

Networked control systems: a perspective from chaos

Guofeng Zhang

Dept. of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta, Canada T6G 2G1
Email: gzhang@math.ualberta.ca

Tongwen Chen

Dept. of Electrical and Computer Engineering
University of Alberta
Edmonton, Alberta, Canada T6G 2V4
Email: tchen@ece.ualberta.ca

Abstract

In this paper, a nonlinear system aiming at reducing the signal transmission rate in a networked control system is constructed by adding nonlinear constraints to a linear feedback control system. Its stability is investigated in detail. It turns out that this nonlinear system exhibits very interesting dynamical behaviors: in addition to local stability, its trajectories may converge to a non-origin equilibrium or be periodic or just be oscillatory. Furthermore it exhibits sensitive dependence on initial conditions — a sign of chaos. Complicated bifurcation phenomena are exhibited by this system. After that, control of the chaotic system is discussed. All these are studied under scalar cases in detail. Some difficulties involved in the study of this type of systems are analyzed. Finally an example is employed to reveal the effectiveness of the scheme in the framework of networked control systems.

Keywords: stability, attractor, nonlinear constraint, chaos, bifurcation, tracking, networked control systems.

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1 Introduction

1.1 Limited information related control

In the past decade great interest has been devoted to the study of *limited information related* control problems. Limited information related control is defined as follows: Given a physical plant G and a set of performance specifications such as tracking, design a controller C based on limited information such that the resulting closed-loop system meets the prespecified performance specifications. There are generally two sources of limited information, one is signal quantization, and the other is signal transmission through various networks.

In designing a digital control system, signal quantization induced by signal converters such as A/D, D/A and computer finite word-length limitation is unavoidable. To compensate this, traditional design methods generally proceed like this: First design a controller ignoring the effect of signal quantization, then model it as external white noise and analyze its effect on the designed system. If the performance is acceptable, it is okay; otherwise, adjust controller parameters such as the sampling frequency, or do redesign (including the choice of converters) until satisfactory performance is obtained. Recently the following problems have been asked:

1. How to study the effect of signal quantization more rigorously? More precisely, how will it genuinely affect the performance of the underlying control system?
2. If there are positive answers to the above question, can one design better controllers based on this knowledge?

To address these two problems, stability, the fundamental requirement of a control system, has been studied recently in somewhat detail. Delchamps [1990] studied the problem of stabilizing an *unstable* linear time-invariant discrete-time system via state feedback where the state is quantized by an arbitrarily given quantizer of *fixed* quantization sensitivity. It turned out that there are no state feedback strategies ensuring asymptotic stability of the closed-loop system in the sense of Lyapunov. Instead, the resulting closed-loop system behaves chaotically. Fagnani & Zampieri [2003] continued this research in the context of a linear discrete-time scalar system. Based on the flow information provided by the system invoked by quantization, stabilizing methods based on the Lyapunov approach and chaotic dynamics of the system were discussed. Ishii & Francis [2003] studied the quadratic stabilization of an unstable linear time-invariant continuous-time system by designing a digital controller whose input was the quantized system state; an upper bound of sampling periods was calculated geometrically using state feedback for the system G with a carefully designed quantizer of fixed quantization sensitivity, by which the trajectories of the closed-loop system would enter and stay in a region of attraction around the origin. Clearly in order to achieve asymptotic stability, quantizers with *variable* quantization sensitivities must be adopted. In Brockett & Liberzon [2000], for the system G , by choosing a quantizer q with time-varying sensitivities, a linear time-invariant feedback was designed to yield global asymptotic stability. This problem was also studied in Elia & Mitter [2001] for exponential stability using

logarithmic quantizers. In Nair & Evans [2002], exponential stabilization of the system G with a quantizer is studied under the framework of probability theory. More interestingly, the simultaneous effect of sampling period T and quantization sensitivity was studied in Bamieh [2003], where it is shown via simulation that system performance would become *unbounded* as $T \rightarrow 0$ if a quantizer of fixed sensitivity was inserted into a control loop composed of a system and an unstable controller. Therefore it is fair to say that the problem—performance of quantized systems—is quite complicated as well as challenging. Much research is still required in this area.

Another representation of limited information is signals suffering from time-delays or even loss, which are ubiquitous in the networked control systems (Wong & Brockett [1997], Walsh *et al.* [2001], and Ray [1987]). The fast-developing secure, high speed networks (Varaiya & Walrand [1996] and Peterson & Davie [2000]) make control over networks possible. Compared to the traditional point-to-point connection, the main advantages of connecting various system components such as processes, controllers, sensors and actuators via communication networks are wire reduction, low cost and easy installation and maintenance, etc. Thanks to these merits, networked control systems have been built successfully in various fields such as automobiles (Krtolica *et al.* [1994], Ozguner *et al.* [1992]), aircrafts (Ray [1987] and Sparks [1997]), robotic controls (Malinowshi *et al.* [2001], Safaric *et al.* [1999]) and so on. In addition, in the field of distributed control, networks may provide distributed subsystems with more information so that performance can be improved (Ishii & Francis [2002]). However, networks inevitably introduce time delays and packet dropouts due to network propagation, signal computation and coding, congestion, etc., which lead to limited information for the system to be controlled as well as the controller, thus complicating the design of controllers and degrading the performance of control systems or even destabilizing them (Zhang *et al.* [2001]). Therefore it is very desirable to reduce time delays and packet dropouts when implementing a networked control system. For the limitation of space, for now we will concentrate on discussing a network protocol proposed by Walsh, Beldiman, Bushnell, and Hong, *et al.* (Walsh *et al.* [1999, 2001, 2002a, 2002b]) since our proposed one is in the same spirit as theirs. For a more complete review on networked control systems and more references, please refer to Zhang & Chen [2003].

1.2 Network based control

One effective way to avoid large time delays and high probability of packet dropouts is by reducing network traffic. In a series of papers published by Walsh, Beldiman, Bushnell, and Hong, *et al.* (Walsh *et al.* [1999, 2001, 2002a, 2002b]), a network protocol called try-once-discard (TOD) is proposed. In that scheme, there is a network along the route from a MIMO plant to its controller. At each transmission time, each sensor node calculates the importance of its current value by comparing it with the latest one, the larger the difference is, the more important the current value is, then the most important one gets access to the network. For this scheme, based on the Lyapunov method and the perturbation theory, a minimal time within which there must have at least one network transmission to guarantee stability of networked control systems is derived.

This network protocol, TOD, essentially belongs to the category of dynamical schedulers. In comparison with static schedulers such as token rings, it allocates network resources more effectively. However, a supervisor computer, i.e., a central controller, is required to compare those differences and decide which node should get access to the network at each transmission time. It is therefore complicated and possibly difficult to implement. In this paper, we introduce another technique aiming at reducing network traffic.

1.3 A new networked control technique

Consider the feedback system in Fig. 1, where G is a discrete-time system of the form:

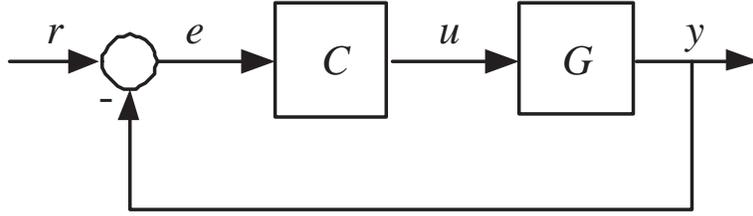


Fig. 1. A standard feedback system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \quad (1)$$

with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}^m$, the output $y \in \mathbb{R}^p$ and the reference input $r \in \mathbb{R}^p$ respectively; C is a stabilizing controller:

$$\begin{aligned} x_d(k+1) &= A_d x_d(k) + B_d e(k), \\ u(k) &= C_d x_d(k) + D_d e(k), \\ e(k) &= r(k) - y(k), \end{aligned} \quad (2)$$

with its state $x_d \in \mathbb{R}^{n_c}$. Let $\xi = \begin{bmatrix} x \\ x_d \end{bmatrix}$, then the closed-loop system from r to e can be modeled by

$$\begin{aligned} \xi(k+1) &= \begin{bmatrix} A - BD_d C & BC_d \\ -B_d C & A_d \end{bmatrix} \xi(k) + \begin{bmatrix} BD_d \\ B_d \end{bmatrix} r(k), \\ e(k) &= \begin{bmatrix} -C & 0 \end{bmatrix} \xi(k) + r(k). \end{aligned} \quad (3)$$

Now we add nonlinear constraints on both u and y . Specifically, consider the system in Fig. 2. The nonlinear constraint H_1 is defined as, for a given $\delta_1 > 0$, let $v(-1) = 0$, and for $k \geq 0$,

$$v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } \|u_c(k) - v(k-1)\|_\infty > \delta_1, \\ v(k-1), & \text{otherwise.} \end{cases} \quad (4)$$

Similarly H_2 is defined as, for a given $\delta_2 > 0$, let $z(-1) = 0$, and for $k \geq 0$,

$$z(k) = H_2(y_c(k), z(k-1)) = \begin{cases} y_c(k), & \text{if } \|y_c(k) - z(k-1)\|_\infty > \delta_2, \\ z(k-1), & \text{otherwise.} \end{cases} \quad (5)$$

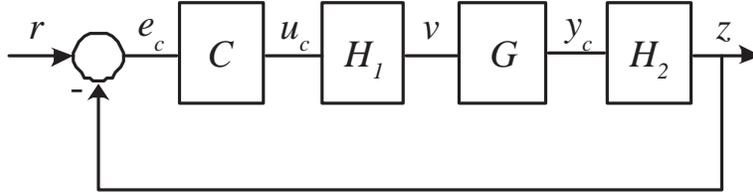


Fig. 2. A constrained feedback system

It can be shown that $\|H_1\|$, the induced norm of H_1 , equals 2, so is $\|H_2\|$.

In a networked control system, there are normally computer networks along the routes from the controller C to the system G and from G to C . These networks (usually shared by other clients) will introduce time delays into the closed-loop system. It is quite appealing to compensate this adverse effect. If we regard H_1 as a component of C and H_2 of G , G (resp. C) contains previous version of u_c (resp. y_c), then there will have no signal transmission from C to G and (or) from G to C if the inequalities in Eqs. (4)-(5) are not satisfied, suggesting that we are reducing network traffic. We expect this will benefit the overall system connected by the common networks. One example will be given in Sec. 3 to illustrate this point.

Similar work is done in Otanez *et al.* [2003] where *adjustable* deadbands are proposed to reduce network traffics. In that formulation, the closed-loop system with deadbands is modeled as a perturbed system, then its exponential stability follows that of the original system [Khalil 1996]. The constraints proposed here are fixed (δ_1 and δ_2), we will see the stability of the system in Fig. 2 is quite complicated (e.g., only local stability can be obtained). However, the advantage of fixed deadbands is that it will reduce network traffic more effectively. Furthermore, the stability region can be scaled as large as desired. This is one advantage of our proposed scheme. Moreover, we find out that the system in Fig. 2 has rather complex dynamics — it appears chaotic. As is known chaotic behavior will in general provide more system dynamics, i.e., more information of the underlying system, therefore we hope we can achieve better control in the framework of Fig. 2. We will address this problem more rigorously in Sec. 2.2.

For the “constrained” system in Fig. 2, let p denote the state of the system G , and p_d denote the state of the controller C , then

$$\begin{aligned} p(k+1) &= Ap(k) + Bv(k), \\ y_c(k) &= Cp(k), \end{aligned}$$

and

$$\begin{aligned} p_d(k+1) &= A_d p_d(k) + B_d e_c(k), \\ u_c(k) &= C_d p_d(k) + D_d e_c(k), \\ e_c(k) &= r(k) - z(k). \end{aligned}$$

Let $\eta = \begin{bmatrix} p \\ p_d \end{bmatrix}$, then the closed-loop system from r to e is

$$\begin{aligned} \eta(k+1) &= \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix} + \begin{bmatrix} 0 \\ B_d \end{bmatrix} r(k), \\ e_c(k) &= \begin{bmatrix} -C & 0 \end{bmatrix} \eta(k) + r(k), \end{aligned} \quad (6)$$

where v and z are given in Eqs. (4)-(5).

To test whether the scheme adopted here is useful in the framework of networked control systems, we have to address at least the following two concerns:

- The stability of the system in Fig. 2. Since stability is fundamental to any control system, the first question about this system is its stability. In this paper, the Lyapunov stability is studied in detail:
 1. Given that both G and C are stable, The system is *locally* exponentially stable (Lemma 1).
 2. However, the behavior of the state trajectory (p, p_d) , starting outside the stability region, is hard to predict. A scalar case is studied in detail to illustrate various dynamics the system can exhibit (Sec. 2.1): Its trajectory may converge to an equilibrium which is not necessarily the origin (Proposition 1, Corollary 1), or be periodic (Theorem 3, Theorem 4), or aperiodic (Theorem 1, Theorem 2), which can either be quasiperiodic or exhibit sensitive dependence on initial conditions — a sign of chaos, advocating novel control method — chaotic control.
 3. For higher-order cases, a positively invariant set is constructed (Theorem 5).
 4. Finally it is proved that the set of all initial points $\eta(0)$ whose closed-loop trajectories tend to an equilibrium as $k \rightarrow \infty$ has Lebesgue measure zero if either G or C is unstable (Theorem 6).
- This research is mainly devoted to the study of networked control systems (NCSs), hence it is natural and necessary to analyze its effectiveness in the framework of networked control systems. An example is used to illustrate the efficacy of our scheme (Sec. 3).

The outline of this paper as follows. Sec. 2 is devoted to the study of stability. An example is constructed to show the effectiveness of our scheme in Sec. 3. Some concluding remarks are in Sec. 4.

2 Stability

In this section, we discuss the stability of the system in Eq. (6). Firstly a sufficient condition ensuring local exponential stability is derived. Secondly concentrated mainly on scalar cases, the intriguing behavior of the dynamics of the system is studied in detail. It appears that the system behaves chaotically. Finally it is proven that the Lebesgue measure of the set of trajectories converging to a certain equilibrium is zero if either the system G or the controller C is unstable.

Letting $r = 0$, the system in Eq. (6) becomes

$$\begin{aligned} \eta(k+1) &= \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix} \eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix}, \\ \begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} &= \begin{bmatrix} 0 & C_d \\ C & 0 \end{bmatrix} \eta(k) + \begin{bmatrix} 0 & D_d \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix}, \\ \begin{bmatrix} v(k) \\ -z(k) \end{bmatrix} &= \begin{bmatrix} H_1(u_c(k), v(k-1)) \\ -H_2(y_c(k), z(k-1)) \end{bmatrix}, \quad k \geq 0. \end{aligned} \tag{7}$$

Then, we have the following result regarding local stability.

Lemma 1 *If both the system G and the controller C are stable, then the origin is locally exponentially stable.*

Proof: Define

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & A_d \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} 0 & C_d \\ C & 0 \end{bmatrix}.$$

Since both G and C are stable, $\rho(\tilde{A}) < 1$ where $\rho(M)$ is the spectral radius of a square matrix M . Then for any given $\varepsilon > 0$ satisfying $\rho(\tilde{A}) + \varepsilon < 1$, there exists a matrix norm $\|\cdot\|_*$ such that $\|\tilde{A}\|_* \leq \rho(\tilde{A}) + \varepsilon$ [Huang, 1984]. Furthermore, this matrix norm satisfies $\|MN\|_* \leq \|M\|_* \|N\|_*$ for any two matrices M and N of dimension $n + n_c$. Therefore, for a vector x of dimension $n + n_c$, one can define a vector norm $|x|_*$ such that $|Mx|_* \leq \|M\|_* |x|_*$. One way to define such a norm is the following: Let \mathcal{O} denote the zero vector of dimension $n + n_c$, define

$$|x|_* := \left\| \begin{bmatrix} x, \underbrace{\mathcal{O}, \dots, \mathcal{O}}_{n+n_c-1} \end{bmatrix} \right\|_*,$$

then

$$|Mx|_* = \|[Mx, \mathcal{O}, \dots, \mathcal{O}]\|_* \leq \|M\|_* \|[x, \mathcal{O}, \dots, \mathcal{O}]\|_* = \|M\|_* |x|_*.$$

For a vector ω of dimension $\nu < n + n_c$, denote by \mathcal{O} the zero vector of dimension $n + n_c - \nu$, define $|\omega|_* := \left\| \begin{bmatrix} \omega' & \mathcal{O}' \end{bmatrix}' \right\|_*$, then $|\cdot|_*$ is a norm on the vector space $\mathbb{R}^{\nu \times 1}$. We treat a matrix of dimension less than $n + n_c$ in the similar way.

Let $\|\cdot\|_1$ be the induced matrix norm of the vector norm $\|\cdot\|_\infty$, then there exist positive constants c_1 and c_2 such that $c_1 \|M\|_* \leq \|M\|_1 \leq c_2 \|M\|_*$ for any matrix $M \in \mathbb{R}^{n+n_c}$. Let $\delta := \min\{\delta_1, \delta_2\}$, then $\|M\|_1 \leq \delta$ if $\|M\|_* \leq \delta/c_2$. Hence, in the sequel we concentrate on the matrix norm $\|\cdot\|_*$ and the upper bound δ/c_2 . Now we are ready to derive the local stability of the system in Eq. (7). We claim that the stability region contains a ball centered at the origin with radius

$$rd := \min \left\{ \frac{\delta_1}{c_2 \left\| \begin{bmatrix} 0 & C_d \end{bmatrix} \right\|_*}, \frac{\delta_2}{c_2 \left\| \begin{bmatrix} C & 0 \end{bmatrix} \right\|_*} \right\} \tag{8}$$

(denoted $\mathfrak{B}(0, rd)$).

Suppose $|\eta(0)|_* \leq rd$, by Eq. (7),

$$|y_c(0)|_* \leq \|[C \ 0]\|_* |\eta(0)|_* \leq \frac{\delta_2}{c_2},$$

then

$$\|y_c(0)\|_\infty \leq \delta_2,$$

hence

$$z(0) = H_2(y_c(0), z(k-1)) = z(-1) = 0.$$

Therefore

$$|u_c(0)|_* \leq \|[0 \ C_d]\|_* |\eta(0)|_* \leq \frac{\delta_1}{c_2},$$

which means

$$\|u_c(0)\|_\infty \leq \delta_1,$$

and

$$v(0) = H_1(u_c(0), v(k-1)) = v(-1) = 0.$$

Then

$$\eta(1) = \tilde{A}\eta(0).$$

Similarly,

$$\begin{aligned} |y_c(1)|_* &\leq \|[C \ 0]\|_* |\eta(1)|_* = \|[C \ 0]\|_* \|\tilde{A}\|_* |\eta(0)|_* \\ &\leq \|[C \ 0]\|_* \left(\rho(\tilde{A}) + \varepsilon\right) |\eta(0)|_* \leq \frac{\delta_2}{c_2}, \end{aligned}$$

$$\|y_c(1)\|_\infty \leq \delta_2,$$

$$H_2(y_c(1), z(0)) = z(0) = 0.$$

Moreover,

$$\begin{aligned} |u_c(1)|_* &\leq \|[0 \ C_d]\|_* |\eta(1)|_* = \|[0 \ C_d]\|_* \|\tilde{A}\|_* |\eta(0)|_* \\ &\leq \|[0 \ C_d]\|_* \left(\rho(\tilde{A}) + \varepsilon\right) |\eta(0)|_* \leq \frac{\delta_1}{c_2}, \end{aligned}$$

which means

$$\|u_c(1)\|_\infty \leq \delta_1,$$

and

$$v(1) = H_1(u_c(1), v(0)) = v(0) = 0.$$

Then

$$\eta(2) = \tilde{A}\eta(1) = \tilde{A}^2\eta(0)$$

implying there is no updating for the inputs to G and C . Following this process, we see

$$\eta(k) = \tilde{A}^k \eta(0)$$

converges to zero as k tends to ∞ . ■

Remark 1: Though this system is locally exponentially stable, it is hard to find the exact stability region except for a scalar system controlled by a static feedback. However, even in this scalar case, very complex dynamics can be exposed by the system. This is the topic of the next subsection.

2.1 Scalar case

In this part, the definitions of such concepts as (positively) invariant sets, topological transitivity, structural stability, invariant sets and ω -limit sets, etc., are adopted from Robinson [1995] or Robinson [2004] unless otherwise specified.

To get a flavor of the complexity that the system in Fig. 2 may exhibit, we first study a simple one-dimensional system:

$$\begin{aligned} x(k+1) &= ax(k) + bv(k), \\ u_c(k) &= x(k), \end{aligned} \tag{9}$$

with $v(-1) \in \mathbb{R}$ without loss of generality, and for $k \geq 0$,

$$v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } |u_c(k) - v(k-1)| > \delta_1, \\ v(k-1), & \text{else,} \end{cases}$$

where $\delta_1 = 0.01$. The system in Eq. (9) is a static state feedback system with feedback gain equal to 1. Note that in this example there is no constraint on the output of the system G . Now let $a = 9/10$ and $b = -3/10$. By choosing different initial values $(v(-1), x(0))$, Figs. 3–4 are obtained. In these two figures, the horizontal axis

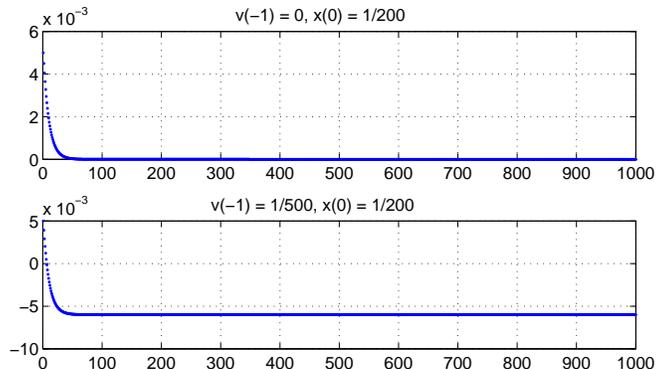


Fig. 3. Two trajectories converging to two different fixed points

stands for the iteration time k , and the vertical axis denotes the value of x . It is clear from these two figures

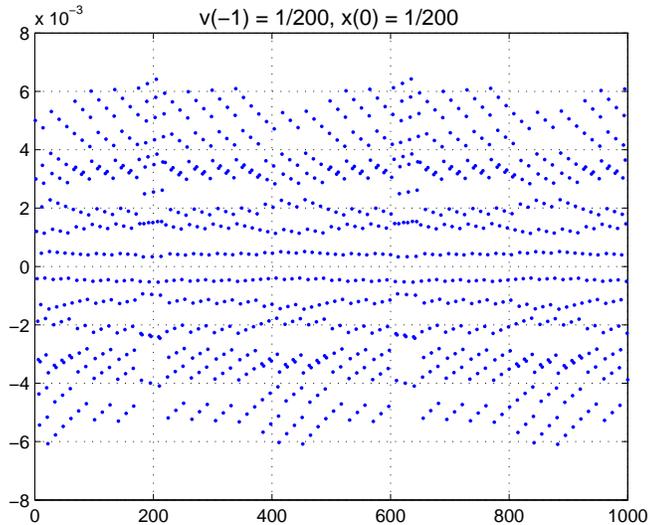


Fig. 4. An aperiodic trajectory

that different initial conditions give rise to significantly different types of trajectories: the first converging to the origin and the second converging to a non-origin point and the last just oscillating. Furthermore, the system in Eq. (9) is actually able to exhibit “chaotic” behavior, i.e., sensitive dependence on initial conditions. Fig. 5 reveals this phenomenon clearly. Is the trajectory in the lower part of Fig. 5 aperiodic? Fig. 6 is its spectrum

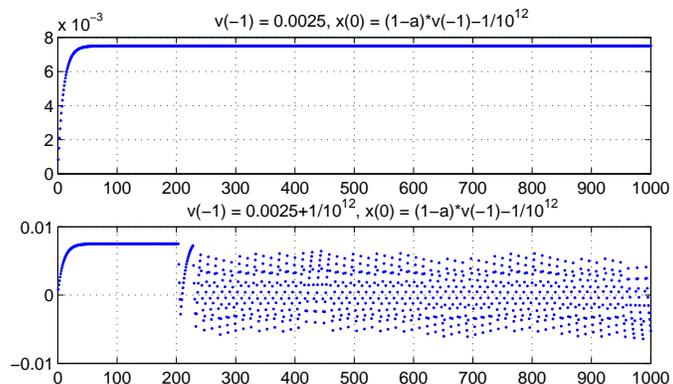


Fig. 5. Sensitive dependence on initial conditions

produced using the function “pmtm” in Matlab. One can see that this trajectory contains a broad band of frequencies.

Next let $a = 1$ and $b = -3/10$, and we get Figs. 7-8 where the horizontal axis denotes $v(k - 1)$ and the vertical axis stands for $x(k)$. The first two (in Fig. 7) are eventually periodic orbits of different periods, the third one (in Fig. 8) is aperiodic.

The complicated behavior of the system in Fig. 2 is due to its nonlinearity. To some extent, invariant sets provide some measure of how complex the dynamics of a system is. According to the above examples, the

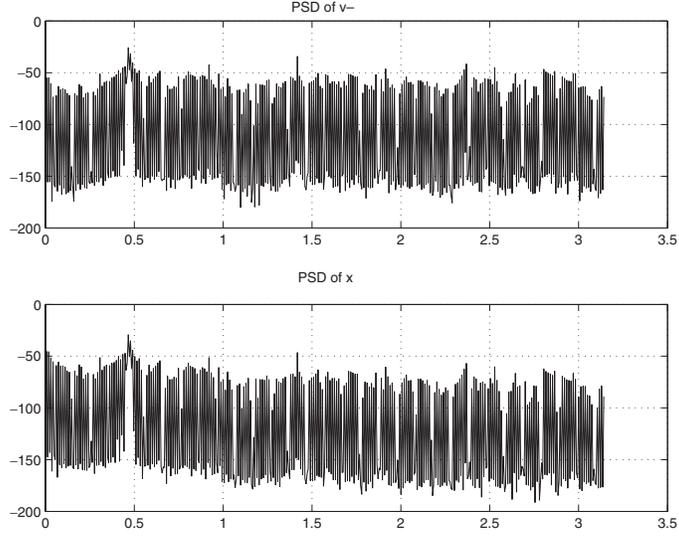


Fig. 6. Spectrum of an aperiodic orbit

invariant sets of the system in Eq. (9) contain not only the origin, non-origin fixed points (Fig. 3), but also periodic (Fig. 7) and aperiodic orbits (Fig. 8). Furthermore, it may contain a strange attractor if chaos is indeed present in the system. In the rest of this subsection, we will analyze the dynamics of this system. We always assume that $|a + b| < 1$ which guarantees the boundedness of trajectories of the system.

2.1.1 Case 1: $|a| < 1$

For convenience, define

$$\xi(k) := \begin{bmatrix} v(k-1) \\ x(k) \end{bmatrix},$$

then the system can be written as

$$\begin{aligned} \xi(k+1) &= \begin{bmatrix} 1 & 0 \\ b & a \end{bmatrix} \xi(k) + s_k \begin{bmatrix} -1 & 1 \\ -b & b \end{bmatrix} \xi(k) \\ &:= (A + s_k B) \xi(k) := F(\xi(k)), \quad \forall k \geq 0, \end{aligned} \quad (10)$$

and

$$\begin{aligned} s_k &= 1 \text{ if } |x(k) - v(k-1)| > \delta; \\ s_k &= 0 \text{ if } |x(k) - v(k-1)| \leq \delta. \end{aligned} \quad (11)$$

Based on this representation, the fixed points of the system are the line segment:

$$x = \frac{b}{1-a} v_-, \quad (12)$$

within the region:

$$|x - v_-| \leq \delta. \quad (13)$$

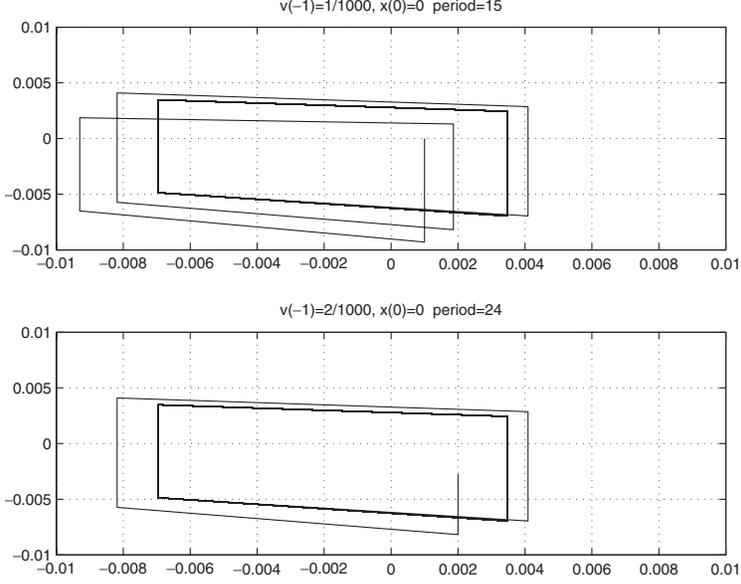


Fig. 7. Two periodic orbits

(Note v_- indicates that v is one step behind x .) For the local stability of fixed points, we have the following result.

Proposition 1 *For the system in Eq. (10) with $|a| < 1$, a local stability region, denoted by $R_{loc} \subset \mathbb{R}^2$, of its fixed points is the region encircled by*

$$|x - v_-| = \delta, \quad (14)$$

and

$$|v_-| = \frac{1 - |a|}{1 - (a + b)} \delta. \quad (15)$$

Proof: Given an initial point $(v(-1), x(0)) \in R_{loc}$, we have

$$x(1) = ax(0) + bv(-1).$$

In general,

$$x(k) = a^k x(0) + \sum_{i=0}^{k-1} a^i b v(-1), \quad (16)$$

provided that

$$|x(k) - v(-1)| \leq \delta, \quad \forall k > 0. \quad (17)$$

Now we show that Eq. (17) indeed holds.

Since

$$\begin{aligned} x(k) - v(-1) &= a^k x(0) + \sum_{i=0}^{k-1} a^i b v(-1) - v(-1) \\ &= a^k (x(0) - v(-1)) + \left(1 - a^k\right) \frac{b + a - 1}{1 - a} v(-1), \end{aligned}$$

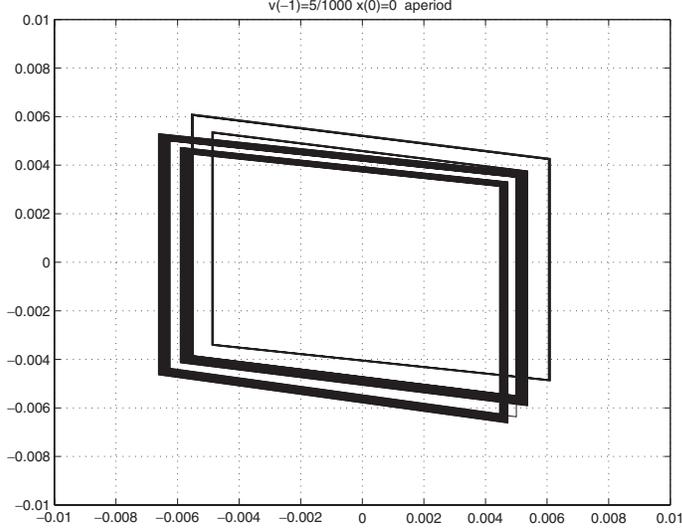


Fig. 8. An aperiodic orbit

one has

$$|x(k) - v(-1)| \leq |a^k| |x(0) - v(-1)| + (1 - a^k) \frac{1 - (a+b)}{1-a} |v(-1)|.$$

If $0 \leq a < 1$, then

$$\begin{aligned} |x(k) - v(-1)| &\leq a^k \delta + (1 - a^k) \frac{1 - (a+b)}{1-a} \frac{1-a}{1-(a+b)} \delta \\ &= \delta. \end{aligned}$$

If $-1 < a < 0$ and $a^k > 0$, then

$$\begin{aligned} |x(k) - v(-1)| &\leq a^k \delta + (1 - a^k) \frac{1 - (a+b)}{1-a} \frac{1+a}{1-(a+b)} \delta \\ &= \left(a^k + (1 - a^k) \frac{1+a}{1-a} \right) \delta \\ &\leq (a^k + (1 - a^k)) \delta = \delta. \end{aligned}$$

If $-1 < a < 0$ and $a^k < 0$, then

$$\begin{aligned} |x(k) - v(-1)| &\leq -a^k \delta + (1 - a^k) \frac{1 - (a+b)}{1-a} \frac{1+a}{1-(a+b)} \delta \\ &= \left(-a^k + (1 - a^k) \frac{1+a}{1-a} \right) \delta \\ &= \frac{1+a - 2a^k}{1-a} \delta. \end{aligned}$$

Therefore it suffices to show that

$$\frac{1+a - 2a^k}{1-a} \leq 1.$$

However, it is equivalent to

$$a \leq a^k,$$

which holds for $-1 < a < 0$ and $a^k < 0$. By taking limit in Eq. (16) with respect to k , $(v(k-1), x(k))$ converges to a fixed point defined by Eqs. (12)-(13). The proof is completed. ■

Having identified a local stability region, next we will study the following problem: Can the actual stability region of the fixed points be larger than the region given in Proposition 1? We will see that this problem is actually a difficult one in that it heavily depends on system parameters. Before doing so, we first concentrate on the “one-dimensional case”, i.e., the dynamics of x , to get the globally attracting region of x of the system in Eq. (9). We have the following result.

Proposition 2 *The globally attracting region of x is given by*

$$|x| \leq \frac{|b|}{1 - |a + b|} \delta. \quad (18)$$

Furthermore, it is positively invariant.

Proof: According to Eq. (9),

$$\begin{aligned} x(1) &= ax(0) + bv(0) = (a + b)x(0) + b(v(0) - x(0)), \\ x(2) &= ax(1) + bv(1) = (a + b)^2 x(0) + (a + b)b(v(0) - x(0)) + b(v(1) - x(1)), \\ &\vdots \\ x(k) &= (a + b)^k x(0) + \sum_{i=0}^{k-1} (a + b)^i b(v(k-1-i) - x(k-1-i)), \end{aligned}$$

hence

$$|x(n)| \leq |a + b|^n |x(0)| + \frac{1 - |a + b|^n}{1 - |a + b|} |b| \delta, \quad \forall n \geq 1. \quad (19)$$

By taking limit on both sides, one gets Eq. (18). Moreover, if

$$|x(0)| \leq \frac{|b|}{1 - |a + b|} \delta,$$

then

$$|x(n)| \leq \frac{|b|}{1 - |a + b|} \delta, \quad \forall n \geq 1,$$

which means that the region given by Eq. (18) is positively invariant. ■

Based on this observation, we are ready to derive a positive invariant set for the system in Eq. (10).

Theorem 1 *For the system in Eq. (10), if*

$$\frac{|b|}{1 - |a + b|} > \frac{1 - |a|}{1 - (a + b)},$$

then region defined by

$$|x| \leq \frac{|b|}{1 - |a + b|} \delta$$

and

$$|v_-| \leq \frac{|b|}{1 - |a + b|} \delta$$

is a positively invariant set. Otherwise, the region defined by

$$|x| \leq \frac{1 - |a|}{1 - (a + b)} \delta$$

and

$$|v_-| \leq \frac{1 - |a|}{1 - (a + b)} \delta$$

is globally attracting, which indicates that the fixed points given by Eqs. (12)-(13) are the only invariant set of the system (For convenience, we call such a system a generic system).

Proof: It readily follows from Proposition 1 and Proposition 2. ■

The following result is an immediate consequence of Theorem 1.

Corollary 1 *If the system in Eq. (10) satisfies either of*

- $a > 0$ and $b > 0$;
- $a < 0$ and $b < 0$,

then it is a generic system.

Proof: Suppose $a > 0$ and $b > 0$. Then

$$\frac{|b|}{1 - |a + b|} = \frac{b}{1 - (a + b)} \leq \frac{1 - a}{1 - (a + b)} = \frac{1 - |a|}{1 - (a + b)}.$$

Hence the system is generic. On the other hand, given $a < 0$ and $b < 0$,

$$\frac{|b|}{1 - |a + b|} = \frac{-b}{1 + (a + b)}, \quad \frac{1 - |a|}{1 - (a + b)} = \frac{1 + a}{1 - (a + b)}.$$

Since

$$\frac{-b}{1 + (a + b)} \leq \frac{1 + a}{1 - (a + b)}$$

is equivalent to

$$a^2 \leq 1 + b^2,$$

which says

$$\frac{|b|}{1 - |a + b|} \leq \frac{1 - |a|}{1 - (a + b)},$$

i.e., the system is generic. ■

Theorem 1 tells us that, in order to have complex dynamics,

$$\frac{|b|}{1 - |a + b|} > \frac{1 - |a|}{1 - (a + b)} \quad (20)$$

must be satisfied. However, this is *not* a sufficient condition. For the case when

$$a = 9/10, \quad b = -3/10,$$

(which satisfies Eq. (20)), we have already known that the system exhibits complicated dynamics (see Figs. 3-6). However, for the case when

$$a = 3/10, \quad b = -9/10,$$

which also satisfies Eq. (20), there is no complex dynamic behavior, i.e., the system is generic. The following argument provides a sufficient proof for this specific system.

Given $(v(-1), x(0))$ satisfying

$$|x(0) - v(-1)| > \delta,$$

one has

$$\begin{aligned} x(1) &= (a + b)x(0), \\ v(0) &= x(0). \end{aligned}$$

Suppose

$$|x(1) - v(0)| > \delta,$$

then

$$|x(0)| > \frac{\delta}{1 - (a + b)}, \quad (21)$$

and

$$\begin{aligned} x(2) &= (a + b)x(1) = (a + b)^2 x(0), \\ v(1) &= x(1) = (a + b)x(0). \end{aligned}$$

If

$$|x(2) - v(1)| \leq \delta, \quad (22)$$

and

$$|v(1)| \leq \frac{1 - |a|}{1 - (a + b)} \delta,$$

then the trajectory will converge to some fixed point. Meanwhile,

$$|x(0)| \leq \frac{1 - |a|}{1 - (a + b)} \frac{1}{|a + b|} \delta. \quad (23)$$

Note that Eq. (22) holds given Eq. (21). Therefore, only

$$\frac{\delta}{1 - (a + b)} \leq \frac{1 - |a|}{1 - (a + b)} \frac{1}{|a + b|} \delta \quad (24)$$

is required. Moreover, Eq. (24) is equivalent to

$$-b \leq 1. \quad (25)$$

Systems with

$$a = 3/10, b = -9/10, \quad (26)$$

and

$$a = 8/10, b = -9/10, \quad (27)$$

both satisfy Eq. (25). However, for a sufficiently large time k , any trajectory $(v(k-1), x(k))$ governed by Eq. (26) will satisfy

$$|x(k) - v(k-1)| > \delta,$$

and

$$|x(k)| > \frac{\delta}{1 - (a + b)}.$$

consequently, it will converge to a fixed point. However, any trajectory $(v(k-1), x(k))$ governed by Eq. (27) violates these two conditions, predicting complex dynamics, see Fig. 11 below.

We have already analyzed three cases:

- $a > 0$ and $b > 0$;
- $a < 0$ and $b < 0$;
- $a > 0$ and $b < 0$.

What about the case when $a < 0$ and $b > 0$? Next we will prove that such a system is generic. It is easy to see that the transition matrix of the system in Eq. (9) is some combination of $(a + b)^k$ and $\left(a^m + \sum_{i=0}^{m-1} a^i b\right)$ with scalar multiplication as the involved operation, where $k \geq 0$ and $m > 1$, since $|a + b| < 1$, if

$$\left| a^m + \sum_{i=0}^{m-1} a^i b \right| < 1 \quad (28)$$

for all $m > 1$, the state x will tend to the origin unless it reaches a fixed point. In this case, the system is generic. Via simple manipulation, Eq. (28) is equivalent to

$$0 \leq (1 - a^m) \frac{1 - (a + b)}{1 - a} \leq 2. \quad (29)$$

Given $a < 0$ and $b > 0$, define

$$f(m) := (1 - a^m) \frac{1 - (a + b)}{1 - a}, \quad \forall m > 1,$$

then

$$f(m) \geq 0, \quad \forall m > 1,$$

and

$$f(3) = \max_{m>1} f(m).$$

However,

$$\begin{aligned} f(3) - 2 &= (1 - a^3) \frac{1 - (a + b)}{1 - a} - 2 \\ &= (1 + a + a^2)(1 - (a + b)) - 2 \\ &\leq 0. \end{aligned}$$

hence, Eq. (29) (then Eq. (28)) holds for all $m > 1$, which means the system is generic.

In the rest of this part, we will concentrate on a specific system and study its complexity. Consider the system in Fig. 2, where

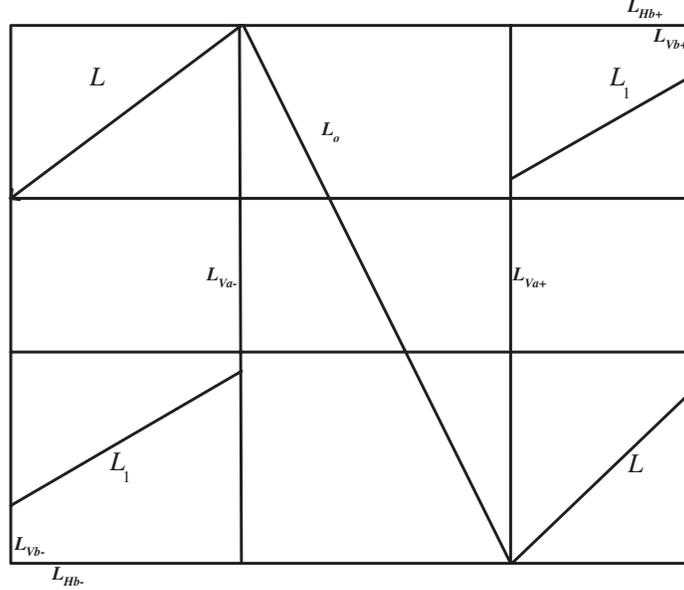


Fig. 9. Diagram for the case of $a = 0.9$ and $b = -0.3$

$$\begin{aligned} x(k+1) &= ax(k) + bv(k), \\ u_c(k) &= x(k), \end{aligned} \tag{30}$$

with $a = 0.9$, $b = -0.3$, $v(-1) \in \mathbb{R}$, and for $k \geq 0$,

$$v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } |u_c(k) - v(k-1)| > 0.01, \\ v(k-1), & \text{else.} \end{cases}$$

We make the following definitions (Fig. 9):

$$\begin{aligned}
L_{Hb+} &:= \left\{ \left(v_-, \frac{-b}{1-|a+b|} \delta \right) : |v_-| \leq \frac{-b}{1-|a+b|} \delta \right\}, \\
L_{Hb-} &:= \left\{ \left(v_-, \frac{b}{1-|a+b|} \delta \right) : |v_-| \leq \frac{-b}{1-|a+b|} \delta \right\}, \\
L_{Vb+} &:= \left\{ \left(\frac{-b}{1-|a+b|} \delta, x \right) : |x| \leq \frac{-b}{1-|a+b|} \delta \right\}, \\
L_{Vb-} &:= \left\{ \left(\frac{b}{1-|a+b|} \delta, x \right) : |x| \leq \frac{-b}{1-|a+b|} \delta \right\}, \\
I_b &:= \left\{ (v_-, x) : |v_-| \leq \frac{-b}{1-|a+b|} \delta, |x| \leq \frac{-b}{1-|a+b|} \delta \right\}. \\
\\
L_{Va+} &:= \left\{ \left(\frac{1-|a|}{1-(a+b)} \delta, x \right) : |x| \leq \frac{-b}{1-(a+b)} \delta \right\}, \\
L_{Va-} &:= \left\{ \left(-\frac{1-|a|}{1-(a+b)} \delta, x \right) : |x| \leq \frac{-b}{1-(a+b)} \delta \right\}, \\
I_a &:= \left\{ (v_-, x) \in I_b : |v_-| \leq \frac{1-|a|}{1-(a+b)} \delta, |x| \leq \frac{-b}{1-|a+b|} \delta \right\}. \\
\\
L_o &:= \left\{ (v_-, x) \in I_a : x = \frac{b}{1-a} v_- \right\}, \\
L_{\delta+} &:= \{ (v_-, x) \in I_b : x - v_- = \delta \}, \\
L_{\delta-} &:= \{ (v_-, x) \in I_b : x - v_- = -\delta \}, \\
L_{1\delta+} &:= \left\{ (v_-, x) \in I_b : x = (a+b) v_-, v_- > \frac{1-|a|}{1-(a+b)} \delta \right\}, \\
L_{1\delta-} &:= \left\{ (v_-, x) \in I_b : x = (a+b) v_-, v_- < -\frac{1-|a|}{1-(a+b)} \delta \right\}.
\end{aligned}$$

Clearly, L_o is the set of equilibria, I_a is a local stability region of L_o , and I_b is a globally attracting region and is also positively invariant. Denote the two endpoints of L_o by E^+ and E^- , i.e., $E^+ = \left(-\frac{1-|a|}{1-(a+b)} \delta, -\frac{1-|a|}{1-(a+b)} \delta + \delta \right)$ and $E^- = \left(\frac{1-|a|}{1-(a+b)} \delta, \frac{1-|a|}{1-(a+b)} \delta - \delta \right)$. Define

$$E_s := L_o \setminus \{E^+, E^-\} := \{ (v_-, x) \in L_o : (v_-, x) \notin \{E^+, E^-\} \}.$$

Then each point in E_s is stable in the sense of Lyapunov, however it is *not* asymptotically. As for the stability of E^+ (resp. E^-), each trajectory starting from a point in I_b on $v_- = \frac{1-|a|}{1-(a+b)} \delta$ (resp. $v_- = -\frac{1-|a|}{1-(a+b)} \delta$) will converge to E^+ (resp. E^-). How about trajectories starting from points in $I_b \setminus I_a$ sufficiently close to E^+ (resp. E^-)? It turns out that they never converge to E^+ (or E^-); therefore the two equilibria E^+ (resp. E^-) are *not* stable. To wit, we need more preparations.

For convenience, we regard the system in Eq. (30) as a map, i.e., adopt the notation defined in Eq. (10):

$$\xi(k+1) = F(\xi(k)).$$

Given a set $\Omega \subset I_b$, define

$$\text{pre}^n(\Omega) := \{(v_-, x) \in I_b : F^n((v_-, x)) \subset I_b\}, \quad \forall n \geq 0, \quad (31)$$

where $F^0((v_-, x)) = (v_-, x)$, iteratively $F^n((v_-, x)) = F^{n-1}((v_-, x))$ for $n \geq 1$.

Then

$$\begin{aligned} L_{\delta-} \setminus \left\{ \left(\frac{1-|a|}{1-(a+b)}\delta, \frac{1-|a|}{1-(a+b)}\delta - \delta \right) \right\} &\subset \text{pre}^2(L_{1\delta-}), \\ L_{\delta-} \setminus \left\{ \left(-\frac{1-|a|}{1-(a+b)}\delta, -\frac{1-|a|}{1-(a+b)}\delta + \delta \right) \right\} &\subset \text{pre}^2(L_{1\delta-}). \end{aligned}$$

Based on this observation, we have

$$F(I_b \setminus \{L_{Va-} \cup L_{Va+}\}) \subset I_b \setminus \{L_{Va-} \cup L_{Va+}\},$$

i.e., $I_b \setminus \{L_{Va-} \cup L_{Va+}\}$ is positively invariant. As a result, trajectories starting from points in $I_b \setminus \{L_{Va-} \cup L_{Va+}\}$, no matter how close to E^+ (resp. E^-) they are, will *not* converge to E^+ (resp. E^-), indicating that neither E^+ nor E^- is locally stable.

Moreover, for a given set $\Omega \subset I_b \setminus \{L_{Va-} \cup L_{Va+}\}$, define

$$\begin{aligned} \text{Img}^n(\Omega) &:= \{F^n(\Omega)\}, \\ \Psi(\Omega) &:= \cup_{n=0}^{\infty} \text{Img}^n(\Omega), \end{aligned}$$

then it is easy to verify that

$$\begin{aligned} F(\Psi(L_{1\delta+})) &\subset \Psi(L_{1\delta+}), \\ F(\Psi(L_{1\delta-})) &\subset \Psi(L_{1\delta-}), \end{aligned}$$

which furthermore imply all trajectories starting within $I_b \setminus \{L_{Va-} \cup L_{Va+}\}$ will eventually move along the line segment $\Psi(L_{1\delta+}) = \Psi(L_{1\delta-})$. For a point $\xi \in I_b \setminus \{L_{Va-} \cup L_{Va+}\}$, let $\omega(x)$ denote its ω -limit set, define

$$\omega(I_b \setminus \{L_{Va-} \cup L_{Va+}\}) := \left\{ \cup_{\xi \in I_b \setminus \{L_{Va-} \cup L_{Va+}\}} \omega(\xi) \right\},$$

then

$$\omega(I_b \setminus \{L_{Va-} \cup L_{Va+}\}) \subset \Psi(L_{1\delta+}).$$

Obviously

$$\omega(I_a) = L_o.$$

Thus we get a characterization of the ω -limit sets of the system in Eq. (30). However, we have to admit that this characterization is somewhat crude because all trajectories starting within $I_b \setminus \{L_{Va-} \cup L_{Va+}\}$ will eventually move along merely a part of each line segment in $\Psi(L_{1\delta+})$ instead of the whole line segment. Fig.

10 given later will visualize this observation. Now the problem of finding the exact ω -limit set of the system is still under our study. Nevertheless, adopting the argument on pp. 24 in Robinsion [1995], it is easy, though not straightforward due to the nature of the map F , to show that this ω -limit set is indeed *invariant*. By extensive simulation, we find that this ω -limit set is also *topological transitive*, however, up to now we have not been able to build solid theoretic background to support it.

Based on the above analysis, it is fair to say that the dynamics of the system in Eq. (30) is remarkably complicated: It indeed exhibits the feature of sensitive dependence on initial conditions, this sensitivity locates only on $\cup_{n=0}^{\infty} (\text{pre}^n(L_{\delta-}) \cup \text{pre}^n(L_{\delta+}))$, a subset of $\cup_{n=0}^{\infty} (\text{pre}^n(L_{1\delta-} \cup L_{1\delta+}))$. Hence it is weakly chaotic. Next we will calculate its generalized topological entropy in the spirit of Kopf [2000] and Galatolo [2003].

Denote by $I_{\text{binv}+}$ the region encircled by the lines L_{Va+} , L_{Vb+} , $L_{1\delta+}$, $\text{Img}^1(L_{\delta-})$. Similarly denote the region encircled by the lines L_{Va-} , L_{Vb-} , $L_{1\delta-}$, $\text{Img}^1(L_{\delta+})$ by $I_{\text{binv}-}$, based on the above analysis, we have the following claim:

Claim 1: The steady state of the system will settle in the region $I_{\text{binv}+} \cup I_{\text{binv}-}$.

This claim is a straightforward application of the foregoing analysis, however it plays an important role in the calculation of the topological entropy of the system.

For the definition of topological entropy for piecewise monotone transformations with discontinuities, please refer to Kopf [2000]. Now we will give a construction in order to compute the topological entropy for our system, which is clearly piecewise monotone (under some metric defined on the system rather than under the usual Euclidean metric; however, this is not essential.) with discontinuities.

Define

$$\begin{aligned}
\text{Pim}_F(0) &:= \{L_{\delta-}, L_{\delta+}\}, \\
\text{Pim}_F(1) &:= \{L : L \cap \text{Pim}_F(0) = \phi, F(L) \subset \text{Pim}_F(0)\}, \\
&\vdots \\
\text{Pim}_F(m) &:= \{L : L \cap (\cup_{i=0}^{m-1} \text{Pim}_F(i)) = \phi, F^m(L) \subset \text{Pim}_F(0)\}, \quad m \geq 1,
\end{aligned} \tag{32}$$

where ϕ stands for the empty set. Note that the elements in each $\text{Pim}_F(m)$ are line segments.

Denote by $\#(\text{Pim}_F(m))$ the number of elements in $\text{Pim}_F(m)$.

Before calculating the topological entropy, we need to pay a bit more attention to the mapping F . Clearly, according to Fig. 9 there exists a positive integer M such that

$$\begin{aligned}
\text{pre}^{M+l}(L_{\delta+}) \cap L_{1\delta-} &\neq \phi, \\
\text{pre}^{M+l}(L_{\delta-}) \cap L_{1\delta+} &\neq \phi, \quad \forall l \geq 1.
\end{aligned} \tag{33}$$

In fact, each set of intersections contains exactly one element (one line segment). For each given integer $n > 0$, define

$$\#F(n) := \sum_{m=0}^n \#(\text{Pim}_F(m)),$$

and define the *topological entropy* of F as

$$\aleph(F) := \lim_{n \rightarrow \infty} \frac{\log \#F(n)}{n}, \quad (34)$$

which is well-defined (see the proof below). Then we have

Theorem 2 *For the system in Eq. (30), the following statements hold:*

- For $m \leq M$,

$$\#(\text{Pim}_F(m)) = 2. \quad (35)$$

- For $m > M$,

$$\#(\text{Pim}_F(m)) = 2 + 2 \cdot (m - M), \quad (36)$$

and

$$\aleph(F) = 0. \quad (37)$$

Proof: Eq. (35) is self-evident, Eq. (36) follows from Claim 1 restricting $\#(\text{Pim}_F(m))$ on $I_{binv+} \cup I_{binv-}$ for $m > M$ and the analysis above. Then for sufficiently large n ($n \geq M$),

$$\#F(n) = 2(M + 1) + 2 \frac{(n - M)(n - M + 1)}{2},$$

thus

$$\begin{aligned} \aleph(F) &:= \lim_{n \rightarrow \infty} \frac{\log \#F(n)}{n} \\ &= \lim_{n \rightarrow \infty} \frac{\log(2(M + 1) + (n - M)(n - M + 1))}{n} \\ &= 0. \end{aligned}$$

■

Remark 2: In light of this result, from the perspective of topological entropy, our system is a weakly chaotic system.

The above discussion is mainly for the case of $|b| < a$. For example, given $a = 0.9$ and $b = -0.3$, Fig. 10 plots a trajectory at large time instants, i.e., its asymptotic behavior. Now consider the case when $a = 0.8$ and $b = -0.9$, hence $|b| < a$, and we also draw its asymptotic behavior in Fig. 11 from the same initial point. We observe that their asymptotic behavior is different. The reason is still unclear up to now.

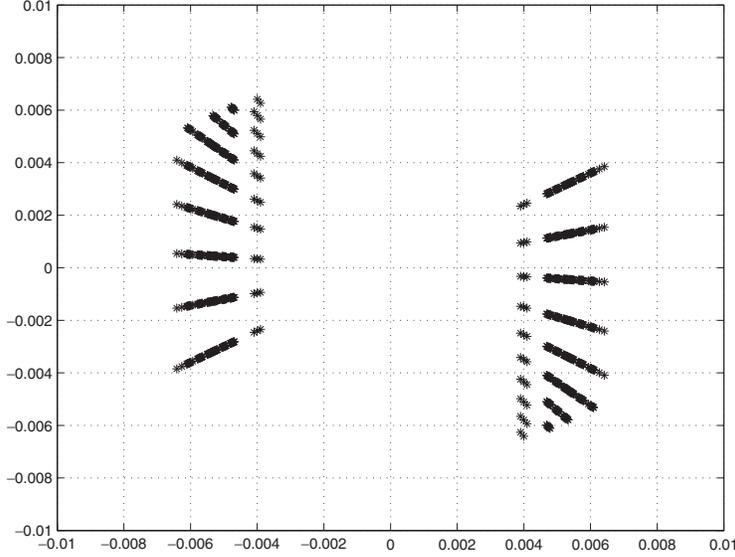


Fig. 10. A trajectory at large time instants for $|b| < a$

2.1.2 Case 2: $a = 1$

Consider the system

$$\begin{aligned} x(k+1) &= x(k) + bv(k), \\ u_c(k) &= x(k), \end{aligned} \tag{38}$$

where $|1+b| < 1$. Let $v(-1) \in \mathbb{R}$, and for $k \geq 0$,

$$v(k) = H_1(u_c(k), v(k-1)) = \begin{cases} u_c(k), & \text{if } |u_c(k) - v(k-1)| > 0.01, \\ v(k-1), & \text{otherwise.} \end{cases}$$

Figs. 7-8 show that the dynamics of the system in Eq. (38) can be fairly complicated. In the rest of this subsection we mainly study the problem when the system will have periodic orbits. From now on, we assume

$$-1 \leq b < 0.$$

For this case, L_{Va-} , L_{Va+} and L_o in Fig. 9 now become one line segment

$$L := \{(0, x) : |x| \leq \delta\}.$$

Define

$$\Gamma_{in} := \{(v_-, x) \in I_b : |x - v_-| \leq \delta\},$$

and

$$\Gamma_{ex} := I_b \setminus \Gamma_{in},$$

the following is a necessary condition for the existence of periodic orbits.

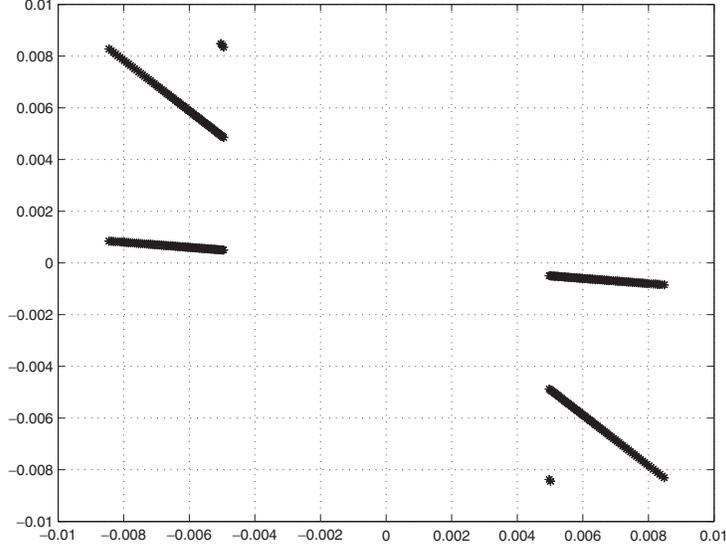


Fig. 11. A trajectory at large time instants $|b| > a$

Theorem 3 *If the system in Eq. (38) has periodic orbits, then there exist an even integer $n > 0$ and integers $K_i > 0$ such that*

$$\prod_{i=1}^n (1 + K_i b) = 1. \quad (39)$$

Without loss of generality, we here prove the case of $n = 2$. The following Lemma is used in the proof of Theorem 3:

Lemma 2 *Suppose $\xi_0 \in \Gamma_{ex}$ is a point on a periodic orbit at time K_0 , it will be inside Γ_{in} at $K_0 + 1$.*

Proof: Let $\xi_0 = \begin{bmatrix} v_- \\ x \end{bmatrix}$. Then

$$\xi_1 = \begin{bmatrix} v_1 \\ x_1 \end{bmatrix},$$

where,

$$x_1 = (a + b)x,$$

$$v_1 = x.$$

Hence

$$|x_1 - v_1| = |1 - (a + b)| |x| = |b| |x|.$$

Since $\begin{bmatrix} v_- & x \end{bmatrix}'$ is the steady state (a periodic point),

$$|x| \leq \frac{|b| \delta}{1 - (a + b)} = \delta.$$

One obtains

$$|x_1 - v_1| = |1 - (a + b)| |x| \leq |b| \delta \leq \delta,$$

i.e., $\xi_1 \in \Gamma_{in}$. ■

Note that Lemma 2 is not trivial because there are systems such as the case when $a = 3/10$ and $b = -9/10$ violating this property.

Proof of Theorem 3: Without loss of generality, suppose the periodic orbit begins with $\xi_0 \in \Gamma_{ex}$, and by Lemma 2,

$$\xi_1 = A_1 \xi_0 \in \Gamma_{in}, \tag{40}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 + b \end{bmatrix}.$$

Assume after a time K_1 the state

$$\xi_2 = (A_1 + A_2)^{K_1-1} \xi_1$$

is in Γ_{ex} , where

$$A_2 = \begin{bmatrix} 1 & -1 \\ b & -b \end{bmatrix}.$$

Then

$$\xi_3 = A_1 \xi_2 \in \Gamma_{in}.$$

After a time K_2 the state

$$\xi_4 = (A_1 + A_2)^{K_2-1} \xi_3$$

returns to ξ_0 . Then

$$(A_1 + A_2)^{K_2-1} A_1 (A_1 + A_2)^{K_1-1} A_1 \xi_0 = \xi_0. \tag{41}$$

By straightforward algebraic computations, one gets

$$\begin{aligned} dcx &= x, \\ dx &= v_-, \end{aligned} \tag{42}$$

where

$$d = 1 + K_2 b, \quad c = 1 + K_1 b.$$

Since ξ_0 is outside the sector,

$$|x - v_-| > \delta.$$

According to Eq. (42), note that $c \neq 0$, $d \neq 0$, then

$$cd = 1. \tag{43}$$

Given $a = 1$,

$$K_1 + K_2 = K_1 K_2 (-b), \quad (44)$$

which is equivalent to Eq. (39) for $n = 2$. ■

Remark 3: Theorem 3 provides a necessary condition for having periodic orbits. Interestingly for the case of $n = 2$, extensive experiments imply that there are periodic orbits of period $K_1 + K_2$ if K_1 and K_2 satisfy Eq. (39), and there are no periodic orbits if there are no such K_1 and K_2 that satisfy Eq. (39). Based on the observation, Theorem 3 is not severely conservative.

Following the above analysis, we immediately have

Corollary 2 *Suppose a is rational and b is irrational in Eq. (10), then there are no periodic orbits.*

Proof: Following the proof above, it suffices to show that

$$\prod_{i=1}^n \left(a^{K_i} + \sum_{j=0}^{K_i-1} a^j b \right) = 1 \quad (45)$$

has no positive integer solutions for any given even number $n > 0$. This can be easily verified. ■

Though the above result is simple, its significance can not be underestimated: If a system has periodic orbits, then it is *not* structurally stable.

Now suppose a system has periodic orbits, how to find them? And how to determine their periods? we first consider an example.

Example 1: In the case when $a = 1$ and $b = -0.3$, there are two periodic solutions. One is of period 24 corresponding to $K_1 = 4$ and $K_2 = 20$, the other is of period 15 corresponding to $K_1 = 5$ and $K_2 = 10$. Observe that

$$-b = 0.3 = \frac{3}{2 \cdot 5} := \frac{p}{q_1 \cdot q_2},$$

where $p = 3$, $q_1 = 2$, $q_2 = 5$. Interestingly

$$\begin{aligned} 4 &= \frac{q_2 + 1}{p} q_1, & 20 &= \frac{q_2 + 1}{p} q_1 \cdot q_2, \\ 5 &= \frac{q_1 + 1}{p} q_2, & 10 &= \frac{q_1 + 1}{p} q_1 \cdot q_2. \end{aligned}$$

Based on this observation, we propose a necessary condition for Theorem 3 for the case when $n = 2$.

Theorem 4 *Given*

$$a = 1 \quad \text{and} \quad b = -\frac{p}{q},$$

suppose positive integers p and q satisfy $1 < p < q$, $p \neq 2$ (which is the trivial case), and $\gcd(p, q) = 1$, i.e., the greatest common divisor of p and q is 1. Define

$$\Delta := \{q_i : q_i \text{ is a prime number, } q_i | q\}. \quad (46)$$

Then if $p|(q_i + 1)$,

$$\left(\frac{q_i + 1}{p} \frac{q}{q_i}, \frac{q_i + 1}{p} q \right)$$

is a solution of Eq. (39).

Proof: Obviously, given $p|(q_i + 1)$, $\left(\frac{q_i + 1}{p} \frac{q}{q_i}, \frac{q_i + 1}{p} q \right)$ is a solution of Eq. (39). Now we show how the set Δ is constructed in the above way. Given $q_i \in \Delta$, If there are two positive numbers m and n satisfying

$$\frac{1}{m} + \frac{1}{n} = \frac{p}{q_i}, \quad (47)$$

then $\left(m \frac{q}{q_i}, mq \right)$ is a solution to Eq. (39). Hence we need only to pay attention to solutions to Eq. (47). Suppose (m, n) is a solution of Eq. (47), then either $\gcd(m, q_i) = 1$ or $\gcd(n, q_i) = 1$ (otherwise, $p = 2$ or does not exist). For convenience, we always assume $\gcd(m, q_i) = 1$. According to Eq. (47),

$$\frac{m+n}{nm} = \frac{p}{q_i},$$

i.e.,

$$(m+n)q_i = pmn.$$

Then $q_i|pmn$. Since $\gcd(p, q_i) = \gcd(m, q_i) = 1$, $q_i|n$. Let

$$n = kq_i, \quad (48)$$

which leads to

$$\frac{1}{m} + \frac{1}{kq_i} = \frac{p}{q_i}.$$

Consequently,

$$m(pk - 1) = kq_i,$$

hence $m|kq_i$. Since $\gcd(m, q_i) = 1$, $m|k$. In light of Eq. (48), we set

$$n = ml.$$

Substituting it into Eq. (47), one has

$$\frac{1}{m} + \frac{1}{ml} = \frac{p}{q_i},$$

equivalently,

$$mpl = q_i(l + 1),$$

which means $q_i|mpl$. i.e., $q_i|l$. Similarly, $l|q_i(l + 1)$, hence $l = q_i$. Therefore,

$$m = \frac{q_i + 1}{p}.$$

If $p|(q_i + 1)$, then

$$\left(\frac{q_i + 1}{p}, \frac{q_i + 1}{p} q_i \right)$$

solves Eq. (47), and

$$\left(\frac{q_i + 1}{p} \frac{q}{q_i}, \frac{q_i + 1}{p} q \right)$$

is a solution of Eq. (39). ■

The above theorem provides a construction for the solutions to Eq. (39). However, this is somewhat inadequate. For example, for $a = 1$ and $b = -\frac{3}{7}$, the set Δ is empty. There are no positive integers satisfying Eq. (39) for $n = 2$ either. This is good for us. However for $a = 1$ and $b = -\frac{3}{2 \cdot 5 \cdot 11}$, we have the following observations (Table 1):

Table 1. Some periodic orbits

(v_-, x_0)	$(\frac{7}{1000}, 0)$	$(\frac{2}{1000}, 0)$	$(\frac{1}{1000}, 0)$	$(\frac{1}{2000}, 0)$
Periods	165	147	243	480
(v_-, x_0)	$(\frac{1}{2000}, \frac{1}{2000})$	$(\frac{1}{2000}, \frac{1}{8000})$	$(\frac{1}{2000}, \frac{1}{8500})$	$(\frac{1}{3000}, 0)$
Periods	264	480	243	1083

Periods 165, 264 and 480 can be obtained based on Theorem 4, however, others can not. Actually there are more periodic and aperiodic orbits (Table 2):

Table 2. More periodic orbits and aperiodic orbits

(v_-, x_0)	$(\frac{\delta}{1000}, \frac{(a+b)\delta}{1000})$	$(\frac{\delta}{100}, \frac{(a+b)\delta}{100})$	$(\frac{\delta}{10}, \frac{(a+b)\delta}{10})$	$(\frac{2\delta}{10}, \frac{2(a+b)\delta}{10})$
Periods	4107	264	243	165
(v_-, x_0)	$(\frac{4\delta}{10}, \frac{4(a+b)\delta}{10})$	$(\frac{9\delta}{20}, \frac{9(a+b)\delta}{20})$	$(\frac{5\delta}{10}, \frac{5(a+b)\delta}{10})$	$(\frac{6\delta}{10}, \frac{6(a+b)\delta}{10})$
Periods	aperiodic	aperiodic	147	aperiodic
(v_-, x_0)	$(\frac{7\delta}{10}, \frac{7(a+b)\delta}{10})$	$(\frac{8\delta}{10}, \frac{8(a+b)\delta}{10})$	$(\frac{9\delta}{10}, \frac{9(a+b)\delta}{10})$	$(\frac{10\delta}{10}, \frac{10(a+b)\delta}{10})$
Periods	165	165	243	4107

Furthermore, we observed that there are at least 55 solutions to Eq. (39) for $n = 4$.

The foregoing analysis tells us:

- There may exist periodic orbits of very large periods.
- There are always aperiodic orbits.

Inspired by the proof of Corollary 2, especially by Eq. (45), now we attempt to construct systems with $|a| < 1$ that have periodic orbits. First choose $n = 2$, $a = 9/10$, choose $K_1 = 15$ and $K_2 = 7$. Then

$$b = -\frac{9015229097816388767119}{41428905812371212328810}$$

solves Eq. (45). Surprisingly the trajectory starting from

$$(v_-, x_0) = \left(\frac{-b * \delta}{1 - |a + b|} - \frac{1}{10^4}, 0 \right)$$

will become a periodic orbit of period $22(= K_1 + K_2)$ after some iterations, i.e., it is an eventually periodic orbit. It can be shown this periodic orbit is locally stable. However, a trajectory starting outside the stability region, say, from

$$(v_-, x_0) = \left(\frac{-b * \delta}{1 - |a + b|} - \frac{1}{10^4}, \frac{1}{10^3} \right)$$

is aperiodic. For the case when $|a| > 1$, suppose $a = 11/10$, choose $K_1 = 7$ and $K_2 = 5$. Then

$$b = -\frac{2138428376721}{5792012767210}$$

solves Eq. (45). And the trajectory starting from

$$(v_-, x_0) = \left(\frac{\delta}{10^5}, \frac{(a + b) * \delta}{10^5} \right)$$

will become a periodic orbit of period $12(= K_1 + K_2)$ after some iterations, i.e., it is an eventually periodic orbit. It can be shown this periodic orbit is also locally stable and there are aperiodic orbits too.

Remark 4: From this construction, one finds out that most systems with $|a| < 1$ or $|a| > 1$ will be unlikely to have periodic orbits.

2.1.3 Case 3: $|a| > 1$

This case is analogous to that of $|a| < 1$ except that all the fixed points are unstable. The Fig. 12 is one trajectory at sufficiently large time instants.

The complex dynamics exhibited by our system is due to its nonlinearity. This is different from a quantized system. The complicated behavior of an unstable quantized scalar system is extensively studied in Delchamps [1988, 1989, 1990], Fagnani & Zampieri [2003], etc. In Delchamps [1990], it is mentioned that given that the system parameter a is stable, a quantized system may have many fixed points as well as many periodic orbits which are all asymptotically stable. However, for our constrained systems, most of them will not possess periodic orbits. For the systems with $a = 1$, periodic orbits are locally stable, this is not the case of for a quantized system (Delchamps [1990]). Given that a is unstable, the ergodicity of the quantized system is studied in Delchamps [1990]. In essence, related results there depend heavily on the affine representation of the system by which the system is piecewise expanding, i.e., the absolute value of derivative of the piecewise affine map in each interval is greater than 1. Based on this crucial property, the main theorem (Theorem 1) in Lasota & Yorke [1973] and then that of Li & Yorke [1978] are employed to show there exists a unique invariant

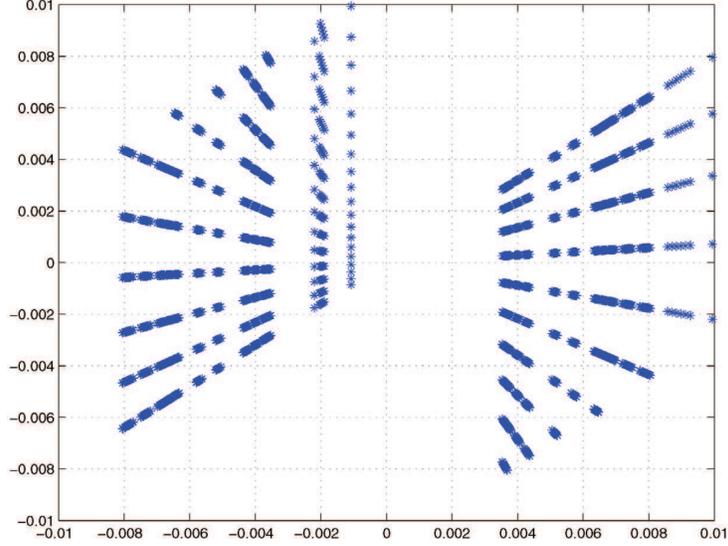


Fig. 12. A trajectory at large time instants

measure under the affine mapping and which is also ergodic with respect to that mapping. Therefore ergodicity is established for scalar unstable quantized systems. However, this is not the case for our system. Though the system is piecewise linear, it is **singular** with respect to the Lebesgue measure and furthermore, the derivative of the system in the region Γ_{in} is $(a + b)$, whose absolute value is strictly less than 1. Hence the results in Lasota & Yorke [1973] and Li & Yorke [1978] are not applicable here. By extensive experiments, we strongly believe that the system indeed has the property of ergodicity, however, the problem still remains open.

To appreciate what qualitative behavior of a higher dimensional system can have, we give the following example.

Example 2: Suppose the system G in Fig. 2 is given by

$$\begin{aligned} x(k+1) &= 2x(k) + 3v(k), \\ y_c(k) &= x(k), \end{aligned}$$

and the controller C is given by

$$\begin{aligned} x_d(k+1) &= -2x_d(k) + 1.5e_c(k), \\ u_c(k) &= x_d(k), \\ e_c(k) &= r(k) - z(k), \end{aligned}$$

where $v(k)$ and $z(k)$ are outputs of Eqs. (4)-(5) respectively. We set $r \equiv 0$. We call the resulting system Σ_ρ . It is easy to see that the closed-loop system without the constraints H_1 and H_2 is asymptotically stable. Under H_1 and H_2 , three figures, Figs. 13–15, are drawn. The first is for $(v(k-1), x(k))$, the second for $(z(k-1), x_d(k))$ and the last for $(x(k), x_d(k))$.

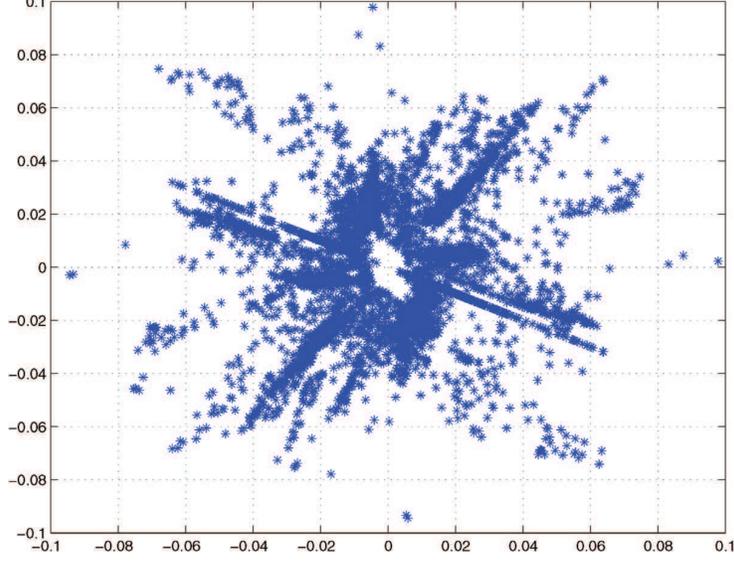


Fig. 13. Plot of $(v(k-1), x(k))$ at large time instants (≥ 95000)

Next we analyze the chaotic behavior of system Σ_o using nonlinear data analysis. First we show sensitive dependence on initial conditions. Choose an initial condition

$$[v(-1), x(0), z(-1), x_d(0)] = [-1/1000, 1/1000, 2/1000, -1/1000],$$

set the iteration number to be 600000, then we get trajectory of x ; perturb the initial condition above slightly to $[-1/1000, 1/1000 + 1/10^{13}, 2/1000, -1/1000]$, under the same iteration, we get another trajectory of x , the following plot (Fig.16) is the difference between these two x of the last 1200 points of the iteration: From this figure, one can clearly see sensitive dependence on initial conditions. In general, the spectra of a chaotic orbit will be continuous. Here we draw the spectrum of x starting from $[-1/1000, 1/1000, 2/1000, -1/1000]$ (Fig. 17): What about the Lyapunov exponents? based on the last 10000 point of x , using the software “Chaos Data Analyzer”, choosing parameters $D = 3$, $n = 3$ and $A = 10^{-4}$, we get the largest Lyapunov exponent 0.407 ± 0.027 , indicating the trajectory is indeed a chaotic one.

Now we look at the dynamics of Example 2 geometrically. For a given dynamical system, generally complicated manifold structure will lead to complex dynamics. we now indicate that the manifold structure of system Σ_o is indeed very complicated. To simplify the discussion, suppose there is no constraint H_2 in Fig. 2, i.e., $v(k) \equiv u_c(k)$. The fixed points of the system Σ_o is given by

$$\left\{ (x, x_d, z_-) : x = \frac{3}{2}z_-, x_d = -\frac{1}{2}z_-, |z_-| \leq 2\delta \right\} \quad (49)$$

Define

$$\mathbf{x} := \begin{bmatrix} x & x_d & z_- \end{bmatrix}',$$

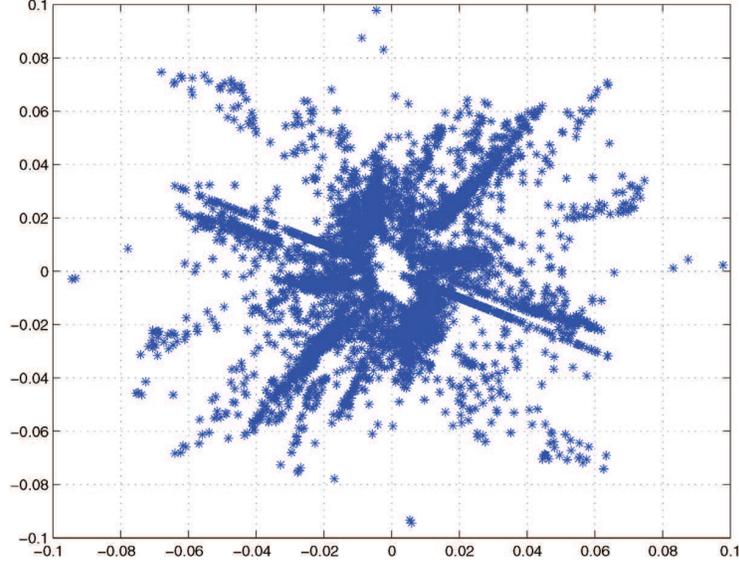


Fig. 14. Plot of $(z(k-1), x_d(k))$ at large time instants (≥ 95000)

$$T := \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{\mathbf{x}} := [\tilde{x} \quad \tilde{x}_d \quad \tilde{z}_-]' = T\mathbf{x}.$$

Then the system under new coordinates is

$$\Sigma_{n1} : \tilde{\mathbf{x}}(k+1) = \begin{bmatrix} \frac{1}{2} & 3 & -\frac{3}{4} \\ -1 & -2 & -\frac{1}{2} \\ 1 & 0 & \frac{3}{2} \end{bmatrix} \tilde{\mathbf{x}}(k)$$

under

$$|\tilde{x} - \tilde{z}_-| > \delta, \quad (50)$$

and

$$\Sigma_{n2} : \tilde{\mathbf{x}}(k+1) = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tilde{\mathbf{x}}(k)$$

under

$$|\tilde{x} - \tilde{z}_-| \leq \delta. \quad (51)$$

For convenience, we denote this system by Σ_n . It is easy to see that the fixed points of Σ_n are

$$\{(0, 0, \tilde{z}_-) : |\tilde{z}_-| \leq 2\delta\}. \quad (52)$$

Some comments are appropriate here:

- The subsystem Σ_{n1} is a stable system.

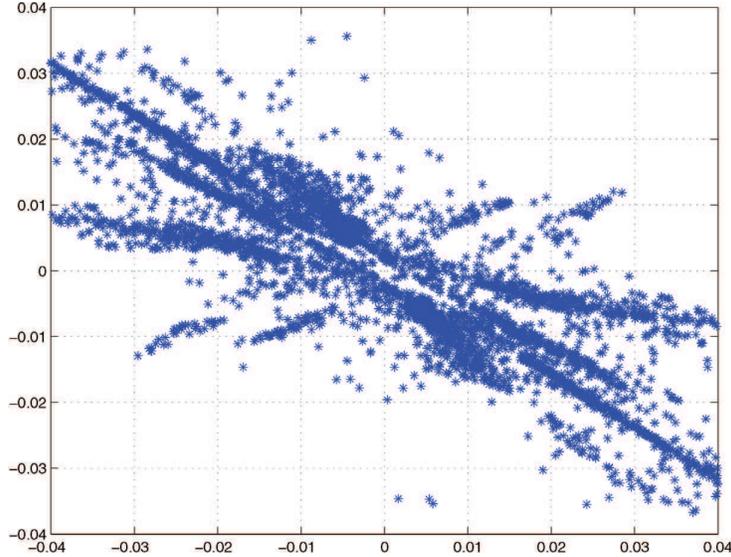


Fig. 15. Plot of $(x(k), x_d(k))$ at large time instants (≥ 95000)

- If Eq. (51) is satisfied, a trajectory (governed by Σ_{n2}) will move on a surface

$$\tilde{z}_- = \gamma$$

for some $\gamma \in [-2\delta, 2\delta]$. We call such a surface Ω_γ . The point $(0, 0, \gamma)$ is the origin of Σ_{n2} on Ω_γ . Furthermore, the line

$$\Gamma_{s,\gamma} : \tilde{x} = 0, \quad \tilde{z}_- = \gamma \quad (53)$$

is the stable manifold of Σ_{n2} and similarly, the line

$$\Gamma_{u,\gamma} : \tilde{x}_d = 0, \quad \tilde{z}_- = \gamma \quad (54)$$

is the unstable manifold of Σ_{n2} .

Suppose a trajectory Γ of the system Σ_n starts from a point p and is governed by Σ_{n2} , if $p \in \Gamma_{u,\gamma}$ (or in general $p \notin \Gamma_{s,\gamma}$) on some surface Ω_γ , then the trajectory will contract along \tilde{x}_d -axis and stretch along \tilde{x} -axis. Due to the Eq. (51), after some time, Γ will move according to the stable subsystem Σ_{n1} . At this moment, Γ will leave the surface Ω_γ , and move toward the origin $(0, 0, 0)$. Due to the Eq. (50), after some time, it will move again on some surface $\Omega_{\gamma'}$ for some $\gamma' \in [-2\delta, 2\delta]$. If it is not exactly on the line $\Gamma_{s,\gamma'}$, it will once more contract along \tilde{x}_d -axis and stretch along \tilde{x} -axis and repeat the above behavior. So normally a trajectory never settles done, indicating its intriguing behavior.

2.2 Chaotic control?

The complex dynamical behavior of the system in Fig. 2 has been studied in detail in the foregoing sections, compared to the standard control scheme such as that in Fig. 2, whose dynamics can only be either converging

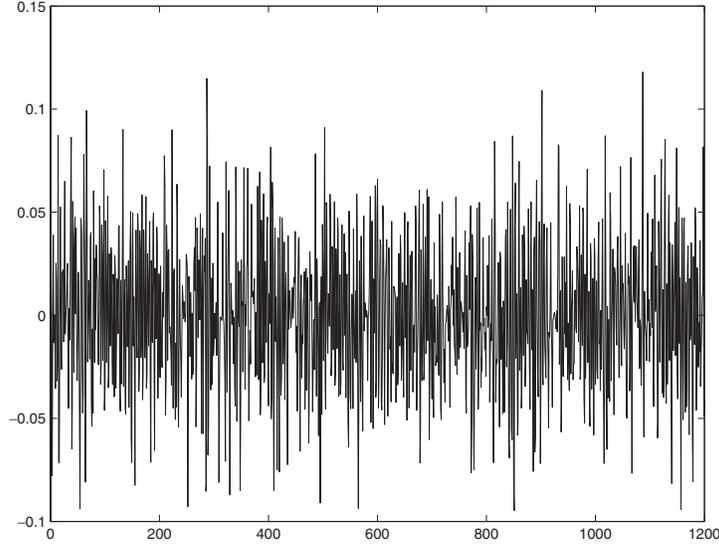


Fig. 16. Sensitive dependence on initial conditions

to the origin, or being periodic or unbounded trajectories, the scheme adopted in Fig. 2 provides much more dynamical properties. Of course this means that a control engineer has more flexibility at his/her disposal. This is particularly attracting from the viewpoint of multi-purpose control. We believe this is the main merit this control scheme can provide. In this subsection, we will study the following problem: Given a control performance specification, can we achieve it by possibly adjusting the system parameters? We will discuss two control specifications:

- (1) The system has one unique fixed point.
- (2) A periodic orbit is desirable.

For item (1), without loss of generality, assume that the desirable unique fixed point is the origin. If the parameter a in the system in Eq. (9) satisfies $|a| < 1$, then we can achieve asymptotic stability with respect to the origin by adjusting the nonlinear block H_1 , though the system itself has no such property. According to Fig. 9, we need merely to let the value $v(k-1)$ stored in H_1 be 0 when $|x(k)| < \delta$ (This feature is illustrated in Figs. 10-11). Then the trajectory will move along the x -axis toward the origin, i.e., the asymptotic stability of the origin is achieved. If the parameter a in the system in Eq. (9) satisfies $|a| \geq 1$, we can not expect asymptotic stability of the origin because it itself is unstable. However, we can keep the trajectory arbitrarily close to the origin at large time instants, by adopting the following scheme: Suppose it is desirable to keep the trajectory within the distance ϵ around the origin, then choose δ small enough so that $x(k^*)$ satisfies $|x(k^*)| < \epsilon/|a|^2$ at some time instant k^* (this can be realized, see Fig. 12). Next let $v(k^*-1) = 0$ when $|x(k^*)| < \epsilon/|a|$. If $|a| = 1$, then the trajectory will stay at $(0, x(k^*))$ forever. The goal is achieved. On the other hand, assume $a > 1$. If $x(k^*) > 0$, we first let the trajectory move along the x -axis until we get

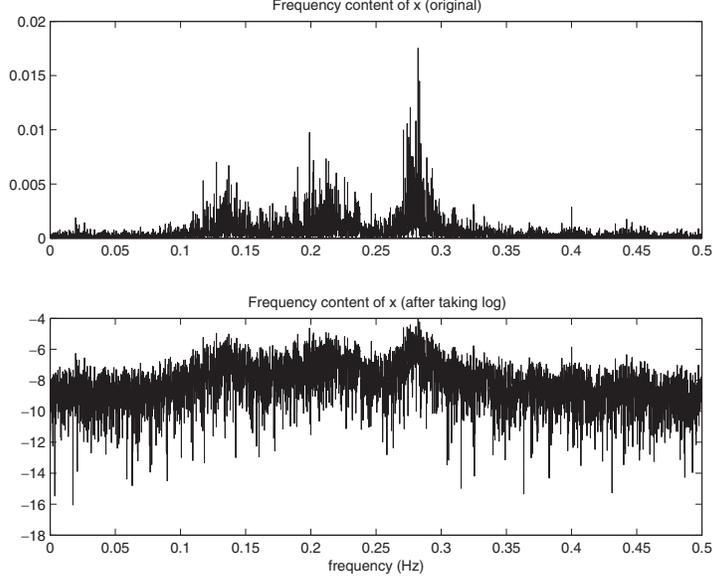


Fig. 17. Sensitive dependence on initial conditions

$x(k^* + 1) < \epsilon/|a|$, then choose $v(k^*) > 0$. In this way $(x(k^* + 1), v(k^*))$ is below the line segment of fixed point (then $(x(k^* + 1), v(k^*))$ will move downward at the next step) such that

$$x(k^* + 2) > 0,$$

and

$$x(k^* + 2) = ax(k^* + 1) + bv(k^*) < \epsilon/|a|^2.$$

(Note this is guaranteed by the property of the vector field of the system.) Then let $v(k^* + 1) = 0$, and repeat the above procedure. Similarly if $x(k^*) < 0$, all we need to do is to choose suitable $v(k^*) < 0$ such that $(x(k^* + 1), v(k^*))$ is above the line segment of fixed points, then follow the above procedure. In this way we can keep the trajectory within the distance ϵ of the origin. Based on the above analysis, we observe that the instability of the parameter a poses a difficulty for implementing our scheme, there are more discussions from the perspective of higher dimensional systems (e.g. Theorem 6 below). The foregoing discussion is reminiscent of Proposition 2.2 in Delchamps [1990], however our scheme is better since the K_1 in that paper can be ∞ here. Moreover, our algorithm is simpler too.

For the item (2), suppose it is desired that the system operates on a periodic orbit Γ of period T . If $a = 1$, according to Theorem 4, by suitably choosing b , a periodic orbit of period T can be built. If $a \neq 1$, then following the discussion at the end of Sec. 2.1.2, it is also possible to construct a periodic orbit of period T . Then the real question is: Can we really find an initial condition which produce or converge to the desirable periodic orbit Γ ? If Γ is within a strange attractor, then from almost all initial points, trajectories will be within an arbitrarily small neighborhood of Γ at some time k ; this is the property of a stranger attractor. So we can just pick up such an initial condition, let the system run automatically first, and apply control similarly to the

case in item (1) when the trajectory is sufficiently close to Γ , and keep it remain within a small neighborhood of Γ . Therefore the problem boils down to constructing a strange attractor containing Γ . This is the problem we are currently studying. Note that our chaotic system seems different than many known chaotic systems, which have strange attractors within which there are periodic orbits of any periods. However in light of Corollary 2, there may be no periodic orbits at all when a is rational and b is irrational. This annoying fact may probably be due to the scheme we are proposing involves discontinuities. We have already known that there may be a great variety of dynamics this scheme can produce, which brings more freedom to a control engineer, and especially suitable for multi-purpose controller design. However in order to make the proposed scheme more useful, a thorough study of this scheme has to be conducted.

We have to acknowledge that the preceding analysis is naive, nevertheless, it illustrates that by using the trajectories of the system, i.e., some extra information in addition to system parameters, we can achieve better control in some sense. For chaotic control, interested readers may refer to Schuster [1999]. These will be our future research directions. Here we still adhere to classic control theory.

2.3 Stability analysis of higher-dimensional systems

Now we return to our analysis of higher dimensional system in Fig. 2. We will find a positively invariant set for this system. For simplicity, let $D_d = 0$. Define

$$\check{A} := \begin{bmatrix} A & BC_d \\ -B_d C & A_d \end{bmatrix}, \quad \check{B} := \begin{bmatrix} B & 0 \\ 0 & -B_d \end{bmatrix}, \quad \check{B} := \check{B}\check{C} = \begin{bmatrix} 0 & BC_d \\ -B_d C & 0 \end{bmatrix}.$$

Since the controller C is stabilizing, the closed-loop system in Fig. 1 is asymptotically stable. Then there exists a Lyapunov function $v(\xi(k)) = \xi'(k)P\xi(k)$ with $P = \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix} > 0$ such that

$$\begin{aligned} \Delta v(\xi(k)) &= \xi'(k+1)P\xi(k+1) - \xi'(k)P\xi(k) \\ &= \xi'(k) \left(\check{A}'P\check{A} - P \right) \xi(k) \\ &= -\|\xi(k)\|_2^2 \text{ for all } \xi(k). \end{aligned}$$

Correspondingly, define $v_c(\eta(k)) = \eta'(k)P\eta(k)$, then

$$\begin{aligned} \Delta v_c(\eta(k)) &= \eta'(k+1)P\eta(k+1) - \eta'(k)P\eta(k) \\ &= \eta'(k) \left(\check{A}'P\check{A} - P \right) \eta(k) + 2\eta'(k)\check{A}'P\check{B} \left(\begin{bmatrix} H_1(u_c(k), v(k-1)) \\ H_2(y_c(k), z(k-1)) \end{bmatrix} - \begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} \right) \\ &\quad + \left(\begin{bmatrix} H_1(u_c(k), v(k-1)) \\ H_2(y_c(k), z(k-1)) \end{bmatrix} - \begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} \right)' \check{B}' \cdot \\ &\quad P\check{B} \left(\begin{bmatrix} H_1(u_c(k), v(k-1)) \\ H_2(y_c(k), z(k-1)) \end{bmatrix} - \begin{bmatrix} u_c(k) \\ y_c(k) \end{bmatrix} \right) \\ &\leq -\|\eta(k)\|_2^2 + 2\|\eta(k)\|_2 \cdot \left\| \check{A}'P\check{B} \right\|_{\infty} \cdot \gamma \cdot \bar{\delta} + (\gamma \cdot \bar{\delta})^2 \cdot \left\| \check{B}'P\check{B} \right\|_{\infty}, \end{aligned}$$

where the positive constant $\gamma = \sqrt{m+p}$ and $\bar{\delta} = \max\{\delta_1, \delta_2\}$. Hence $\Delta v_c(\eta(k)) < 0$ if

$$\|\eta(k)\|_2 > \gamma \cdot \bar{\delta} \left\| \check{A}' P \check{B} \right\|_{\infty} + \gamma \cdot \bar{\delta} \sqrt{\left\| \check{A}' P \check{B} \right\|_{\infty}^2 + \left\| \check{B}' P \check{B} \right\|_{\infty}}.$$

For convenience, define

$$\begin{aligned} r_1 &:= \gamma \cdot \bar{\delta} \left\| \check{A}' P \check{B} \right\|_{\infty} + \gamma \cdot \bar{\delta} \sqrt{\left\| \check{A}' P \check{B} \right\|_{\infty}^2 + \left\| \check{B}' P \check{B} \right\|_{\infty}}, \\ r_2 &:= \left\| \check{A} \right\|_{\infty} r_1 + \left\| \check{B} \right\|_{\infty} \bar{\delta}, \end{aligned}$$

then we have

Theorem 5 *The set Ω defined by*

$$\Omega := \left\{ \eta \mid \eta'(k) P \eta(k) \leq \max\{\bar{\sigma}(P) r_1^2, \bar{\sigma}(P) r_2^2\} \right\}$$

is a positively invariant set, where $\bar{\sigma}(P)$ is the largest singular value of P .

Proof: We need only to show that for each $\eta(0) \in \Omega$, $\eta(k) \in \Omega$ for all $k \geq 1$. Suppose for some integer $k_0 > 0$, we have $\|\eta(k_0)\|_2 \leq r_1$, and $\|\eta(k_0+1)\|_2 > r_1$, then $\Delta v_c(\eta(k_0+1)) < 0$, which means $\eta'(k_0+2) P \eta(k_0+2) < \eta'(k_0+1) P \eta(k_0+1)$. Furthermore, the trajectory will eventually fall into the set $\left\{ \eta \mid \eta'(k) P \eta(k) \leq \bar{\sigma}(P) r_1^2 \right\}$. Therefore it suffices to show $\eta(k_0+1) \in \Omega$. Since

$$\|\eta(k_0+1)\|_2 \leq \left\| \check{A} \right\|_{\infty} r_1 + \left\| \check{B} \right\|_{\infty} \bar{\delta},$$

one has

$$\eta'(k_0+1) P \eta(k_0+1) \leq \bar{\sigma}(P) r_2^2,$$

which gives $\eta(k_0+1) \in \Omega$. ■

The preceding result ascertains the existence of a positively invariant set for the system in Fig. 2, the system behavior insider this invariant set may be very complex. The next result gives an upper bound for all equilibria of the system in Eq. (7).

Defining

$$\Phi := - \begin{bmatrix} I & -C_d(I - A_d)^{-1} B_d + D_d \\ -C(I - A)^{-1} B & I \end{bmatrix}, \quad (55)$$

then we have:

Corollary 3 *For the system in Eq. (7), supposing both G and C are stable, if the matrix $(C_d(I - A_d)^{-1} B_d - D_d) C(I - A)^{-1} B$ has no eigenvalue at $(-1, 0)$, then $\left\| (I - \tilde{A})^{-1} \check{B} \right\|_1 \cdot \|\Phi^{-1}\|_1 \bar{\delta}$ is an upper bound for all equilibria of this system.*

Proof: Suppose \bar{x} is an equilibrium of the system in Eq. (7), then there are an integer $K > 0$ and some vector ϖ such that

$$\eta(k+1) = \tilde{A}\eta(k) + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \varpi \quad (56)$$

for all $k > K$. Letting $k \rightarrow \infty$, we get

$$\bar{x} = \tilde{A}\bar{x} + \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \varpi,$$

then

$$\bar{x} = (I - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & B_d \end{bmatrix} \varpi,$$

and

$$\left\| \tilde{C}\bar{x} + \begin{bmatrix} 0 & D_d \\ 0 & 0 \end{bmatrix} \varpi - \varpi \right\|_{\infty} = \|\Phi\varpi\|_{\infty}.$$

Because the matrix $(C_d(I - A_d)^{-1}B_d - D_d)C(I - A)^{-1}B$ has no eigenvalue at $(-1, 0)$, Φ is invertible. Furthermore, since $\left\| \tilde{C}\bar{x} + \begin{bmatrix} 0 & D_d \\ 0 & 0 \end{bmatrix} \varpi - \varpi \right\|_{\infty} \leq \bar{\delta}$, $\|\varpi\|_{\infty} \leq \|\Phi^{-1}\|_1 \|\Phi\varpi\|_{\infty} \leq \bar{\delta}$, we have

$$\|\bar{x}\|_{\infty} \leq \left\| (I - \tilde{A})^{-1} \tilde{B} \right\|_1 \cdot \|\Phi^{-1}\|_1 \bar{\delta}. \quad (57)$$

Because \bar{x} is arbitrarily chosen, the result follows. ■

In particular, assume we have a scalar system with a static state feedback:

$$\begin{aligned} x(k+1) &= ax(k) + bv(k), \\ u(k) &= -fx(k), \\ v(k) &= H_1(u(k), v(k-1)), \end{aligned} \quad (58)$$

where $|a - bf| < 1$. Then following the above procedure, $|\bar{x}| \leq (\delta|bf|)/(1 - |a - bf|)$ where \bar{x} can be any equilibrium.

An upper bound has been found for all equilibria. Will any of these equilibria be stable if either G or C is unstable? We have a result reminiscent of that in Delchamps [1990]

Theorem 6 *Assume either G or C is unstable, and \tilde{A} is invertible, then the set of all initial points η_0 whose closed-loop trajectories tend to an equilibrium as $k \rightarrow \infty$ has Lebesgue measure zero.*

Proof: Denote this set by U . Let E^s be the generalized stable eigenspace of Eq. (7), Then the Lebesgue measure of E^s is zero since Eq. (7) is unstable. Suppose $\eta(0) \in U$, following the process in the proof of Corollary 1, there exist $K > 0$ and some vector ϖ such that

$$\eta(K) = \begin{bmatrix} A - BD_dC & BC_d \\ -B_dC & A_d \end{bmatrix} \varpi, \quad (59)$$

and Eq. (56) holds for all $k > K$. Since \tilde{A} is unstable, $\eta(k) \in E^s$ for all $k \geq K$. Furthermore, the invertibility of \tilde{A} implies that ϖ is uniquely determined by $\eta(K)$. Due to the uniqueness of the state trajectory the system in Eq. (7), note also that this system is essentially a system with unit time delay, the trajectory starting from $(\eta(-1) = 0, \eta(0))$ is identical to that starting from $(\varpi, \eta(K))$. Define a mapping F as

$$F : U \rightarrow E^s, \quad (60)$$

$$\eta(0) \mapsto \eta(K),$$

where $\eta(0)$ and $\eta(K)$ satisfying Eqs. (59) and (56), then F is injective. Therefore the Lebesgue measure of U is zero. ■

3 An Example

In this section, one example will be used to illustrate the effectiveness of the scheme proposed in this paper. In this example, the networked control system consists of two subsystems, (each composed of a system and its controller), the outputs of the controlled systems will be sent respectively to controllers via a network. For the ease of notation, we denote the two systems, their controllers and their outputs by G_1, G_2, C_1, C_2, y_1 and y_2 respectively. Here two transmission methods will be compared: one is just letting the outputs transmitted sequentially, i.e., the communication order is $[y_1(0), y_2(0), y_1(1), y_2(1), \dots]$. Another method is adding the nonlinear constraint H_2 to the subsystem composed of G_1 and C_1 , if the difference between the two adjacent signals are greater than $\delta_2 = 0.01$, then this subsystem gets access to the network; otherwise the other gets access. Here, we will compare the tracking errors produced under these two schemes respectively. For convenience, we call the first method the *regular static scheduler* and the second the *modified static scheduler*.

The controlled system G_1 is:

$$x_1(k+1) = \begin{bmatrix} 1.0017 & 0.1000 & 0.0250 & 0.0009 \\ 0.0500 & 1.0000 & 0.5000 & 0.0259 \\ 0.2000 & -0.0003 & 1.0000 & 0.1052 \\ -0.0034 & -0.2103 & -0.0517 & 1.1034 \end{bmatrix} x_1(k)$$

$$+ \begin{bmatrix} 0.0050 \\ 0.0991 \\ -0.0052 \\ -0.1155 \end{bmatrix} w(k) + \begin{bmatrix} -0.0050 & -0.0000 \\ -0.1000 & -0.0001 \\ 0.0000 & -0.0005 \\ 0.0103 & -0.0105 \end{bmatrix} u_1(k),$$

$$z_1(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} x_1(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(k),$$

$$y_1(k) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_1(k),$$

and G_2 is:

$$\begin{aligned}
 x_2(k+1) &= \begin{bmatrix} 1.0000 & 0.0100 & 0.0002 & 0.0000 \\ 0.0005 & 1.0000 & 0.0500 & 0.0003 \\ 0.0200 & -0.0000 & 1.0000 & 0.0101 \\ -0.0000 & -0.0201 & -0.0005 & 1.0100 \end{bmatrix} x_2(k) \\
 &+ \begin{bmatrix} 0.0000 \\ 0.0100 \\ -0.0001 \\ -0.0102 \end{bmatrix} w(k) + \begin{bmatrix} -0.0000 & -0.0000 \\ -0.0100 & -0.0000 \\ 0.0000 & -0.0000 \\ 0.0001 & -0.0010 \end{bmatrix} u_2(k), \\
 z_2(k) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} x_2(k) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} w(k), \\
 y_2(k) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_2(k),
 \end{aligned}$$

where w is a unit step. z_1 and z_2 are tracking errors. Controllers C_1 and C_2 can be obtained using the technique in Chen & Francis [1995]. Denote the first element of y_1 by y_{11} and that of y_2 by y_{21} ; the second element of y_1 by y_{12} and that of y_2 by y_{22} , then the subsystem with variables $x_1, x_2, z_1, z_2, y_{11}, y_{21}$ is G_1 controlled by C_1 and the subsystem with variables $x_1, x_2, z_1, z_2, y_{12}, y_{22}$ is G_2 controlled by C_2 . The simulation results are in Figs. 18–19.

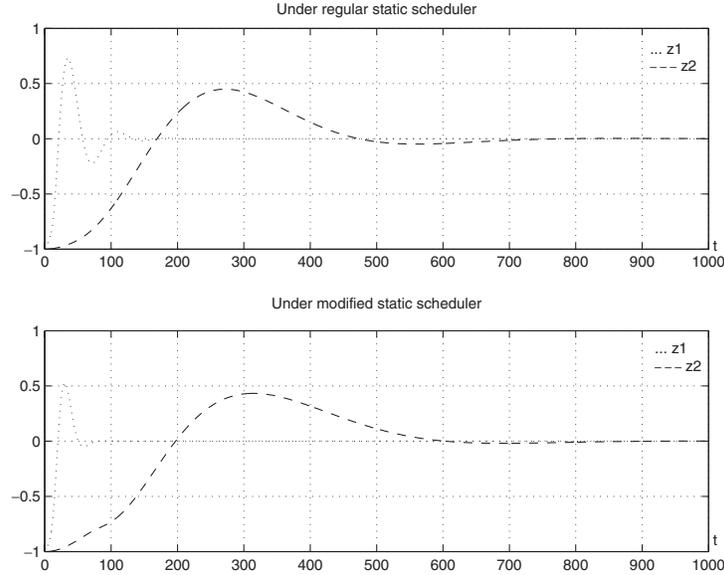


Fig. 18. The first elements of z_1 and z_2

From these two figures, one finds that the tracking errors approach zero faster under the modified static scheduler than under the regular one. Note that both systems are unstable. If one of the two systems is stable, one can expect better convergence rate. In essence, our scheme is based on the following principle: Allocate access to the network to the systems with faster dynamics first, then take care of the systems of slower dynamics. In this way, we hope we can improve system performance. Interestingly, a similar idea is explored

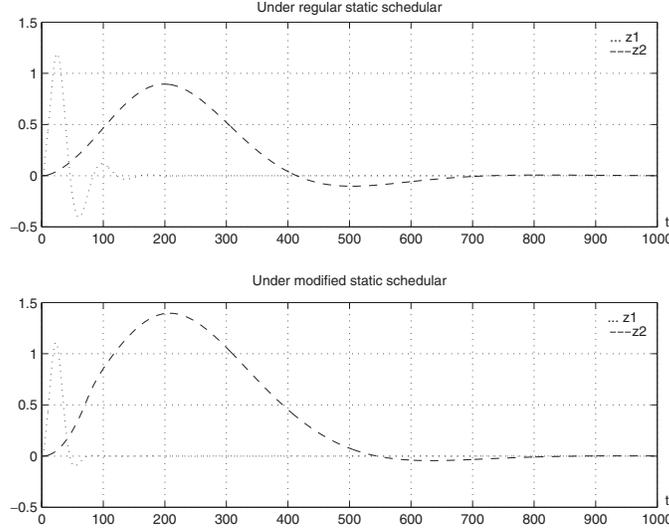


Fig. 19. The second elements of z_1 and z_2

in Hristu & Morgansen [1999].

4 Conclusion

In this paper, a new networked control technique is proposed and its effectiveness is illustrated via simulations. The complicated dynamics of this type of systems is studied both numerically and theoretically. A simulation shows that the scheme proposed here has possible application in networked control systems. There are several problems guiding our further research: 1) Continuity of state trajectories with respect to the initial points under space partition induced by the discontinuities of the system. 2) How to find a precise characterization of the attracting set for our system, and is it topologically transitive (i.e., is it a chaotic attractor)? Topological transitivity, an indispensable feature of a chaotic attractor, is closely related to ergodicity of a map. As discussed in Sec. 2.1.3, the proof of topological transitivity or ergodicity is difficult for our system from the point of view of measure theory due to the singularity of the map and its violation of conditions in Lasota & Yorke [1973]. However, this investigation is unavoidable should one want to find the chaotic attractor inherited in the system studied. 3) For different system parameters, different aperiodic orbits can be obtained, what are the differences among these orbits? In particular, given two aperiodic orbits, one generated from a system having no periodic orbits and the other generated by a system having periodic orbits, is there any essential difference between them? 4) In Sec. 2.1.2, periodic orbits are constructed for some originally stable ($|a| < 1$) and originally unstable ($|a| > 1$) systems. However given a system, how to determine if there are periodic orbits, and if so, how to find all of them is still an unsolved problem. 5) How to effectively design controllers based on chaotic control? Obviously the solution of this problem depends on the forgoing ones. 6) How to incorporate properly the scheme proposed in this paper into the framework of networked control systems? The simulation in Sec. 3

is naive, more research is required here to make the proposed scheme practical.

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