# VARIATIONAL FORMULATION FOR THE KPZ AND RELATED KINETIC EQUATIONS

Horacio S. Wio

Instituto de Física de Cantabria, Universidad de Cantabria & CSIC, E-39005 Santander, Spain

We present a variational formulation for the Kardar-Parisi-Zhang (KPZ) equation that leads to a thermodynamic-like potential for the KPZ as well as for other related kinetic equations. For the KPZ case, with the knowledge of such a potential we prove some global shift invariance properties previously conjectured by other authors. We also show a few results about the form of the stationary probability distribution function for arbitrary dimensions. The procedure used for KPZ was extended in order to derive more general forms of such a functional leading to other nonlinear kinetic equations, as well as cases with density dependent surface tension.

Keywords: Kinetic equations, KPZ, Lyapunov functional, Reaction-diffusion models.

#### 1. Introduction

Phenomena far from equilibrium are ubiquitous in nature, including among many other, turbulence in fluids, interface and growth problems, chemical reactions, biological systems, as well as economical and sociological structures. During the last decades the focus on statistical physics research has shifted towards the study of such systems. Among those studies, the understanding of growing kinetics at a microscopic as well as on a mesoscopic level constitutes a major challenge in physics and material science [Tong & Williams, 1994; Barabási & Stanley, 1995; Halpin-Healy & Zhang, 1995; Marsili *et al.*, 1996]. Some recent papers have shown how the methods and know-how from static critical phenomena have been exploited within nonequilibrium phenomena of growing interfaces, obtaining scaling properties, symmetries, morphology of pattern formation in a driven state, etc [Hentschel, 1994; Prähofer & Spohn, 2004; López *et al.*, 2005; Fogedby, 2006; Ma *et al.*, 2007; Castro *et al.*, 2007].

Even though it was (briefly) discussed in [Cross & Hohenberg, 1993], there is still a common belief that the nontrivial spatial-temporal behavior occurring in several nonequilibrium systems, originates from the non-potential (or nonvariational) character of the dynamics, meaning that there is no Lyapunov functional for the dynamics. However, Graham and co-workers have shown in a series of papers [Graham, 1987; Graham & Tel, 1990] that a Lyapunov-like functional exists for a very general dynamical system, the complex Ginzburg-Landau equation. Such a functional is formally defined as the solution of a Hamilton-Jacobi-like equation, or obtained in a small gradient expansion [Descalzi & Graham, 1994]. The confusion associated with the qualification of *nonvariational* dynamics comes from the idea that the dynamics of systems having nontrivial attractors (limit cycle, chaotic) cannot be deduced from the minimization of a potential playing the same role as the free energy in equilibrium systems [Cross & Hohenberg, 1993]. Nevertheless, this does not preclude the existence of a Lyapunov functional for the dynamics that will have local minima identifying the attractors of the system [Montagne et al., 1996; Wio, 1997; Wio et al., 2002; Wio & Deza, 2007]. However, once the system has reached an attractor that is not a fixed point, the dynamics proceeds inside the attractor driven by *nonvariational* contributions to the dynamical flow, that do not change the value of the Lyapunov functional, implying that the dynamics is not completely determined once the indicated functional is known. This situation has known examples even in equilibrium statistical mechanics [Hohenberg & Halperin, 1977]. Hence, the Lyapunov functional, or nonequilibrium potential (NEP) [Graham, 1987; Graham & Tel, 1990; Wio, 1997; Ao, 2004], plays the role in nonequilibrium situations of a thermodynamical-like potential characterizing the global properties of the dynamics: attractors, relative (or nonlinear) stability of these attractors, height of the barriers separating attractions basins, offering the possibility of studying transitions among the attractors due to the effect of (thermal) fluctuations.

In a recent series of papers we have shown several results related to the obtention of the indicated NEP's for other system's classes: scalar and non-scalar reaction-diffusion systems [Bouzat & Wio, 1999; von Haeften *et al.*, 2004; von Haeften *et al.*, 2005; von Haeften & Wio, 2007]. In particular we have exploited those results for the study of stochastic resonance [Gammaitoni *et al.*, 1998] in extended systems [Wio, 1997; Bouzat & Wio, 1999; von Haeften *et al.*, 2000; von Haeften *et al.*, 2004; von Haeften *et al.*, 2005; von Haeften *et al.*, 2005; von Haeften *et al.*, 2007; Wio & Deza, 2007;]. In those works, we have analyzed problems of stochastic resonance in scalar and activator-inhibitor systems, in systems with local and nonlocal interactions, studied system-size stochastic resonance, etc.

Here, and related to the kinetics of growing interfaces, we discuss the case of the Kardar-Parisi-Zhang equation (KPZ) [Kardar, Parisi & Zhang, 1986; Medina *et al.*, 1989]. This equation, that describes the evolution of  $h(\bar{x}, t)$ , a field that corresponds to the height of a fluctuating interface, reads

$$\frac{\partial h(\bar{x},t)}{\partial t} = \nu \nabla^2 h(\bar{x},t) + \frac{\lambda}{2} \left( \nabla h(\bar{x},t) \right)^2 + K_o + \xi(\bar{x},t), \tag{1}$$

Opposing to a claim in a recent paper [Fogedby, 2006]: The KPZ equation is in fact a genuine kinetic equation describing a nonequilibrium process in the sense that the drift  $\nu \nabla^2 h + \frac{\lambda}{2} \nabla h \cdot \nabla h - F$  cannot be derived from an effective free energy; in [Wio, 2007] it was shown that such a **nonequilibrium thermodynamic-like potential** (NETLP) for the KPZ equation exists.

In this paper we present the approach to obtain the NETLP for the KPZ equation, and also show how it is possible to obtain such a NETLP for other related kinetic equations, as well as extend the procedure to other general situations. The organization is as follows. In the next Section we show how to obtain the NETLP for the KPZ case and, exploiting its knowledge, we discuss conjectures advanced in [Hentschel, 1994] and how they are fulfilled. Next we discuss about the form of the stationary (or asymptotic) probability distribution function, *valid for any dimension*. In the following Section we discuss the situation with a non-local interaction and symmetric kernels, showing how this form of extending the derivation procedure allows to obtain more general forms of kinetic equations. The following Section discusses the case of density dependent surface tension, while the last one presents some final comments.

#### 2. Kardar-Parisi-Zhang case

#### 2.1 Derivation

In order to show how to obtain the above indicated functional, we start considering the following simple (general) scalar reaction-diffusion equation with multiplicative noise

$$\frac{\partial}{\partial t}\phi(\bar{x},t) = \nu\nabla^2\phi(\bar{x},t) + f(\phi(\bar{x},t)) + \phi(\bar{x},t)\eta(\bar{x},t),\tag{2}$$

where  $f(\phi(\bar{x}, t))$  is a general nonlinear function. The noise  $\eta(\bar{x}, t)$  is a Gaussian white noise, of zero mean and intensity  $\sigma$ . For this equation we assume the Stratonovich interpretation. It is known that the system in Eq. (2) has the following NETLP

$$\mathcal{F}[\phi] = \int_{\Omega} \left\{ -\int_{0}^{\phi(\bar{x},t)} f(\varphi) \, d\varphi + \frac{\nu}{2} \left( \nabla \phi(\bar{x},t) \right)^{2} \right\} d\bar{x},$$

where  $\Omega$  indicates the integration range, and

$$\frac{\partial}{\partial t}\phi(\bar{x},t) = -\frac{\delta\mathcal{F}[\phi]}{\delta\phi(\bar{x},t)} + \phi(\bar{x},t)\,\eta(\bar{x},t);\tag{4}$$

where the contribution from the boundaries is null, due to the variation  $\delta\phi$  being fixed (= 0) at these boundaries, as usual. As has been shown in previous works [Izús *et al.*, 1998; Bouzat & Wio, 1998], it also fulfills the Lyapunov characteristic  $\frac{\partial}{\partial t} \mathcal{F}[\phi] \leq 0$  (it is worth here commenting that, as a matter of fact, this conditions is only valid in a weak noise limit).

Exploiting the so called Hopf-Cole transformation we now define a new field,  $h(\bar{x}, t)$  that, as indicated before, corresponds to an interface height,

$$h(\bar{x},t) = \frac{2\nu}{\lambda} \ln \phi(\bar{x},t), \tag{5}$$

with the inverse

$$\phi(\bar{x},t) = e^{\frac{\lambda}{2\nu}h(\bar{x},t)}.$$

As  $\phi(\bar{x},t) \geq 0$ ,  $h(\bar{x},t)$  is always well defined. The transformed equation reads

$$\partial_t h(\bar{x},t) = \nu \nabla^2 h + \frac{\lambda}{2} \left( \nabla h \right)^2 + \frac{\lambda}{2\nu} e^{-\frac{\lambda h}{2\nu}} \hat{f}(h) + \xi(\bar{x},t), \tag{6}$$

where  $\hat{f}(h) = f(\phi) = f(e^{\frac{\lambda}{2\nu}h})$ . Now, in order to reduce our general result to the one shown in Eq. (6), we assume  $f(\phi(\bar{x},t)) = a\phi(\bar{x},t)$  (with a some constant). In this way Eq. (2) becomes Eq. (1), with  $a = \frac{\lambda}{2\nu}K_o$  and  $\sigma = \left(\frac{\lambda}{2\nu}\right)^2 \varepsilon$ . However, the noise term that in the original equation (Eq. (2)) has a multiplicative character, in the transformed equation (Eq. (1)) becomes additive.

(3)

If we now apply the same transformation to the NETLP indicated in Eq.(3), restricting ourselves to  $f(\phi(\bar{x},t)) = a\phi(\bar{x},t)$ , we obtain

$$\mathcal{F}[h] = \int_{\Omega} e^{\frac{\lambda}{\nu}h(\bar{x},t)} \frac{\lambda}{2\nu} \Big[ -K_o + \frac{\lambda}{4} \left( \nabla h(\bar{x},t) \right)^2 \Big] d\bar{x}.$$
<sup>(7)</sup>

It is easy to prove that this functional fulfills both, the relation

$$\frac{\partial}{\partial t}h(\bar{x},t) = -\Gamma[h]\frac{\delta\mathcal{F}[h]}{\delta h(\bar{x},t)} + \xi(\bar{x},t); \tag{8}$$

as well as the Lyapunov characteristic  $\frac{\partial}{\partial t} \mathcal{F}[h] \leq 0$ , where the function  $\Gamma[h]$  is given by

$$\Gamma[h] = \left(\frac{2\nu}{\lambda}\right)^2 e^{-\frac{\lambda}{\nu}h(\bar{x},t)}.$$

Hence we have a **free energy-like functional** from where the KPZ kinetic equation can be obtained through functional derivation. Clearly, the contribution to the variation coming from the boundaries is again null.

It is worth to consider once more Eq. (2), but now assuming  $f(\phi(\bar{x},t)) = a\phi(\bar{x},t) - b\phi(\bar{x},t)^3$ , i.e. we include a limiting (or saturating) term. In such a case, the reaction-diffusion equation corresponds to a form of the so called Schlögl model [Mikhailov, 1990; Wio, 1994]. The "potential" contribution to the NETLP in Eq. (3) has the form

$$-\int_{\Omega} \left(\frac{a}{2}\phi(\bar{x},t)^2 - \frac{b}{4}\phi(\bar{x},t)^4\right) \, d\bar{x}$$

Applying again the indicated Hopf-Cole transformation, in Eq. (1) a new associated term arises, having the form  $-\gamma e^{\frac{\lambda}{\nu}h(\bar{x},t)}$ , with  $b = \frac{\lambda}{2\nu}\gamma$ . The new kinetic equation corresponds to a form of the so called *bounded-KPZ* [Grinstein *et al.*, 1996; Tu *et al.*, 1997; de los Santos *et al.*, 2007]. Clearly, we will also have an extra term in the associated NETLP (Eq. (7)). However, here we restrict ourselves to the case b = 0, and only analyze the more "usual" form of the KPZ equation indicated by Eq. (1).

#### 2.2 Some Properties

Let us now check some of the properties assumed previously for such a functional. According to the analysis of global shift invariance in [Hentschel, 1994], it is easy to see that the relations indicated by Eq. (9) in the indicated paper are fulfilled. That is, we can prove that if l is an arbitrary (constant) shift

$$\mathcal{F}[h+l] = K[l]\mathcal{F}[h]$$
  

$$\Gamma[h+l] = K[l]^{-1}\Gamma[h], \qquad (9)$$

with  $K[l] = e^{\frac{\lambda}{D}l} = \left(\frac{2\nu}{\lambda}\right)^2 \Gamma[l]^{-1}$ .

To prove other conjectures also indicated in [Hentschel, 1994], we introduce the free energy-like density  $\widetilde{\mathcal{F}}[h, \nabla h]$ , defined through

$$\mathcal{F}[h] = \int d\bar{x} \, \widetilde{\mathcal{F}}[h, \nabla h]$$

Hence,  $\widetilde{\mathcal{F}}[h, \nabla h] = \frac{\lambda}{2\nu} e^{\frac{\lambda}{\nu}h(\bar{x},t)} [-K_o + \frac{\lambda}{4} (\nabla h(\bar{x},t))^2]$ . The relations we refer are

$$\mathcal{F}[h, \nabla h] = e^{sh} \mathcal{F}_1[(\nabla h)^2]$$
  

$$\Gamma[h, \nabla h] = e^{-sh} \Gamma_1[(\nabla h)^2], \qquad (10)$$

and according to the form of  $\widetilde{\mathcal{F}}[h, \nabla h]$ , it is clear that the first relation above results obviously true, while for the second relation we have that  $\Gamma[h, \nabla h] = e^{-sh(\bar{x},t)}\Gamma_o$ , where  $\Gamma_o = 1$ , and  $s = \frac{\lambda}{\nu}$ , as  $\Gamma[h]$  is not function of  $\nabla h$ . In addition, it can be also proved that the indicated NETLP is also invariant under the nonlinear Galilei transformation that, as discussed in [Fogedby, 2006], are fulfilled by the KPZ equation.

It is here adequate to make a warning. We have found that, from the indicated *free energy*-like functional for the KPZ kinetic equation and by a functional derivative, we can obtain a form that resembles a (relaxation) model **A** according to the classification in [Hohenberg & Halperin, 1977]. In one hand, in the standard "model A" it is known that the

dynamics can be seen as a superposition of modes that decay exponentially towards a steady state. In addition, it is also known that the time-dependent correlations obey certain constraints such as positivity. On the other hand, in the KPZ problem it is known that the relaxation of perturbations decay in a stretched exponential manner [Schwartz & Edwards, 2002; Edwards & Schwartz, 2002; Colaiori & Moore, 2002; Katzav & Schwartz, 2004]. Hence, even though Eq. (8) looks similar to "model A", its behavior is far from trivial, and we have no a-priori intuition to what its dynamic could be. Clearly, this is a point to have in mind when suggesting any kind of Antsatz for the temporal behavior.

### 3. Probability Distribution Function for KPZ

#### 3.1 General Aspects

The knowledge of the NETLP indicated in the previous section, allows us to readily write the asymptotic long time probability distribution (pdf). We start writing the (functional) Fokker-Planck equation (FPE) associated to Eq. (1), that reads

$$\frac{\partial}{\partial t}\mathcal{P}[h(\bar{x},t)] = \int_{\Omega} d\bar{x} \frac{\delta}{\delta h} \Big\{ -\Big[\nu \nabla^2 h(\bar{x},t) + \frac{\lambda}{2} \left(\nabla h(\bar{x},t)\right)^2\Big] \mathcal{P}[h(\bar{x},t)] + \varepsilon \frac{\delta}{\delta h} \mathcal{P}[h(\bar{x},t)] \Big\}; \tag{11}$$

where, in order to focus in the most relevant aspects and to simplify, we adopted  $K_o = 0$ .

In Chap. 6 of [Barabási & Stanley, 1995] as well as in Sect. 3.5 of [Halpin-Healy & Zhang, 1995] the form of the probability distribution function (pdf) for the one dimensional KPZ equation was discussed. The form of such a pdf for an arbitrary dimension, so far, is not known. However, here we show the general form of such a pdf.

The knowledge of such a NETLP for the KPZ equation allows us to readily write the asymptotic long time probability distribution function (pdf), valid for (worth to be remarked) **any dimension**, which (due to the "diagonal" character of  $\Gamma[h]$ ) is given by

$$\mathcal{P}_{as}[h(\bar{x})] \sim \exp\left\{-\frac{1}{\varepsilon}\int d\bar{x} \int_{h_{ref}}^{h(\bar{x})} d\psi \,\Gamma[\psi] \frac{\delta \mathcal{F}[\psi]}{\delta \psi}\right\}$$
$$\sim \exp\left\{-\frac{1}{\varepsilon}\int d\bar{x} \left(\Gamma[h]\widetilde{\mathcal{F}}[h] - \int_{h_{ref}}^{h(\bar{x})} d\psi \,\frac{\delta\Gamma[\psi]}{\delta \psi}\widetilde{\mathcal{F}}[\psi]\right)\right\}$$
$$\sim \exp\left\{-\frac{\nu}{2\varepsilon}\int d\bar{x} \left(\nabla h\right)^2 + \frac{\lambda}{2\varepsilon}\int d\bar{x} \int_{h_{ref}}^{h(\bar{x})} d\psi \left(\nabla \psi\right)^2\right\}$$
$$\sim \exp\left\{-\frac{\Phi[h]}{\varepsilon}\right\}, \tag{12}$$

where, we reiterate, we assumed  $K_o = 0$ , and  $h_{ref}$  is an arbitrary reference state. The second line results by using functional methods (see for instance [Hänggi, 1985]). The third line shows a nice structure, where we can identify a contribution, with a Gaussian dependence on the slope, plus a "correction" term proportional to  $\lambda$ . The above indicated result is the valid (albeit "formal") solution for arbitrary dimension irrespective of boundary conditions, while for the one-dimensional case, and the adequate (periodic) boundary conditions, it is possible to show that this pdf reduces to the well known Gaussian result [Barabási & Stanley, 1995; Halpin-Healy & Zhang, 1995].

Clearly, a relevant point is related to the interpretation of the integral over the function  $\psi(\bar{x})$  in Eq. (12). In order to simplify we consider the 1-d case, and focus only on the first term. Using functional techniques, we have

$$\nu \int dx \int_{h_{ref}}^{h(x)} d\psi \, \frac{\partial^2}{\partial x^2} \psi(x) = -\frac{\nu}{2} \int dx \int_{h_{ref}}^{h(x)} d\psi \frac{\delta}{\delta \, \psi(x)} \left( \int dx' \left( \frac{\partial \psi(x')}{\partial x'} \right)^2 \right) = \frac{\nu}{2} \int dx \left( \frac{\partial h}{\partial x} \right)^2, \tag{13}$$

where we have used that  $\frac{\delta \psi(x')}{\delta \psi(x)} = \delta(x - x')$ . Another way to grasp it is via a discrete representation

$$\approx \sum_{j} \int^{h_{j}} d\psi_{j} (\psi_{j+1} - 2\psi_{j} + \psi_{j-1})$$
$$\approx \frac{\nu}{2} \sum_{j} \int^{h_{j}} d\psi_{j} \frac{\partial}{\partial \psi_{j}} \sum_{l} [\psi_{l+1} - \psi_{l}]^{2}$$

that, as only the l = j and l = j - 1 terms survive, yields

$$\approx \frac{\nu}{2} \sum_{j} [h_{j+1} - h_j]^2.$$
(14)

In both cases we obtain the known result.

The interpretation for the second (KPZ) term is analogous to the one in the example indicated above. However, for this case the situation is a little more delicate. Several recent papers have discussed different alternatives to the discrete form of such a contribution in the KPZ equation in 1-d [Newman & Bray, 1996; Lam & Shin, 1998; Ma, Jiang & Yang, 2007]. We will not come into these details here, that will be deeply discussed in [Revelli & Wio, 2008].

#### 3.2 Nonequilibrium Potential for KPZ

The last line in Eq. (12) defines the functional  $\Phi[h]$ 

$$\Phi[h] = \frac{\nu}{2} \int d\bar{x} \left(\nabla h\right)^2 - \frac{\lambda}{2} \int d\bar{x} \int_{h_{ref}}^{h(\bar{x})} d\psi \left(\nabla \psi\right)^2.$$
(15)

The variation of this functional give us

$$\frac{\partial}{\partial t}h(\bar{x},t) = -\frac{\delta\Phi[h]}{\delta h(\bar{x},t)} + \xi(\bar{x},t).$$
(16)

This functional also fulfills the Lyapunov condition  $\frac{\partial}{\partial t} \Phi[h] = -\left(\frac{\delta \Phi[h]}{\delta h(\bar{x},t)}\right)^2 \leq 0.$ 

Hence, such a functional could be identified as the *nonequilibrium potential* [Graham, 1987; Graham & Tel, 1990; Wio *et al.*, 2002] for the KPZ case. More, even though it could be also identified with a Hamiltonian for KPZ, it is worth to remark that it differs from the form indicated in Eq. (3.4) of [Halpin-Healy & Zhang, 1995].

The last functional form (albeit so far only formal) could, in principle, allow us to exploit several known techniques [Graham, 1987; Graham & Tel, 1990; Wio *et al.*, 2002]. It also shows that the claim by previous authors indicated at the Introduction is not true.

### 3.3 NEP's application: A simple example

Here we discuss a simple example in order to show the possibilities that offers the knowledge of such a NETLP. For this example we analyze a sightly different situation than the one studied in Eq. (1) and its associated NEP in Eq. (15). We only consider a spatial quenched noise ("disorder") instead of the spatial-temporal noise considered so far. The equation associated to such a problem is

$$\frac{\partial}{\partial t}h(\bar{x},t) = \nu \nabla^2 h(\bar{x},t) + \frac{\lambda}{2} \left(\nabla h(\bar{x},t)\right)^2 + K_o + \vartheta(\bar{x}),\tag{17}$$

where, as in previous studies [Nattermann & Renz, 1989; Krug & Halpin-Healy, 1993; Ramasco, López & Rodriguez, 2006; Szendro, López & Rodriguez, 2007],  $\vartheta(\bar{x})$  is a quenched, Gaussian distributed, noise. For this case we have that

$$\frac{\partial}{\partial t}h(\bar{x},t) = -\Gamma[h]\frac{\delta\mathcal{F}[h]}{\delta h(\bar{x},t)},\tag{18}$$

with the same form of  $\mathcal{F}[h]$  as in Eq. (7), but  $K_o$  replaced by  $K_o + \vartheta(\bar{x})$ .

The indicated studies have shown that the front profile presents a triangular structure [Ramasco, López & Rodriguez, 2006; Szendro, López & Rodriguez, 2007] and, clearly, it is of relevance to determine the slope of such structures. Using the known form of the NETLP, we can minimize this free-energy-like functional and obtain, in the 1-d case, that such a slope is  $\alpha = \left(\frac{16}{\nu\lambda}\right)^{1/3}$ , a value that agrees quite well with the numerical evaluations.

#### 4. Other Kinetic Equations: Non Locality

We can go still further and look for the possibility of deriving a NETLP for more general forms of kinetic equations. To this end, let us assume that we have the following non-local reaction-diffusion equation [Wio *et al.*, 2002]

$$\frac{\partial}{\partial t}\phi(\bar{x},t) = \nu \nabla^2 \phi(\bar{x},t) + a\phi(\bar{x},t) - \beta \int_{\Omega} d\bar{x'} \mathbf{G}(\bar{x},\bar{x'})\phi(\bar{x'},t) + \phi(\bar{x},t)\,\eta(\bar{x'},t),\tag{19}$$

where, as discussed in [Bouzat & Wio, 1999; von Haeften *et al.*, 2004; von Haeften & Wio, 2007], the kernel  $\mathbf{G}(\bar{x}, \bar{x'})$  could be of a very general character, and  $\beta$  is the interaction intensity. It was shown that the form of the associated NETLP is

$$\mathcal{F}[\phi] = \int_{\Omega} \left[ -\frac{a}{2} \phi(\bar{x}, t)^2 + \frac{\nu}{2} \left( \nabla \phi(\bar{x}, t) \right)^2 + \beta \int_{\Omega} d\bar{x'} \,\phi(\bar{x}, t) \,\mathbf{G}(\bar{x}, \bar{x'}) \,\phi(\bar{x'}, t) \right] d\bar{x}. \tag{20}$$

As we have done before, using the Hopf-Cole transformation we obtain a *generalized-non-local* form of the KPZ equation

$$\frac{\partial}{\partial t}h(\bar{x},t) = \nu\nabla^2 h(\bar{x},t) + \frac{\lambda}{2}\left(\nabla h(\bar{x},t)\right)^2 + K_o - \beta e^{-\frac{\lambda}{2\nu}h(\bar{x},t)} \int_{\Omega} d\bar{x'} \mathbf{G}(\bar{x},\bar{x'}) e^{\frac{\lambda}{2\nu}h(\bar{x'},t)} + \xi(\bar{x},t). \tag{21}$$

Even though the nonlocal contribution indicated above differs from those discussed in [Mukherji & Bhattacharjee, 1997; Katzav, 2003], it is clear that such form of a nonlocal term is also of great interest. Repeating the previous procedure we find the associated NETLP

$$\mathcal{F}[h] = \int_{\Omega} d\bar{x} e^{\frac{\lambda}{\nu}h(\bar{x},t)} \Big[ -\frac{\lambda}{2\nu} K_o + \left(\frac{\lambda^2}{8\nu}\right) \left(\nabla h(\bar{x},t)\right)^2 + \beta e^{-\frac{\lambda}{2\nu}h(\bar{x},t)} \int_{\Omega} d\bar{x'} \mathbf{G}(\bar{x},\bar{x'}) e^{\frac{\lambda}{2\nu}h(\bar{x'},t)} \Big]. \tag{22}$$

At this stage it is required to make some assumptions about the kernel.

We assume that the nonlocal kernel has translational invariance, that is  $\mathbf{G}(\bar{x}, \bar{x'}) = \mathbf{G}(\bar{x} - \bar{x'})$ . Also, that it is of (very) "short" range, that allows us to expand it as

$$\mathbf{G}(\bar{x} - \bar{x'}) = \sum_{n=0}^{\infty} A_{2n} \delta^{(2n)}(\bar{x} - \bar{x'}), \qquad (23)$$

with  $\delta^{(n)}(\bar{x} - \bar{x'}) = \nabla_{x'}^n \delta(\bar{x} - \bar{x'})$ , and symmetry properties taken into account. Exploiting this form of the kernel, we arrive to the following contributions in Eq. (21)

$$e^{-\frac{\lambda}{2\nu}h(\bar{x},t)}\beta \int_{\Omega} d\bar{x'} \mathbf{G}(\bar{x}-\bar{x'})e^{\frac{\lambda}{2\nu}h(\bar{x'},t)} \approx \beta \left\{ A_0 + A_2 \left[ \left(\frac{\lambda}{2\nu}\right)^2 (\nabla h)^2 + \frac{\lambda}{2\nu} \nabla^2 h \right] + A_4 \left[ \left(\frac{\lambda}{2\nu}\right)^4 (\nabla h)^4 + 6 \left(\frac{\lambda}{2\nu}\right)^3 (\nabla h)^2 \nabla^2 h + 2 \left(\frac{\lambda}{2\nu}\right)^2 \nabla^2 (\nabla h)^2 - \left(\frac{\lambda}{2\nu}\right)^2 (\nabla^2 h)^2 + \frac{\lambda}{2\nu} \nabla^4 h \right] \right\} + A_6 \dots, \quad (24)$$

where the last term indicates contributions of order  $n \ge 3$  (2n = 6). The parameter  $\beta$  could (in principle) be positive or negative, indicating an inhibitor or an activator role for the nonlocal interaction term, respectively.

These contributions have the same form of those ones that arose in several previous works, where scaling properties, symmetry arguments, etc, have been used to discuss the possible contributions to a general form of the kinetic equation [Hentschel, 1994; Linz *et al.*, 2000; Lopez *et al.*, 2005]. Clearly, the different contributions that arose in Eq. (24) are tightly related to several of other previously studied equations, like the Kuramoto-Sivashinsky [Sivashinsky, 1977; Kuramoto, 1978], the Sun-Guo-Grant equation [Sun *et al.*, 1989], and others [Hentschel, 1994].

#### 5. Density Dependent Surface Tension

We can also go further in another direction. Let us consider the case of *density dependent diffusion (surface tension)*. Following the same original approach, we start considering the following reaction-diffusion equation with a density dependent diffusion for a field  $\phi(x,t)$ , as studied in [von Haeften *et al.*, 2000; Tessone & Wio, 2007]

$$\partial_t \phi(x,t) = \nabla \left( \nu(\phi) \nabla \phi \right) + f(\phi) + \phi(x,t) \eta(x,t).$$
<sup>(25)</sup>

As before,  $f(\phi)$  is a general nonlinear function, that we collapse to  $f(\phi) = a\phi(x,t)$ . As in Eq. (2),  $\eta(x,t)$  is a delta correlated white noise of intensity  $\sigma$ . We assume  $\nu(\phi) = \nu_o g(\phi)$ , that –using the Hopf-Cole transformation– yields  $\tilde{\nu}(h) = \nu_o \tilde{g}(h)$ .

The transformed equation reads

$$\partial_t h(x,t) = \nabla \left( \tilde{\nu}(h) \nabla h \right) + \frac{\tilde{\lambda}(h)}{2} \left( \nabla h \right)^2 + K_o + \xi(x,t), \tag{26}$$

with  $\tilde{\lambda}(h) = \lambda_o \tilde{g}(h)$  and  $a = \frac{\lambda_o}{2\nu_o} K_o$ . This equation has a KPZ-like form, with both a density dependent surface tension term and a density dependent nonlinear coupling, with both nonlinearities having the same functional dependence.

In order to see the associated NETLP, let us remember its form for the original reaction-diffusion equation (Eq. (25)), it reads

$$\mathcal{F}[\phi] = \int_{\Omega} \left\{ -\int_{0}^{\phi} \nu(\phi') f(\phi') \, d\phi' + \frac{1}{2} \left(\nu(\phi) \nabla \phi\right)^{2} \right\} \, dx, \tag{27}$$

that fulfills [von Haeften *et al.*, 2000]

$$\frac{\partial}{\partial t}\phi(x,t) = -\frac{1}{\nu(\phi)}\frac{\delta\mathcal{F}[\phi]}{\delta\phi} + \phi\,\eta(x,t).$$

Applying once more the Hopf-Cole transformation, we obtain

$$\mathcal{F}[h] = \int_{\Omega} dx \, \left(\frac{\lambda_o}{2\nu_o}\right)^2 \left\{ -\int^h du \, e^{\frac{\lambda_o u}{2\nu_o}} \tilde{\nu}(u) \, K_o + \frac{1}{2} \, \left(\tilde{\nu}(h) e^{\frac{\lambda_o h}{2\nu_o}} \nabla h\right)^2 \right\}.$$
(28)

From this NETLP we can prove that

$$\frac{\partial}{\partial t}h(\bar{x},t) = -\tilde{\Gamma}[h]\frac{\delta\mathcal{F}[h]}{\delta h(x,t)} + \xi(x,t);$$
(29)

where  $\xi(x,t)$  is a white noise of intensity  $\varepsilon$  (with  $\sigma = \left(\frac{\lambda_o}{2\nu_o}\right)^2 \varepsilon$ ), and the function  $\tilde{\Gamma}[h]$  is given by

$$\tilde{\Gamma}[h] = \left(\frac{2\nu_o}{\lambda_o}\right)^2 \frac{e^{-\frac{\lambda_o}{\nu_o}h(\bar{x},t)}}{\tilde{\nu}(h)}.$$

As before, if we have  $f(\phi(\bar{x},t)) = a\phi(\bar{x},t) - b\phi(\bar{x},t)^3$ , we get a generalized form of the "bounded KPZ" equation.

#### 6. Conclusions

The previous results indicates that, for a very general form of a KPZ-like equation, we can device a NETLP  $\dot{a}$  la carte. Consider the following equation

$$\partial_t h(\bar{x}, t) = \nu \nabla^2 h + \frac{\lambda}{2} \left( \nabla h \right)^2 + F(h) + \xi(\bar{x}, t)$$
(30)

where F(h) is a general nonlinear function of h. According to the previous results we could readily write the associate NETLP that has the form

$$\mathcal{F}[h] = \int \left\{ -\frac{\lambda}{2\nu} \int_{h_{ref}}^{h} e^{\frac{\lambda u}{\nu}} F(u) du + \frac{\lambda^2}{8\nu} \left( e^{\frac{\lambda h}{2\nu}} \nabla h \right)^2 \right\} dx.$$
(31)

Clearly, it fulfills the Lyapunov condition  $\frac{\partial}{\partial t} \mathcal{F}[h(\bar{x}, t)] \leq 0$ , and also, through functional derivation, allows us to obtain the kinetic equation as

$$\frac{\partial}{\partial t}h(\bar{x},t) = -\Gamma[h]\frac{\delta\mathcal{F}[h]}{\delta h(x,t)} + \xi(x,t),$$
(32)

with  $\Gamma[h] = \left(\frac{2\nu}{\lambda}\right)^2 e^{-\frac{\lambda}{\nu}h(\bar{x},t)}$ . Hence, it is clear that we can write the NETLP for a very general KPZ-like form, independently of the actual form of F(h).

Summarizing, we have here found the form of the Lyapunov functional or NETLP for the KPZ equation. More, we have devised a way to extent the procedure to derive it, and in such a way we were able to derive more general forms, including several kinetic equations studied in the literature of interface growing phenomena. Even the case of density dependent surface tension, have been discussed. From this NETLP, and through a functional derivative, we have obtained either the KPZ as well as other generalized kinetic equations. We have also shown that the NETLP for KPZ fulfills global shift properties, as well as other ones anticipated for such an unknown functional. More, we have found the exact expression, valid for any dimension, for the probability distribution function, and have commented on the result of a simple example that indicate the usefulness of such a functional.

As indicated in the literature, dynamic renormalization group techniques, being useful and powerful, in many cases only offers incomplete results, having no access to the strong coupling phase [Barabási & Stanley, 1995; Wiese, 1998]. Hence, it is clear the need of alternative ways to analyze the KPZ and related problems, as for instance the selfconsistent expansion [Katzav & Schwartz, 1999; Katzav, 2003]. The present results open new possibilities of making non-perturbational studies for the KPZ problem. For instance, through the analysis of long time mean values of  $h(\bar{x}, t)$ . In a similar way, it would be possible to obtain correlations, and from them to extract information about scaling exponents. Such study will be the subject of forthcoming work.

## Acknowledgments

The author thanks R. Cuerno, H. Fogedby, J.M. López, M.A. Muñoz, L. Pesquera, J. Revelli, M.A. Rodriguez, R. Toral for fruitful discussions and/or valuable comments. He also acknowledges financial support from MEC, Spain, through Grant No. CGL2007-64387/CLI; and the award of a *Marie Curie Chair*, from the European Commission, during part of the development of this work.

## References

Ahlers V. & Pikovsky A. [2002] "Critical Properties of the Synchronization Transition in Space-Time Chaos", *Phys. Rev. Lett.* 88, 254101.

Ao P. [2004] "Potential in stochastic differential equations: novel construction", J. Phys. A 37, L25-L30.

Barabási A.-L. & Stanley H.E. [1995] Fractal concepts in surface growth (Cambridge U,P., Cambridge).

Bouzat S. & Wio H.S. [1998] "Nonequilibrium potential and pattern formation in a three-component reaction-diffusion system", *Phys. Lett. A* 247, 297-302.

Bouzat S. & Wio H.S. [1999] "Stochastic resonance in extended bistable systems: The role of potential symmetry", *Phys. Rev. E* 59, 5142-5149.

Castro M., et al. [2007] "Generic equations for pattern formation in evolving interfaces", New J. Phys. 9, 102.

Colaiori F. & Moore M.A. [2002] "Numerical solution of the mode-coupling equations for the Kardar-Parisi-Zhang equation in one dimension", *Phys. Rev. E* **65**, 017105.

Cross M.C. & Hohenberg P.C. [1993] "Pattern formation outside of equilibrium", Rev. Mod. Phys. 65, 851-1112.

de los Santos F., Romera E., Al Hammal O. & Muñoz M.A. [2007] "Critical wetting of a class of nonequilibrium interfaces: A mean-field picture", *Phys. Rev. E* **75**, 031105.

Descalzi O. & Graham R. [1994] "Nonequilibrium potential for the Ginzburg-Landau equation in the phase-turbulent regime", Z. Phys. B 93, 509-513.

Edwards S.F. & Schwartz M. [2002] "Lagrangian statistical mechanics applied to non-linear stochastic field equations ", *Physica A* **303**, 357-386.

Fogedby H.C. [2006] "Kardar-Parisi-Zhang equation in the weak noise limit: Pattern formation and upper critical dimension", *Phys. Rev. E* **73**, 031104.

Gammaitoni L., Hänggi P., Jung P. & Marchesoni F. [1998] "Stochastic resonance", Rev. Mod. Phys. 70, 223-287.

Graham R. [1987], in *Instabilities and Nonequilibrium Structures*, Eds.E.Tirapegui and D.Villaroel (D.Reidel, Dordrecht)

Graham R. & Tel T. [1990] "Steady-state ensemble for the complex Ginzburg-Landau equation with weak noise", *Phys. Rev. A* 42, 4661-4667.

Grinstein G., Muñoz M.A. & Tu Y. [1996] "Phase Structure of Systems with Multiplicative Noise", *Phys. Rev. Lett.* **76**, 4376 - 4379.

Hänggi P. [1985], "The functional derivative and its use in the description of noisy dynamical systems" in *Stochastic Processes Applied to Physics*, Pesquera L. & Rodriguez M.A., Eds, (World Scientific, Singapore) pgs. 69-94.

Halpin-Healy T. & Zhang Y.-Ch. [1995], Phys. Rep. 254, 215-414.

Hayase, Y. [1997] "Collision and Self-Replication of Pulses in a Reaction Diffusion System", J. Phys. Soc. Jpn. 66, 2584-2587.

Hentschel H.G.E. [1994] "Shift invariance and surface growth", J. Phys. A 27, 2269-2276.

Hohenberg P. & Halperin B. [1977] "Theory of dynamic critical phenomena", Rev. Mod. Phys. 49, 435-479.

Izús G., Deza R. & Wio H.S. [1998] "Exact nonequilibrium potential for the FitzHugh-Nagumo model in the excitable and bistable regimes", *Phys. Rev. E* 58, 93-98.

Kardar M., Parisi G. & Zhang Y.C. [1986] "Dynamic Scaling of Growing Interfaces", *Phys. Rev. Lett.* **56**, 889-892. Katzav E. [2003] "Self-consistent expansion results for the nonlocal Kardar-Parisi-Zhang equation", *Phys. Rev. E* **68**, 046113.

Katzav E. & Schwartz M. [1999] "Self-consistent expansion for the Kardar-Parisi-Zhang equation with correlated noise", *Phys. Rev. E* **60**, 5677-5680.

Katzav E. & Schwartz M. [2004] "Numerical evidence for stretched exponential relaxations in the Kardar-Parisi-Zhang equation", *Phys. Rev. E* 69, 052603.

Krug J. & Halpin-Healy T. [1993] "Directed polymers in the presence of columnar disorder", J. Phys. I 3, 2179-2198. Kuramoto Y. [1978] "Diffusion-Induced Chaos in Reaction Systems", Suppl. Prog. Theor. Phys. 64, 346-367.

Lam C.-H. & Shin F.G. [1998], "Improved discretization of the Kardar-Parisi-Zhang equation", *Phys. Rev. E* 58, 5592.

Linz S.J., Raible M. & Hänggi P. [2000], in *Stochastic Processes in Physics*, Freund J.A. & Pöschel T., Eds., *Lecture Notes in Physics*, **557**, 473-483 (Springer-Verlag, Berlin).

López J.M., Castro M. & Gallego R. [2005] "Scaling of Local Slopes, Conservation Laws, and Anomalous Roughening in Surface Growth", *Phys. Rev. Lett.* **94**, 166103.

Ma K., Jiang J. & Yang C.B. [2007] "Scaling behavior of roughness in the two-dimensional KardarParisiZhang growth", *Physica A* **378**, 194-200.

Marsili M., Maritain A., Toigo F. & Banavar J.R. [1996] "Stochastic growth equations and reparametrization invariance", *Rev. Mod. Phys.* 68, 963-983.

Medina E., Hwa T., Kardar M. & Zhang Y.C. [1989] "Burgers equation with correlated noise: Renormalization-group analysis and applications to directed polymers and interface growth", *Phys. Rev. A* **39**, 3053-3075.

Mikhailov A.S. [1990], Foundations of Synergetics I (Springer-Verlag, Berlin).

Montagne R., Hernández-Garcia E. & San Miguel M. [1996] "Numerical study of a Lyapunov functional for the complex Ginzburg-Landau equation", *Physica D* 96, 47-65.

Mukherji S. & Bhattacharjee S.M. [1997] "Nonlocality in Kinetic Roughening", Phys. Rev. Lett. 79, 2502-2505.

Nattermann T. & Renz W. [1989] "Diffusion in a random catalytic environment, polymers in random media, and stochastically growing interfaces", *Phys. Rev. A* 40, 4675-4681.

Newman T,J, & Bray A.J. [1996], "Strong coupling behaviour in discrete KardarParisiZhang equations", J. Phys. A 29, 7917.

Prähofer M. & Spohn H. [2004] "Exact Scaling Functions for One-Dimensional Stationary KPZ Growth", J. Stat. Phys. 115, 255-279.

Ramasco J.J., López J.M. & Rodriguez M.A. [2006] "Interface depinning in the absence of an external driving force", *Phys. Rev. E* 64, 066109.

Revelli J.A. & Wio H.S. [2008], "Some discretization problems in the one dimensional KPZ equation", to be submitted. Schwartz M. & Edwards S.F. [2002] "Stretched exponential in non-linear stochastic field theories", *Physica A* **312**, 363-368.

Sivashinsky G. [1977] "Nonlinear analysis of hydrodynamic instability in laminar flamesI. Derivation of basic equations", Acta Astronautica 4, 1177-1206.

Sun T., Guo H. & Grant M. [1989] "Dynamics of driven interfaces with a conservation law", *Phys. Rev. A* 40, 6763-6766.

Szendro I.G., López J.M. & Rodriguez M.A. [2007] "Localization in disordered media, anomalous roughening, and coarsening dynamics of faceted surfaces", *Phys. Rev. E* **76**, 011603.

Tessone C. & Wio H.S. [2007] "Stochastic Resonance in an Extended FitzHugh-Nagumo System: The role of Selective Coupling", *Physica A* **374**, 46-54.

Tong W.M. & Williams R.S. [1994], "Kinetics of surface growth", Annu. Rev. Phys. Chem. 45, 401-438.

Tu Y., Grinstein G. & Muñoz M.A. [1997] "Systems with Multiplicative Noise: Critical Behavior from KPZ Equation and Numerics", *Phys. Rev. Lett.* **78**, 274-277.

von Haeften B., Deza R. & Wio H.S. [2000] "Enhancement of stochastic resonance in distribute systems due to selective coupling", *Phys. Rev. Lett.* **84**, 404-407.

von Haeften B., Izús G., Mangioni S., Sánchez A.D. & Wio H.S. [2004] "Stochastic resonance between dissipative structures in a bistable noise-sustained dynamics", *Phys. Rev. E* 69, 021107.

von Haeften B., Izús G. & Wio H.S. [2005] "System Size Stochastic Resonance between Dissipative Structures in a Bistable Noise-induced Dynamics", *Phys. Rev. E* 72, 021101.

von Haeften B. & Wio H.S. [2007] "Optimal Range of a Non-Local Kernel for Stochastic Resonance in Extended Systems", *Physica A* **376**, 199-207.

Wiese K.J. [1998] "On the Perturbation Expansion of the KPZ Equation", J. Stat. Phys. 93, 143-154.

Wio H.S. [1994], An Introduction to Stochastic Processes and Nonequilibrium Statistical Physics (World Scientific, Singapore).

Wio H.S. [1997], "Nonequilibrium Potential in Reaction-Diffusion Systems", in 4th Granada Lectures in Computational Physics, Eds Garrido P.L. & Marro J., (Springer-Verlag), pg.135-193.

Wio H.S. [2007], "Nonequilibrium Free Energy-Like Functional for the KPZ Equation", arXiv:0709.4439, submitted to *Europhys. Lett.*.

Wio H.S., Bouzat S. & von Haeften B. [2002] "Stochastic Resonance in Spatially Extended Systems: the Role of Far From Equilibrium Potentials", *Physica A* **306C** 140-156.

Wio H.S. & Deza R. [2007] "Aspects of stochastic resonance in reaction–diffusion systems: The nonequilibrium-potential approach", *Europ. Phys. J - Special Topics* **146**, 111-126.