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LIMIT CYCLES OF DISCONTINUOUS PIECEWISE LINEAR DIFFERENTIAL SYSTEMS

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Abstract. We study the bifurcation of limit cycles from the periodic orbits of a 2−dimensional (respectively 4−dimensional) linear center in $\mathbb{R}^n$ perturbed inside a class of discontinuous piecewise linear differential systems. Our main result shows that at most 1 (respectively 3) limit cycle can bifurcate up to first-order expansion of the displacement function with respect to the small parameter. This upper bound is reached. For proving these results we use the averaging theory in a form where the differentiability of the system is not needed.

1. Introduction

Discontinuous piecewise linear differential systems appear in a natural way in control theory and in the study of electrical circuits. These systems can present complicated dynamical phenomena such as those exhibited by general nonlinear differential systems. One of the main ingredients in the qualitative description of the dynamical behavior of a differential system is the number and the distribution of its limit cycles.

The goal of this paper is to study, in $\mathbb{R}^n$ for all $n \geq 2$, the existence of limit cycles of the control system of the form

$$\dot{x} = A_0 x + \varepsilon F(x), \tag{1}$$

with $|\varepsilon| \neq 0$ a sufficiently small real parameter, where A_0 is equal to

$$A_0^1 = \begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \text{ or } A_0^2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

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and $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is given by $F(x) = Ax + \varphi_0(k^T x)b$, with $A \in \mathcal{M}_n(\mathbb{R})$, $k, b \in \mathbb{R}^n \backslash \{0\}$ and $\varphi_0 : \mathbb{R} \longrightarrow \mathbb{R}$ the discontinuous function

$$\varphi_0(x_1) = \begin{cases} -1, & \text{if } x_1 \in (-\infty, 0), \\ \ \ 1, & \text{if } x_1 \in (0, \infty), \end{cases} \tag{2}$$

where $x = (x_1, \ldots, x_n)^T$. The independent variable is denoted by t, vectors of $\mathbb{R}^n$ are column vectors, and k^T denotes a transposed vector.

For this purpose first we will study the problem

$$\dot{x} = A_0 x + \varepsilon F_\alpha(x), \tag{3}$$

where F_α is equal to F replacing φ_0 by the piecewise linear function $\varphi_\alpha : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\varphi_\alpha(x_1) = \begin{cases} -1 & \text{if } x_1 \in (-\infty, -\alpha), \\ \dfrac{x_1}{\alpha} & \text{if } x_1 \in [-\alpha, \alpha], \\ \ \ 1 & \text{if } x_1 \in (\alpha, \infty), \end{cases} \tag{4}$$

where $\alpha > 0$, and after we will tend α to 0. The graphics of φ_0 and φ_α are illustrated in Figures 1 and 2, respectively.

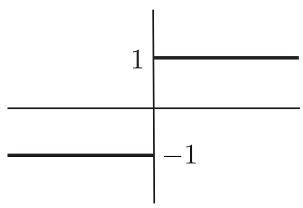

FIGURE 1. Graphic of φ_0.

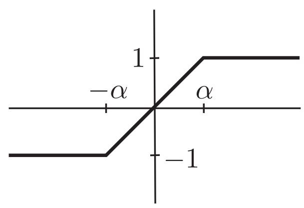

FIGURE 2. Graphic of φ_α.

For $\varepsilon = 0$ and $A_0 = A_0^1$, system (3) becomes

$$\dot{x}_1 = -x_2,\ \dot{x}_2 = x_1,\ \dot{x}_i = 0 \text{ for } i = 3, \ldots, n \tag{5}$$

and for $\varepsilon = 0$ and $A_0 = A_0^2$, system (3) becomes

$$\dot{x}_1 = -x_2,\ \dot{x}_2 = x_1,\ \dot{x}_3 = -x_4,\ \dot{x}_4 = x_3,\ \dot{x}_i = 0 \text{ for } i = 5, \ldots, n. \tag{6}$$

We note that the origin of every plane $x_k =$ constant for $k = 3, \ldots, n$ in the first case, and of every hyperplane $k = 5, \ldots, n$ in the second case, is a *global isochronous center* for (5) and (6), respectively, i.e. all the orbits contained in such a plane or hyperplane different from the origin are periodic with the same period 2π.

A *limit cycle* of a differential system is an isolated periodic orbit in the set of all periodic orbits of the system. The *Poincaré map* (or equivalently, the *displacement map*) is a suitable tool for studying limit cycles of autonomous systems (detailed explanations can be found in [5] or [6]; see also Section 2). We recall that a limit cycle of a system corresponds to an isolated zero of the displacement function.

Our main results are the following.

Theorem 1. *Consider $A_0 = A_0^1$. For $n \geq 2$ at most one limit cycle of the piecewise linear differential system* (3) *bifurcates from the periodic orbits of system* (5)*, up to first-order expansion of the displacement function of* (3) *with respect to the small parameter ε. Moreover there are systems* (3) *having exactly one limit cycle bifurcating from a circle centered at the origin of the plane $x_3 = \ldots = x_n = 0$.*

Corollary 1.1. *Consider $A_0 = A_0^1$. For $n \geq 2$ at most one limit cycle of the discontinuous piecewise linear differential system* (1) *bifurcates from the periodic orbits of system* (5)*, up to first-order expansion of the displacement function of* (1) *with respect to the small parameter ε. Moreover there are systems* (1) *having exactly one limit cycle.*

Theorem 2. *Consider $A_0 = A_0^2$. For $n \geq 4$ at most three limit cycles of the piecewise linear differential system* (3) *bifurcate from the periodic orbits of system* (6)*, up to first-order expansion of the displacement function of* (3) *with respect to the small parameter ε. Moreover there are systems* (3) *having exactly three limit cycles bifurcating from circles centered at the origin of the plane $x_5 = \ldots = x_n = 0$.*

Corollary 2.1. *Consider $A_0 = A_0^2$. For $n \geq 4$ at most three limit cycles of system* (1) *bifurcates from the periodic orbits of the discontinuous piecewise linear differential system* (6)*, up to first-order expansion of the displacement function of* (1) *with respect to the small parameter ε. Moreover there are systems* (1) *having exactly three limit cycles.*

Theorems 1 and 2 are an extensions of the main results of [7] and [2] respectively, where is done the case $\alpha = 1$ for $\mathbb{R}^4$.

We emphasize that the bifurcation from $\varepsilon = 0$ to $\varepsilon \neq 0$ in Theorems 1 and 2 takes place for $\varepsilon > 0$ and for $\varepsilon < 0$ sufficiently small, i.e. on both sides of the value $\varepsilon = 0$. We remark that in a Hopf bifurcation the limit cycle only appears on one side of the bifurcation value of the parameter, but in our case in which the limit cycles bifurcate from periodic orbits of the period annulus of a center they appear on both sides of the parameter.

The proofs of Theorems 1 and 2 are based on the first-order averaging method. We will present this method in Section 2, in the form obtained in [1]. The advantage of this result is that the smoothness assumptions for the vector field of the differential system are minimal. In particular, it can be applied to piecewise linear differential systems, which are not C^2 (not even C^1), as required in its classical version, see for instance, Theorem 11.5 of [8]. This non-differential application of the averaging method to control systems was used for the first time in [2]. This method has been used frequently for computing periodic orbits; see for instance [4]. From the paper [3] we can study the stability of the limit cycles of Theorems 1 and 2; for more details see remarks 4 and 5.

The proofs of Theorems 1 and 2 will be the subject of Sections 3 and 4 respectively. The first step in the study of system (3), Lemmas 1 and 4, is to reduce the number of parameters by a linear change of variables. The next objective is to transform the system into one which is in the standard form for applying the averaging theory. This is accomplished in Lemmas 2 and 5 through a change of variables related on the first integrals of systems (5) and (6). The computation of the averaged function (see equation (8)) will be also a special task. After that we must determine the number of its isolated zeros. The relation between the averaging method and the displacement function will be also discussed.

Reference [9] can be seen for a theoretical discussion about suitable transformations of high dimensional differential systems which are small perturbations of a center, into the standard form for averaging. The general idea is to relate this change of variables on the first integral of the center.

We would like to add some comments related to our approach to the problem of counting the limit cycles of piecewise linear differential systems. We have chosen here to study bifurcation with respect to a small parameter from the periodic orbits of a center, up to first-order expansion of the displacement map. For some values of the coefficients, this is sufficient for finding the exact number of limit cycles. But in some cases the first-order expansion of the displacement map can be identically zero, then a higher order averaging theory is needed. The study can be done by using second-, third-, ... order averaging theory. A key point in these studies is the relation between the averaging theory and the displacement map due to the fact that the displacement map of a piecewise linear differential system is analytic in a neighborhood of a limit cycle.

2. First-Order Averaging Method

The aim of this section is to present the first-order averaging method as obtained in [1]. Differentiability of the vector field is not needed. The specific conditions for the existence of a simple isolated zero of the averaged function are given in terms of the Brouwer degree. In fact, the Brouwer degree theory is the key point in the proof of this theorem. We remind here that continuity of some finite dimensional function is a sufficient condition for the existence of its Brouwer degree (see [10] for precise definitions).

Theorem 3. *We consider the following differential system*

$$\dot{x}(t) = \varepsilon H(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{7}$$

where $H : \mathbb{R} \times D \longrightarrow \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \longrightarrow \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$. *We define* $h : D \longrightarrow \mathbb{R}^n$ *as*

$$h(z) = \frac{1}{T}\int_0^T H(s,z)ds, \tag{8}$$

and assume that:

(i) *H and R are locally Lipschitz with respect to x;*

(ii) *for* $a \in D$ *with* $h(a) = 0$*, there exists a neighborhood* V *of* a *such that* $h(z) \neq 0$ *for all* $z \in \overline{V} \backslash \{a\}$ *and* $d_B(h, V, 0) \neq 0$ *(here* $d_B(h, V, 0)$ *denote the Brouwer degree of* h *at 0).*

Then, for $|\varepsilon| > 0$ *sufficiently small, there exists an isolated T-periodic solution* $\psi(., \varepsilon)$ *of system* (7) *such that* $\psi(0, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

Here we will need some facts from the proof of Theorem 3. Hypothesis (i) assures the existence and uniqueness of the solution of each initial value problem on the interval $[0, T]$. Hence, for each $z \in D$, it is possible to denote by $x(., z, \varepsilon)$ the solution of (7) with the initial value $x(0, z, \varepsilon) = z$. We consider also the displacement function $\zeta : D \times (-\varepsilon_f, \varepsilon_f) \longrightarrow \mathbb{R}^n$ defined by

$$\zeta(z, \varepsilon) = \int_0^T [\varepsilon H(t, x(t, z, \varepsilon)) + \varepsilon^2 R(t, x(t, z, \varepsilon), \varepsilon)]dt. \tag{9}$$

From the proof of Theorem 3 we extract the following facts.

Remark 1. *For every* $z \in D$ *the following relation holds*

$$x(T, z, \varepsilon) - x(0, z, \varepsilon) = \zeta(z, \varepsilon).$$

The function ζ *can be written in the form*

$$\zeta(z, \varepsilon) = \varepsilon h(z) + \varepsilon^2 O(1),$$

where h *is given by* (8) *and the symbol* $O(1)$ *denotes a bounded function on every compact subset of* $D \times (-\varepsilon_f, \varepsilon_f)$*. Moreover, for* $|\varepsilon|$ *sufficiently small,* $z = \psi(0, \varepsilon)$ *is an isolated zero of* $\zeta(., \varepsilon)$*.*

Note that from Remark 1 it follows that a zero z of the displacement function $\zeta(z, \varepsilon)$ at time T provides initial conditions for a periodic orbit of the system of period T. We also remark that $h(z)$ is the displacement function up to terms of order ε. Consequently the zeros of $h(z)$, when $h(z)$ is not identically zero, also provides periodic orbits of period T.

For a given systems there is the possibility that the function ζ is not globally differentiable, but the function h is C^1 when $z > \alpha$, as we shall see in Sections 3 and 4. In fact, only differentiability in some neighborhood of a fixed isolated zero of h could be enough. When this is the case, one can use the following remark in order to verify the hypothesis (ii) of Theorem 3.

Remark 2. *Let* $h : D \longrightarrow \mathbb{R}^n$ *be a* C^1 *function, with* $h(a) = 0$*, where* D *is an open subset of* $\mathbb{R}^n$ *and* $a \in D$*. Whenever* a *is a simple zero of* h *(i.e. the Jacobian* $J_h(a)$ *of* h *at* a *is not zero), then there exists a neighborhood* V *of* a *such that* $h(z) \neq 0$ *for all* $z \in \overline{V} \backslash \{a\}$ *and* $d_B(h, V, 0) \in \{-1, 1\}$*.*

3. Proof of Theorem 1

In this section we assume that $A_0 = A_0^1$. The next lemma shows that through a linear change of variables, it is possible to reduce the number of parameters of system (3).

Lemma 1. *Assume that $k_1^2 + k_2^2 \neq 0$ if $n = 2$, and $(k_1^2 + k_2^2)k_3 \neq 0$ if $n > 2$. Then by a linear change of variables system* (3) *can be transformed into the system*

$$\dot{x} = A_1 x + \varepsilon \overline{A} x + \varepsilon \varphi_\alpha(x_1)\overline{b}, \tag{10}$$

where $\overline{A} \in M_n(\mathbb{R})$ and $\overline{b} \in \mathbb{R}^n$ are convenient functions of A and b. Moreover

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

if $n = 2$, and

$$A_1 = \begin{pmatrix} 0 & -1 & \varepsilon & 0 & \dots & 0 \\ 1 & 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

if $n > 2$.

For a proof see Lemma 1 of [7].

A system equivalent to system (10), which will be in the standard form for applying the averaging theory, will be obtained in the next lemma by a proper change of the variables.

Lemma 2. *Changing the variables $(x_1, x_2, x_3, \dots, x_n)$ to $(\theta, r, x_3, \dots, x_n)$ by using $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, system* (10) *is transformed into a system of the form*

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, x_3, \dots, x_n) + O(\varepsilon^2), \\ \frac{dx_j}{d\theta} &= \varepsilon H_{j-1}(\theta, r, x_3, \dots, x_n) + O(\varepsilon^2), \end{aligned} \tag{12}$$

for $j = 3, \dots, n$ where

$$H_1 = \cos\theta F_1 + \sin\theta F_2,$$

$$H_j = a_{j1} r\cos\theta + a_{j2} r\sin\theta + b_j \varphi_\alpha(r\cos\theta) + \sum_{k=3}^{n} a_{jk} x_k,$$

and for $i = 1, 2$ we have that $F_i = a_{i1} r\cos\theta + a_{i2} r\sin\theta + \varphi_\alpha(r\cos\theta) b_i + \sum_{k=3}^{n} a_{ik} x_k$. We take ε_0 sufficiently small, n arbitrarily large and $D_n = (1/n, n)$. Then the vector field of system (12) *is well defined and continuous on $\mathbb{R} \times D_n \times \mathbb{R}^{n-2} \times (-\varepsilon_0, \varepsilon_0)$. Moreover this system is 2π-periodic with respect to the variable θ and locally Lipschitz with respect to the variable r.*

Proof. System (10) in the variables (θ, r) becomes

$$\begin{aligned} \dot{r} &= \varepsilon H_1(\theta, r, x_3, \dots, x_n), \\ \dot{\theta} &= 1 + \frac{\varepsilon}{r}(\cos\theta F_2 - \sin\theta F_1), \\ \dot{x}_j &= \varepsilon H_{j-1}(\theta, r, x_3, \dots, x_n) \text{ for } j = 3, \dots, n. \end{aligned}$$

We note that for $|\varepsilon|$ sufficiently small $\dot{\theta}(t) > 0$ for each t when $(\theta, r) \in \mathbb{R}\times D_n$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is easy to see that the right-hand side of the new system, for every fixed α, is well defined and continuous on $\mathbb{R}\times D_n\times(-\varepsilon_0, \varepsilon_0)$, it is 2π-periodic with respect to the independent variable θ and locally Lipschitz with respect to r. Form (12) is obtained after an expansion with respect to the small parameter ε. $\square$

Here $\overline{A} = (a_{ij})$ if $n = 2$, and if $n > 2$ then a_{ij} is the element of the row i and column j of the $n \times n$ matrix

$$\overline{A} + \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Our next step is to find the corresponding function (8), so we must compute

$$h_j(r, x_3, \dots, x_n) = \frac{1}{2\pi}\int_0^{2\pi} H_j(\theta, r, x_3, \dots, x_n)d\theta, \tag{13}$$

for $j = 1, \dots, n-1$.

For each $r > 0$ we define

$$\begin{aligned} I_0(r) &= \int_0^{2\pi} \varphi_\alpha(r\cos\theta)d\theta, \\ I_1(r) &= \int_0^{2\pi} \varphi_\alpha(r\cos\theta)\cos\theta d\theta, \\ I_2(r) &= \int_0^{2\pi} \varphi_\alpha(r\cos\theta)\sin\theta d\theta, \end{aligned}$$

where φ_α is the piecewise linear function given by (4).

Lemma 3. *The integrals I_0, I_1 and I_2 satisfy $I_0(r) = I_2(r) = 0$ for all $r > 0$, and*

$$I_1(r) = \begin{cases} \dfrac{\pi r}{\alpha} & \text{if } r \le \alpha, \\ \dfrac{\pi r}{\alpha} + \dfrac{2}{r}\sqrt{r^2-\alpha^2} - 2r\arctan\left(\dfrac{\sqrt{r^2-\alpha^2}}{\alpha}\right) & \text{if } r > \alpha. \end{cases} \tag{14}$$

Proof. Whenever $0 < r \le \alpha$ we have that $|r\cos\theta| \le \alpha$ and $|r\sin\theta| \le \alpha$ for all $\theta \in [0, 2\pi)$. Then $\varphi_\alpha(r\cos\theta) = r\cos\theta/\alpha$ for every θ. Thus

$$\begin{array}{rclcl} I_0(r) & = & \dfrac{r}{\alpha}\displaystyle\int_0^{2\pi} \cos\theta d\theta & = & 0, \\ I_1(r) & = & \dfrac{r}{\alpha}\displaystyle\int_0^{2\pi} \cos^2\theta d\theta & = & \dfrac{\pi r}{\alpha}, \\ I_2(r) & = & \dfrac{r}{\alpha}\displaystyle\int_0^{2\pi} \sin\theta\cos\theta d\theta & = & 0. \end{array}$$

We fix now $r > \alpha$ and consider $\theta_c \in (0, \pi/2)$ such that $\cos\theta_c = \alpha/r$. Then we can write

$$\begin{array}{rcl} I_0(r) & = & \displaystyle\int_0^{\theta_c} d\theta + \frac{r}{\alpha}\int_{\theta_c}^{\pi-\theta_c} \cos\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} d\theta + \\ & & \displaystyle\frac{r}{\alpha}\int_{\pi+\theta_c}^{2\pi-\theta_c} \cos\theta d\theta + \int_{2\pi-\theta_c}^{2\pi} d\theta, \\ I_1(r) & = & \displaystyle\int_0^{\theta_c} \cos\theta d\theta + \frac{r}{\alpha}\int_{\theta_c}^{\pi-\theta_c} \cos^2\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} \cos\theta d\theta + \\ & & \displaystyle\frac{r}{\alpha}\int_{\pi+\theta_c}^{2\pi-\theta_c} \cos^2\theta d\theta + \int_{2\pi-\theta_c}^{2\pi} \cos\theta d\theta, \\ I_2(r) & = & \displaystyle\int_0^{\theta_c} \sin\theta d\theta + \frac{r}{\alpha}\int_{\theta_c}^{\pi-\theta_c} \sin\theta\cos\theta d\theta - \int_{\pi-\theta_c}^{\pi+\theta_c} \sin\theta d\theta + \\ & & \displaystyle\frac{r}{\alpha}\int_{\pi+\theta_c}^{2\pi-\theta_c} \sin\theta\cos\theta d\theta + \int_{2\pi-\theta_c}^{2\pi} \sin\theta d\theta. \end{array}$$

Straightforward computations lead to the following expressions:

$$\begin{array}{rcl} I_0(r) & = & 0, \\ I_1(r) & = & \dfrac{\pi r}{\alpha} + \dfrac{2}{r}\sqrt{r^2-\alpha^2} - 2r\arctan\left(\dfrac{\sqrt{r^2-\alpha^2}}{\alpha}\right), \\ I_2(r) & = & 0, \end{array}$$

where we use that $\sin\theta_c = \sqrt{r^2-\alpha^2}/r$ and $\theta_c = \arctan(\sqrt{r^2-\alpha^2}/\alpha)$. $\square$

In short, from (13) we get that

$$\begin{array}{rcl} h_1(r, x_3, \ldots, x_n) & = & \dfrac{1}{2\pi}[\pi r a_{11} + b_1 I_1(r) + \pi r a_{22}], \\ h_{j-1}(r, x_3, \ldots, x_n) & = & a_{j3}x_3 + \ldots + a_{jn}x_n, \end{array} \tag{15}$$

for $j = 3, \ldots, n$.

Proposition 1. *Suppose that*

(i) *the determinant of the minor of the matrix $\overline{A} = (a_{ij})$ erasing the first two rows and the first two columns is not zero (of course this condition is only required if $n > 2$), and*

(ii) $\dfrac{(a_{11}+a_{22}+(b_1/\alpha))}{b_1} \in (K_\alpha, 1)$, *where* $K_\alpha = 0$ *when* $M_\alpha = \max\left\{\alpha, \alpha\sqrt{\dfrac{2}{1+\alpha}}\right\}$ *is* α, *or* $K_\alpha = \dfrac{2}{\pi}\arctan\left(\sqrt{\dfrac{1-\alpha}{1+\alpha}}\right)$ *when* $M_\alpha = \alpha\sqrt{\dfrac{2}{1+\alpha}}$.

Then system (10) *for* $|\varepsilon| \neq 0$ *sufficiently small has exactly one limit cycle bifurcating from the circle of radius* $\overline{r}_\alpha$ *centered at the origin of the plane* $x_3 = \ldots = x_n = 0$, *where* $\overline{r}_\alpha$ *is the unique solution in the interval* (M_α, ∞) *of the equation*

$$\arctan\left(\frac{\sqrt{r^2-\alpha^2}}{\alpha}\right) - \frac{\sqrt{r^2-\alpha^2}}{r^2} = \frac{\pi(a_{11}+a_{22}+(b_1/\alpha))}{2b_1}.$$

Proof. In order to have that the system $h_i(r, x_3, \ldots, x_n) = 0$ for $i = 1, \ldots, n-1$ has isolated solutions − otherwise the Jacobian on them becomes zero and we cannot apply Theorem 3 for studying the limit cycles of system (10) for $|\varepsilon| \neq 0$ sufficiently small − it is required for $n > 2$ that the determinant of the matrix B obtained from the minor of the matrix $\overline{A}$ by erasing the first two rows and the first two columns is not zero. Then from the equations $h_i(r, x_3, \ldots, x_n) = 0$ for $i = 2, \ldots, n-1$ we get that $x_3 = \ldots = x_n = 0$. For $r > \alpha$, from equation $h_1(r, x_3, \ldots, x_n) = 0$ we obtain

$$f_\alpha(r) = \arctan\left(\frac{\sqrt{r^2-\alpha^2}}{\alpha}\right) - \frac{\sqrt{r^2-\alpha^2}}{r^2} = \frac{\pi(a_{11}+a_{22}+(b_1/\alpha))}{2b_1}.$$

Note that with the previous hypotheses,

$$\det\left(\frac{\partial(h_1, \ldots, h_{n-1})}{\partial(r, x_3, \ldots, x_n)}\right) = 0 \tag{16}$$

if and only if $h_1'(r, x_3, \ldots, x_n).\det(B) = 0$, or equivalently $r = r_\alpha^0 = \alpha\sqrt{\dfrac{2}{1+\alpha}}$.

Also $f'(r) = 0$ if and only if $r = r_\alpha^0$.

Take $\alpha < 1$. So the function

$$f_\alpha : (r_\alpha^0, \infty) \to \left(\arctan\left(\sqrt{\frac{1-\alpha}{1+\alpha}}\right) - \frac{\sqrt{1-\alpha^2}}{2\alpha}, \frac{\pi}{2}\right)$$

is a diffeomorphism and there exist a unique solution $\overline{r}_\alpha > r_\alpha^0$ such that

$$f_\alpha(\overline{r}_\alpha) = \frac{\pi(a_{11}+a_{22}+(b_1\ \alpha))}{2b_1}.$$

For $\alpha \geq 1$ we have $f_\alpha : (\alpha, \infty) \to (0, \pi/2)$ and the result follows as above. $\square$

Therefore Theorem 1 follows directly from Theorem 3 and Proposition 1.

Remark 3. *Note that for $r \leq \alpha$ we have $h_1(r, x_3, \ldots, x_n) = 0$ if and only if $a_{11} + a_{22} + (b_1/\alpha) = 0$. Since in the hypothesis (ii) of Proposition 1 we exclude this possibility then there is not limit cycles for $r \leq \alpha$.*

Proof of Corollary 1.1. Using the previous notations let α tends to 0. So we have

$$\lim_{\alpha \to 0} \overline{r}_\alpha \geq \lim_{\alpha \to 0} r_\alpha^0 = 0. \tag{17}$$

But $\lim_{\alpha \to 0} \overline{r}_\alpha \neq 0$. In fact, consider the function

$$g_\alpha(r) = f_\alpha(r) - \frac{\pi(a_{11} + a_{22} + (b_1/\alpha))}{2b_1}.$$

Remember that $\overline{r}_\alpha$ is a zero of this function. So

$$\lim_{\alpha \to 0} \overline{r}_\alpha = 0 = \lim_{\alpha \to 0} r_\alpha^0 \Rightarrow 0 = \lim_{\alpha \to 0} g_\alpha(\overline{r}_\alpha) = \lim_{\alpha \to 0} g_\alpha(r_\alpha^0),$$

but a straight calculus shows that

$$\lim_{\alpha \to 0} g_\alpha(r_\alpha^0) = -\infty,$$

which is a contradiction, and so $\lim_{\alpha \to 0} \overline{r}_\alpha \neq 0$.

We can conclude that there exists a unique limit cycle of radius $\overline{r}_0 > 0$ for the discontinuous system (1) with $A_0 = A_0^1$. Note that $\varphi_0 = \lim_{\alpha \to 0} \varphi_\alpha$. $\square$

Remark 4. *Using the main result of* [3] *the stability of the limit cycles associated with the solution $(\overline{r}_\alpha, 0, \ldots, 0)$ is given by the eigenvalues of the matrix*

$$\frac{\partial(h_1, \ldots, h_{n-1})}{\partial(r, x_3, \ldots, x_n)}\bigg|_{(r, x_3, \ldots, x_n) = (\overline{r}_\alpha, 0, \ldots, 0)}.$$

4. Proof of Theorem 2

In this section we assume that $A_0 = A_0^2$. As in section 3, the next lemma shows that through a linear change of variables, it is possible to reduce the number of parameters os system (3).

Lemma 4. *Assume that for $n > 4$, $(k_1^2 + k_2^2)k_5 \neq 0$ or $(k_3^2 + k_4^2)k_5 \neq 0$. Then by a linear change of variables, system* (3) *can be transformed into the system*

$$\dot{x} = A_1 x + \varepsilon \overline{A} x + \varepsilon \varphi_\alpha(x_1) \overline{b}, \tag{18}$$

where $\overline{A} \in M_n(\mathbb{R})$ is an arbitrary matrix and $\overline{b} = e_1$ or $\overline{b} = e_3$. Moreover

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

if $n = 4$, *and*

$$A_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & \varepsilon & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & 0 & -1 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

if $n > 4$.

Proof. For the case $n = 4$ see the proof of Lemma 3.1 of [2]. So, we consider $n > 4$. Since the linear change of variables $x = Jy$, with J an invertible matrix, transforms system (3) into

$$\dot{y} = J^{-1}A_0Jy + \varepsilon J^{-1}AJy + \varepsilon\varphi_\alpha(k^TJy)J^{-1}b,$$

we have to find J such that

$$J^{-1}A_0J = A_1, \tag{19}$$

$$k^T = e_1^TJ^{-1}, \tag{20}$$

$$J^{-1}b = \overline{b}. \tag{21}$$

We denote by z_{ij}, for $i, j = 1, \dots, n$ the elements of the matrix J^{-1}. Using equations (19) and (20), easy computations show that J^{-1} is given by

$$J^{-1} = \begin{pmatrix} k_1 & k_2 & k_3 & k_4 & k_5 & k_6 & k_7 & \dots & k_n \\ -k_2 & k_1 & -k_4 & k_3 & -k_5 & -k_6 & -k_7 & \dots & -k_n \\ z_{31} & z_{32} & z_{33} & z_{34} & k_5 & k_6 & k_7 & \dots & k_n \\ -z_{32} & z_{31} & -z_{34} & z_{33} & -k_5 & -k_6 & -k_7 & \dots & -k_n \\ 0 & 0 & 0 & 0 & -k_5\varepsilon^{-1} & -k_6\varepsilon^{-1} & -k_7\varepsilon^{-1} & \dots & -k_n\varepsilon^{-1} \\ 0 & 0 & 0 & 0 & z_{65} & z_{66} & z_{67} & \dots & z_{6n} \\ 0 & 0 & 0 & 0 & z_{75} & z_{76} & z_{77} & \dots & z_{7n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & z_{n5} & z_{n6} & z_{n7} & \dots & z_{nn} \end{pmatrix}.$$

If we take $z_{31} = z_{32} = z_{34} = 0$, $z_{33} = z_{66} = \dots = z_{nn} = 1$, $z_{ij} = 0$ for $i = 6, \dots, n$, $j = 5, \dots, n$ (except for $z_{66}, \dots, z_{nn}$), $k_1 = b_1/(b_1^2 + b_2^2)$ and $k_2 = b_2/(b_1^2 + b_2^2)$ in the expression of J^{-1}, then equation (21) is satisfied with $\overline{b} = e_1$. In this case we obtain a matrix J^{-1} whose determinant is $-(k_1^2 + k_2^2)k_5/\varepsilon$.

On the other hand, if we take $z_{32} = z_{33} = z_{34} = 0$, $z_{31} = z_{66} = \dots = z_{nn} = 1$, $z_{ij} = 0$ for $i = 6, \dots, n$, $j = 5, \dots, n$ (except for $z_{66}, \dots, z_{nn}$), $k_3 = b_3/(b_3^2 + b_4^2)$ and $k_4 = b_4/(b_3^2 + b_4^2)$ in the expression of J^{-1}, then equation (21) is satisfied with $\overline{b} = e_3$. In this case we obtain a matrix J^{-1} whose determinant is $-(k_3^2 + k_4^2)k_5/\varepsilon$.

By hypothesis at least one of the above expressions for the determinant of J^{-1} is nonzero. Hence there exists the change of variables $x = Jy$. This completes the proof of Lemma 4. $\square$

An equivalent system in the standard form for applying the averaging theory, will be found in the next lemma doing a convenient change of variables.

Lemma 5. *Changing the variables* $(x_1, x_2, \ldots, x_n)$ *to* $(\theta, r, \rho, s, x_5, \ldots, x_n)$ *by using*

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_3 = \rho\cos(\theta+s), \quad x_4 = \rho\sin(\theta+s),$$

system (18) *is transformed into a system of the form*

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, \rho, s, x_5, \ldots, x_n) + O(\varepsilon^2), \\ \frac{d\rho}{d\theta} &= \varepsilon H_2(\theta, r, \rho, s, x_5, \ldots, x_n) + O(\varepsilon^2), \\ \frac{ds}{d\theta} &= \varepsilon H_3(\theta, r, \rho, s, x_5, \ldots, x_n) + O(\varepsilon^2), \\ \frac{dx_j}{d\theta} &= \varepsilon H_{j-1}(\theta, r, , \rho, s, x_5, \ldots, x_n) + O(\varepsilon^2), \end{aligned} \tag{22}$$

for $j = 5, \ldots, n$ *where*

$$\begin{aligned} H_1 &= \cos\theta F_1 + \sin\theta F_2, \\ H_2 &= \cos(\theta+s)F_3 + \sin(\theta+s)F_4, \\ H_3 &= \frac{1}{r}\cos\theta F_2 - \frac{1}{r}\sin\theta F_1 - \frac{1}{\rho}\cos(\theta+s)F_4 + \frac{1}{\rho}\sin(\theta+s)F_3, \\ H_{j-1} &= a_{j1}r\cos\theta + a_{j2}r\sin\theta + a_{j3}\rho\cos(\theta+s) + a_{j4}\rho\sin(\theta+s) + \\ &\quad + b_j\varphi_\alpha(r\cos\theta) + \sum_{k=5}^{n} a_{jk}x_k, \end{aligned}$$

and for $i = 1, \ldots, 4$ *we have that* $F_i = a_{i1}r\cos\theta + a_{i2}r\sin\theta + a_{i3}\rho\cos(\theta+s) + a_{i4}\rho\sin(\theta+s) + \varphi_\alpha(r\cos\theta)b_i + \sum_{k=5}^{n} a_{ik}x_k$. *We take* ε_0 *sufficiently small,* n *arbitrarily large and* $D_n = (1/n, n) \times (1/n, n) \times \mathbb{R}$. *Then the vector field of system* (22) *is well defined and continuous on* $\mathbb{R} \times D_n \times \mathbb{R}^{n-4} \times (-\varepsilon_0, \varepsilon_0)$. *Moreover this system is* 2π*-periodic with respect to the variable* θ *and locally Lipschitz with respect to the variable* r.

Proof. System (18) in the variables $(\theta, r, \rho, s, x_5, \ldots, x_n)$ becomes

$$\begin{aligned} \dot\theta &= 1 + \frac{\varepsilon}{r}(\cos\theta F_2 - \sin\theta F_1), \\ \dot r &= \varepsilon H_1(\theta, r, \rho, s, x_5, \ldots, x_n), \\ \dot\rho &= \varepsilon H_2(\theta, r, \rho, s, x_5, \ldots, x_n), \\ \dot s &= \varepsilon H_3(\theta, r, \rho, s, x_5, \ldots, x_n), \\ \dot x_j &= \varepsilon H_{j-1}(\theta, r, \rho, s, x_5, \ldots, x_n) \text{ for } j = 5, \ldots, n. \end{aligned}$$

We note that for $|\varepsilon|$ sufficiently small $\dot\theta(t) > 0$ for each t when $(\theta, r, \rho, s, x_5, \ldots, x_n) \in \mathbb{R} \times D_n \times \mathbb{R}^{n-4}$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is easy to see that the right-hand side of the new system, for every fixed α, is well defined and continuous on $\mathbb{R} \times D_n \times \mathbb{R}^{n-4} \times (-\varepsilon_0, \varepsilon_0)$, it is 2π-periodic with respect to the independent variable θ and locally Lipschitz with respect to r. Form (22) is obtained after an expansion with respect to the small parameter ε. □

Here $\overline{A} = (a_{ij})$ if $n = 4$, and if $n > 4$ then a_{ij} is the element of the row i and column j of the $n \times n$ matrix

$$\overline{A} + \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We will apply Theorem 3 to system (22). Now we will find the corresponding function (8). We will denote it by $h : D_n \times \mathbb{R}^{n-4} \to \mathbb{R}^{n-1}$, $h = (h_1, h_2, \dots, h_{n-1})^T$. For each $i = 1, \dots, n-1$ the component h_i is defined by the formula

$$h_i(r, \rho, s, x_5, \dots, x_n) = \int_0^{2\pi} H_i(\theta, r, \rho, s, x_5, \dots, x_n) d\theta$$

where H_i are like in Lemma 5.

In order to calculate the expression of h, we will use the following formulas

$$\int_0^{2\pi} \cos^2 \theta d\theta = \pi, \quad \int_0^{2\pi} \sin^2 \theta d\theta = \pi, \quad \int_0^{2\pi} \cos(\theta + s) \sin(\theta + s) d\theta = 0,$$

$$\int_0^{2\pi} \cos\theta \cos(\theta + s) d\theta = \pi \cos s, \quad \int_0^{2\pi} \cos\theta \sin(\theta + s) d\theta = \pi \sin s,$$

$$\int_0^{2\pi} \sin\theta \cos(\theta + s) d\theta = -\pi \sin s, \quad \int_0^{2\pi} \sin\theta \sin(\theta + s) d\theta = \pi \cos s,$$

$$\int_0^{2\pi} \cos^2(\theta + s) d\theta = \pi, \quad \int_0^{2\pi} \sin^2(\theta + s) d\theta = \pi, \quad \int_0^{2\pi} \cos\theta \sin\theta d\theta = 0.$$

For each $r > 0$ we consider the functions I_0, I_1 and I_2 defined in section 3. Their expressions were obtained in Lemma 3. Thus we obtain the following expressions for the components of h

$$
\begin{array}{rcl}
h_1 & = & c_1 r + (c_2 \cos s + c_3 \sin s)\rho + b_1 I_1(r), \\
h_2 & = & (c_5 \cos s + c_6 \sin s) r + c_7 \rho + b_3 \cos s I_1(r), \\
h_3 & = & c_4 + (c_2 \sin s - c_3 \cos s)\dfrac{\rho}{r} + (c_5 \sin s - c_6 \cos s)\dfrac{r}{\rho} + \\
 & & + b_3 \sin s \dfrac{I_1(r)}{\rho}, \\
h_{j-1} & = & a_{j5} x_5 + \ldots + a_{jn} x_n,
\end{array} \tag{23}
$$

for $j = 5, \ldots, n$, where the coefficients c_i are given by $c_1 = (a_{11} + a_{22})\pi$, $c_2 = (a_{13} + a_{24})\pi$, $c_3 = (a_{14} - a_{23})\pi$, $c_4 = (a_{21} - a_{12} - a_{43} + a_{34})\pi$, $c_5 = (a_{31} + a_{42})\pi$, $c_6 = (a_{41} - a_{32})\pi$ and $c_7 = (a_{33} + a_{44})\pi$.

In order to have that the system $h_i(\theta, r, \rho, s, x_5, \ldots, x_n) = 0$ for $i = 1, \ldots, n-1$ has isolated solutions it is required for $n > 4$ that the determinant of the matrix B obtained from the minor of the matrix $\overline{A}$ by erasing the first four rows and the first four columns is not zero. Then from the equations $h_i = 0$ for $i = 4, \ldots, n-1$ we get that $x_5 = \ldots = x_n = 0$.

Our next step is to study the solvability of the system

$$
\begin{array}{l}
h_1(r, \rho, s, x_5, \ldots, x_n) = 0, \\
h_2(r, \rho, s, x_5, \ldots, x_n) = 0, \\
h_3(r, \rho, s, x_5, \ldots, x_n) = 0.
\end{array} \tag{24}
$$

Of course any isolated 2π-periodic solution of (22) with $|\varepsilon| \neq 0$ sufficiently small, corresponds to a limit cycle of (18).

Lemma 5 states that the hypotheses of Theorem 3 are fulfilled for system (22), where the function h is given by (23). Using also Remark 2 we conclude that, for $|\varepsilon|$ sufficiently small, and for each simple zero $(r^*, \rho^*, s^*, 0, \ldots, 0) \in D_n \times \mathbb{R}^{n-4}$ of h, there exists an isolated 2π-periodic solution $\varphi(., \varepsilon)$ of system (22) such that $\varphi(0, \varepsilon) \to (r^*, \rho^*, s^*, 0, \ldots, 0)$ as $\varepsilon \to 0$. In short we have proved the next result.

Proposition 2. *From each periodic orbit of system* (6)*, which corresponds to a simple zero of h in $D_n \times \mathbb{R}^{n-4}$, a branch of limit cycles bifurcates from system* (18)*.*

For a proof of the next result see Lemma 3.4 of [2].

Lemma 6. *The displacement function of system* (18) *for the transversal section $x_2 = 0$, written in the coordinates of Lemma 5, has the form*

$$\epsilon h(r, \rho, s, x_5, \ldots, x_n) + O(\varepsilon^2).$$

The proof of the following result will be the subject of the next section. We note that, in order to find the zeros of h in $D_n \times \mathbb{R}^{n-4}$ (or equivalently, to find the zeros $(r^*, \rho^*, s^*) \in D_n$ of system (24), because $x_5 = \cdots = x_n = 0$), it is sufficient to look for them in $(0, \infty) \times (0, \infty) \times [0, 2\pi)$. This is due to the fact that n can be chosen arbitrarily large, and h, as well as the transformation of Lemma 5 are 2π-periodic with respect to the variable s.

Proposition 3. *Let $\tilde{h} : (M_\alpha, \infty) \times (0, \infty) \times \mathbb{R} \to \mathbb{R}^3$ be the function $\tilde{h} = (h_1, h_2, h_3)$ whose components are given by* (23)*, where*
$M_\alpha = \max\left\{\alpha, \alpha\sqrt{\dfrac{2}{1+\alpha}}\right\}$, *$c_i$, for $i = 1, \ldots, 7$ are arbitrary real parameters, $b_1, b_3 \in \{0, 1\}$ with $b_1 b_3 = 0$ and $b_1^2 + b_3^2 \neq 0$. Then*

(i) *$\tilde{h}$ is of class C^1;*

(ii) *the maximum number of isolated zeros of $\tilde{h}$ in $(M_\alpha, \infty) \times (0, \infty) \times \mathbb{R}$ is three.*

(iii) *Suppose that $0 < \alpha < 1$. For*

$$c_1 = -2 + \frac{8(-1+\alpha)\pi}{9\alpha}, \quad c_2 = c_1, \quad c_3 = 0, \quad c_4 = -\frac{21c_1}{20}, \quad c_5 = 0,$$
$$c_6 = -\frac{c_1}{2}, \quad c_7 = \frac{c_1}{2}, \quad b_1 = 1, \quad b_2 = b_3 = b_4 = 0,$$

the function $\tilde{h}$ has exactly three simple zeros.

(iv) *Suppose that $\alpha \geq 1$ and $\alpha \neq 3$. For*

$$c_1 = 1 - \frac{3}{\alpha}, \quad c_2 = c_1, \quad c_3 = 0, \quad c_4 = -\frac{21c_1}{20}, \quad c_5 = 0,$$
$$c_6 = -\frac{c_1}{2}, \quad c_7 = \frac{c_1}{2}, \quad b_1 = 1, \quad b_2 = b_3 = b_4 = 0,$$

the function $\tilde{h}$ has exactly three simple zeros.

(v) *Suppose that $\alpha = 3$. For*

$$c_1 = -\frac{1}{2}, \quad c_2 = c_1, \quad c_3 = 0, \quad c_4 = -\frac{21c_1}{20}, \quad c_5 = 0,$$
$$c_6 = -\frac{c_1}{2}, \quad c_7 = \frac{c_1}{2}, \quad b_1 = 1, \quad b_2 = b_3 = b_4 = 0,$$

the function $\tilde{h}$ has exactly three simple zeros.

The conclusion of Theorem 2 follows from Lemmas 4 and 6 and Propositions 2 and 3.

5. Proof of Proposition 3

Function $\widetilde{h}$ is a composition of some elementary functions and function I_1. A direct study of I_1 shows that it is of class C^1 on (α, ∞). Thus, since $M_\alpha \geq \alpha$, statement (i) holds. We will divide the proof of the statement (ii) into several lemmas. Two auxiliary results will be given at the beginning, followed by a discussion with respect to different values of the coefficients.

The following notations will be used

$$d(s) = b_3 \cos s(c_2 \cos s + c_3 \sin s) - b_1 c_7,$$
$$k_1(s) = (c_5 b_1 - c_1 b_3) \cos s + c_6 b_1 \sin s,$$
$$k_2(s) = -c_2 c_5 \cos^2 s - (c_2 c_6 + c_3 c_5) \cos s \sin s - c_3 c_6 \sin^2 s + c_1 c_7,$$
$$\begin{aligned} f(s) \quad = \quad & c_4 d(s) k_1(s) + (c_2 \sin s - c_3 \cos s) k_1^2(s) + \\ & (c_5 \sin s - c_6 \cos s) d^2(s) + b_3 d(s) k_2(s) \sin s. \end{aligned}$$

For studying the zeros of $\widetilde{h}$ it will be necessary to study the zeros of f.

Lemma 7. *The function $f : [0, 2\pi) \to \mathbb{R}$ can have at most six isolated zeros. Moreover they appear in pairs of the form s^* together with $\xi^* = s^* + \pi($ mod $2\pi)$ and when $b_1 = 0$ and $b_3 = 1$ two of them are $s^* = \pi/2$ and $\xi^* = 3\pi/2$.*

For a proof see Lemma 4.1 of [2].

The following result will be needed in the sequel.

Lemma 8. *We consider the equation*

$$I_1(r) = \frac{c}{\alpha} r, \quad r > 0, \tag{25}$$

where $I_1(r)$ is like before and c is a real parameter. Then we have the following situations

(I) *Assume that $\alpha \geq 1$.*

(I-i) If $\pi(1 - \alpha) < c < \pi$, then (25) has a unique solution $r^* > \alpha$.

(I-ii) If $c = \pi$, then (25) has the interval $(0, \alpha]$ as the set of solutions.

(I-iii) If $c > \pi$ or $c \leq \pi(1 - \alpha)$, then (25) has no solution.

(II) *Assume that $\alpha < 1$ and denote by $Q(\alpha)$ the number*

$$Q(\alpha) = \pi + (1 + \alpha)\sqrt{\frac{1 - \alpha}{(1 + \alpha)}} - 2\alpha \arctan \sqrt{\frac{1 - \alpha}{(1 + \alpha)}}.$$

(II-i) If $\pi(1 - \alpha) < c < \pi$, then (25) has a unique solution $r^* > \alpha\sqrt{2/(1 + \alpha)}$.

(II-ii) If $c = \pi$, then (25) has $(0, \alpha] \cup \{r^*\}$, where $r^* > \alpha\sqrt{2/(1 + \alpha)}$, as the set of solutions.

(II-iii) If $\pi < c < Q(\alpha)$, then (25) has exactly two solutions $r_1^* < \alpha\sqrt{2/(1 + \alpha)}$ and $r_2^* > \alpha\sqrt{2/(1 + \alpha)}$.

(II-iv) If $c = Q(\alpha)$, then (25) has a unique solution $r^* = \alpha\sqrt{2/(1 + \alpha)}$.

(II-v) If $c > Q(\alpha)$ or $c \leq \pi(1 - \alpha)$, then (25) has no solution.

Proof. It is easy to see that the result holds for $r \in (0, \alpha]$, since in this case $I_1(r) = \dfrac{\pi}{\alpha} r$. For $r > \alpha$ equation (25) becomes

$$\frac{\pi r}{\alpha} + \frac{2}{r}\sqrt{r^2 - \alpha^2} - 2r \arctan\left(\frac{\sqrt{r^2 - \alpha^2}}{\alpha}\right) = \frac{c}{\alpha} r.$$

Introducing the new variable $u = \sqrt{r^2 - \alpha^2}/\alpha$, we obtain the equivalent equation

$$\alpha \arctan u - \frac{(\pi - c)}{2} - \frac{u}{1 + u^2} = 0, \quad u > 0.$$

We study the graphic of this function on the interval $(0, \infty)$. Then we can see:

(I) If $\alpha \geq 1$ then the equation has solution if and only if $\pi(1-\alpha) < c < \pi$. In this case the solution $u^* > 0$ is unique.

(II) If $\alpha < 1$ then the equation has solution if and only if $\pi(1-\alpha) < c \leq Q(\alpha)$. Moreover, if $\pi(1-\alpha) < c \leq \pi$, then the solution $u^* > \sqrt{(1-\alpha)}/(1+\alpha)$ is unique. If $\pi < c < Q(\alpha)$, then there exists two solutions $u_1^* < \sqrt{(1-\alpha)}/(1+\alpha)$ and $u_2^* > \sqrt{(1-\alpha)}/(1+\alpha)$. If $c = Q(\alpha)$ then the solution $u^* = \sqrt{(1-\alpha)}/(1+\alpha)$ is unique.

This completes the proof of the lemma. □

The next three lemmas study the zeros of $\tilde{h}$.

Lemma 9. *The function $\tilde{h}$ can have at most three isolated zeros $(r^*, \rho^*, s^*) \in (M_\alpha, \infty) \times (0, \infty) \times \mathbb{R}$, where M_α is like in Proposition 3, with $d(s^*) \neq 0$. When $b_1 = 0$, $b_3 = 1$ and $c_1 = 0$, then $\tilde{h}$ has at most two such zeros. If $b_1 = 0$, $b_3 = 1$ and $c_1 = 0$, then $\tilde{h}$ has no such zeros.*

Proof. We have that the two first equations of system (24) and (**??**) are equivalent to

$$\rho = \frac{k_1(s)}{d(s)} r, \tag{26}$$

$$I_1(r) = \frac{k_2(s)}{d(s)} r, \tag{27}$$

where d, k_1 and k_2 are defined at the beginning of this section. Replacing (26) and (27) in h_3 we obtain that $h_3 = 0$ is equivalent to $f(s) = 0$, where f also was defined at the beginning of this section.

Fix a zero s^* of f. We wish to study the solvability of (27) with respect to $r > M_\alpha$.

By Lemma 8 there exists an isolated solution $r > M_\alpha$ of (27) if and only if

$$\frac{\pi}{\alpha} - \pi < \frac{k_2(s^*)}{d(s^*)} < \frac{\pi}{\alpha} \text{ when } \alpha \geq 1, \text{ or} \tag{28}$$

$$\frac{\pi}{\alpha} - \pi < \frac{k_2(s^*)}{d(s^*)} < \frac{Q(\alpha)}{\alpha} \text{ when } \alpha < 1, \tag{29}$$

and in these cases it is unique. Note that we are excluding the case $c = Q(\alpha)$. Also, for a fixed r^*, corresponding to some s^*, whenever

$$\frac{k_1(s^*)}{d(s^*)} > 0, \tag{30}$$

we can uniquely find ρ^* satisfying (26). We will verify that condition (30) is satisfied only for at most half of the zeros of f. In order to do this, we note that $k_1(s + \pi) = -k_1(s)$, $d(s + \pi) = d(s)$, and we remind that the zeros of

f appear in pairs, s^* together with $s^*+\pi$. Thus, condition (30) is satisfied either for s^*, or for $s^*+\pi$, unless $k_1(s^*)=0$. Hence, function $\tilde{h}$ can have at most three isolated zeros $(r^*,\rho^*,s^*)\in(M_\alpha,\infty)\times(0,\infty)\times\mathbb{R}$ with $d(s^*)\neq 0$.

If $b_1=0$, $b_3=1$ and $c_1=0$ then $k_1=0$, so (30) cannot be satisfied for any s. If $b_1=0$, $b_3=1$ and $c_1\neq 0$, then $k_1(s)=-c_1\cos s$. By Lemma (7) two of the zeros of f are $s^*=\pi/2$ and $s^*=3\pi/2$, which cannot provide any solution since (30) does not hold. □

Lemma 10. *When $b_1=1$, $b_3=0$, $c_7=0$, the function $\tilde{h}$ can have at most two isolated zeros in $(M_\alpha,\infty)\times(0,\infty)\times\mathbb{R}$.*

Proof. In this case $d=0$, so we cannot apply Lemma 9. We can prove that system (24) is equivalent to

$$I_1(r)=\Big(-c_1-q(s)\frac{\rho}{r}\Big)r, \tag{31}$$

$$k_1(s)=0, \tag{32}$$

$$p_1(s)\Big(\frac{\rho}{r}\Big)^2+c_4\frac{\rho}{r}+p_2(s)=0, \tag{33}$$

where $q(s)=c_2\cos s+c_3\sin s$, $p_1(s)=c_2\sin s-c_3\cos s$ and $p_2(s)=c_5\sin s-c_6\cos s$. For a fixed s, equation (33) provides at most two isolated values of ρ/r. By Lemma 8 for fixed s and fixed ρ/r, if $r>M_\alpha$ then equation (31) gives at most one isolated value for r. If $k_1=0$ then system (31)–(33) has no isolated solution. If $k_1\neq 0$, then we denote by s^* and $\xi^*=s^*+\pi$, the zeros of k_1. Substituting s^* and ξ^* in (33) we obtain

$$p_1(s^*)\Big(\frac{\rho}{r}\Big)^2+c_4\frac{\rho}{r}+p_2(s^*)=0$$

and

$$p_1(s^*)\Big(\frac{\rho}{r}\Big)^2-c_4\frac{\rho}{r}+p_2(s^*)=0,$$

respectively. These two equations can have together at most two positive solutions ρ/r. We conclude that $\tilde{h}$ can have at most two isolated zeros. □

Lemma 11. *When $b_1=0$, $b_3=1$ and $c_7\neq 0$ the function $\tilde{h}$ can have at most two isolated zeros $(r^*,\rho^*,s^*)\in(M_\alpha,\infty)\times(0,\infty)\times\mathbb{R}$ with $d(s^*)=0$. Moreover either $c_1=0$, or $\tilde{h}$ has at most one isolated zero.*

Proof. Whenever $b_1=0$, system (24) is equivalent to

$$q(s)\frac{\rho}{r}=-c_1, \tag{34}$$

$$\cos s\frac{I_1(r)}{r}=-\tilde{q}(s)-c_7\frac{\rho}{r}, \tag{35}$$

$$c_4+p_1(s)\frac{\rho}{r}+p_2(s)\frac{r}{\rho}+\sin s\frac{I_1(r)}{\rho}=0, \tag{36}$$

where $\tilde{q}(s) = c_5 \cos s + c_6 \sin s$ and q, p_1 and p_2 are like in the proof of Lemma 10. In this case $d(s) = q(s)\cos s$. We study the cases when $d(s^*) = 0$, i.e. either $q(s^*) = 0$ and equation (34) is degenerate, or $\cos s^* = 0$ and equation (35) is degenerate. If (34) is degenerate, but (35) is not, then we can write $I_1(r)/r$ as a function of s and ρ/r. Discussion is similar now to the one done in the proof of Lemma 10. The conclusion will be that $\tilde{h}$ can have at most two isolated solutions (r^*, ρ^*, s^*) with $r^* > M_\alpha$ and $d(s^*) = 0$, and they can be found knowing that

$$\begin{aligned} c_2 \cos s^* + c_3 \sin s^* &= 0, \\ \cos s^* p_1(s^*)\Big(\frac{\rho}{r}\Big)^2 + p_3(s^*)\frac{\rho}{r} - c_6 &= 0, \\ \frac{I_1(r)}{r} &= -\frac{\tilde{q}(s^*)}{\cos s^*} - \frac{c_7}{\cos s^*}\frac{\rho}{r}, \end{aligned} \tag{37}$$

where $p_3(s) = c_4 \cos s - c_7 \sin s$. In this case, equations (34) and (37) say that necessary conditions for the existence of isolated solutions are $c_1 = 0$ and $c_2^2 + c_3^2 \neq 0$, respectively.

It remains to study when equation (35) is degenerate, i.e. if $s^* = \pi/2$ or $s^* = 3\pi/2$. When we replace these values in (34) and (35) we obtain

$$c_3\frac{\rho}{r} = -c_1, \quad c_7\frac{\rho}{r} = -c_6,$$

and

$$c_3\frac{\rho}{r} = c_1, \quad c_7\frac{\rho}{r} = c_6,$$

respectively. From each system we can obtain at most one isolated value of ρ/r, but it is easy to see that only one of them is positive. Thus, $\tilde{h}$ can have at most one zero with $s^* = \pi/2$ or $3\pi/2$, and $c_1 \neq 0$ is a necessary condition for the existence of such zero. □

In order to conclude that $\tilde{h}$ can have at most three zeros in D_n, three different cases will be considered.

(i) If $b_1 = 1$, $b_3 = 0$, $c_7 \neq 0$, then $d(s) = c_7 \neq 0$ for all $s \in [0, 2\pi)$. The conclusion follows by Lemma 9;
(ii) If $b_1 = 1$, $b_3 = 0$, $c_7 = 0$, then we apply Lemma 10;
(iii) If $b_1 = 0$, $b_3 = 1$, $c_7 \neq 0$ then the conclusion holds by Lemmas 9 and 11.

We will now prove statements (iii)$-$(v) of Proposition 3.

For the values of the coefficients given in statement (iii), the components of function $\tilde{h}$ are

$$\begin{aligned} h_1 &= \Big(-2 + \frac{4(-1+\alpha)\pi}{5\alpha}\Big)(r + \rho\cos s) + I_1(r), \\ h_2 &= \Big(-1 + \frac{2(-1+\alpha)\pi}{5\alpha}\Big)(\rho - r\sin s), \\ h_3 &= \Big(-2 + \frac{4(-1+\alpha)\pi}{5\alpha}\Big)\Big(-\frac{21}{20} + \frac{r\cos s}{2\rho} + \frac{\rho\sin s}{r}\Big), \end{aligned}$$

and the equation $f(s) = 0$ is equivalent to

$$21 \sin s - 20 \sin^3 s - 10 \cos s = 0.$$

With the notation $x = \cos s$, equation $f(s) = 0$ becomes

$$1 - 61x^2 + 360x^4 - 400x^6 = 0.$$

This polynomial equation has six solutions $x_{1,2} = \pm\sqrt{5}/5$, $x_{3,4,5,6} = \pm\sqrt{(7 \pm 2\sqrt{11})/20}$, and all are in the interval $(-1, 1)$.

We obtain that f has the following zeros, $s_1 = \arccos(\sqrt{5}/5)$, $s_{2,3} = \arccos(\sqrt{(7 \pm 2\sqrt{11})/20})$, $\xi_{1,2,3} = s_{1,2,3} + \pi$.

One can prove that condition (29) is satisfied for every zero of f, but condition (30) is satisfied only for the zeros s_1, s_2 and s_3 of f. Then, $\tilde{h}$ has exactly three zeros. Since $\rho^* = r^* \sin s^*$ and $r^* > \alpha$ is the unique solution of (27) we conclude, using the software *Mathematica*, that $J_{\tilde{h}}(r^*, \rho^*, s^*) = 0$ if and only if α is approximately 4.89655. But in the statement (iii) we have that $0 < \alpha < 1$ and so $J_{\tilde{h}}(r^*, \rho^*, s^*) \neq 0$.

For the values of the coefficients given in statement (iv), the components of function $\tilde{h}$ are

$$h_1 = \frac{\alpha - 3}{\alpha}\Big(r + \rho \cos s\Big) + I_1(r),$$
$$h_2 = \frac{3 - \alpha}{2\alpha}\Big(r \sin s - \rho\Big),$$
$$h_3 = \frac{3 - \alpha}{\alpha}\Big(\frac{21}{20} - \frac{\rho}{r}\sin s - \frac{r}{2\rho}\cos s\Big),$$

and the equation $f(s) = 0$ is equivalent to

$$21 \sin s - 20 \sin^3 s - 10 \cos s = 0.$$

The zeros of f were obtained above.

One can prove that condition (28) is satisfied for every zero of f, but condition (30) is satisfied only for the zeros s_1, s_2 and s_3 of f. Then, $\tilde{h}$ has exactly three zeros. Since $\rho^* = r^* \sin s^*$ and $r^* > \alpha$ is the unique solution of (27) we conclude, using the software *Mathematica*, that $J_{\tilde{h}}(r^*, \rho^*, s^*) = 0$ if and only if $\alpha = 3$. But in the statement (iv) we have that $\alpha > 1$ and $\alpha \neq 3$ so $J_{\tilde{h}}(r^*, \rho^*, s^*) \neq 0$.

For the values of the coefficients given in statement (v), the components of $\tilde{h}$ are given by

$$h_1 = -\frac{r}{2} - \frac{\rho \cos s}{2} + I_1(r),$$
$$h_2 = -\frac{\rho}{4} + \frac{r \sin s}{4},$$
$$h_3 = \frac{21}{40} - \frac{\rho \sin s}{2r} - \frac{r \cos s}{4\rho},$$

and the equation $f(s) = 0$ also is equivalent to

$$21 \sin s - 20 \sin^3 s - 10 \cos s = 0.$$

The zeros of f were obtained above.

One can prove that conditions (28) (for $\alpha = 3$) is satisfied for every zero of f, but condition (30) is satisfied only for the zeros s_1, s_2 and s_3 of f. Then $\tilde{h}$ has exactly three zeros. Using the software *Mathematica* we conclude that $J_{\tilde{h}}(r^*, \rho^*, s^*) \neq 0$ which the hypothesis of the statement (v).

This concludes the proof of Proposition 3. □

Taking $x_i = 0$, for $i = 5, \ldots, n$, we conclude directly that the map h has exactly three zeros in $(M_\alpha, \infty) \times (0, \infty) \times \mathbb{R} \times \mathbb{R}^{n-4}$.

Remark 5. *Using the main result of* [3] *the stability of the limit cycles associated with the solution* $(r^*, \rho^*, s^*, 0, \ldots, 0)$ *is given by the eigenvalues of the matrix*

$$\frac{\partial(h_1, \ldots, h_{n-1})}{\partial(r, \rho, s, x_5, \ldots, x_n)}\Big|_{(r,\rho,s,x_5,\ldots,x_n)=(r^*,\rho^*,s^*,0,\ldots,0)}.$$

Proof of Corollary 2.1. By Lemma 9 we have that $\overline{r}_i^\alpha > M_\alpha$ where $\overline{r}_i^\alpha$ for $i = 1, 2, 3$ is the radius of the limit cycles when $\varepsilon \to 0$. Now let α tends to 0. So we have

$$\lim_{\alpha \to 0} \overline{r}_i^\alpha \geq \lim_{\alpha \to 0} M_\alpha = 0. \tag{38}$$

But $\lim_{\alpha \to 0} \overline{r}_i^\alpha \neq 0$. In fact consider the function

$$g_\alpha(r) = \alpha \arctan\left(\frac{\sqrt{r^2 - \alpha^2}}{\alpha}\right) - \frac{(\pi - c)}{2} - \frac{\alpha\sqrt{r^2 - \alpha^2}}{r^2}$$

given in Lemma 8. Remember that $\overline{r}_i^\alpha$ for $i = 1, 2, 3$ are zeros of this function. So

$$\lim_{\alpha \to 0} \overline{r}_i^\alpha = 0 = \lim_{\alpha \to 0} M_\alpha \Rightarrow 0 = \lim_{\alpha \to 0} g_\alpha(\overline{r}_i^\alpha) = \lim_{\alpha \to 0} g_\alpha(M_\alpha) = \frac{-\pi + c - 1}{2},$$

and this last number is equal to zero if and only if $c = \pi + 1$. Looking again at Lemma 8 we can see that, when α tends to zero, we have $\pi < c \leq \pi + 1$, and $c = \pi + 1$ implies that $c = Q(0)$. But this case was excluded in the proof of Lemma 9. So $c \neq \pi + 1$ and $\lim_{\alpha \to 0} \overline{r}_i^\alpha \neq 0$.

We can conclude that, for each $\overline{r}_i^\alpha$ there exists a unique limit cycle of radius $\overline{r}_i^0 > 0$ (when $\varepsilon \to 0$), $i = 1, 2, 3$, for the discontinuous system (1) with $A_0 = A_0^2$ and $\varphi_0 = \lim_{\alpha \to 0} \varphi_\alpha$. □

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References

[1] A. Buica and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math., **128** (2004), 7–22.

[2] A. Buica and J. Llibre, *Bifurcation of limit cycles from a four-dimensional center in control systems*, International Journal of Bifurcation and Chaos, Vol. 15, No. 8 (2005), 2653–2662.

[3] A. Buica, J. Llibre and O. Makarenkov, *Asymptotic stability of periodic solutions for nonsmooth differential equations with application to the nonsmooth van der Pol oscillator*, SIAM J. Math. Anal. 40 (2009), 2478–2495.

[4] C.A. Buzzi, J. Llibre, J.C. Medrado and J. Torregrosa, *Bifurcation of limit cycles from a center in $\mathbb{R}^4$ in resonance* $1 : N$, Dynamical Systems, Vol. 24, No. 1 (2009), 123–137.

[5] S.N. Chow and J. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, Berlin 1982.

[6] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*, Springer-Verlag, Berlin 1983.

[7] J. Llibre and A. Makhlouf, *Bifurcation of limit cycles from a two-dimensional center inside* $\mathbb{R}^n$, Nonlinear Analysis 72, (2010), 1387–1392.

[8] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, 2^{nd} edition, Universitext, Springer 1996.

[9] M. Han, K. Jiang and D. Green Jr., *Bifurcations of periodic orbits, subharmonic solutions and invariant Tori of high-dimensional systems*, Nonlin. Anal., **36** (1999), 319–329.

[10] N. G. Lloyd, *Degree Theory*, Cambridge University Press 1978.

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