On the non-equivalence of Lorenz System and Chen System*

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Abstract: In this paper, we prove that the Chen system with a set of chaotic parameters is not smoothly equivalent to the Lorenz system with any parameters.

Key words: Lorenz system, Chen system, smooth equivalence, topological equivalence.

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1 Introduction

Nonlinear science had experienced an unprecedented and vigorous development particularly during the second half of the 20th century, and it was considered "the third revolution" in natural science in the history. The main subjects in the study of nonlinear science include chaos, bifurcation, fractals, solitons and complexity. Because of the important significance to unveil the essence of chaos and wide potential application prospects of chaos theory in many fields, research on chaos always carries a heavy weight in nonlinear science. H. Poincaré [1] and C. Maxwell [2] both had some vague concepts of chaos in their times. In the earlier 1960s, E. N. Lorenz [3] discovered the now-famous Lorenz system, which actually produces visible chaos. Lorenz system is the first mathematical and physical model of chaos, thereby becoming the starting point and foundation stone for later research on chaos theory. Since

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the 1960s, particularly with this model, mathematicians, physicists and engineers from various fields have thoroughly studied the essence of chaos, characteristics of chaotic systems, bifurcations, routes to chaos, and many other related topics [4]. There are also some chaotic systems of great significance that are closely related to the Lorenz system, where a particular example in point is the Chen system. Since the Chen system was first found in 1999 [5,6], hundreds of papers have been published on this new chaotic system with deep and comprehensive results obtained. A monograph on the Lorenz systems family including the Chen system has also been published [7]. To further understand the interesting Chen system, one fundamental question has to be answered: are the Chen system and the Lorenz system non-equivalent, either topologically or smoothly? The purpose of this paper is to prove that the Lorenz system and the Chen system are indeed non-equivalent smoothly.

2 Results and Proofs

The dynamical system ϕ_t^{abc} defined by

$$\begin{cases} \dfrac{dx}{dt} = a(y-x), \\ \dfrac{dy}{dt} = cx - xz - y, \\ \dfrac{dz}{dt} = xy - bz, \end{cases} \tag{2.1}$$

is called the **Lorenz system** with parameters a, b, c.

The dynamical system ψ_t^{abc} defined by

$$\begin{cases} \dfrac{dx}{dt} = a(y-x), \\ \dfrac{dy}{dt} = (c-a)x - xz + cy, \\ \dfrac{dz}{dt} = xy - bz, \end{cases} \tag{2.2}$$

is called the **Chen system** with parameters a, b, c.

It is clear that system (2.1) has 3 equilibrium points if $b(c-1) > 0$, i.e.,

$$\begin{aligned} P_1 &= (0,0,0), \\ P_2 &= (-\sqrt{b(c-1)}, -\sqrt{b(c-1)}, c-1), \\ P_3 &= (\sqrt{b(c-1)}, \sqrt{b(c-1)}, c-1), \end{aligned}$$

and system (2.2) has 3 equilibrium points if $b(2c-a) > 0$, i.e.,

$$\begin{aligned} Q_1 &= (0,0,0), \\ Q_2 &= (-\sqrt{b(2c-a)}, -\sqrt{b(2c-a)}, 2c-a), \\ Q_3 &= (\sqrt{b(2c-a)}, \sqrt{b(2c-a)}, 2c-a). \end{aligned}$$

Denote the coordinates of P_i by (x_i, y_i, z_i), $i = 1, 2, 3$, and the coordinates of Q_i by (x_i', y_i', z_i'), $i = 1, 2, 3$, and denote the vector fields on the right sides of (2.1) and (2.2) by $\vec{U}(x, y, z)$ and $\vec{V}(x, y, z)$, respectively. It is clear that their Jacobians are:

$$
\begin{aligned}
D\vec{U}(x,y,z) &= \begin{pmatrix} -a & a & 0 \\ c-z & -1 & -x \\ y & x & -b \end{pmatrix}, \\
D\vec{V}(x,y,z) &= \begin{pmatrix} -a & a & 0 \\ c-a-z & c & -x \\ y & x & -b \end{pmatrix},
\end{aligned}
$$

and their determinants are:

$$
\begin{aligned}
\det D\vec{U}(P_1) &= ab(c-1), \\
\det D\vec{U}(P_2) &= \det D\vec{U}(P_3) = -2ab(c-1),
\end{aligned}
$$

$$
\begin{aligned}
\det D\vec{V}(Q_1) &= ab(2c-a), \\
\det D\vec{V}(Q_2) &= \det D\vec{V}(Q_3) = -2ab(2c-a).
\end{aligned}
$$

In general, let $f(x)$ and $g(x)$ be vector fields on $\mathbb{R}^n$, and

$$\dot{x} = f(x), \;\; x \in \mathbb{R}^n \tag{2.3}$$

$$\dot{y} = g(y), \;\; y \in \mathbb{R}^n \tag{2.4}$$

be two systems of differential equations on $\mathbb{R}^n$.

Definition 2.1 *If there exists a diffeomorphism h on $\mathbb{R}^n$ such that*

$$f(x) = M^{-1}(x) g(h(x)), \tag{2.5}$$

where $M(x)$ is the Jacobian of h at the point x, then (2.3) and (2.4) are said to be **smoothly equivalent**.

Remark 2.2 *If (2.3) and (2.4) are smoothly equivalent, and suppose that x_0 and $y_0 = h(x_0)$ are the corresponding equilibria of $f(x)$ and $g(x)$, $A(x_0)$ and $B(y_0)$ are the Jacobians of $f(x)$ and $g(x)$, respectively, then $A(x_0)$ and $B(y_0)$ are similar, i.e., their characteristic polynomials and eigenvalues are the same.*

Theorem 2.3 *The Chen system and the Lorenz system are not smoothly equivalent, i.e., there exists a Chen system $\psi_t^{a'b'c'}$ which is not smoothly equivalent to any Lorenz system ϕ_t^{abc}.*

Proof Since $M(x) \neq 0, M^{-1}(x) \neq 0$ and due to (2.5), we have

$$f(x) = 0 \Leftrightarrow g(h(x)) = 0,$$

that is, the equilibria x of $f(\cdot)$ correspond to the equilibria $h(x)$ of $g(\cdot)$, therefore a Chen system is smoothly equivalent to a Lorenz system with the same number of equilibria. It suffices to prove that a Chen system with 3 equilibria can not be smoothly equivalent to a Lorenz system with any 3 equilibria.

Suppose that h is a diffeomorphism on $\mathbb{R}^3$ such that $\psi_t^{a'b'c'}$ and ϕ_t^{abc} smoothly equivalent under h. Because $detD\vec{U}(P_1)$ and $detD\vec{V}(Q_1)$ are positive, P_1 corresponds to Q_1 under h. Because

$$D\vec{U}(P_1) = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix},$$

$$D\vec{V}(Q_1) = \begin{pmatrix} -a' & a' & 0 \\ c' - a' & c' & 0 \\ 0 & 0 & -b' \end{pmatrix},$$

the characteristic equation of $D\vec{U}(P_1)$ is:

$$\lambda^3 + (a+b+1)\lambda^2 + (a + ab - ac + b)\lambda - ab(c-1) = 0,$$

and the characteristic equation of $D\vec{V}(Q_1)$ is:

$$\lambda^3 + (a' + b' - c')\lambda^2 + (a'^2 + a'b' - 2a'c' - b'c')\lambda - a'b'(2c' - a') = 0.$$

Let

$$u = a' + b' - c', \tag{2.6}$$
$$v = a'^2 + a'b' - 2a'c' - b'c', \tag{2.7}$$
$$w = -a'b'(2c' - a'). \tag{2.8}$$

By Remark 2.1, we must have

$$a + b + 1 = u, \tag{2.9}$$
$$a + ab - ac + b = v, \tag{2.10}$$
$$-ab(c-1) = w. \tag{2.11}$$

By (2.9), we have $a = u - 1 - b$, so that in combining with (2.11),

$$c = 1 - \frac{w}{ab} = 1 - \frac{w}{b(u-1-b)}.$$

Substituting a, c in (2.10), we have

$$b^3 - ub^2 + va - w = 0. \tag{2.12}$$

It is clear that $D\vec{U}(P_2)$ and $D\vec{U}(P_3)$ have the same characteristic equations, and $D\vec{V}(Q_2)$ and $D\vec{V}(Q_3)$ have the same characteristic equations. Hence, we may assume that P_2 corresponds to Q_2. By comparing the coefficients of their first-order terms, we have

$$ab + bc = b'c'.$$

Substituting them into the formulas of a, c, we get

$$\begin{aligned} & b(u-1-b) + b\left(1 - \frac{w}{b(u-1-b)}\right) \\ = \;& (u-1)b - b^2 + b - \frac{w}{u-1-b} \\ = \;& b'c', \end{aligned}$$

and

$$b^3 - (2u-1)b^2 + [(u-1)^2 + u - 1 + b'c']b - w - (u-1)b'c' = 0.$$

Subtracting this from (2.12), we obtain

$$(u-1)b^2 + (u + v - u^2 - b'c')b + (u-1)b'c' = 0. \tag{2.13}$$

By resultant elimination [8], a necessary and sufficient condition for (2.12) and (2.13) to have same roots is

$$\begin{aligned} & M_0(a', b'c') \\ & = \begin{vmatrix} 1 & -u & v & -w & 0 \\ 0 & 1 & -u & v & -w \\ u-1 & u+v-u^2-b'c' & (u-1)b'c' & 0 & 0 \\ 0 & u-1 & u+v-u^2-b'c' & (u-1)b'c' & 0 \\ 0 & 0 & u-1 & u+v-u^2-b'c' & (u-1)b'c' \end{vmatrix} \\ & = 0. \end{aligned} \tag{2.14}$$

Substituting u, v, w in the above equation by (2.6), (2.7) and (2.8), we get an algebraic equation of a', b', c', as

$$\begin{aligned} & b'(a'-2c')^2(1+c')(a'^3 - a^4 + a'^5 + a'^2b' - a'^3b' + a'b'^2 - 2a'^2b'^2 - 2a'b'^3 + a'^2b'^3 + a'b'^4 \\ & -2a'^2c' + 3a'^3c' - 4a'^4c' - 3a'b'c' + 4a'^2b'c' - 5a'^3b'c' - b'^2c' + 5a'b'^2c' - 2a'^2b'^2c' + 2b'^3c' \\ & -2a'b'^3c' - b'^4c' - 2a'^2c'^2 + 4a'^3c'^2 - 3a'b'c'^2 + 6a'^2b'c'^2 - b'^2c'^2 + 4a'b'^2c'^2 + b'^3c'^2) \\ & = 0, \end{aligned} \tag{2.15}$$

and its solution is given by the union of the following four surfaces:

$$
\begin{aligned}
b' &= 0, \\
a' - 2c' &= 0, \\
1 + c' &= 0,
\end{aligned}
$$

$$
\begin{aligned}
&a'^3 - a^4 + a'^5 + a'^2b' - a'^3b' + a'b'^2 - 2a'^2b'^2 - 2a'b'^3 + a'^2b'^3 + a'b'^4 - 2a'^2c' + 3a'^3c' \\
&-4a'^4c' - 3a'b'c' + 4a'^2b'c' - 5a'^3b'c' - b'^2c' + 5a'b'^2c' - 2a'^2b'^2c' + 2b'^3c' - 2a'b'^3c' \\
&-b'^4c' - 2a'^2c'^2 + 4a'^3c'^2 - 3a'b'c'^2 + 6a'^2b'c'^2 - b'^2c'^2 + 4a'b'^2c'^2 + b'^3c'^2 = 0.
\end{aligned}
$$

Denote the point set of all solutions of (2.14) by C. It is clear that C is a Borel subset of $\mathbb{R}^3$ and its Lebegue measure is 0. So, there are many points not belonging to C, for example, the values of $a' = 45, b' = 5, c' = 28$ give $b'(2c' - a') = 55 > 0, M_0(45, 5, 28) = 2.919 \times 10^{11} \neq 0$, i.e., $(45, 5, 28) \notin C$. This means that the Chen system $\psi_t^{45,5,28}$ is not smoothly equivalent to the Lorenz system ϕ_t^{abc} with any values of a, b, c, while $\psi_t^{45,5,28}$ is chaotic according to [5, 6 or 7 (p.39)].

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