# HILBERT'S 16th PROBLEM FOR QUADRATIC SYSTEMS. NEW METHODS BASED ON A TRANSFORMATION TO THE LIENARD EQUATION 

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Fractionally-quadratic transformations which reduce any two-dimensional quadratic system to the special Lienard equation are introduced. Existence criteria of cycles are obtained.
Keywords: Quadratic system; cycles; Lienard equation; Hilbert's 16th problem.

## 1. Introduction

Within last century the study of cycles of twodimensional quadratic systems was stimulated by 16th Hilbert's problem and its different variants [Hilbert, 1902; Lloyd, 1988; Blows \& Perko, 1994; Ilyashenko, 2002; Chen \& Wang, 1979; Shi, 1980].

The development of asymptotical methods for investigation of cycles of quadratic systems is facilitated by the fractionally-quadratic transformations, which reduce any two-dimensional quadratic system to the Lienard equation of special type with discontinuous nonlinear functions [Leonov, 1997; Leonov, 1998; Leonov, 2006]. In the present paper these new facilities are demonstrated.

The elements of direct Lyapunov method, the small perturbations methods, and the bifurcation theory together permits us to obtain the existence criteria of small cycles in the neighborhoods of stationary points of Lienard equation.

This criteria applied to quadratic systems.
The question arises as to whether in two-dimensional quadratic systems a simultaneous bifurcation of cycles from two weak stable and weak unstable equilibria is possible by variation of one scalar parameter. Here we provide the positive answer to this question.

For the Lienard equation, the classical theorems on the existence of cycles is well known [Lefschetz, 1957; Cesari, 1959]. Here there occurs quite naturally the possibility of generalization of these results and of applying them to the problem on the existence of cycles in quadratic systems. On this way in the space of parameters of quadratic systems, the sets of positive Lebesgue measure, for which the cycles exist, are discovered.

## 2. The transformation of quadratic system to the Lienard equation

Let us consider the quadratic system

$$
\begin{align*}
& \dot{x}=a_{1} x^{2}+b_{1} x y+c_{1} y^{2}+\alpha_{1} x+\beta_{1} y \\
& \dot{y}=a_{2} x^{2}+b_{2} x y+c_{2} y^{2}+\alpha_{2} x+\beta_{2} y \tag{1}
\end{align*}
$$

where $a_{i}, b_{i}, c_{i}, \alpha_{i}, \beta_{i}$ are real numbers.
Here we shall follow the paper [Leonov, 1997; Leonov, 1998; Leonov, 2006].

Proposition 1. Without loss of generality, we can assume that $c_{1}=0$.

Proof. Suppose, for definiteness, that $a_{2} \neq 0$ (otherwise, having performed the change $x \rightarrow y$, $y \rightarrow x$, we obtain at once $c_{1}=0$ ). Introduce the linear transformation $x_{1}=x+\nu y, y_{1}=y$. To prove Proposition 1, it is sufficient to show that for the certain numbers $\rho, \kappa, \nu$ the following identity

$$
\begin{aligned}
(x & +\nu y)^{\bullet}=\left(a_{1}+\nu a_{2}\right) x^{2}+\left(b_{1}+\nu b_{2}\right) x y+ \\
& +\left(c_{1}+\nu c_{2}\right) y^{2}+\left(\alpha_{1}+\nu \alpha_{2}\right) x+\left(\beta_{1}+\nu \beta_{2}\right) y= \\
& =\rho(x+\nu y) y+\kappa(x+\nu y)^{2}+\left(\alpha_{1}+\nu \alpha_{2}\right) x+ \\
& +\left(\beta_{1}+\nu \beta_{2}\right) y
\end{aligned}
$$

is valid. This identity is equivalent to the following system of equations

$$
\begin{align*}
& \kappa=a_{1}+\nu a_{2}, \\
& \kappa \nu^{2}+\rho \nu=c_{1}+\nu c_{2},  \tag{2}\\
& \rho+2 \kappa \nu=b_{1}+\nu b_{2} .
\end{align*}
$$

The above relations are satisfied if

$$
\left(a_{1}+\nu a_{2}\right) \nu^{2}-\nu\left(b_{1}+\nu b_{2}\right)+\left(c_{1}+\nu c_{2}\right)=0 .
$$

Since $a_{2} \neq 0$, this equation with respect to $\nu$ always has a real root. Thus, the system of equations (2) always has a real solution.

Further we assume that $c_{1}=0$.
Proposition 2. Let be $b_{1} \neq 0$. The straight line $\beta_{1}+b_{1} x=0$ on the plane $\{x, y\}$ is invariant or transversal for trajectories of system (1).

Proof. The last statement follows from the relation

$$
\begin{aligned}
& \left(\beta_{1}+b_{1} x\right)^{\bullet}=b_{1}\left[\left(b_{1} x+\beta_{1}\right) y+a_{1} x^{2}+\alpha_{1} x\right]= \\
& =\left[a_{1}\left(\frac{\beta_{1}}{b_{1}}\right)^{2}-\alpha_{1}\left(\frac{\beta_{1}}{b_{1}}\right)\right] b_{1}
\end{aligned}
$$

for $x=-\beta_{1} / b_{1}$. This implies that if

$$
a_{1}\left(\frac{\beta_{1}}{b_{1}}\right)^{2}-\alpha_{1}\left(\frac{\beta_{1}}{b_{1}}\right)=0
$$

then the straight line $\beta_{1}+b_{1} x=0$ is invariant and if

$$
a_{1}\left(\frac{\beta_{1}}{b_{1}}\right)^{2}-\alpha_{1}\left(\frac{\beta_{1}}{b_{1}}\right) \neq 0
$$

then the straight line $\beta_{1}+b_{1} x=0$ is transversal.
Excluding the trivial case that the right-hand side of the first equation of system (1) is independent of $y$, we assume that

$$
\begin{equation*}
\left|b_{1}\right|+\left|\beta_{1}\right| \neq 0 \tag{3}
\end{equation*}
$$

Then by Proposition 2 we obtain that the trajectories of system (1) are also the trajectories of the system

$$
\begin{align*}
& \dot{x}=y+\frac{a_{1} x^{2}+\alpha_{1} x}{\beta_{1}+b_{1} x}, \\
& \dot{y}=\frac{a_{2} x^{2}+b_{2} x y+c_{2} y^{2}+\alpha_{2} x+\beta_{2} y}{\beta_{1}+b_{1} x} . \tag{4}
\end{align*}
$$

Consider the following transformation

$$
\begin{aligned}
& \bar{y}=y+\frac{a_{1} x^{2}+\alpha_{1} x}{\beta_{1}+b_{1} x} \\
& \bar{x}=x .
\end{aligned}
$$

Using these new phase variables (here the bars over the variables $\bar{x} \rightarrow x, \bar{y} \rightarrow y$ are omitted), system (4) can be written as

$$
\begin{equation*}
\dot{x}=y, \quad \dot{y}=-Q(x) y^{2}-R(x) y-P(x) \tag{5}
\end{equation*}
$$

where

$$
\begin{gathered}
Q(x)=\frac{-c_{2}}{\beta_{1}+b_{1} x}, \\
R(x)=-\frac{\left(b_{1} b_{2}-2 a_{1} c_{2}+a_{1} b_{1}\right) x^{2}+\left(b_{2} \beta_{1}+b_{1} \beta_{2}-\right.}{\left(\beta_{1}+b_{1} x\right)^{2}} \\
\\
\frac{\left.-2 \alpha_{1} c_{2}+2 a_{1} \beta_{1}\right) x+\alpha_{1} \beta_{1}+\beta_{1} \beta_{2}}{\left(\beta_{1}+b_{1} x\right)^{2}}, \\
P(x)=-\left(\frac{a_{2} x^{2}+\alpha_{2} x}{\beta_{1}+b_{1} x}-\frac{\left(b_{2} x+\beta_{2}\right)\left(a_{1} x^{2}+\alpha_{1} x\right)}{\left(\beta_{1}+b_{1} x\right)^{2}}+\right. \\
\left.+\frac{c_{2}\left(a_{1} x^{2}+\alpha_{1} x\right)^{2}}{\left(\beta_{1}+b_{1} x\right)^{3}}\right) .
\end{gathered}
$$

Proposition 2 and condition (3) imply that the trajectories of system (5) are also trajectories of the system
$\dot{x}=y e^{p(x)}, \quad \dot{y}=\left[-Q(x) y^{2}-R(x) y-P(x)\right] e^{p(x)}$, where $p(x)$ is a certain integral of the function $Q(x)$.

From this system, using the change $\bar{x}=x, \bar{y}=$ $y e^{p(x)}$, we obtain

$$
\begin{equation*}
\dot{x}=y, \quad \dot{y}=-f(x) y-g(x) \tag{6}
\end{equation*}
$$

Here the bars over the variables $x$ and $y$ are also omitted.

So, for $b_{1} \neq 0$, by means of the above nondegenerate changes we can reduce system (1) to the Lienard equation (6) with the functions

$$
\begin{gathered}
f(x)=R(x) e^{p(x)}=R(x)\left|\beta_{1}+b_{1} x\right|^{q} \\
g(x)=P(x) e^{2 p(x)}=P(x)\left|\beta_{1}+b_{1} x\right|^{2 q}
\end{gathered}
$$

Here for $b_{1} \neq 0, \quad q=-\frac{c_{2}}{b_{1}}$.
Proposition 3. Let be $b_{1} \neq 0, \beta_{1} \neq 0, \alpha_{1} \neq$ 0 . Without loss of generality, we can assume that $b_{1}=\alpha_{1}=\beta_{1}=1$.

Proof. Using the change

$$
x=\frac{\beta_{1}}{b_{1}} \bar{x}, y=\frac{\alpha_{1}}{b_{1}} \bar{y}, t=\frac{\bar{t}}{\alpha_{1}},
$$

we obtain

$$
\dot{\bar{x}}=\frac{\alpha_{1} \beta_{1} a_{1}}{b_{1}} \bar{x}^{2}+\overline{x y}+\bar{x}+\bar{y} .
$$

Let us consider system (6) where

$$
\begin{aligned}
& f(x)=(A x+B) x|x+1|^{q-2} \\
& g(x)=\frac{\left(C_{1} x^{3}+C_{2} x^{2}+C_{3} x+C_{4}\right) x}{(x+1)^{3}}|x+1|^{2 q}
\end{aligned}
$$

Here $A, B, C_{j}(j=1, \ldots, 4), q$ are real numbers.
Proposition 4. For numbers $A, B, C_{j}, q$ of system (6) exist corresponding numbers $a_{1}, b_{1}=1, \alpha_{1}=$ $1, \beta_{1}=1, a_{2}, b_{2}, c_{2}=-q, \alpha_{2}, \beta_{2}=-1$ of system (1) iff

$$
\begin{aligned}
& \frac{(B-A)}{(2 q-1)^{2}}((1-q) B+(3 q-2) A)=2 C_{2}-3 C_{1}-C_{3}, \\
& \frac{(B-A)}{(2 q-1)^{2}}(B+2(q-1) A)=C_{2}-2 C_{1}-C_{4} .
\end{aligned}
$$

Here

$$
\begin{aligned}
& a_{1}=1+\frac{B-A}{2 q-1} \\
& a_{2}=-(q+1) a_{1}^{2}-A a_{1}-C_{1} \\
& b_{2}=-A-a_{1}(2 q+1) \\
& \alpha_{2}=a_{1}^{2}-2 a_{1}+A\left(a_{1}-1\right)+\left(2 C_{1}-C_{2}\right)
\end{aligned}
$$

## 3. Existence criterion for small cycles of the Lienard equation in the neighborhood of equilibrium

Consider the equation

$$
\begin{equation*}
\ddot{z}+z=u(t) \tag{7}
\end{equation*}
$$

For this equation it is well known the following simple result.

Lemma 1 [Arnol'd, 1976]. The solution of equation (7) with initial data $z(0), \dot{z}(0)$ is given by formula

$$
\begin{align*}
z(t) & =\left[z(0)-\int_{0}^{t} u(\tau) \sin \tau d \tau\right] \cos t+ \\
& +\left[\dot{z}(0)+\int_{0}^{t} u(\tau) \cos \tau d \tau\right] \sin t \tag{8}
\end{align*}
$$

For $\dot{z}(0)=0$, taking into account formula (8), we have the following relation

$$
\begin{align*}
\dot{z}(t) & =-z(0) \sin t+\int_{0}^{t} u(\tau) \sin \tau d \tau \sin t+  \tag{9}\\
& +\int_{0}^{t} u(\tau) \cos \tau d \tau \cos t
\end{align*}
$$

Consider the equation

$$
\ddot{x}+F(x, \varepsilon) \dot{x}+G(x, \varepsilon)=0
$$

or the equivalent system

$$
\begin{align*}
& \dot{x}=y \\
& \dot{y}=-F(x, \varepsilon) y-G(x, \varepsilon) \tag{10}
\end{align*}
$$

Here $\varepsilon$ is a positive number, $F(x, 0)=f(x), G(x, 0)=$ $g(x), f\left(x_{0}\right)=g\left(x_{0}\right)=0$, in a certain neighborhood of the point $x=x_{0}, \varepsilon=0$ the functions $F(x, \varepsilon)$ and $G(x, \varepsilon)$ are smooth functions.

Theorem 1. If the inequalities

$$
\begin{align*}
& f^{\prime \prime}\left(x_{0}\right) g^{\prime}\left(x_{0}\right)-g^{\prime \prime}\left(x_{0}\right) f^{\prime}\left(x_{0}\right)<0  \tag{11}\\
& g^{\prime}\left(x_{0}\right)>0, \quad F\left(x_{\varepsilon}, \varepsilon\right)>0
\end{align*}
$$

where $x_{\varepsilon}$ is a zero of the function $G(x, \varepsilon)$ in the neighborhood of the point $x=x_{0}$, are satisfied, then there exists a number $\varepsilon_{0}>0$ such that for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$ system (10) has a cycle.

Proof. We introduce the notation $z(t)=x(t)-$ $x_{0}$ and put

$$
\begin{aligned}
& f(x)=f_{1}\left(x-x_{0}\right)+f_{2}\left(x-x_{0}\right)^{2}+O\left(\left(x-x_{0}\right)^{3}\right) \\
& g(x)=g_{1}\left(x-x_{0}\right)+g_{2}\left(x-x_{0}\right)^{2}+O\left(\left(x-x_{0}\right)^{3}\right)
\end{aligned}
$$

Without loss of generality, it can be assumed that $g_{1}=1$.

Consider the case when $\varepsilon=0$ and $z(0)$ is a small number.

Here we shall use smoothnis of functions $F$ and $G$ and shall follow the first Lyapunov method on finite time interval [Lefschetz, 1957; Cesari, 1959].

The first approximation to the solution $z(t)$ of system (10) is the function

$$
z_{1}(t)=z(0) \cos t
$$

The second approximation $z_{2}(t)$ to the solution $z(t)$ is given by the equation

$$
\begin{equation*}
\ddot{z}_{2}+z_{2}=u(t) \tag{12}
\end{equation*}
$$

where

$$
u(t)=z(0)^{2}\left(f_{1} \cos t \sin t-g_{2}(\cos t)^{2}\right)
$$

By Lemma 1 we obtain that $z_{2}(t)$ has the form

$$
\begin{align*}
& z_{2}(t)=z(0) \cos t+z(0)^{2}\left(-\frac{f_{1}}{3}(\sin t)^{3} \cos t+\right. \\
& +\frac{g_{2}}{3}\left(1-(\cos t)^{3}\right) \cos t+\frac{f_{1}}{3}\left(1-(\cos t)^{3}\right) \sin t- \\
& \left.-g_{2}\left(\sin t-\frac{1}{3}(\sin t)^{3}\right) \sin t\right) \tag{13}
\end{align*}
$$

This implies that for $\dot{z}(0)=0$ the relation holds:

$$
\begin{aligned}
& \dot{z}_{2}(t)=-z(0) \sin t+z(0)^{2}\left(\frac{f_{1}}{3}(\sin t)^{4}-\right. \\
& -\frac{g_{2}}{3}\left(1-(\cos t)^{3}\right) \sin t+\frac{f_{1}}{3}\left(1-(\cos t)^{3}\right) \cos t- \\
& \left.-g_{2}\left(\sin t-\frac{1}{3}(\sin t)^{3}\right) \cos t\right)
\end{aligned}
$$

It follows that the time $T>0$, of the second crossing, of the straight line $\{y=0\}$ by the solution $x(t), y(t)$ of system (10) with the initial data $x(0), y(0)=0$ satisfies the relation

$$
\begin{equation*}
T=2 \pi+O\left(\left(x(0)-x_{0}\right)^{2}\right) \tag{14}
\end{equation*}
$$

Consider now the function

$$
V(x, y)=\left(y+\int_{x_{0}}^{x} f(z) d z\right)^{2}+2 \int_{x_{0}}^{x} g(z) d z
$$

For the derivative of the function $V$ along the solution of system (10) the following relation

$$
\begin{aligned}
\dot{V}(x, y) & =-2 g(x) \int_{x_{0}}^{x} f(z) d z= \\
& =-f_{1}\left(x-x_{0}\right)^{3}-\left(\frac{2}{3} f_{2}+f_{1} g_{2}\right)\left(x-x_{0}\right)^{4}+ \\
& +O\left(\left(x-x_{0}\right)^{5}\right)
\end{aligned}
$$

is valid. Then, taking into account relations (13), (14), we obtain

$$
\begin{aligned}
& V(x(T), y(T))-V(x(0), 0)= \\
& =-\int_{0}^{T}\left(f_{1} z_{2}(t)^{3}+\left(\frac{2}{3} f_{2}+f_{1} g_{2}\right) z_{2}(t)^{4}\right) d t+ \\
& +O\left(z(0)^{5}\right)=-z(0)^{4} \int_{0}^{2 \pi}\left(\frac{2}{3} f_{2}+f_{1} g_{2}\right)(\cos t)^{4}+ \\
& +3 f_{1}(\cos t)^{2}\left(-\frac{f_{1}}{3}(\sin t)^{3} \cos t+\frac{g_{2}}{3}(1-\right. \\
& \left.-(\cos t)^{3}\right) \cos t+\frac{f_{1}}{3}\left(1-(\cos t)^{3}\right) \sin t- \\
& \left.-g_{2}\left(\sin t-\frac{1}{3}(\sin t)^{3}\right) \sin t\right) d t+ \\
& +O\left(z(0)^{5}\right)=-\frac{\left(f_{2}-f_{1} g_{2}\right) \pi}{2} z(0)^{4}+O\left(z(0)^{5}\right)
\end{aligned}
$$

In this case by the theorem on a continuous dependence of solutions of system (10) on a parameter, for sufficiently small parameter $\varepsilon>0$ in comparison with the number $|z(0)|$ we have the inequality

$$
V(x(T), y(T))>V(x(0), 0)
$$

On the other hand, for small $\varepsilon>0$ the equilibrium $x=x_{\varepsilon}, y=0$ of system (10) is a stable focus. These two facts yield that in a certain small neighborhood of the point $x=x_{\varepsilon}, y=0$ there exists a cycle of system (10) (Fig. 1). Theorem is proved.

Example 1. Consider the equation

$$
\ddot{x}+\left(\varepsilon+f_{1} x+f_{2} x^{2}\right) \dot{x}+x+g_{2} x^{2}+g_{3} x^{3}=0
$$

where $f_{1}, f_{2}, g_{2}, g_{3}$ are arbitrary numbers, $f_{2}-f_{1} g_{2}<$ 0.

Theorem 1 implies that for the sufficiently small positive $\varepsilon$ this equation has at least one periodic solution.


## Figure 1.

Consider system (1) and assume that

$$
\begin{aligned}
& b_{1}=\beta_{1}=1, \alpha_{1}+\beta_{2}=\varepsilon, \\
& A=-b_{2}+2 a_{1} c_{2}-a_{1}, \\
& B=-b_{2}-\beta_{2}+2 \alpha_{1} c_{2}-2 a_{1}, \\
& C=-a_{2}-2 \alpha_{2}+\alpha_{1} b_{2}+a_{1} \beta_{2}+\alpha_{1} \beta_{2}-c_{2} \alpha_{1}^{2}, \\
& D=-\alpha_{2}+\alpha_{1} \beta_{2} . \\
& \text { If } D>0 \text { and } \\
& \quad A D-B C+B D\left(1+c_{2}\right)<0
\end{aligned}
$$

then by theorem 1 we obtain that for sufficiently small $\varepsilon>0$ the system (1) has a cycle in some neighborhood of the point $x=y=0$.

Suppose that in system (1) $a_{1}=a_{2}=c_{1}=0$, $b_{1}=\beta_{1}=c_{2}=1, b_{2}=-1, \alpha_{1}=1 / 3-\varepsilon, \alpha_{2}=$ $\beta_{2}=-1 / 3$. We have

$$
\begin{aligned}
& F(x, \varepsilon)=\left(x^{2}+2(1-\varepsilon) x+\varepsilon\right)|1+x|^{-3}, \\
& G(x, \varepsilon)=\left(\frac{1}{9}\left(x^{2}+2 x\right)+\varepsilon x\left(x^{2}+2 x+\frac{1}{3}-\varepsilon x\right)\right) \\
& (1+x)^{-5}, \\
& f^{\prime}(0)=f^{\prime}(-2)=2, g^{\prime}(0)=g^{\prime}(-2)=\frac{2}{9} \\
& f^{\prime \prime}(0)=f^{\prime \prime}(-2)=-10, g^{\prime \prime}(0)=g^{\prime \prime}(-2)=-2 .
\end{aligned}
$$

In the neighborhood of $x=0, x_{\varepsilon}=0$ and in the neighborhood of $x=-2$ the relation $x_{\varepsilon}=$ $-2-3 \varepsilon+o(\varepsilon)$ holds.

For $x_{\varepsilon}=0$, we have

$$
F\left(x_{\varepsilon}, \varepsilon\right)=\varepsilon
$$

and for $x_{\varepsilon}=-2-3 \varepsilon+o(\varepsilon)$,

$$
F\left(x_{\varepsilon}, \varepsilon\right)=\frac{11 \varepsilon+o(\varepsilon)}{\left|1+x_{\varepsilon}\right|^{3}}
$$

It should also be noted that in the considered case system (10) with $\varepsilon=0$ under the change

$$
x=-z-2, \quad y=-\vartheta
$$

preserves its form and the equilibrium $x=-2, y=$ 0 goes into the equilibrium $z=\vartheta=0$.

Thus, we prove the following
Theorem 2. For sufficiently small $\varepsilon<0$, the system

$$
\begin{aligned}
& \dot{x}=x y+\left(\frac{1}{3}-\varepsilon\right) x+y \\
& \dot{y}=-x y+y^{2}-\frac{1}{3} x-\frac{1}{3} y
\end{aligned}
$$

has no less than two cycles. Moreover, each of two equilibria is surrounded by no less than one cycle.

## 4. The generalization of the Lienard theorem

In this section we give the generalization [Leonov, 2006] of a classical result of Lienard concerning the existence of periodic solution for the equation [Lefschetz, 1957]

$$
\begin{equation*}
\ddot{x}+f(x) \dot{x}+g(x)=0 \tag{15}
\end{equation*}
$$

Consider a system

$$
\begin{align*}
& \dot{x}=y \\
& \dot{y}=-f(x) y-g(x) \tag{16}
\end{align*}
$$

which is equivalent to equation (15).
Suppose, the function $f(x)$ and $g(x)$ are continuous on the interval $(a,+\infty)$ and for the certain numbers $a<\nu_{1} \leq x_{0} \leq \nu_{2}$ the following conditions

1) $\lim _{x \rightarrow a} g(x)=-\infty$,
$\lim _{x \rightarrow+\infty} g(x)=+\infty$,

$$
\begin{equation*}
\lim _{x \rightarrow a} \int_{x_{0}}^{x} g(z) d z=\lim _{x \rightarrow+\infty} \int_{x_{0}}^{x} g(z) d z=+\infty \tag{17}
\end{equation*}
$$

2) $\quad f(x)>0, \quad \forall x \in\left(a, \nu_{1}\right) \bigcup\left(\nu_{2},+\infty\right)$,

$$
\begin{equation*}
\int_{\nu_{2}}^{\nu_{1}} f(x) d x \leq 0 \tag{18}
\end{equation*}
$$

are satisfied.
Theorem 3. If conditions 1) and 2) are valid, then in the phase space

$$
\begin{equation*}
\left\{x \in(a,+\infty), y \in R^{1}\right\} \tag{19}
\end{equation*}
$$

there exists a piecewise-smooth transversal closed curve, which intersects the straight line $\{y=0\}$ at the certain points $a<\mu_{1}<\nu_{1}$ and $\mu_{2}>\nu_{2}$ (Fig. 2). If, in addition, in space (19) system (16) has only one unstable by Lyapunov focal equilibrium, then system (16) has a cycle.

Proof. Consider a pair of the numbers $\mu_{1} \in$ $\left(a, \nu_{1}\right)$ and $\mu_{2} \in\left(\nu_{2},+\infty\right)$ such that $\mu_{1}$ is sufficiently close to $a, \mu_{2}$ is sufficiently large, and

$$
\begin{equation*}
\int_{\mu_{1}}^{\mu_{2}} g(x) d x=0 \tag{20}
\end{equation*}
$$

Without loss of generality, we put

$$
\begin{aligned}
& g(x)<0, \quad \forall x \in\left[\mu_{1}, \nu_{1}\right] \\
& g(x)>0, \quad \forall x \in\left[\nu_{2}, \mu_{2}\right]
\end{aligned}
$$

Consider the following functions

$$
\begin{aligned}
V_{1}(x, y) & =y^{2}+2 \int_{x_{0}}^{x} g(z) d z \\
V_{2}(x, y) & =\left(y+\int_{\nu_{1}}^{x} f(z) d z\right)^{2}+2 \int_{x_{0}}^{x} g(z) d z \\
V_{3}(x, y) & =\left(y+\int_{\nu_{2}}^{x} f(z) d z\right)^{2}+2 \int_{x_{0}}^{x} g(z) d z \\
V_{4}(x, y) & =V_{2}(x, y)-\varepsilon\left(x-\nu_{1}\right) \\
V_{5}(x, y) & =V_{3}(x, y)-\varepsilon\left(x-\nu_{2}\right) \\
V_{6}(x, y) & =V_{3}(x, y)+\varepsilon\left(x-\nu_{2}\right) \\
V_{7}(x, y) & =V_{2}(x, y)+\varepsilon\left(x-\nu_{1}\right)
\end{aligned}
$$

Here $\varepsilon$ is a certain sufficiently small number.
Consider now the sets $\Omega_{j}$ (see Fig. 2)

$$
\begin{aligned}
& \Omega_{1}=\left\{x \in\left[\mu_{1}, \nu_{1}\right], y \geq 0, V_{1}(x, y) \leq V_{1}\left(\mu_{1}, 0\right)\right\} \\
& \Omega_{2}=\left\{x \in\left[\nu_{1}, x_{0}\right], y \geq 0, V_{4}(x, y) \leq V_{2}\left(\nu_{1}, y_{1}\right)\right\} \\
& \Omega_{3}=\left\{x \in\left[x_{0}, \nu_{2}\right], y \geq 0, V_{5}(x, y) \leq V_{3}\left(\nu_{2}, y 2\right)\right\} \\
& \Omega_{4}=\left\{x \in\left[\nu_{2}, \mu_{2}\right], y \geq 0, V_{3}(x, y) \leq V_{3}\left(\mu_{2}, 0\right)\right\} \\
& \Omega_{5}=\left\{x \in\left[\nu_{2}, \mu_{2}\right], y \leq 0, V_{1}(x, y) \leq V_{1}\left(\mu_{2}, 0\right)\right\} \\
& \Omega_{6}=\left\{x \in\left[x_{0}, \nu_{2}\right], y \leq 0, V_{6}(x, y) \leq V_{3}\left(\nu_{2}, y 3\right)\right\} \\
& \Omega_{7}=\left\{x \in\left[\nu_{1}, x_{0}\right], y \leq 0, V_{7}(x, y) \leq V_{2}\left(\nu_{1}, y_{4}\right)\right\} \\
& \Omega_{8}=\left\{x \in\left[\mu_{1}, \nu_{1}\right], y \leq 0, V_{2}(x, y) \leq V_{2}\left(\mu_{1}, 0\right)\right\}
\end{aligned}
$$



Figure 2.

Here $y_{1}>0, y_{2}>0, y_{3}<0, y_{4}<0$ are solutions of the square equations

$$
\begin{aligned}
& V_{1}\left(\nu_{1}, y_{1}\right)=V_{1}\left(\mu_{1}, 0\right), \\
& V_{3}\left(\nu_{2}, y_{2}\right)=V_{3}\left(\mu_{2}, 0\right), \\
& V_{1}\left(\nu_{2}, y_{3}\right)=V_{1}\left(\mu_{2}, 0\right), \\
& V_{2}\left(\nu_{1}, y_{4}\right)=V_{2}\left(\mu_{1}, 0\right)
\end{aligned}
$$

For the derivatives of the functions $V_{j}(x, y)$ along the solutions of system (24) we have the following relations

$$
\begin{aligned}
& \dot{V}_{1}=-2 f(x) y^{2}, \quad \dot{V}_{2}=-2 g(x) \int_{\nu_{1}}^{x} f(z) d z, \\
& \dot{V}_{3}=-2 g(x) \int_{\nu_{2}}^{x} f(z) d z \\
& \dot{V}_{4}=-2 g(x) \int_{\nu_{1}}^{x} f(z) d z-\varepsilon y \\
& \dot{V}_{5}=-2 g(x) \int_{\nu_{2}}^{x} f(z) d z-\varepsilon y \\
& \dot{V}_{6}=-2 g(x) \int_{\nu_{2}}^{x} f(z) d z+\varepsilon y \\
& \dot{V}_{7}=-2 g(x) \int_{\nu_{1}}^{x} f(z) d z+\varepsilon y .
\end{aligned}
$$

Therefore under the above assumptions concerning $\mu_{1}$ and $\mu_{2}$ and for $y \neq 0, x \neq \nu_{j}$ we have the inequalities

$$
\begin{array}{lll}
\dot{V}_{1}<0 & \text { on } \Omega_{1} \cup \Omega_{5}, \quad \dot{V}_{4}<0 \quad \text { on } \quad \Omega_{2}, \\
\dot{V}_{5}<0 & \text { on } \Omega_{3}, \quad \dot{V}_{3}<0 \quad \text { on } \Omega_{4}, \\
\dot{V}_{6}<0 & \text { on } \Omega_{6}, \quad \dot{V}_{7}<0 \quad \text { on } \Omega_{7}, \\
\dot{V}_{2}<0 & \text { on } \Omega_{8} .
\end{array}
$$

Note that by conditions 2), (20) for sufficiently small $\varepsilon$ we have inequalities $y_{5}<y_{6}$ and $y_{7}>y_{8}$, where $y_{5}$ is a positive solution of the equation

$$
V_{4}\left(x_{0}, y_{5}\right)=V_{2}\left(\nu_{1}, y_{1}\right),
$$

$y_{6}$ is a positive solution of the equation

$$
V_{5}\left(x_{0}, y_{6}\right)=V_{3}\left(\nu_{2}, y_{2}\right),
$$

$y_{7}$ is a negative solution of the equation

$$
V_{6}\left(x_{0}, y_{7}\right)=V_{3}\left(\nu_{2}, y_{3}\right),
$$

$y_{8}$ is a negative solution of the equation

$$
V_{7}\left(x_{0}, y_{8}\right)=V_{2}\left(\nu_{1}, y_{4}\right)
$$

(Fig. 2).
Thus, we construct a transversal closed curve, which proves the first assertion of theorem. The instability of focal equilibrium gives the existence of cycle.

Let us clarify the fact that the relations $\dot{V}_{4}<$ $0, \dot{V}_{5}<0, \dot{V}_{6}<0, \dot{V}_{7}<0$ are satisfied on the sets $\Omega_{2}, \Omega_{3}, \Omega_{6}, \Omega_{7}$, respectively.

Hold fixed the arbitrary $\varepsilon>0$, we choose $\mu_{1}$ and $\mu_{2}$ so much closer to $a$ and to $+\infty$, respectively, that the minimal values of $|y|$ on the intersection of the closed curve (Fig. 2) and the band $\{x \in$ [ $\left.\left.\nu_{1}, \nu_{2}\right]\right\}$ are more than

$$
\frac{1}{\varepsilon} \max _{x \in\left[\nu_{1}, \nu_{2}\right]} 2\left|g(x) \int_{\nu_{1}}^{x} f(z) d z\right|
$$

and

$$
\frac{1}{\varepsilon} \max _{x \in\left[\nu_{1}, \nu_{2}\right]} 2\left|g(x) \int_{\nu_{2}}^{x} f(z) d z\right| .
$$

This implies the required inequalities $\dot{V}_{j}<0$. Thus, we have here $\mu_{j}=\mu_{j}(\varepsilon)$ and

$$
\lim _{\varepsilon \rightarrow 0} \mu_{1}(\varepsilon)=a, \quad \lim _{\varepsilon \rightarrow 0} \mu_{2}(\varepsilon)=+\infty
$$

Represent now conditions 1) and 2) in terms of quadratic system (1).

We have

$$
a=-\frac{\beta_{1}}{b_{1}}, \quad b_{1} \neq 0 .
$$

Without loss of generality, we put $b_{1}>0$. Conditions 1) and 2) are satisfied if

$$
\begin{gather*}
0<2 c_{2}<b_{1}, \quad \beta_{1}>0, \\
\frac{a_{1} \beta_{1}}{b_{1}}>\alpha_{1} \\
\frac{a_{1}\left(2 c_{2}-b_{1}\right)}{b_{1}}>b_{2} \tag{21}
\end{gather*}
$$

$$
\frac{a_{1}\left(b_{1} b_{2}-a_{1} c_{2}\right)}{b_{1}^{2}}>a_{2}
$$

Besides, from conditions (21) it follows a positive invariance of the half-plane $\{x \geq a\}$ for quadratic system (1).

Note that here $c_{1}=0$ and the parameters $\alpha_{2}$ and $\beta_{2}$ do not enter into conditions (21).

A set of closed piecewise-smooth transversal curves can be constructed similarly. Therefore we have the following

Theorem 4. Let conditions (21) be valid. Then any solution, of system (1) with initial data such that $x(0)>a$, tends as $t \rightarrow+\infty$ to a bounded attractor, placed in the half-plane $\{x>a\}$.

Theorem 4 permits us to localize the search of the limit cycles of quadratic system (1). We remark that if in the half-plane $\{x>a\}$ we have a unique unstable by Lyapunov focal equilibrium of system (1) and for it conditions (21) are valid, then system (1) has a periodic solution, placed in the half-plane $\{x>a\}$.

Besides, under these assumptions we have $f(x)>$ 0 for $x<a$. Then, using the Lyapunov functions

$$
V(x, y)=y^{2}+\int_{2 a}^{x} g(z) d z
$$

we can prove that a solution of system (1) with the initial data $x(0)<a$ either tends as $t \rightarrow+\infty$ to the equilibrium, either to infinity, either leaves in a definite time the half-plane $\{x<a\}$.

Using the above and Theorem 4 we
Theorem 5. Suppose, conditions (21) are valid and in the half-plane $\{x>a\}$ system (1) has a unique unstable by Lyapunov focal equilibrium. Then any solution of system (1) with initial data such that $x(0)<a$ either tends as $t \rightarrow+\infty$ to the equilibrium, either to infinity, either to a bounded attractor, situated in the half-plane $\{x>a\}$. Any solution of system (1) with initial data such that $x(0) \geq a$ tends as $t \rightarrow+\infty$ to a bounded attractor, situated in the half-plane $\{x>a\}$. This attractor has at least one cycle.

Example 2. We put $b_{1}=\beta_{1}=1, \alpha_{1}=0, a_{2}=$ $b_{2}=-1, c_{1}=0, c_{2}=1 / 4, \alpha_{1}=-1, \beta_{2}=2, \alpha_{2}=$ -1000 . Here conditions (21) are fulfilled, $g(x) \neq 0$ for $x \neq 0$ and $x>-1$, equilibrium $x=y=0$ is unstable focus. Therefore system (1) has at least one cycle by theorem 5 .

The conditions of Theorem 5 effectively select in the space of parameters of system (1) a set, of positive Lebesque measure, in which the cycles
exist. What the obtaining of the results of such kind is of present interest is remarked in [Arnol'd, 2005].
V.A.Arnold (see [Arnol'd, 2005]) writes: "To estimate the number of limit cycles of quadratic vector fields on a plane, A.N.Kolmogorov distributed, as a mathematical practice, few hundreds of such fields (with the randomly chosen coefficients of quadratic expressions) among few hundreds students of the mechanics and mathematics faculty of Moscow State University.

Each student had to find the number of limit cycles of his field.

The result of this experiment was perfectly sudden: all fields had none of the limit cycles!

For a small change of coefficients of field, a limit cycle is preserved. Therefore in the space of coefficients, the systems with one, two, three (and even, as was proved later, four) limit cycles make up open sets and if the coefficients of polynomials are drawn at random, then the probability of hitting in this sets are positive.

The fact that this event did not occur suggests that the above-mentioned probabilities are, evidently, small."

But the set of existence of cycles with selected by theorem 5 is not small.

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