained from only minor adjustments to the circuit, this leads to the possibility that bistable regions will have been overlooked in experimentation as the region of bistability is very small. Also, as suggested by Elgar *et al.* [1998] numerical approximations to the actual circuit may have a tendency to retain harmonics which may be damped in electronics due to the physical medium.

It is the aforementioned variety in possible types of behavior that lead to major problems in analyzing the Chua system. The solutions can vary from periodic limit cycle behavior, to quasi-periodic motion around a torus, to chaotic movement around the now familiar double scroll attractor (Fig. 1). Calculating a Poincaré section, as shown in Fig. 1, through this three-dimensional system, will inherently involve lengthy computer runs and some trial and error with the numerical procedures to obtain accurate and useful results.

2. Chua Circuit Equations and Parameter Values

The transformed Chua circuit (in dimensionless form) can have describing equations as follows:

$$\begin{aligned}\dot{x} &= a(y - x) - f(x)\\ \dot{y} &= b(a(x - y) + z)\\ \dot{z} &= -cy\,,\end{aligned}$$

where $f(x)$ represents the piecewise-linear function

$$f(x) = dx + \frac{1}{2}(e - d)(|x + 1| - |x - 1|)\,,$$

characterizing the nonlinear resistance.

In order to analyze any bistable behavior of a system it is only necessary to vary one parameter of the governing equations. Through experimentation it has been found that varying the parameter a, the resistance in the circuit, and fixing the other parameters as follows yields successful results:

$$b = 0.1, \quad c = 0.5, \quad d = -0.5$$

and

$$e = -0.8\,.$$

In all further analysis the parameters b, c, d and e will be as above.

3. Bifurcation Diagram by Generation of First Poincaré Returns

A bifurcation map (Fig. 2) for varying a in the interval $0.535 \leq a \leq 0.575$, was produced using a fourth-order Runge–Kutta scheme with a varying step length h. To guarantee the accuracy of the first returns a value of $h = 0.001$ was used where the trajectory was suitably close to the plane P (see Fig. 1), here using $\pi/8 < \arctan(y/x) < 3\pi/8$. The radius of the trajectory at P was estimated by averaging the solution values from the two discrete points on either side of this plane. Where the trajectory moved away from the plane P, a step length of $h = 0.05$ was used to increase the speed at which the bifurcation map was generated. In addition, only those points where the trajectory crossed the plane to the right of the center of the double scroll were recorded in order to prevent trajectories being included where the solution had not moved through a full cycle.

With the generation of each first return, the value of a was increased by 0.001 and the system integrated forwards until the trajectory again crossed P, when a was once again increased. This scheme (an iterative method for Poincaré returns — see [Lynch, 2000]) closely simulates the attributes of the physical model. This method was applied by linearly increasing a and then linearly decreasing a, continually feeding back the previous results as the initial conditions of the system. The results of ramping up and then ramping down the parameter value a are shown in Fig. 2. In physical applications this would correspond to increasing and then decreasing a resistance in the circuit. There is a small region of bistability in the approximate range, $0.542 < a < 0.548$. As the parameter a is increased, the steady-state remains on the lower branch until there is a step up to the upper branch, and as a is increased further the circuit enters a chaotic regime through a quasi-periodic route to chaos. As the parameter a is decreased, the system passes through the chaotic, quasi-periodic and periodic regions to join the upper branch of the bistable cycle. As a is decreased further there is still a slight ringing in the bistable cycle as the solution overshoots before returning once more to the steady-state at $a \approx 0.542$.

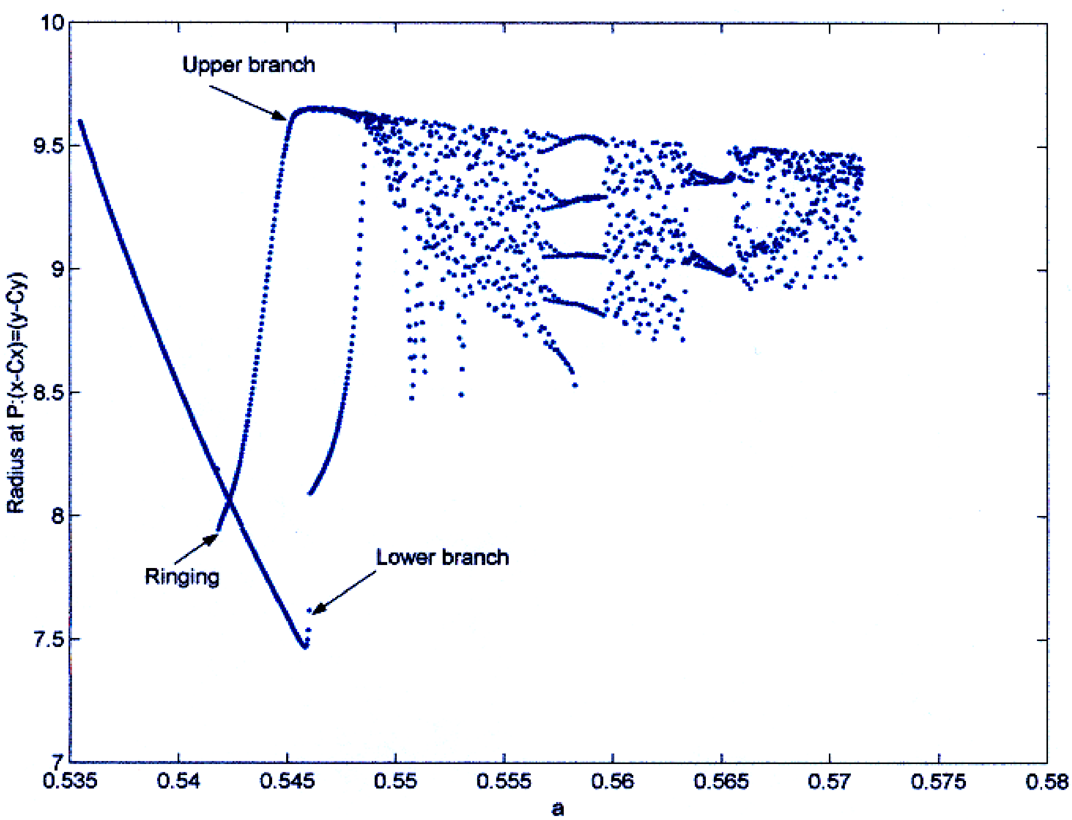


Fig. 2. Bifurcation map for Chua's circuit, with feedback included. There is an isolated counterclockwise bistable region.

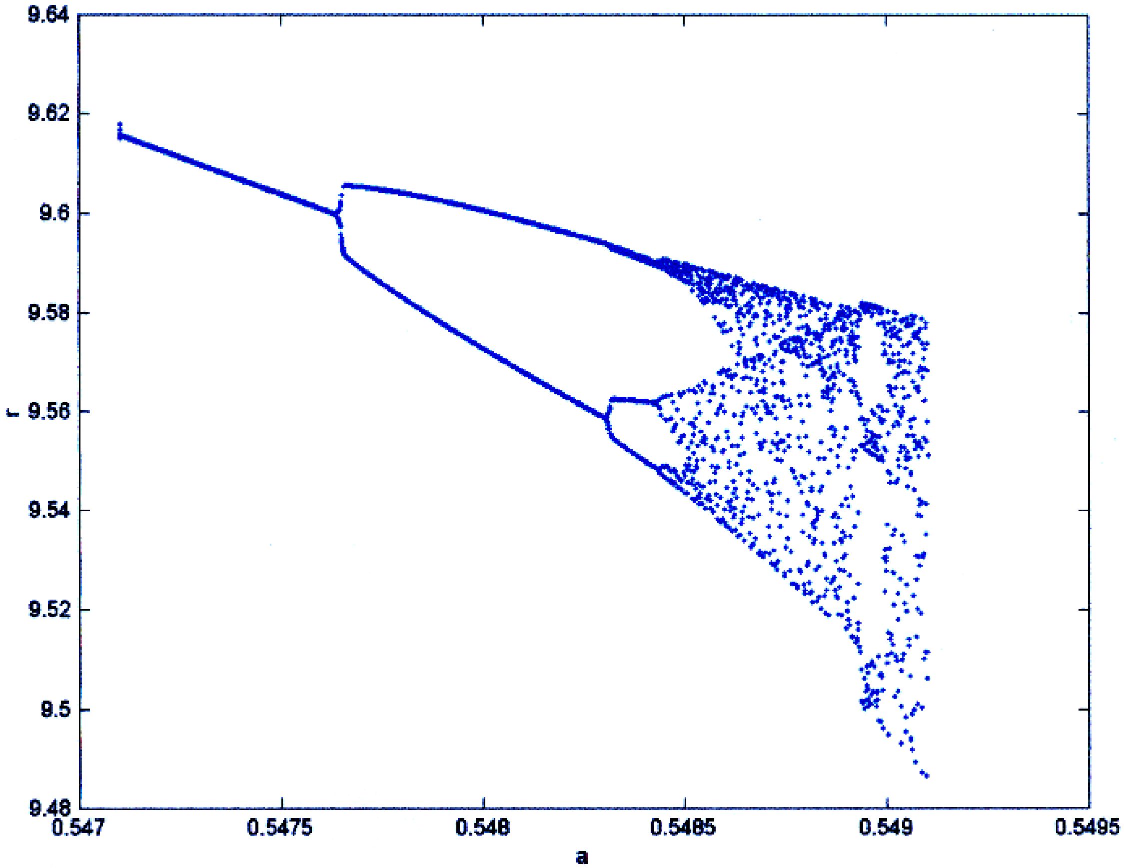
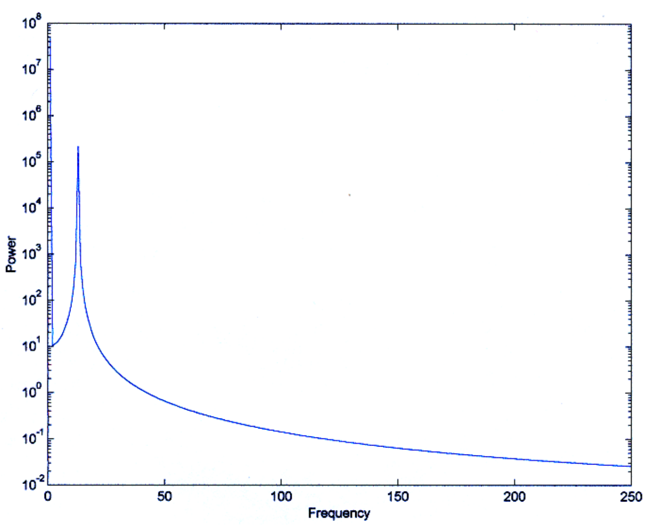
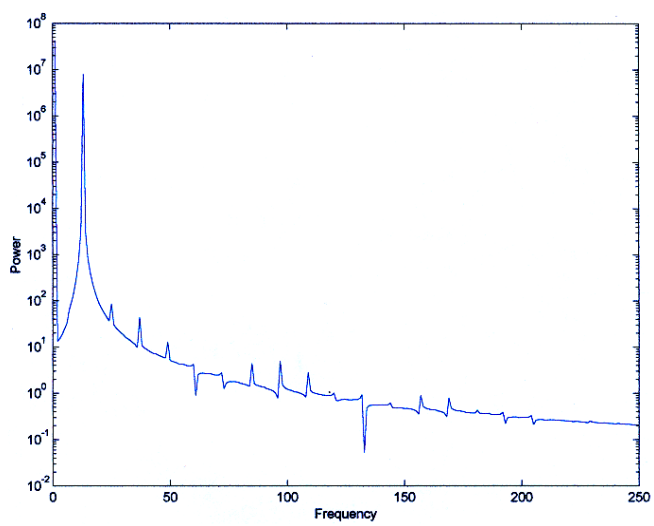


Fig. 3. Bifurcation diagram showing a quasi-periodic route to chaotic behavior.



(a)



(b)

Fig. 4. Power spectra for Chua circuit simulations: (a) $a = 0.544$, (b) $a = 0.546$, (c) $a = 0.548$ and (d) $a = 0.55$.

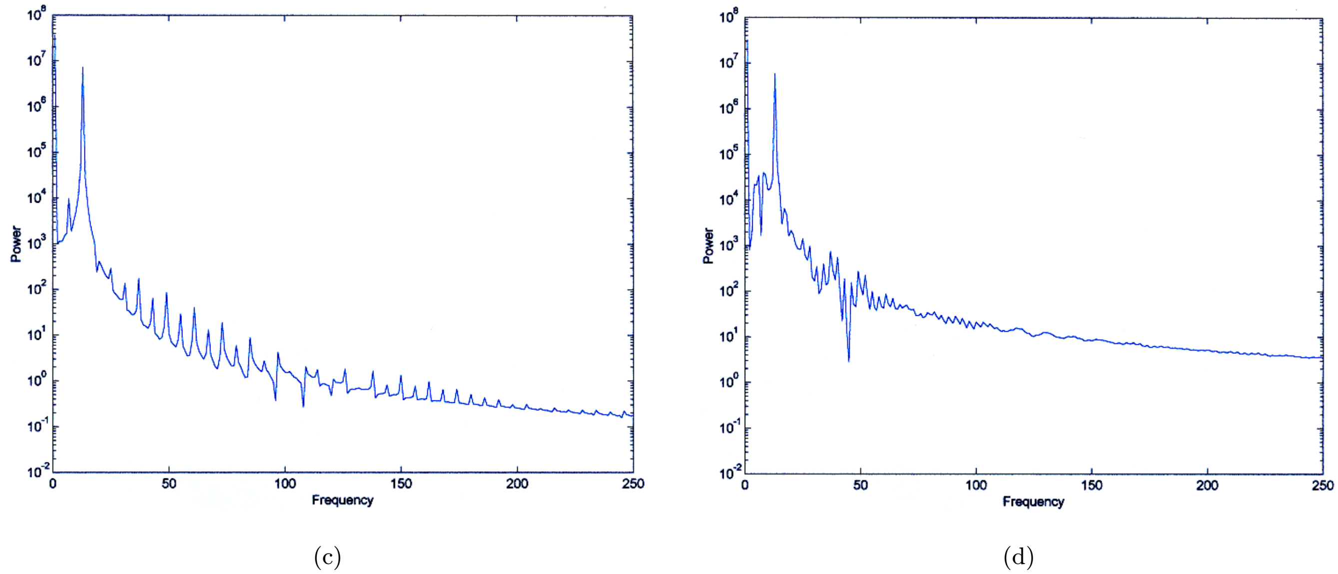


Fig. 4. (*Continued*)

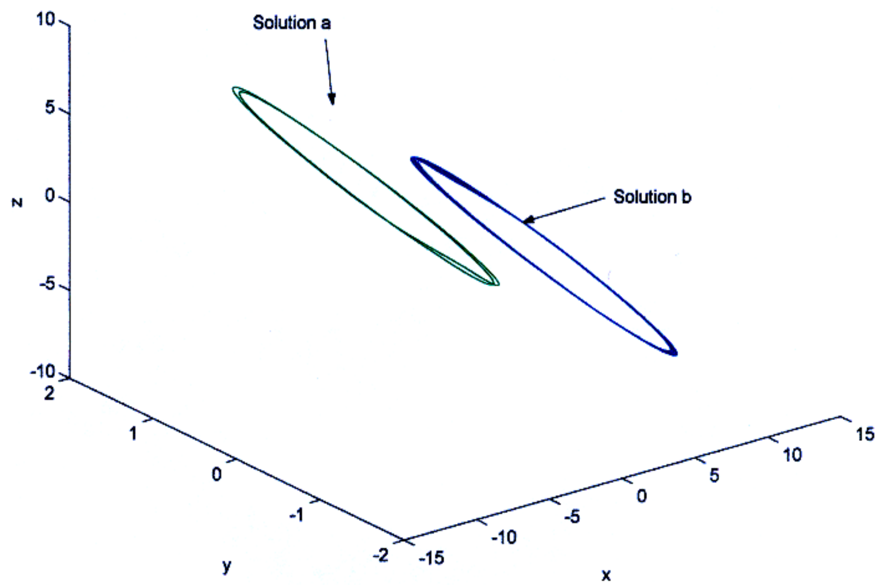


Fig. 5. Multistable behavior for Chua's circuit $a = 0.54809$. The solution trajectory obtained depends on the history of the system.

In addition, a more detailed bifurcation diagram (Fig. 3), for linearly increasing a, $0.5472 < a < 0.5492$, was produced using the method described above. As can be seen, there is a quasi-periodic route to chaos as established by Newhouse *et al.* [1978]. That is, a periodic solution degenerates into a quasi-periodic torus that then bifurcates to a two torus, bifurcates again to a three-torus and then rapidly becomes chaotic. This has also been verified using Floquet's theorem to test for subharmonic instability within this bistable region.

Examining the power spectra for solutions obtained within the bistable region further illustrates their qualitative behavior. Figures 4(a)–4(d) show the power spectra obtained for solutions on the lower branch of the bifurcation map as the value of a is increased. As the value a varies and increases through the bistable region there is a gradual

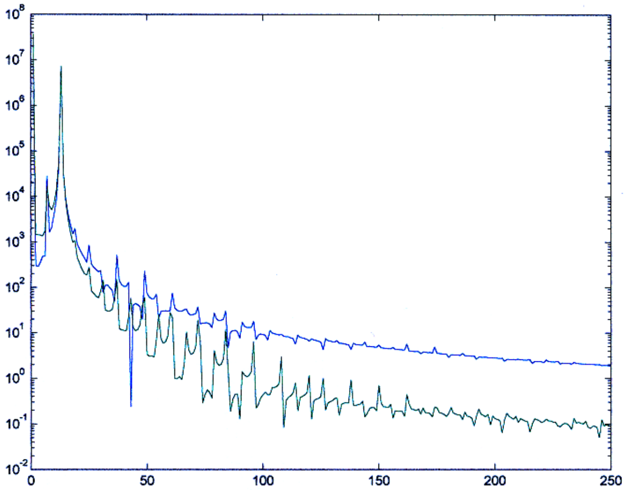


Fig. 6. Power spectra for Chua circuit simulation, $a = 0.54809$. The qualitative nature of the solution depends on the history of the system.

descent into chaos via the route described by Newhouse *et al.* [1978]. As the solutions exhibit more unstable behavior the higher frequencies become more significant until at $a = 0.55$, for example, the solution produced can be said to be truly chaotic.

In order to demonstrate more fully what is actually occurring inside the bistable region, trajectories for $a = 0.54809$, were produced. The first (Fig. 5 — solution a) was generated using initial conditions ($x = 1$, $y = 1$, $z = -1$), a step length $h = 0.1$, and allowing the first 1000 iterates to be discarded to allow the trajectory to settle onto the attractor. The second (Fig. 5 — solution b) was generated using a similar step length, however, the initial value for a was set at 0.54809, and gradually increased by 0.001 every 1000 iterates until $a = 0.56$ (ramping up) and then again ramping down to $a = 0.54809$. Both plots were created using increasing numbers of steps until there was no discernible difference in the plots produced, the final computer runs being continued for over 50,000 points. Both sets were then plotted on the same diagram to allow for easier comparison of the solution sets. The power spectra (Fig. 6) for both solution sets are also included.

4. Conclusions

The method of Poincaré first returns has been applied to produce a bifurcation diagram for the Chua system. By increasing and then decreasing one of the parameter values it has been possible to generate a small isolated counterclockwise hysteresis loop. Within this bistable region the phase trajectories can follow quasi-periodic movement around a torus. In physical applications it is possible that the bistable nature would be missed or the quasi-periodic movement interpreted as steady-state behavior due to the nature of the solutions.

Phase trajectories within this bistable region have been plotted in order to demonstrate the behavior of the system within this region. As far as the authors are aware this is the first time an isolated bistable region for Chua's original equations has been demonstrated.

References

Christopher, C. J. & Lynch, S. [1999] "Small-amplitude limit cycles of Liénard equations with either quadratic or cubic damping or restoring terms," *Nonlinearity* **12**(4), 1099–1112.

Elgar, S. [1998] "Higher order spectra of nonlinear polynomial models for Chua's circuit," *Int. J. Bifurcation and Chaos* **8**(12), 2425–2431.

Kal'yanov, E. V. [1998] "Chaotic oscillations and hysteresis in a bistable system," *J. Commun. Techn. Electron.* **44**(5), 534–542.

Lynch, S., Steele, A. L. & Hoad, J. E. [1998] "Stability analysis of non-linear optical resonators," *Chaos Solit. Fract.* **9**(6), 935–946.

Lynch, S. & Christopher, C. J. [1999] "Limit cycles in highly non-linear differential equations," *J. Sound Vibr.* **224**(3), 505–517.

Lynch, S. [2000] *Dynamical Systems with Applications using MAPLE* (Birkhäuser, Boston).

Lynch, S. & Steele, A. L. [2000] "Controlling chaos in non-linear bistable optical resonators," *Chaos Solit. Fract.* **11**(5), 721–728.

Lynch, S. [2001] "Multistability, bistability and chaos control," *Nonlin. Anal. Th. Meth. Appl.* **47**, 4501–4512.

Madan, R. N. [1993] *Chua's Circuit: A Paradigm for Chaos* (World Scientific, Singapore).

Nejad, L. A. M. & Wagenblast, R. [1999] "Time dependent chemical models of spherical dark clouds," *Astron. Astrophys.* **350**, 204–229.

Newhouse, S., Ruelle, D. & Takens, F. [1978] "Occurrence of strange axiom A attractors near quasi-periodic flows on T^m, $m \geq 3$," *Commun. Math. Phys.* **64**(35).

Scott, S. K. [1994] *Oscillations, Waves and Chaos in Chemical Kinetics* (Oxford Science Publications).

Steele, A. L., Lynch, S. & Hoad, J. E. [1997] "Analysis of optical instabilities and bistability in a non-linear optical fibre loop mirror with feedback," *Opt. Commun.* **137**, 136–142.

Szczyglowski, J. [2001] "Influence of eddy currents on magnetic hysteresis loops in soft magnetic materials," *J. Magn. Magn. Mater.* **223**(1), 97–102.