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About the derivation of the SCA algorithm

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Abstract

In Allefeld & Kurths [2004], we introduced an approach to multivariate phase synchronization analysis in the form of a Synchronization Cluster Analysis (SCA). A statistical model of a synchronization cluster was described, and an abbreviated instruction on how to apply this model to empirical data was given, while an implementation of the corresponding algorithm was (and is) available from the authors. In this letter, the complete details on how the data analysis algorithm is to be derived from the model are filled in.

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Starting point for the derivation of the procedure of Synchronization Cluster Analysis (SCA) is the factorization

$$\varrho_{ij} = \varrho_{iC}\,\varrho_{jC} \quad \text{for} \quad i \neq j$$

(oscillator indices $i, j = 1 \ldots N$). The given input is the empirical estimate of the bivariate synchronization indices, $\bar{R}_{ij}$, which is asymptotically normally distributed:

$$\bar{R}_{ij} \sim \mathrm{N}(\varrho_{ij}, \sigma_{ij}^2).$$

Since the phase difference between oscillators $\Delta\phi_{ij} = \Delta\phi_j - \Delta\phi_i$ is (according to the model) the difference of two independent random variables, it is approximately wrapped normally distributed, and consequently

$$\sigma_{ij}^2 = \frac{1}{2n}\left(1 - \varrho_{ij}^2\right)^2,$$

where n is the sample size (number of independent realizations) underlying $\bar{R}_{ij}$. Estimates of the to-cluster synchronization strengths $\hat{\varrho}_{iC}$ are therefore given by the minimum of the sum of normalized square errors

$$S = \sum_{i,j>i} \frac{\left(\bar{R}_{ij} - \varrho_{iC}\,\varrho_{jC}\right)^2}{\sigma_{ij}^2}.$$

This minimum can not be found directly, but via an iterative procedure. To simplify matters, we treat σ_{ij} as constant in the derivation of the iteration equations, but update it later in each step such that the correct minimum is found.

At the position of the minimum, $\varrho_{iC} = \hat{\varrho}_{iC}$ for all $i = 1 \ldots N$, all the partial derivatives of S with respect to ϱ_{kC} for $k = 1 \ldots N$ have to be zero:

$$\frac{\partial}{\partial \varrho_{kC}} S(\varrho_{1C}, \ldots \varrho_{NC})\bigg|_{\varrho_{iC}=\hat{\varrho}_{iC}} = 0$$

Because of the symmetry $\bar{R}_{ij} = \bar{R}_{ji}$ this can be written as [1]

$$0 = \frac{\partial}{\partial \varrho_{kC}} \sum_{i\neq k} \frac{\left(\bar{R}_{ik} - \varrho_{iC}\,\varrho_{kC}\right)^2}{\sigma_{ik}^2}\Bigg|_{\varrho_{iC}=\hat{\varrho}_{iC}} = 2\sum_{i\neq k} \hat{\varrho}_{iC} \frac{\hat{\varrho}_{iC}\,\hat{\varrho}_{kC} - \bar{R}_{ik}}{\sigma_{ik}^2},$$

[1] An index $i \neq k$ means "over all i that are not equal to k".

or equivalently as

$$\sum_i F_{ik}\ \hat{\varrho}_{iC}\ \left(\hat{\varrho}_{iC}\hat{\varrho}_{kC} - \bar{R}_{ik}\right) = 0$$

with

$$F_{ik} = \begin{cases} 0 & i = k \\ 1/(1 - \hat{\varrho}_{iC}^2 \hat{\varrho}_{kC}^2)^2 & i \neq k \end{cases} .$$

This set of N equations defines the optimal estimate. An iteration equation to improve a given estimate can be derived from this by solving for $\hat{\varrho}_{kC}$:

$$\hat{\varrho}'_{kC} = \frac{\sum_i F_{ik}\ \hat{\varrho}_{iC}\ \bar{R}_{ik}}{\sum_i F_{ik}\ \hat{\varrho}_{iC}^2} \quad \text{for all} \quad k = 1 \ldots N.$$

In practice this equation leads to an oscillating behavior, which can be smoothed out by taking the average of given and "improved" estimate as the new one:

$$\hat{\varrho}'_{kC} = \frac{1}{2}\left(\hat{\varrho}_{kC} + \frac{\sum_i F_{ik}\ \hat{\varrho}_{iC}\ \bar{R}_{ik}}{\sum_i F_{ik}\ \hat{\varrho}_{iC}^2}\right).$$

To start the iteration, a useful initial estimate is $\hat{\varrho}_{kC} = \max_{i \neq k} \bar{R}_{ik}$.

References

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