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Papers

HETEROCLINIC ORBITS ARISING FROM COUPLED CHUA'S CIRCUITS

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In this paper, we study the existence of heteroclinic orbits for ordinary differential equations which arise from a one-dimensional array of Chua's circuits. By using the upper and lower solutions method, and a zero-order approximation we show that for a certain set of parameters there exist traveling wave solutions for some given wave speeds.

Keywords: Heteroclinic orbits; upper and lower solution method; Nagumo condition; traveling wave solution.

1. Introduction

Chua's circuit is a simple electronic circuit exhibiting a wide variety of bifurcation and chaotic phenomena. Because of its universality and simplicity, Chua's circuit has captured much interest among researchers in science and engineering. The circuit equations are described by the following set of differential equations:

$$
\begin{aligned}
C_1\frac{dV_{C_1}}{dt} &= \frac{1}{R}(V_{C_2} - V_{C_1}) - F(V_{C_1})\,,\\
C_2\frac{dV_{C_2}}{dt} &= \frac{1}{R}(V_{C_1} - V_{C_2}) + i_L\,,\\
L\frac{di_L}{dt} &= -V_{C_2}\,,
\end{aligned}
$$

where F is nonlinear and the shape of F is shown in Fig. 1. See, for example, [Anishchenko & Safonova, 1992; Belykh & Chua, 1992; Chua, 1992, 1998; Kahan & Sicardi-Schifino, 1999; Kennedy & Roska, 1992; Wu & Pivka, 1993] and [Zhong & Ayrom, 1985a, 1985b] for more details.

By using a change of variables, the circuit equations can be transformed into the following form, see [Chua, 1992; Kocarev & Roska, 1993],

$$
\begin{aligned}
\dot{u} &= \alpha(z - f(u))\,,\\
\dot{z} &= u - z + w\,,\\
\dot{w} &= -\beta z\,,
\end{aligned}
\tag{1}
$$

where α and β are positive constants, $f(u)$ has the shape of the function $u + F(u)$.

In [Perez-Munuzuri *et al.*, 1992, 1993], a finite array of Chua's circuits was considered with Nemunann boundary conditions. The system is as follows:

$$
\begin{aligned}
\dot{u}_k &= \alpha(z_k - f(u_k)) + \overline{D}(u_{k-1} - 2u_k + u_{k+1})\,,\\
\dot{z}_k &= u_k - z_k + w_k\,, \quad k = 0, 1, 2, \ldots, l\,,\\
\dot{w}_k &= -\beta z_k\,,
\end{aligned}
\tag{2}
$$

where $u_0(t) = u_{-1}(t)$, $u_l(t) = u_{l+1}(t)$ and $\overline{D} > 0$ represents the diffusion coefficient of the variable u,

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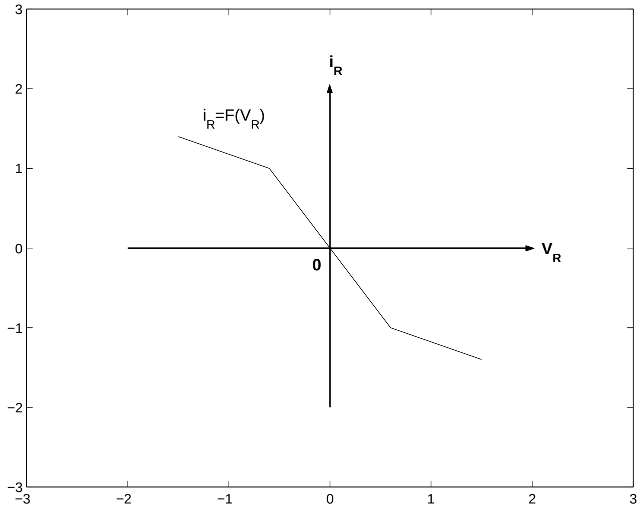

Fig. 1. The shape of $F(u)$.

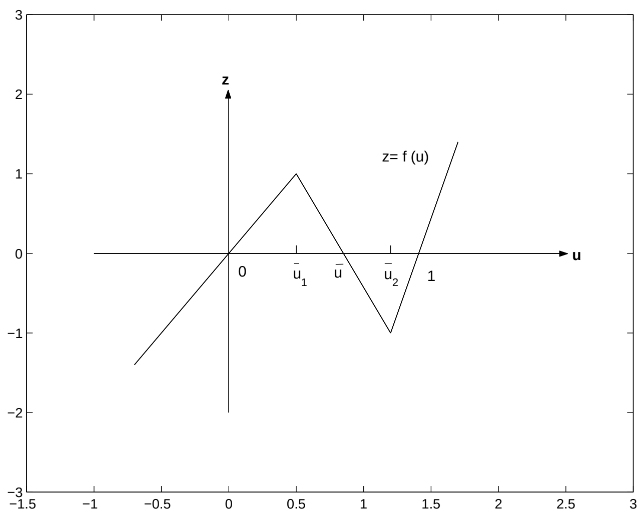

Fig. 2. Diagram of $z = f(u) + u$.

and

$$f(u) = \begin{cases} m_0 u\,, & u \le \overline{u}_1\,, \\ -m_1(u - \overline{u})\,, & \overline{u}_1 < u \le \overline{u}_2\,, \\ m_2(u-1)\,, & u > \overline{u}_2\,, \end{cases}$$

is as shown in Fig. 2. The system (2) is an example of Cellular Neural Networks as described in [Chua, 1992]. In [Perez-Munuzuri *et al.*, 1992, 1993], traveling wave-like solutions were observed numerically when the number of cells l is large. Also the propagation failure had been detected for a certain parameter range.

In this paper we will study an idealized system that consists of an array of infinitely many cells. To do so, we consider the following partial differential equations:

$$\begin{aligned} \frac{\partial U}{\partial t} &= \alpha(Z - f(U)) + D\frac{\partial^2 U}{\partial x^2}\,, \quad D > 0\,, \\ \frac{\partial Z}{\partial t} &= U - Z + W\,, \\ \frac{\partial W}{\partial t} &= -\beta Z\,. \end{aligned} \tag{3}$$

The main purpose of this paper is to show the existence of traveling wave solutions for (3). If such traveling wave solutions exist, then by using the method in [Chow *et al.*, 1998], one could obtain traveling wave solutions for the discrete case with infinitely many lattice points. Thus, the numerical results in [Perez-Munuzuri *et al.*, 1992, 1993], are consistent with our theoretical result.

Let c be the wave speed of the traveling wave solution. By introducing a moving coordinate $t'' = t'/-c = 1/-c(x - ct)$ and setting

$$U(t,x) = u\left(\frac{x}{-c} + t\right), \quad Z(t,x) = z\left(\frac{x}{-c} + t\right),$$
$$W(t,x) = w\left(\frac{x}{-c} + t\right),$$

we arrive at ordinary differential equations for (u, z, w):

$$\begin{cases} \varepsilon\ddot{u} = \dot{u} - \alpha(z - f(u))\,, \\ \dot{z} = u - z + w\,, \\ \dot{w} = -\beta z \end{cases} \tag{4}$$

with the boundary conditions:

$$\begin{aligned} &\lim_{t\to-\infty}(u, \dot{u}, z, w) = (0,0,0,0)\,, \\ &\lim_{t\to+\infty}(u, \dot{u}, z, w) = (1,0,0,-1)\,, \end{aligned} \tag{5}$$

where $\varepsilon = D/c^2$. We will use the upper and lower solution method to show that for any $\varepsilon > 0$, the system (4) with boundary condition (5) has a solution for a certain set of parameter values. This means that (4) has a heteroclinic orbit connecting the equilibrium points $(0,0,0,0)$ and $(1,0,0,-1)$.

Our approach is as follows. In Sec. 2, a Nagumo condition for a differential equation is introduced. For equations satisfying the Nagumo condition, the existence of the solution of the corresponding boundary value problem is shown by using the upper and lower solution method. Then in Sec. 3, a singular perturbation type of approach is used to obtain a zero-order approximation solution. The main result is in Sec. 4 where we show the existence of heteroclinic orbit of Eq. (4). Thus one concludes that for any small $\varepsilon > 0$ and given wave speed c, (3) has a traveling wave solution.

2. An Existence Theorem and Nagumo Condition

In this section we will use the upper and lower solutions to give an existence theorem of a boundary value problem in which the differential equation involves a functional. The functional is related to the singular perturbation problem (4) and the details will be presented in later sections. An existence theorem for equations involving functionals is given. We will then define a Nagumo condition under which our main existence theorem in Sec. 4 is proven. The approach is based on the work in [Bernfeld & Lakshmikantham, 1974; De Coster & Habets, 1996].

Let $\mathcal{T} : C^0([a,b],\mathbb{R}) \to C^2([a,b],\mathbb{R})$ be a bounded linear operator and $f : [a,b] \times \mathbb{R} \times \mathbb{R} \times C^0([a,b],\mathbb{R}) \to \mathbb{R}$ be continuous.

Theorem 2.1. *If $f(t, y, z, [\mathcal{T}\xi])$ is bounded on $[a,b] \times \mathbb{R} \times \mathbb{R} \times C^1([a,b],\mathbb{R})$. Then for and $A, B \in \mathbb{R}$, the boundary value problem*

$$y'' = f(t, y, y', [\mathcal{T}y]),$$
$$y(a) = A, \quad y(b) = B$$

has a solution.

Proof. Let f be bounded by $M > 0$ and choose $Q > 0$ to be sufficiently large so that

$$A \le Q, \quad B \le Q, \quad \frac{|A-B|}{b-a} \le Q$$

and

$$b - a \le \left(\frac{8Q}{M}\right)^{\frac{1}{2}}, \quad b - a \le \frac{2Q}{M}.$$

Consider the Banach space $C^0([a,b],\mathbb{R})$ equipped with the norm $\|y\| = \max_{a \le t \le b} |y(t)|$. Let

$$\mathcal{C} = \{y \in C^1([a,b],\mathbb{R}) : \|y\| \le 2Q, \\ \|y'\| \le 2Q\} \subset C^0([a,b],\mathbb{R}).$$

Notice that $\mathcal{C}$ is a closed, convex and bounded subset of $C^0([a,b],\mathbb{R})$. Define the mapping $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\mathcal{A}y(t) = \int_a^b G(t,s) f(s, y(s), y'(s), [\mathcal{T}y](s))\mathrm{d}s + w(t),$$

where

$$G(t,s) = \begin{cases} \dfrac{(b-t)(s-a)}{a-b}, & a \le s \le t \le b, \\ \dfrac{(b-s)(t-a)}{a-b}, & a \le t \le s \le b \end{cases}$$

is the usual Green's function for the boundary value problem: $w''(t) = 0$, $w(a) = 0$, $w(b) = 0$ and $w(t)$ is the solution for: $w''(t) = 0$, $w(a) = A$, $w(b) = B$. It is easy to see that if $\mathcal{A}$ has a fixed point $y(\cdot)$ in $\mathcal{C}$, then $y(\cdot)$ is a solution of our boundary value problem.

Observe that

$$\int_a^b \left|G(t,s)\right| = \frac{(b-t)(t-a)}{2} \le \frac{(b-a)^2}{8}$$

and

$$\int_a^b \left|G_t(t,s)\right| = \frac{(b-t)^2 + (t-a)^2}{2(b-a)} \le \frac{b-a}{2}.$$

Thus,

$$\|\mathcal{A}y(t)\| \le \frac{(b-a)^2}{8} \cdot M + Q \le 2Q$$

and

$$\|(\mathcal{A}y)'(t)\| \le \frac{b-a}{2} \cdot M + Q \le 2Q.$$

Hence $\mathcal{A}$ maps $\mathcal{C}$ into itself. Since

$$\|(\mathcal{A}y)''(t)\| \le |f(t, y(t), y'(t), [\mathcal{T}y](t))| \le M,$$

$\mathcal{A}$ is completely continuous. By Schauder's fixed point theorem, $\mathcal{A}$ has a fixed point in $\mathcal{C}$, which is a solution of our boundary value problem. This completes the proof. ■

Now we consider the following boundary value problem

$$y'' = f(t, y, y', [\mathcal{T}\xi](t)), \tag{6}$$

$$y(a) = A, \quad y(b) = B, \tag{7}$$

where $f : [a, b]\times\mathbb{R}\times\mathbb{R}\times C^0([a, b], \mathbb{R}) \to \mathbb{R}$ is continuous, $\mathcal{T} : C^0([a, b], \mathbb{R}) \to C^2([a, b], \mathbb{R})$ is bounded and linear, and $\xi \in C^0([a, b])$. The definitions of the upper and lower solutions and the Nagumo condition related to Eq. (6) are given below.

Definition 2.2. Let $\underline{\omega}$ and $\overline{\omega}$ be continuous and piecewise C^2 functions with $\underline{\omega}(t) \le \overline{\omega}(t)$ on $[a, b]$. That is, there is a finite partition $\{t_i\}$, $i = 0, \dots, n$, of $[a, b]$ with $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, such that on each closed subinterval $[t_{i-1}, t_i]$, $\underline{\omega}$ and $\overline{\omega}$ are twice continuously differentiable. At the partition point t_i, the right- and left-handed derivatives satisfy the following:

$$\underline{\omega}'(t_i^-) < \underline{\omega}'(t_i^+)\,, \quad \overline{\omega}'(t_i^-) > \overline{\omega}'(t_i^+)\,,$$
$$i = 1, 2, \dots, n-1\,.$$

In addition, if on each open subinterval (t_{i-1}, t_i) the following inequalities are satisfied for any $\xi(t) \in C^0([a, b])$ satisfying $\underline{\omega}(t) \le \xi(t) \le \overline{\omega}(t)$,

$$\underline{\omega}''(t) \ge f(t, \underline{\omega}(t), \underline{\omega}'(t), [\mathcal{T}\xi](t))\,,$$
$$\overline{\omega}''(t) \le f(t, \overline{\omega}(t), \overline{\omega}'(t), [\mathcal{T}\xi](t))$$

then $\underline{\omega}(t)$ and $\overline{\omega}(t)$ are called the piecewise C^2 lower and upper solutions, respectively, of the differential equation:

$$y'' = f(t, y, y', [\mathcal{T}\xi(t)])\,.$$

Definition 2.3. Let $\underline{\omega}, \overline{\omega} \in C^0([a, b], \mathbb{R})$ with $\underline{\omega}(t) \le \overline{\omega}(t)$ on $[a, b]$ and $h \in C^0(\mathbb{R}^+, \mathbb{R}^+)$ satisfy

$$\int_\lambda^\infty \frac{s ds}{h(s)} > \max_{t\in[a,b]} \overline{\omega}(t) - \min_{t\in[a,b]} \underline{\omega}(t)\,,$$

where

$$\lambda(b - a) = \max\{|\underline{\omega}(a) - \overline{\omega}(b)|, |\underline{\omega}(b) - \overline{\omega}(a)|\}\,.$$

The differential equation (6) is said to satisfy a Nagumo condition on $[a, b]$ with respect to the pair $\underline{\omega}, \overline{\omega}$ if the following inequality is satisfied for any $t \in [a, b]$, $\underline{\omega}(t) \le u \le \overline{\omega}(t)$, $v \in \mathbb{R}$ and any $\xi \in C^0([a, b], \mathbb{R})$ with $\underline{\omega}(t) \le \xi(t) \le \overline{\omega}(t)$,

$$|f(t, u, v, [\mathcal{T}\xi](t))| \le h(|v|)\,.$$

The following is the main result of this section.

Theorem 2.4. *Suppose that $\underline{\omega}$ and $\overline{\omega}$ are piecewise C^2 lower and upper solutions of the differential equation*

$$y'' = f(t, y, y', [\mathcal{T}\xi](t))\,,$$

where $f : [a, b] \times \mathbb{R} \times \mathbb{R} \times C^0([a, b], \mathbb{R})$ is continuous and $\mathcal{T} : C^0([a, b], \mathbb{R}) \to C^2([a, b], \mathbb{R})$ is a bounded linear operator. Assume that $y'' = f(t, y, y', [\mathcal{T}\xi](t))$ satisfies a Nagumo condition on $[a, b]$ with respect to the pair $\underline{\omega}$ and $\overline{\omega}$. Then for any A, B with $\underline{\omega}(a) \le A \le \overline{\omega}(a)$ and $\underline{\omega}(b) \le B \le \overline{\omega}(b)$, the following boundary value problem

$$y'' = f(t, y, y', [\mathcal{T}y])\,, \tag{8}$$

$$y(a) = A, \quad y(b) = B\,, \tag{9}$$

has a solution $y(t)$ with $\underline{\omega}(t) \le y(t) \le \overline{\omega}(t)$ on $[a, b]$. Moreover, let $N_0 > 0$ be determined by:

$$\int_\lambda^{N_0} \frac{s ds}{h(s)} = \max_{t\in[a,b]} \overline{\omega}(t) - \min_{t\in[a,b]} \underline{\omega}(t)\,,$$

then $|y'(t)| \le N_0$, for all $a \le t \le b$.

Proof. First of all, We will construct a modified differential equation of Eq. (6). Let $N_1 = \max_{a\le x\le b} |\underline{\omega}'(x)|$, $N_2 = \max_{a\le x\le b} |\overline{\omega}'(x)|$, where at the partition points the maximum is taken over the right- and left-handed derivatives. Let $N = \max\{N_0, N_1, N_2\} + 1$. Define

$$F(t, y, z, [\mathcal{T}\xi]) = f(t, R_1(y), R_2(z), [\mathcal{T}(R_3(\xi))]) + R_4(y),$$

where

$$R_1(y) = \begin{cases} \overline{\omega}(t)\,, & y > \overline{\omega}(t)\,, \\ y\,, & \underline{\omega}(t) \le y \le \overline{\omega}(t)\,, \\ \underline{\omega}(t)\,, & y < \underline{\omega}(t)\,, \end{cases}$$

$$R_2(z) = \begin{cases} N\,, & z > N\,, \\ z\,, & |z| \le N\,, \\ -N\,, & z < -N\,, \end{cases}$$

and

$$R_3(\xi)(t) = \begin{cases} \overline{\omega}(t)\,, & \xi(t) > \overline{\omega}(t)\,, \\ \xi(t)\,, & \underline{\omega}(t) \le \xi(t) \le \overline{\omega}(t)\,, \\ \underline{\omega}(t)\,, & \xi(t) < \underline{\omega}(t)\,, \end{cases}$$

$$R_4(y) = \begin{cases} \dfrac{y - \overline{\omega}(t)}{1 + |y - \overline{\omega}(t)|}\,, & y > \overline{\omega}(t)\,, \\ 0\,, & \underline{\omega}(t) \le y \le \overline{\omega}(t)\,, \\ \dfrac{y - \underline{\omega}(t)}{1 + |y - \underline{\omega}(t)|}\,, & y < \underline{\omega}(t)\,. \end{cases}$$

Thus, $F(t,y,z,[\mathcal{T}\xi])$ is continuous and bounded on $[a,b]\times\mathbb{R}\times\mathbb{R}\times C^0([a,b]$. Hence, it follows from Theorem 2.1 that there exists a solution $y(t)$ of

$$y'' = F(x,y,y',[\mathcal{T}y])$$

$$y(a) = A\,, \quad y(b) = B\,.$$

It remains to show that this solution $y(t)$ has all of the properties stated in the Theorem.

First, we show that $\underline{\omega}(t) \le y(t) \le \overline{\omega}(t)$ on $[a,b]$. We prove only that $y(t) \le \overline{\omega}(t)$ on $[a,b]$. Assume on the contrary that $y(t) > \overline{\omega}(t)$ at some points in $[a,b]$. Since $\underline{\omega}(a) \le A \le \overline{\omega}(a)$ and $\underline{\omega}(b) \le B \le \overline{\omega}(b)$, it must be the case that $m(t) = y(t) - \overline{\omega}(t)$ has a positive maximum at some point $\overline{t} \in (a,b)$. There are two cases to be considered. For the first case, assume $\overline{t} = t_i$ for some $i = 1,2,\ldots,n-1$. Since $y'(\overline{t}) - \overline{\omega}'(\overline{t}^-) \ge 0 \ge y'(\overline{t}) - \overline{\omega}'(\overline{t}^+)$, thus $\overline{\omega}'(\overline{t}^+) \ge \overline{\omega}'(\overline{t}^-)$. This contradicts the assumption that $\overline{\omega}'(t_i^-) > \overline{\omega}'(t_i^+)$. For the second case, assume $\overline{t} \in (t_{i-1},t_i)$ for some fixed i. We have $y(\overline{t}) > \overline{\omega}(\overline{t}), y'(\overline{t}) = \overline{\omega}'(\overline{t})$ and

$$\begin{aligned} m''(\overline{t}) &= y''(\overline{t}) - \overline{\omega}''(\overline{t}) \\ &= F(\overline{t},y(\overline{t}),y'(\overline{t}),[\mathcal{T}y](\overline{t})) - \overline{\omega}''(\overline{t}) \\ &= f(\overline{t},\overline{\omega}(\overline{t}),\overline{\omega}'(\overline{t}),[\mathcal{T}(R_3(y))](\overline{t})) \\ &\quad + \frac{y(\overline{t}) - \overline{\omega}(\overline{t})}{1+|y(\overline{t}) - \overline{\omega}(\overline{t})|} - \overline{\omega}''(\overline{t}) \\ &\ge \frac{(y(\overline{t}) - \overline{\omega}(\overline{t}))}{1+|y(\overline{t}) - \overline{\omega}(\overline{t})|} > 0\,, \end{aligned}$$

which is impossible at a maximum of $m(t)$. Thus we conclude that $y(t) \le \overline{\omega}(t)$ on $[a,b]$. A similar argument can be applied to obtain the other inequality. Thus we have $\underline{\omega}(t) \le y(t) \le \overline{\omega}(t)$ on $[a,b]$ and conclude that y satisfies $y'' = f(x,y,R_2(y'),[\mathcal{T}y](t))$.

Next, we shall show that $|y'(x)| \le N$ for $t \in [a,b]$. Observe that $\underline{\omega}(x) \le y(x) \le \overline{\omega}(x)$ on $[a,b]$, therefore by the Mean Value Theorem there exists a $t_0 \in (a,b)$ such that $|(y(a)-y(b))/(a-b)| = |y'(t_0)| \le \max\{(|\underline{\omega}(b) - \overline{\omega}(a)|)/b-a, (|\underline{\omega}(a) - \overline{\omega}(b)|)/b-a\} = \lambda < N_0 < N$.

If we assume that $|y'(t)| > N$, for some $t \in [a,b]$, then there exists an interval $[t_1,t_2] \subset [a,b]$ such that the following cases hold:

(1) $y'(t_1) = \lambda$, $y'(t_2) = N$, and $\lambda < y'(t) < N$, $t \in (t_1,t_2)$,
(2) $y'(t_1) = N$, $y'(t_2) = \lambda$, and $\lambda < y'(t) < N$, $t \in (t_1,t_2)$,
(3) $y'(t_1) = -\lambda$, $y'(t_2) = -N$, and $-N < y'(t) < -\lambda$, $t \in (t_1,t_2)$,
(4) $y'(t_1) = -N$, $y'(t_2) = -\lambda$, and $-N < y'(t) < -\lambda$, $t \in (t_1,t_2)$.

Let us consider the case (1). We note that on $[t_1,t_2]$, $y''(t) = F(t,y,y',[\mathcal{T}y]) = f(t,y,y',[\mathcal{T}y])$. By using the Nagumo condition, we obtain

$$\begin{aligned} y''(t)y'(t) &\le |y''(t)|y'(t) \\ &= |f(t,y(t),y'(t),[\mathcal{T}y](t))|y'(t) \\ &\le h(|y'(t)|)y'(t) \end{aligned}$$

for some $h \in C^0(\mathbb{R}^+,\mathbb{R}^+)$. Integrating the above equation over the interval $[t_1,t_2]$, one has

$$\begin{aligned} \int_{t_1}^{t_2} \frac{y''(s)y'(s)ds}{h(|y'(s)|)} &\le \int_{t_1}^{t_2} \frac{|y''(s)|y'(s)ds}{h(|y'(s)|)} \\ &\le \int_{t_1}^{t_2} y'(s)ds = y(t_2) - y(t_1)\,. \end{aligned}$$

This shows that

$$\begin{aligned} y(t_2) - y(t_1) &\ge \int_{\lambda}^{N} \frac{sds}{h(s)} \\ &> \int_{\lambda}^{N_0} \frac{sds}{h(s)} \\ &= \max_{t\in[a,b]} \overline{\omega}(t) - \min_{t\in[a,b]} \underline{\omega}(t)\,. \end{aligned}$$

which is a contradiction to that $y(t)$ is trapped between $\underline{\omega}(t)$ and $\overline{\omega}(t)$.

The other cases can be treated by using similar arguments. Therefore, $|y'(t)| \le N$ on $[a,b]$, and consequently from the manner in which F was defined, $y(t)$ is a solution of

$$y'' = f(x,y,y',[\mathcal{T}y](t))$$

$$y(a) = A\,, \quad y(b) = B\,.$$

Moreover, only a slight modification of the last argument above will show that $|y'(t)| \le N_0$ on $[a,b]$. ■

3. Zero-Order Approximation

Observe that the differential equation (4) with the boundary condition (5) is a singular perturbation problem. A general method to consider this kind of problem is to first set $\varepsilon = 0$ and find the two outer layer expansions that will fit the two end boundary conditions. Then a time scalaring method is used to find the inner layer expansions. After that a matching of the coefficients on all functions will be performed to obtain the final solution. However for the system (4) this approach is not suitable because of the large number of equations involved and insufficent number of boundary conditions. In this paper we will present another method based on a zero-order approximation of the solution.

Let $\varepsilon = 0$ in Eq. (4):

$$\begin{cases} \dot{u} = \alpha(z - f(u))\,, \\ \dot{z} = u - z + w\,, \\ \dot{w} = -\beta z\,, \end{cases} \tag{10}$$

which is the equation given by the traditional Chua's circuits. By definition of outer expansion, it is a heteroclinic solution to Eq. (10). There are numerical studies which indicate the existence of such heteroclinic orbits (see, for example, [Wu & Pivka, 1993]). However, we are not aware of any rigorous mathematical existence of proof. Thus, we use an approximate solution to Eq. (10) and call this the zero-order approximation. In order for the solution to satisfy the following boundary condition

$$\begin{gathered} \lim_{t\to-\infty}(u, z, w) = (0, 0, 0)\,, \\ \lim_{t\to+\infty}(u, z, w) = (1, 0, -1)\,, \end{gathered} \tag{11}$$

we let

$$\begin{gathered} u_0(t) = 0\,, \quad z_0(t) = 0\,, \\ w_0(t) = 0\,, \quad \text{for } t \le 0, \end{gathered} \tag{12}$$

and

$$\begin{gathered} u_0(t) = 1\,, \quad z_0(t) = 0\,, \\ w_0(t) = -1\,, \quad \text{for } t > 0 \end{gathered} \tag{13}$$

be solutions of (10) and (11) on half lines. Let $\tau = t/\varepsilon$, then Eq. (10) becomes

$$\begin{cases} u'' = u' - \varepsilon\alpha(z - f(u))\,, \\ z' = \varepsilon(u - z + w)\,, \\ w' = -\varepsilon\beta z\,, \end{cases} \tag{14}$$

where $' = d/d\tau$. Again by letting $\varepsilon = 0$ in (14), we have

$$\begin{cases} u'' = u', \\ z' = 0\,, \\ w' = 0\,. \end{cases} \tag{15}$$

The solution of (15) is given by:

$$\begin{gathered} \tilde{u}_0(\tau) = c_0 e^{\tau} + c_1\,, \quad \tilde{w}_0(\tau) = c_2\,, \\ \tilde{z}_0(\tau) = c_3\,, \end{gathered} \tag{16}$$

with c_0, c_1, c_2 and c_3 as constants to be determined.

By comparing the functions in (12) and (13) and the function defined by (16), one can see that only $u_0(t)$ can be uniquely determined. Thus we consider the following zero-order composite expansion

$$u_0(t) = \begin{cases} u_0^L(t) = e^{\frac{t}{\varepsilon}}, & t \le 0\,, \\ u_0^R(t) = 1\,, & t > 0\,. \end{cases}$$

And solve the following system of differential equation

$$\begin{cases} \dot{z} = -z + w + u_0(t)\,, \\ \dot{w} = -\beta z\,, \end{cases}$$

with boundary condition

$$\lim_{t\to-\infty}(z, w) = (0, 0)\,, \quad \lim_{t\to+\infty}(z, w) = (0, -1)\,,$$

where $u_0(t)$ is given above. We obtain the following solution

$$z_0(t) = \begin{cases} z_0^L(t) = \dfrac{\varepsilon}{1 + \varepsilon + \beta\varepsilon^2}\, e^{\frac{t}{\varepsilon}}, & t \le 0\,, \\ z_0^R(t) = \left[\dfrac{\varepsilon}{1 + \varepsilon + \beta\varepsilon^2}\, \cos(\overline{\beta}t) + \left(\dfrac{1}{\overline{\beta}} - \dfrac{(1 + 2\beta\varepsilon)\varepsilon}{\overline{\beta}(1 + \varepsilon + \beta\varepsilon^2)}\right)\sin(\overline{\beta}t)\right] e^{-\frac{1}{2}t}, & t > 0\,, \end{cases}$$

$$w_0(t) = \begin{cases} w_0^L(t) = -\dfrac{\beta\varepsilon^2}{1 + \varepsilon + \beta\varepsilon^2}\, e^{\frac{t}{\varepsilon}}, & t \le 0\,, \\ w_0^R(t) = \left[\dfrac{1 + \varepsilon}{1 + \varepsilon + \beta\varepsilon^2}\, \cos(\overline{\beta}t) + \dfrac{1 + (1 - 2\beta)\varepsilon}{2\overline{\beta}(1 + \varepsilon + \beta\varepsilon^2)}\, \sin(\overline{\beta}t)\right] e^{-\frac{1}{2}t} - 1\,, & t > 0\,, \end{cases}$$

where $\overline{\beta} = \sqrt{\beta - 1/4}$.

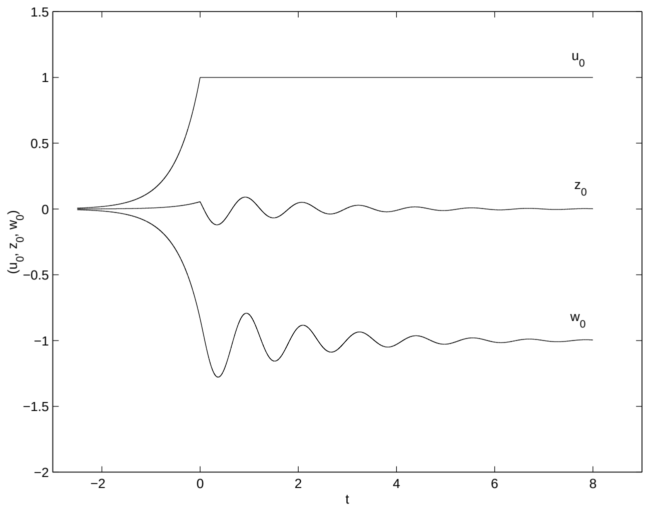

Fig. 3. The graphs of $u_0(t)$, $z_0(t)$, $w_0(t)$.

We will call $(u_0(t), z_0(t), w_0(t))$ the zero-order approximation of the solutions of (4) and (5). In Fig. 3, we show the graphs of $u_0(t)$, $z_0(t)$, $w_0(t)$ when $\varepsilon = 0.5$, $\beta = 30$.

4. Main Result

Our main result is given in Theorem 4.3 below. The general outline of the proof of the main result is as follows. First, we will use the zero-order approximation obtained in Sec. 3 as an approximation for the solution. By a change of variables on the entire system we obtain a system of three equations with different boundary conditions. After a linear subsystem is solved, a functional operator, which is considered in Sec. 2, will be defined. Then upper and lower solutions are determined for a second-order ordinary differential equation that satisfies a Nagumo condition. Finally the existence of solution can be verified by the result in Sec. 2.

The entire process is divided into several lemmas to avoid lengthy argument.

Let a, b, K and M be positive constants. Let

$$\begin{aligned}\Omega = \{U \in C^0(\mathbb{R}, \mathbb{R}); |U(t)| \le Ke^{at}\,,\\ t \le 0, |U(t)| \le Ke^{-bt}\,, \quad t > 0\}\\ \Gamma = \{z \in C^0(\mathbb{R}, \mathbb{R}); |z(t)| \le Me^{at}\,,\\ t \le 0, |z(t)| \le Me^{-bt}\,, \quad t > 0\}\end{aligned}$$

be normed linear spaces equipped with usual sup norm $\|x\| = \sup_{t\in\mathbb{R}} |x(t)|$. Let

$$A = \begin{bmatrix} -1 & 1 \\ -\beta & 0 \end{bmatrix}.$$

Assume $\beta > 1/4$. Thus, there exist $L > 0$ and $0 < \overline{c} < 1/2$ such that

$$\|e^{At}\| \le Le^{-\overline{c}t}, \quad t \ge 0$$

The constants above will be used in the following lemma.

Lemma 4.1. *Consider the following boundary value problem*

$$\begin{cases} \dot{z} = -z + w + \eta \\ \dot{w} = -\beta z \\ \lim\limits_{t\to-\infty} (z, w) = (0, 0), \ \lim\limits_{t\to\infty} (z, w) = (0, 0) \end{cases} \tag{17}$$

where $\eta \in \Omega$. Let $T : \Omega \to \Gamma \times \Gamma$ be defined by

$$T(\eta)(t) = \int_{-\infty}^{t} e^{A(t-s)} \begin{pmatrix} \eta(s) \\ 0 \end{pmatrix} ds\,.$$

Then for $\beta < 1/4$, $b < \overline{c}$ and $0 < a$, $T(\eta)(t)$ is the unique solution of (17) and is a compact operator from Ω to $\Gamma \times \Gamma$.

Proof. It follows from the variation of constant formula, the solution of (17) is given by

$$\begin{aligned}\varphi(t) &= \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}\\ &= e^{At} \begin{pmatrix} z_0 \\ w_0 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} \eta(s) \\ 0 \end{pmatrix} ds\end{aligned}$$

For any η in Ω , it is clear that $\lim_{t\to-\infty}(z(t), w(t)) = (0, 0)$, if and only if

$$\varphi(t) = \int_{-\infty}^{t} e^{A(t-s)} \begin{pmatrix} \eta(s) \\ 0 \end{pmatrix} ds\,.$$

It remains to show that φ lies in $\Gamma \times \Gamma$ for some

$M > 0$. First when $t \le 0$, we have

$$\begin{aligned}
|\varphi(t)| &\le \int_{-\infty}^{t} Le^{-\overline{c}(t-s)} K e^{as} ds\,,\\
&= KLe^{-\overline{c}t} \int_{-\infty}^{t} e^{s(a+\overline{c})} ds\,,\\
&= KLe^{-\overline{c}t} \frac{1}{a+\overline{c}}\, e^{s(a+\overline{c})}\Big|_{-\infty}^{t}\\
&\le \frac{KL}{a+\overline{c}}\, e^{-\overline{c}t+at+\overline{c}t}\,,\\
&= \frac{KL}{a+\overline{c}}\, e^{at}.
\end{aligned}$$

On the other hand when $t > 0$, $\overline{c} > b$, we have

$$\begin{aligned}
|\varphi(t)| &= \left| \int_{-\infty}^{0} e^{A(t-s)} \eta(s) ds + \int_{0}^{t} e^{A(t-s)} \eta(s) ds \right|\\
&\le \int_{-\infty}^{0} Le^{-\overline{c}(t-s)} K e^{as} ds\\
&\quad + \int_{0}^{t} Le^{-\overline{c}(t-s)} K e^{-bs} ds\\
&= LKe^{-\overline{c}t} \int_{-\infty}^{0} e^{s(a+\overline{c})} ds + LKe^{-\overline{c}t} \int_{0}^{t} e^{(\overline{c}-b)} ds\\
&= LKe^{-\overline{c}t} \frac{1}{a+\overline{c}} e^{s(a+\overline{c})}\Big|_{-\infty}^{0}\\
&\quad + LKe^{-\overline{c}t} \frac{1}{\overline{c}-b} e^{s(\overline{c}-b)}\Big|_{0}^{t}\\
&< LKe^{-\overline{c}t} \frac{1}{a+\overline{c}} + \frac{LK}{\overline{c}-b} e^{-bt}\\
&< LK \left(\frac{1}{a+\overline{c}} + \frac{1}{\overline{c}-b} \right) e^{-bt}
\end{aligned}$$

Thus by taking $M = (1/(a+\overline{c}) + 1/(\overline{c}-b))LK$ and $b < \overline{c} < 1/2$, one can see that $\varphi(t)$ lies in $\Gamma \times \Gamma$ and it satisfies $\lim_{t\to\infty}(z, w) = (0, 0)$. It can be easily seen that the solution of system (17) is unique, therefore the operator $T : \Omega \to \Gamma \times \Gamma$ defined by

$$T(\eta)(t) = \int_{-\infty}^{t} e^{A(t-s)} \begin{pmatrix} \eta(s) \\ 0 \end{pmatrix} ds$$

is a linear compact operator. ■

Let $u_0(t)$, $z_0(t)$ be the zero-order approximation as mentioned in Sec. 3, $K = 1 - \overline{u}_2$, M be the constant as defined in Lemma 4.1, and $g(t) : [-n, n] \to \mathbb{R}$ be any continuous function satisfing $|g(t)| \le Me^{at}$, $t \le 0$, and $|g(t)| \le Me^{-bt}$, $t \ge 0$. Now, we shall construct upper and lower solutions.

Lemma 4.2. *Consider the following differential equation*

$$\varepsilon \ddot{y} = \dot{y} + \alpha f(y + u_0(t)) - \alpha g(t) - \alpha z_0(t)\,. \tag{18}$$

Let

$$\overline{\omega}(t) = \begin{cases} \overline{\omega}^L(t) = Ke^{at}, & t \le 0\,,\\ \overline{\omega}^R(t) = Ke^{-bt}, & t \ge 0\,, \end{cases}$$

$$\underline{\omega}(t) = \begin{cases} \underline{\omega}^L(t) = -Ke^{at}, & t \le 0\,,\\ \underline{\omega}^R(t) = -Ke^{-bt}, & t \ge 0\,, \end{cases}$$

then $\overline{\omega}(t)$ and $\underline{\omega}(t)$ are piecewise C^2 upper and lower solutions of the differential equation (18) on the interval $[-n, n]$.

Proof. For each $n \ge 1$ and $t \in [-n, 0]$, let

$$\overline{\omega}^L(t) = Ke^{at},$$
$$\underline{\omega}^L(t) = -Ke^{at}.$$

Then

$$\begin{aligned}\underline{\omega}^L(t) = -Ke^{-at} \le 0 \le Ke^{-at}\\ \le \overline{\omega}^L(t), t \in [-n, 0]\end{aligned}$$

Since for any $t \le 0$,

$$e^{\frac{t}{\varepsilon}} < e^{at}, \qquad \text{if} \quad \varepsilon < \frac{1}{a}\,,$$
$$e^{-at} \le \frac{K+1}{\tilde{u}}\,, \qquad \text{if} \quad \tilde{u} \le Ke^{at} + e^{\frac{t}{\varepsilon}},$$

we have

$$\begin{aligned}
f(\overline{\omega}^L(t) + u_0^L(t)) &= f(Ke^{at} + e^{\frac{t}{\varepsilon}})\\
&= \begin{cases} m_0(Ke^{at} + e^{\frac{t}{\varepsilon}}) \ge m_0 Ke^{at}, & Ke^{at} + e^{\frac{t}{\varepsilon}} \le \overline{u}_1\,,\\ -m_1(Ke^{at} + e^{\frac{t}{\varepsilon}} - \overline{u}) \ge -m_1(K+1)e^{at}, & \overline{u}_1 \le Ke^{at} + e^{\frac{t}{\varepsilon}} \le \overline{u}_2\,,\\ m_2(Ke^{at} + e^{\frac{t}{\varepsilon}} - 1) \ge m_2 Ke^{at} - m_2 \cdot \frac{K+1}{\overline{u}_2} e^{at}, & Ke^{at} + e^{\frac{t}{\varepsilon}} \ge \overline{u}_2\,, \end{cases}
\end{aligned}$$

$$f(\underline{\omega}^L(t)+u_0^L(t)) = f(-Ke^{at}+e^{\frac{t}{\varepsilon}})$$
$$= \begin{cases} m_0(-Ke^{at}+e^{\frac{t}{\varepsilon}}) \le -m_0Ke^{at}+m_0e^{at}, & -Ke^{at}+e^{\frac{t}{\varepsilon}} \le \overline{u}_1\,, \\ -m_1(-Ke^{at}+e^{\frac{t}{\varepsilon}}-\overline{u}) \le m_1Ke^{at}+m_1\overline{u}\cdot\dfrac{K+1}{\overline{u}_1}\,e^{at}, & \overline{u}_1 \le -Ke^{at}+e^{\frac{t}{\varepsilon}} \le \overline{u}_2\,, \\ m_2(-Ke^{at}+e^{\frac{t}{\varepsilon}}-1) \le -m_2Ke^{at}+m_2e^{at}, & -Ke^{at}+e^{\frac{t}{\varepsilon}} \ge \overline{u}_2\,. \end{cases}$$

Let

$$m = \max\left\{|m_0K|, |m_1(K+1)|, \left|m_2K-\frac{m_2(K+1)}{\overline{u}_2}\right|, |m_0(K-1)|, \left|m_1K+\frac{m_1\overline{u}(K+1)}{\overline{u}_1}\right|, |m_2(1-K)|\right\}.$$

Then

$$f(\overline{\omega}^L(t)+u_0^L(t)) \ge -me^{at},$$
$$f(\underline{\omega}^L(t)+u_0^L(t)) \le me^{at}.$$

Notice that

$$z_0^L(t) = \frac{\varepsilon}{1+\varepsilon+\beta\varepsilon^2}\,e^{\frac{t}{\varepsilon}} \le \varepsilon e^{at}.$$

Now if one chooses a to satisfy

$$a > \frac{\alpha}{K}\,[M+m]+\frac{1}{K}\,,$$

then for any sufficiently small $\varepsilon > 0$ and $\varepsilon < \min\{1/a, 1/\alpha\}$, one has

$$\begin{aligned} \varepsilon\ddot{\overline{\omega}}^L &\le \dot{\overline{\omega}}^L + \alpha f(\overline{\omega}^L+u_0^L(t)) \\ &\quad -\alpha g(t)-\alpha z_0^L(t)\,,\quad t\le 0\,, \\ \varepsilon\ddot{\underline{\omega}}^L &\ge \dot{\underline{\omega}}^L + \alpha f(\underline{\omega}+u_0^L(t)) \\ &\quad -\alpha g(t)-\alpha z_0^L(t)\,,\quad t\le 0\,. \end{aligned} \tag{19}$$

On the other hand, when $0 \le t \le n$, let

$$\overline{\omega}^R(t) = Ke^{-bt},$$
$$\underline{\omega}^R(t) = -Ke^{-bt},$$

then

$$\underline{\omega}^R(t) = -Ke^{-t} \le 0 \le Ke^{-t} \le \overline{\omega}^R(t)\,,$$

since

$$\overline{\omega}^R(t)+u_0^R(t) = Ke^{-bt}+1 \ge 1 \ge \overline{u}_2\,,$$
$$\underline{\omega}^R(t)+u_0^R(t) = -Ke^{-bt}+1 \ge 1-K = \overline{u}_2\,,$$

when $t > 0$, we have

$$f(\overline{\omega}^R(t)+u_0^R(t)) = f(Ke^{-bt}+1) = m_2Ke^{-bt},$$
$$f(\underline{\omega}^R(t)+u_0^R(t)) = f(-Ke^{-bt}+1) = -m_2Ke^{-bt}.$$

Also from the definition of $z_0^R(t)$, one can have the following estimate

$$\begin{aligned} |z_0^R(t)| &= \left|\left[\frac{\varepsilon}{1+\varepsilon+\beta\varepsilon^2}\,\cos(\overline{\beta}t) \right.\right. \\ &\qquad \left.\left. + \left(\frac{1}{\overline{\beta}}-\frac{(1+2\beta\varepsilon)\varepsilon}{\overline{\beta}(1+\varepsilon+\beta\varepsilon^2)}\right)\sin(\overline{\beta}t)\right]e^{-\frac{1}{2}t}\right| \\ &\le \left[\frac{\varepsilon}{1+\varepsilon+\beta\varepsilon^2}\right. \\ &\qquad \left. + \left(\frac{1}{\overline{\beta}}+\frac{(1+2\beta\varepsilon)\varepsilon}{\overline{\beta}(1+\varepsilon+\beta\varepsilon^2)}\right)\right]\left|e^{-\frac{1}{2}t}\right. \\ &\le \left[\frac{1}{\overline{\beta}}+\left(1+\frac{1}{\overline{\beta}}\right)\varepsilon+\frac{2\beta}{\overline{\beta}}\,\varepsilon^2\right]e^{-bt}. \end{aligned}$$

Choose $\varepsilon < 1/\beta$, so that $\gamma = [1/\overline{\beta}+(1+1/\overline{\beta})\varepsilon+2\beta/(\overline{\beta})\varepsilon^2]$ is bounded. Then if

$$m_2 > \frac{\gamma}{\beta K}+\frac{M}{K}+\frac{b}{\alpha}\,,$$

and $\varepsilon > 0$ to be sufficiently small, we have

$$\begin{aligned} \varepsilon\ddot{\overline{\omega}}^R &\le \dot{\overline{\omega}}^R + \alpha f(\overline{\omega}^R+u_0^R(t)) \\ &\quad -\alpha g(t)-\alpha z_0^R(t),\quad t>0\,, \\ \varepsilon\ddot{\underline{\omega}}^R &\ge \dot{\underline{\omega}}^R + \alpha f(\underline{\omega}^R+u_0^R(t)) \\ &\quad -\alpha g(t)-\alpha z_0^R(t),\quad t>0\,. \end{aligned} \tag{20}$$

It follows from (19),(20) that $\overline{\omega}(t)$ and $\underline{\omega}(t)$ are the piecewise C^2 upper and lower solutions of the differential equation (18) on the interval $[-n, n]$. ■

Our main result is as follows.

Theorem 4.3. *Consider the following differential* *equations*

$$\begin{cases} \varepsilon\ddot{u} = \dot{u} - \alpha(z - f(u))\,, \\ \dot{z} = u - z + w\,, \\ \dot{w} = -\beta z\,, \end{cases} \tag{21}$$

with the boundary conditions

$$\begin{aligned} &\lim_{t\to-\infty}(u,\dot{u},z,w) = (0,0,0,0)\,, \\ &\lim_{t\to+\infty}(u,\dot{u},z,w) = (1,0,0,-1)\,. \end{aligned} \tag{22}$$

If $\varepsilon > 0$ is sufficiently small, $\beta > 1/4$ and $a > 0$ are sufficiently large, then for any $0 < b \ll \overline{c} < 1/2$ the boundary value problem (21),(22) has solution $(u, \dot{u}, z, w)$.

Moreover, they satisfy the following estimates

$$|(u,z,w) - (u_0^L(t), z_0^L(t), w_0^L(t))| \le Me^{at}, \quad t \le 0,$$
$$|(u,z,w) - (u_0^R(t), z_0^R(t), w_0^R(t))| \le Me^{-bt}, \quad t > 0,$$

where M is the constant defined in Lemma 4.1.

Proof. Let $u_0(t)$, $z_0(t)$ and $w_0(t)$ be the zero-order approximation as defined in Sec. 3 and u, z, w be the solution of Eq. (21). Consider the following change of variables

$$U = u - u_0(t)\,, \quad Z = z - z_0(t)\,, \quad W = w - w_0(t)\,,$$

and replace (U, Z, W) by (u, z, w), then (u, z, w) satisfy the following differential equations

$$\begin{cases} \varepsilon\ddot{u} = \dot{u} + \alpha f(u + u_0(t)) - \alpha z - \alpha z_0(t)\,, \\ \dot{z} = u - z + w\,, \\ \dot{w} = -\beta z\,, \end{cases} \tag{23}$$

with the boundary conditions

$$\begin{aligned} &\lim_{t\to-\infty}(u,\dot{u},z,w) = (0,0,0,0)\,, \\ &\lim_{t\to+\infty}(u,\dot{u},z,w) = (0,0,0,0)\,. \end{aligned} \tag{24}$$

For any integer $n > 0$, let

$$\Omega_n = \Omega \cap \{u : u \in C^0([-n,n],\mathbb{R}), u(t) = 0, \; |t| \ge n\}\,;$$

be the Banach space equipped with the supremum norm.

Let T be the operator defined in Lemma 4.1 and $\mathcal{T}$ be the first component of T. Since the first component of T satisfies a system of linear equations, thus we have $\mathcal{T} : C^0([-n,n],\mathbb{R}) \to C^2([-n,n],\mathbb{R})$ is a bounded linear operator. When $K = 1 - \overline{u}_2$ and the domain is being restricted to Ω_n, the corresponding operators are denoted by T_n and $\mathcal{T}_n$. Then for any $u \in \Omega_n$, we have $T_n u = (z(u), w(u)) = (z, w)$ and satisfies the following differential equations and inequalities

$$\begin{cases} \dot{z} = -z + w + u \\ \dot{w} = -\beta z \end{cases} \tag{25}$$

$$\begin{aligned} |z(u)(t)| &\le Me^{at}, \quad t \le 0\,, \\ |z(u)(t)| &\le Me^{-bt}, \quad t \ge 0\,. \end{aligned} \tag{26}$$

Now, replace the z in Eq. (23) by $z(u) = \mathcal{T}(u)$ and consider the following boundary value problem

$$\begin{aligned} \varepsilon\ddot{u} = \dot{u} &+ \alpha f(u + u_0(t)) \\ &- \alpha\mathcal{T}(u)(t) - \alpha z_0(t)\,, \end{aligned} \tag{27}$$

$$u(-n) = 0, \quad u(n) = 0 \tag{28}$$

Let $\overline{\omega}(t)$ and $\underline{\omega}(t)$ be the functions defined in Lemma 4.2, then for any $u \in \Omega_n$, we have $\underline{\omega}(t) \le u(t) \le \overline{\omega}(t)$ for any $t \in [-n, n]$. Also it follows from inequality (26) that $|\mathcal{T}(u)(t)| < M$ for all t. Hence if we let

$$\begin{aligned} g(t, u, \dot{u}, \mathcal{T}(u)) = \dot{u} &+ \alpha f(u + u_0(t)) \\ &- \alpha\mathcal{T}(u)(t) - \alpha z_0(t)\,, \end{aligned}$$

then g becomes continuous and satisfies

$$|g(t, u, \dot{u}, \mathcal{T}(u))| \le |\dot{u}| + C_1\,,$$

where C_1 can be chosen independent of n. This is due to the fact that $\mathcal{T}(u)(t) = z(u)(t)$ is uniformly bounded. Hence the differential equation (27) satisfies a Nagumo condition with respect to the pair of functions $\underline{\omega}(t)$, $\overline{\omega}(t)$ and the function $h(s) = (s + C_1)/\varepsilon$.

Observe that $\underline{\omega}(-n) < 0 < \overline{\omega}(-n)$ and $\underline{\omega}(n) < 0 < \overline{\omega}(n)$. Therefore for any $u(t)$ which lies between $\underline{\omega}(t)$ and $\overline{\omega}(t)$, it follows from Theorem 2.4 that the boundary value problem (27),(28) has a solution $u_n(t)$. We note that $u_n(t)$ lies between $\underline{\omega}(t)$ and $\overline{\omega}(t)$ and satisfies

$$|\dot{u}_n(t)| \le N_n\,, \quad -n \le t \le n\,,$$

where

$$\int_{\lambda_n}^{N_n} \frac{sds}{h(s)} = \max_{[-n,n]} \overline{\omega}(t) - \min_{[-n,n]} \underline{\omega}(t) = 2K\,.$$

And

$$\lambda_n = \frac{1}{2n} \max(|\overline{\omega}(-n) - \underline{\omega}(n)|, |\overline{\omega}(n) - \underline{\omega}(-n)|)$$
$$= \frac{K}{2n}(e^{-an} + e^{-bn})\,.$$

Now, since $\lim_{n\to-\infty} \lambda_n = 0$ and the right-hand side of the previous equation does not depend on n, thus one can show that N_n has a finite upper bound N. Therefore, we have the following estimates

$$\begin{aligned} &|u_n(t)| \le Ke^a t\,, && -n \le t \le 0 \\ &|u_n(t)| \le Ke^{-bt}, && 0 \le t \le n \\ &|\dot{u}_n(t)| \le N && -n \le t \le n\,. \end{aligned}$$

From a simple computation, one obtains

$$|\ddot{u}_n(t)| \le \frac{|\dot{u}_n(t)| + C_1}{\varepsilon} \le \frac{N + C_1}{\varepsilon}\,, \quad -n \le t \le n\,,$$

where C_1 is the constant. It follows from the argument above that $\{u_n\}$, $\{\dot{u}_n\}$ and $\{\ddot{u}_n\}$ are uniformly bounded sequence and one concludes that $\{u_n\}$ and $\{\dot{u}_n\}$ are also equicontinuous. Therefore an application of the Arzela–Ascoli theorem shows that $\{u_n\}$ and $\{\dot{u}_n\}$ have a convergent subsequence, let them be denoted by $\{u_n\}$ and $\{\dot{u}_n\}$ again.

From the compactness of the operator T_n, we have $\{z(u_n)\}$ and $\{w(u_n)\}$ are uniformly bounded. Since for each $n > 0$, $z(u_n)$ and $w(u_n)$ are the solutions of the system (25), we have $\dot{z}(u_n)$ and $\dot{w}(u_n)$ are uniformly bounded. Hence we can conclude that both $\{z(u_n)\}$ and $\{w(u_n)\}$ are equicontinuous. Again by using the Arsela–Ascoli theorem, we have $\{(u_n(t)),\, \dot{u}_n(t),\, z(u_n)(t),\, w(u_n)(t))\}$ that admits a uniformly convergent subsequence. Let them be denoted by u_n, $\dot{u}_n$, $z(u_n)$ and $w(u_n)$ again, then we obtain the following

$$\lim_{n\to+\infty} (u_n(t), \dot{u}_n(t), z(u_n)(t), w(u_n)(t))$$
$$= (u(t), \dot{u}(t), z(u)(t), w(z)(t))\,.$$

It can be easily seen that this limit function $(u(t),$ $\dot{u}(t)$, $z(u)(t)$, $w(u)(t))$ is a solution of Eq. (23) with the boundary condition (24). Now replace u by $u + u_0$, $z(u)$ by $z + z_0$ and $w(u)$ by $w + w_0$, then (u, z, w) is a solution of the boundary value problem (21),(22). This completes the proof of our main result. ■

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References

Anishchenko, V. S., Safonova, M. A. & Chua, L. O. [1992] "Stochastic resonance in Chua's circuit," *Int. J. Bifurcation and Chaos* **2**, 397–401.

Belykh, V. N. & Chua, L. O. [1992] "New type of strange attractor from a geometric model of Chua's circuit," *Int. J. Bifurcation and Chaos* **2**, 697–704.

Bernfeld, S. R. & Lakshmikantham, V. [1974] *An Introduction to Nonlinear Boundary Value Problems* (Academic Press, NY).

Chow, S.-N., Mallet-Paret, J. & Shen, W. [1998] "Traveling wave in lattice dynamical systems," *J. Diff. Eqs.* **149**, 248–291.

Chua, L. O. [1992] "The genesis of Chua's circuit," *Archiv für Elektronik und Ubertragungstechnik* **46**, 250–257.

Chua, L. O. [1998] *CNN: A Paradigm for Complexity*, World Scientific Series on Nonlinear Science, Series A, Vol. 31, (World Scientific, Singapore).

De Coster, C. & Habets, P. [1996] "Upper and lower solution in the theory of ODE boundary value problem: Classical and recent results," in *Nonlinear Analysis and BVP for ODE*, ed. Zanolin, F. (Springer Wien, NY) Vol. 371, pp. 1–78.

Kahan, S. & Sicardi-Schifino, A. C. [1999] "Homoclinic bifurcations in Chua's circuit," *Physica* **A262**, 144–152.

Kennedy, M. [1992] "Robust op amp realization of Chua's circuit," *Frequenz* **46**, 66–80.

Kocarev, L. & Roska, T. [1993] "Dynamics of the Lorenz equation and Chua's equation: A tutorial," *Chua's Circuit: A Paradigm for Chaos* (World Scientific, Singapore), pp. 25–55.

Perez-Munuzuri, V., Perez-Villar, V. & Chua, L. O. [1992] "Propagation failure in linear arrays of Chua's circuits," *Int. J. Bifurcation and Chaos* **2**, 403–406.

Perez-Munuzuri, V., Perez-Villar, V. & Chua, L. O. [1993] "Traveling wave front and its failure in a one-dimensional array of Chua's circuit," in *Chua's Circuit: A Paradigm for Chaos* (World Scientific, Singapore), pp. 336–350.

Wu, C. W. & Pivka, L. [1993] "From Chua's circuit to Chua's oscillator: A picture book of attractors," *Nonlinear Dynamics of Electronic Systems, Proc. Workshop NDEs'93*, pp. 15–79.

Zhong, G. H. & Ayrom, F. [1985a] "Experimental confirmation of chaos from Chua's circuit," *Int. J. Circuit Th. Appl.* **13**, 93–98.

Zhong, G. H. & Ayrom, F. [1985b] "Periodicity and chaos in Chua's circuit," *IEEE Trans. Circuits Syst.* **CAS 32**, 501–501.