This is a preprint of: "Polynomial first integrals for the Chen and Lu systems", Jaume Llibre, Clàudia Valls, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 22(11), 125022 (7 pages), 2012.
DOI: [10.1142/S0218127412502628]

POLYNOMIAL FIRST INTEGRALS FOR THE CHEN AND LÜ SYSTEMS

JAUME LLIBRE
Departament de Matemàtiques
Universitat Autònoma de Barcelona, 08193, Bellaterra, Barcelona, Catalonia, Spain, e-mail: jllibre@mat.uab.es

CLAUDIA VALLS
Departamento de Matemática
Instituto Superior Técnico (U.T.L.). Av. Rovisco Pais 1049-001 Lisboa, Portugal, e-mail:cvalls@math.ist.utl.pt

June 27, 2011

We characterize all the values of the parameters for which the Chen and Lü systems have polynomial first integrals by using weight homogeneous polynomials and the method of characteristics for solving partial differential equations. We improve previous results which were not complete.

1. Introduction and statement of the main results

The following real differential system

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= (c - a)x - xz + cy, \\ \dot{z} &= xy - bz, \end{aligned} \tag{1}$$

where $a, b, c \in \mathbb{R}$ are parameters is known as the *Chen system* [Chen & Ueta, 1999]. It exhibits chaotic phenomena which resembles some familiar features from both the Lorenz and the Rössler attractors, for suitable choices of the parameters. Despite of its similar structure to the Lorenz system, it is not topologically equivalent. This is why Lü and Chen investigated the real differential system

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= -xz + cy, \\ \dot{z} &= xy - bz, \end{aligned} \tag{2}$$

where $a, b, c \in \mathbb{R}$ are parameters, which now is usually called the *Lü system* [Lü & Chen, 2002]. The Lü system connects the Lorenz system and the Chen system and represents a transition from one to the other. For more details see [Lü & Chen, 2002]. Moreover, recently Lü and Zhang [Lü & Zhang, 2007] and Lü [Lü, 2009] characterize the invariant algebraic surfaces of the Chen and of the Lü systems, respectively. Furthermore, Lü in [Lü, 2007] characterize the Darboux first integrals of the Chen system. These recent years the dynamics of the Chen system has been analyzed from many different points of view. See for instance [Bashkirtseva et al., 2010, Cafagna & Grassi, 2008, Cai et al., 2009, Cao et al., 2008, Chen & Wang, 2007, Chen & Zhou, 2009, Chowdhury & Hashim, 2009, Denquan & Zhixiang, 2009, Fallahi et al., 2008, Hou et al., 2008, Mahmoud et al., 2007, Yao & Liu, 2010].

The vector field associated to (1) is

$$X = a(y-x)\frac{\partial}{\partial x} + ((c-a)x - xz + cy)\frac{\partial}{\partial x} + (xy - bz)\frac{\partial}{\partial x},$$

and the vector field associated to (2) is

$$X = a(y-x)\frac{\partial}{\partial x} + (-xz + cy)\frac{\partial}{\partial x} + (xy - bz)\frac{\partial}{\partial x}.$$

Let U be an open subset of $\mathbb{R}^3$ such that $\mathbb{R}^3 \setminus U$ has zero Lebesgue measure. We say that a real function $H = H(x,y,z)\colon U \subset \mathbb{R}^3 \to \mathbb{R}$ is a *first integral* if $H(x(t), y(t), z(t))$ is constant for all values of the solution $(x(t), y(t), z(t))$ of X where it is defined, i.e., if $XH = 0$.

Two functions $f_1(x,y,z)$ and $f_2(x,y,z)$ are *independent* if their gradients are linearly independent vectors for all $(x,y,z) \in \mathbb{R}^3$ except perhaps for a set of zero Lebesgue measure.

If the vector field X has two independent first integrals H_1 and H_2, we say that it is *completely integrable*. In this case the orbits of X are contained in the curves $\{H_1(x,x,y,z) = h_1\} \cap \{H_2(x,y,z) = h_2\}$, when h_1 and h_2 vary in $\mathbb{R}$.

When $a = 0$ the Chen and Lü differential systems are completely integrable as it is shown in the following result. Its proof is tedious but it follows easily from direct computations.

Theorem 1.1. *When $a = 0$ the Chen and Lü systems are integrable with the first integrals $H_1 = x$ and H_2.*

(a) For the Chen system

$$\begin{aligned} H_2 = \quad & x \exp\left(\frac{2(b-c)\arctan\left(\frac{b-c+\frac{2A}{bz-xy}}{\sqrt{4B-(b-c)^2}}\right)}{\sqrt{4B-(b-c)^2}} \right) \\ & (B(xy^2 + xz^2 - cxz - (b+c)yz) + \\ & c(c-b)x^2y + cx(b^2z - x^2z + cx^2)), \end{aligned}$$

where $A = c(x^2+bz) - x^2z$ and $B = x^2 - bc$.

(b) For the Lü system

$$\begin{aligned} H_2 = \quad & z \exp\left(\frac{-(b-c)\arctan\left(\frac{2xy-(b+c)z}{z\sqrt{4x^2-(b+c)^2}}\right)}{\sqrt{4x^2-(b+c)^2}} \right) \\ & \sqrt{x + \frac{xy^2}{z^2} - \frac{(b+c)y}{z}}. \end{aligned}$$

In view of Theorem 1.1 from now on we consider only the case $a \neq 0$ in both the Chen and the Lü systems.

Our main result for the Chen and Lü systems are the following two theorems.

Theorem 1.2. *The Chen differential system with $a \neq 0$ has a polynomial first integral if and only if $b = c = 0$. In this case any polynomial first integral is a polynomial in the variable $y^2 + z^2 + 2az$.*

Theorem 1.2 provides a counterexample to statement (b) of Theorem 1 of [Lü, 2007], where the author claims that the Chen system has no Darboux polynomial first integrals and of course the polynomial first integral $y^2 + z^2 + 2az$ is a Darboux polynomial for the Chen system.

Theorem 1.3. *The Lü system with $a \neq 0$ has a polynomial first integral if and only if $b = c = 0$. In this case any polynomial first integral is a polynomial in the variable $y^2 + z^2$.*

The result of Theorem 1.3 is well known, see [Llibre & Zhang, 2002, Lü, 2009].

The proof of Theorem 1.2 is given in section 2, but we only provide an sketch of the proof of Theorem 1.3 in section 3. We note that in the proof of Theorems 1.2 we use similar arguments to the ones used in the papers [Lü & Zhang, 2007] and [Lü, 2009].

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. For simplifying the computations we introduce the following weight change of variables

$$x = \mu^{-1}X, \quad y = \mu^{-2}Y, \quad z = \mu^{-2}Z, \quad t = \mu T, \tag{3}$$

with $\mu \in \mathbb{R} \setminus \{0\}$. Then the Chen system (1) becomes

$$\begin{aligned} X' &= a(Y - \mu X), \\ Y' &= -XZ + \mu c Y + \mu^2(c-a)X, \\ Z' &= XY - \mu b Z, \end{aligned}$$

where the prime denotes the derivative of the variables with respect to T.

A polynomial $G(X,Y,Z)$ is said to be *weight homogeneous* of degree $m \in \mathbb{N}$ with respect to the weight exponent $s = (s_1, s_2, s_3)$ for all $\mu \in \mathbb{R} \setminus \{0\}$ we have

$$G(\mu^{s_1} X, \mu^{s_2} Y, \mu^{s_3} Z) = \mu^m G(X, Y, Z).$$

Here $\mathbb{N}$ is the set of positive integers.

Let $g(x, y, z)$ be a polynomial first integral of system (1). Set

$$\begin{aligned} G(X,Y,Z) = & \ \mu^m g(\mu^{-1}X, \mu^{-2}Y, \mu^{-2}Z) \\ = & \sum_{i=0}^{m} \mu^i G_i(X,Y,Z), \end{aligned} \quad (4)$$

where G_i is the weight homogeneous part with weight degree-$m-i$ of G, and m is the weight degree of G with the weight exponent $s = (1, 2, 2)$.

From the definition of a polynomial first integral we have

$$\begin{aligned} & a(y-\mu x) \sum_{i=0}^{m} \mu^i \frac{\partial G_i}{\partial x} + \\ & (-xz + \mu cy + \mu^2 (c-a)x) \sum_{i=0}^{m} \mu^i \frac{\partial G_i}{\partial y} + \\ & (xy - \mu bz) \sum_{i=0}^{m} \mu^i \frac{\partial G_i}{\partial z} = 0, \end{aligned} \quad (5)$$

where we still use x, y, z instead of X, Y, Z.

Equating in (5) the terms with μ^i for $i = 0, 1, \ldots, m+2$, we get

$$\begin{aligned} L[G_0] = & \ 0, \\ L[G_1] = & \ ax\frac{\partial G_0}{\partial x} - cy\frac{\partial G_0}{\partial y} + bz\frac{\partial G_0}{\partial z}, \\ L[G_j] = & \ ax\frac{\partial G_{j-1}}{\partial x} - cy\frac{\partial G_{j-1}}{\partial y} + \\ & bz\frac{\partial G_{j-1}}{\partial z} - (c-a)x\frac{\partial G_{j-2}}{\partial y}, \end{aligned} \quad (6)$$

for $j = 2, 3, \ldots, m+2$, where $G_j = 0$ for $j > m$ and L is the linear partial differential operator of the form

$$L = ay\frac{\partial}{\partial x} - xz\frac{\partial}{\partial y} + xy\frac{\partial}{\partial z}. \quad (7)$$

The characteristic equations associated to the linear partial differential operator are

$$\frac{dx}{dz} = \frac{a}{x}, \quad \frac{dy}{dz} = -\frac{z}{y}.$$

This system has the general solution

$$x^2 - 2az = d_1, \quad y^2 + z^2 = d_2,$$

where d_1 and d_2 are constants of integration. According to this, we make the change of variables

$$u = x^2 - 2az, \quad v = y^2 + z^2, \quad w = z. \quad (8)$$

Its inverse transformation is

$$x = \pm\sqrt{u + 2aw}, \quad y = \pm\sqrt{v - w^2}, \quad z = w. \quad (9)$$

In the following, for simplicity, we only consider the case $x = \sqrt{u + 2aw}$, $y = \sqrt{v - w^2}$, $z = w$. Under the changes of variables (8) and (9), the first equation of (6) becomes the following ordinary differential equation (for fixed u, v):

$$\sqrt{u + 2aw}\,\sqrt{v - w^2}\,\frac{d\bar{G}_0}{dw} = 0,$$

where $\bar{G}_0$ is G_0 written in the variables u, v and w. In what follows we always use the notation $\bar{\theta}$ to denote $\theta(x, y, z)$ written in the variables u, v, w. The last equation has the following general solution $\bar{G}_0 = \bar{T}_0(u, v)$, where $\bar{T}_0$ is an arbitrary smooth function in u and v. So,

$$G_0(x, y, z) = \bar{G}_0(u, v, w) = T_0(x^2 - 2az, y^2 + z^2).$$

Since G_0 is a weight homogeneous polynomial and the weight degrees of u and v in the variables x, y, z are 2 and 4, respectively, G_0 should have weight degree either $m = 4n$, or $m = 4n - 2$ for some convenient $n \in \mathbb{N}$. So G_0 has the form

$$G_0 = \sum_{i=0}^{n} a_i (x^2 - 2az)^{2i} (y^2 + z^2)^{n-i}, \quad (10)$$

with the weight degree $4n$, or

$$G_0 = \sum_{i=1}^{n} a_i (x^2 - 2az)^{2i-1} (y^2 + z^2)^{n-i}, \quad (11)$$

with the weight degree $4n - 2$.

Case 1: G_0 has the form (11). Substituting G_0 into the second equation of (6), we can prove that

$$\begin{aligned} L[G_1] = & \sum_{i=1}^{n} [2a(2i-1) - 2c(n-i)] \\ & a_i (x^2 - 2az)^{2i-1} (y^2 + z^2)^{n-i} \\ & + \sum_{i=1}^{n} [(4a^2 - 2ab)(2i-1)] \\ & a_i (x^2 - 2az)^{2i-2} (y^2 + z^2)^{n-i} z \\ & + \sum_{i=1}^{n} 2(b+c)(n-i) \\ & a_i (x^2 - 2az)^{2i-1} (y^2 + z^2)^{n-i-1} z^2. \end{aligned}$$

Using the transformations (8) and (9) and working in a similar way to solve $\bar{G}_0$ we get the following ordinary differential equation (for fixed u and v):

$$
\begin{aligned}
&\sqrt{u+2aw}\,\sqrt{v-w^2}\,\frac{d\bar{G}_1}{dw} = \\
&\sum_{i=1}^{n}[2a(2i-1)-2c(n-i)]a_i u^{2i-1}v^{n-i} \\
&+\sum_{i=1}^{n}[(4a^2-2ab)(2i-1)]a_i u^{2i-2}v^{n-i}w \\
&+\sum_{i=1}^{n}2(b+c)(n-i)a_i u^{2i-1}v^{n-i-1}w^2.
\end{aligned}
$$

Using

$$
\frac{d}{dw}(\sqrt{u+2aw}\sqrt{v-w^2}) = \frac{-uw+a(v-3w^2)}{\sqrt{u+2aw}\sqrt{v-w^2}},
$$

it is easy to deduce that the integration of the previous equation with respect to w is

$$
\begin{aligned}
\bar{G}_1 = \ & \sum_{i=1}^{n}\frac{1}{a}[2a(2i-1)-2c(n-i)] \\
& a_i u^{2i-1}v^{n-i-1}\sqrt{u+2aw}\sqrt{v-w^2} \\
& +\sum_{i=0}^{n}[(4a^2-2ab)(2i+1)a_{i+1} \\
& +(2a(2i-1)-2c(n-i))a_i]u^{2i}v^{n-i-1}\cdot \\
& \cdot\int\frac{w\,dw}{\sqrt{u+2aw}\sqrt{v-w^2}} \\
& +\sum_{i=1}^{n}[2(b+c)(n-i)+\frac{3}{a}(2a(2i-1) \\
& -2c(n-i))]a_i u^{2i-1}v^{n-i-1}\cdot \\
& \cdot\int\frac{w^2\,dw}{\sqrt{u+2aw}\sqrt{v-w^2}}+\bar{T}_1(u,v),
\end{aligned}
$$

where $\bar{T}_1(u,v)$ is an arbitrary smooth function in u and v. Since $\bar{G}_1$ is a weight homogeneous polynomial of weight degree $4n-3$, we must have $\bar{T}_1(u,v)=0$, and

$$
\begin{aligned}
&(4a^2-2ab)(2i+1)a_{i+1}+ \\
&(2a(2i-1)-2c(n-i))a_i=0, \\
&i=0,\ldots,n, \\
&(2(b+c)(n-i)+\frac{3}{a}(2a(2i-1)- \\
&2c(n-i)))a_i=0, \\
&i=1,\ldots,n,
\end{aligned}
\tag{12}
$$

where $a_0=a_{n+1}=0$. It is easy to prove that since $a\neq 0$, conditions (12) are equivalent to one of the following conditions, either $b=2a=-c$, $n=1/2$ and there exists $i_0\in\{1,\ldots,n-1\}$ such that $a_{i0}\neq 0$, or $b=2a$, $n=1/2$, $G_0=a_n(x^2-2az)^{2n-1}$, or $G_0=0$. Since n is a natural number the two first cases are not possible, and therefore only the third case is possible. In this last case it is obvious that the Chen system has no polynomial first integral of the given form.

Case 2: G_0 has the form (10). Substituting G_0 into the second equation of (6) we get that

$$
\begin{aligned}
L[G_1] = \ & \sum_{i=0}^{n}[4ai-2c(n-i)]a_i \\
& (x^2-2az)^{2i}(y^2+z^2)^{n-i} \\
& +\sum_{i=0}^{n}2[4a^2-2ab]ia_i \\
& (x^2-2az)^{2i-1}(y^2+z^2)^{n-i}z \\
& +\sum_{i=0}^{n}2(b+c)(n-i)a_i \\
& (x^2-2az)^{2i}(y^2+z^2)^{n-i-1}z^2.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\bar{G}_1 = \ & \sum_{i=0}^{n}\frac{1}{a}[4ai-2c(n-i)]a_i \\
& u^{2i}v^{n-i-1}\sqrt{u+2aw}\sqrt{v-w^2} \\
& +\sum_{i=0}^{n-1}\Big(\frac{1}{a}[4ai-2c(n-i)]a_i+ \\
& 4a(2a-b)(i+1)]a_{i+1}\Big)u^{2i+1}v^{n-i-1}\cdot \\
& \cdot\int\frac{w}{\sqrt{u+2av}\sqrt{v-w^2}}\,dw \\
& +4ana_n u^{2n+1}v^{-1}\int\frac{w}{\sqrt{u+2av}\sqrt{v-w^2}}\,dw \\
& +\sum_{i=0}^{n-1}[12ai+2(b-2c)(n-i)]a_i u^{2i}v^{n-i-2}\cdot \\
& \cdot\int\frac{w^2}{\sqrt{u+2av}\sqrt{v-w^2}}\,dw+\bar{T}_2(u,v).
\end{aligned}
$$

In order that G_1 be a weight homogeneous polynomial of weight degree $4n-1$, we must have $\bar{T}_2(u,v)=0$, and

$$
\begin{aligned}
&4ana_n=0, \\
&(4ai-2c(n-i))a_i+ \\
&\quad 4a^2(2a-b)(i+1)a_{i+1}=0, \\
&(12ai+2(b-2c)(n-i))a_i=0,
\end{aligned}
\tag{13}
$$

with $i=0,\ldots,n-1$. We can easily prove that since $a\neq 0$, conditions (13) are equivalent to one of the following conditions:

$$\begin{aligned}
&b = 2a,\ n = 0,\ a_n \neq 0,\ \text{and}\\
&\quad a_i = 0 \text{ for } i = 0, 1, \ldots, n-1;\\
&b = 2a = -c,\ n = 0,\ \text{and there exists}\\
&\quad i_0 \in \{0, \ldots, n-1\} \text{ such that } a_{i_0} \neq 0;\\
&b = -c \neq 2a,\ 2cn = 0,\ a_0 \neq 0,\ \text{and}\\
&\quad a_i = 0 \text{ for } i = 1, \ldots, n;\\
&b = 6a + 2c \neq -c,\ n = 0,\\
&\quad a_{n-i}(-4a^2)^i \begin{pmatrix} n \\ i \end{pmatrix} a_n,\ \text{and } a_n \neq 0.
\end{aligned}$$

Since $n \geq 1$ the unique possibility is $b = c = 0$, $a \neq 0$, $a_0 \neq 0$ and $a_i = 0$ for $i = 1, \ldots, n$. Since $G_0 = a_0(y^2 + z^2)^n$ and $G_1 = 0$, from (6) we get

$$\begin{aligned}
G_2 = &\ 2ana_0(y^2+z^2)^{n-1}z\\
&+\sum_{i=1}^{n} a_i^{(2)}(x^2-2az)^{2i-1}(y^2+z^2)^{n-i}\\
= &\ a_0\begin{pmatrix} n \\ 1 \end{pmatrix}(y^2+z^2)^{n-1}(2az)\\
&+\sum_{i=1}^{n} a_i^{(2)}(x^2-2az)^{2i-1}(y^2+z^2)^{n-i}.
\end{aligned}$$

Then again from (6) with $j = 3$, we can prove that

$$\begin{aligned}
\bar{G}_3 = &\ \sum_{i=1}^{n}(2a(2i-1))a_i^{(2)}u^{2i-1}v^{n-i}\cdot\\
&\cdot\int\frac{dw}{\sqrt{u+2aw}\sqrt{v-w^2}}\\
&+4a^2\sum_{i=1}^{n}(2i-1)a_i^{(2)}u^{2i-2}v^{n-i}\cdot\\
&\cdot\int\frac{w}{\sqrt{u+2aw}\int v-w^2}\,dw+\bar{T}_3(u,v).
\end{aligned}$$

In order that G_3 be a weight homogeneous polynomial of weight degree $4n-3$, we must have $\bar{T}_3(u,v) = 0$, and

$$\begin{aligned}
&2a(2i-1)a_i^{(2)} = 0, \quad i = 1, \ldots, n,\\
&4a^2a_1^{(2)} = 0,\\
&2a^2(2i-1)a_i^{(2)} = 0, \quad i = 2, \ldots, n.
\end{aligned}\tag{14}$$

Consequently $G_3 = 0$. Indeed, from conditions (14) and since $a \neq 0$ is equivalent to $a_i^{(2)} = 0$ for $i = 1, \ldots, n$, it follows that $G_3 = 0$. Hence

$$G_2 = a_0\begin{pmatrix} n \\ 1 \end{pmatrix}(y^2+z^2)^{n-1}(2az).$$

Furthermore, substituting G_2 and G_3 in (6) with $j = 4$, we get that

$$G_4 = a_0n(n-1)(y^2+z^2)^{n-2}(2a)^2z+\sum_{i=0}^{n}a_i^{(4)}u^{2i}v^{n-i-1}.$$

Substituting G_3 and G_4 into (6) with $j = 5$, we obtain that

$$\begin{aligned}
\bar{G}_5 = &\ 4aia_i^{(4)}u^{2i}v^{n-i-1}\int\frac{1}{\sqrt{u+2aw}\sqrt{v-w^2}}\,dw\\
&+\sum_{i=0}^{n}8a^2ia_i^{(4)}u^{2i-1}v^{n-i-1}\cdot\\
&\cdot\int\frac{w}{\sqrt{u+2aw}\sqrt{v-w^2}}\,dw+\bar{T}_5(u,v).
\end{aligned}$$

Since G_5 must be a weight homogeneous polynomial of degree $4n-5$, we must have $\bar{T}_5(u,v) = 0$, and

$$4aia_i^{(4)} = 0, \quad 8aia_i^{(4)} = 0, \quad i = 0, \ldots, n.$$

These equations imply that $i = 0$ and $a_i^{(4)} = 0$ for $i = 1, \ldots, n$. Thus we have that $G_5 = 0$ and

$$G_4 = a_0\begin{pmatrix} n \\ 2 \end{pmatrix}(y^2+z^2)^{n-2}(2a)^2.$$

By recursive calculations we can prove that

$$\begin{aligned}
G_{2i} &= a_0\begin{pmatrix} n \\ i \end{pmatrix}(y^2+z^2)^{n-i}(2az)^i,\\
G_{2i+1} &= 0,
\end{aligned}$$

for $i \geq 0$ and $2i \leq m$. Hence the polynomial first integral is

$$\begin{pmatrix} n \\ i \end{pmatrix}(y^2+z^2)^{n-i}(2az)^i = a_0(y^2+z^2+2az)^n.$$

The generator of the polynomial first integral is $y^2 + z^2 + 2az$. This completes the proof of Theorem 1.2.

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3. We introduce the weight change of variables in (3) and system (2) becomes

$$\begin{aligned}
X' &= a(Y-\mu X),\\
Y' &= -XZ+\mu cY,\\
Z' &= XY-\mu bZ,
\end{aligned}$$

where again the prime denotes the derivative with respect to T.

Let $g(x, y, z)$ be a polynomial first integral of system (2) and set $G(X, Y, Z)$ as in (4). We have

$$a(y-\mu x)\sum_{i=0}^{m}\mu^i\frac{\partial G_i}{\partial x}+(-xz+\mu cy)\sum_{i=0}^{m}\mu^i\frac{\partial G_i}{\partial y}$$
$$+(xy-\mu bz)\sum_{i=0}^{m}\mu^i\frac{\partial G_i}{\partial z}=0,$$

where again we still use x, y, z instead of X, Y, Z.

From the previous equality the coefficients of μ^i for $i = 0, 1, \ldots, m+1$ are

$$L[G_0]=0,$$
$$L[G_j]=ax\frac{\partial G_{j-1}}{\partial x}-cy\frac{\partial G_{j-1}}{\partial y}+bz\frac{\partial G_{j-1}}{\partial z}, \tag{15}$$

for $j = 2, 3, \ldots, m+1$, where $G_j = 0$ for $j > m$ and L is the linear partial differential operator in (7). Proceeding as in the proof of Theorem 1.2 we get that G_0 has either the form (10) with weight degree $4m$, or the form (11) with weight degree $4m-2$. Proceeding analogously to Case 1 in the proof of Theorem 1.2 we get that the Lü system has no polynomial first integrals if G_0 is of the form (11). Again, proceeding as in Case 2 of the proof of Theorem 1.2 we get that $G_1 = 0$ and again from (13) and since $a \neq 0$, the unique possibility is $b = c = 0$, $a_0 \neq 0$ and $a_i = 0$ for $i = 1, \ldots, n$. Therefore $G_0 = a_0(y^2+z^2)^n$, and

$$G_2 = \sum_{i=1}^{n} a_i^{(2)}(x^2-2az)^{2i-1}(y^2+z^2)^{n-i}$$
$$= \sum_{i=1}^{n} a_i^{(2)}(x^2-2az)^{2i-1}(y^2+z^2)^{n-i}.$$

Then from (15) with $j = 3$, and proceeding as in the proof of Theorem 1.2 we get that $a_i^{(2)} = 0$ for $i = 1, \ldots, n$, and consequently $G_2 = G_3 = 0$. By recursive calculations, and proceeding as in Case 2 in the proof of Theorem 1.2 we can prove that $G_i = 0$ for $i = 1, \ldots, n$. Hence the generator of the polynomial first integrals is y^2+z^2. This completes the proof of Theorem 1.3.

Acknowledgements

The first author is partially supported by the MICIIN/FEDER grant MTM2008–03437, the Generalitat de Catalunya grant 2009SGR-410 and ICREA Academia. The second author is partially supported by FCT through CAMGDS, Lisbon.

References

Bashkirtseva, I., Chen, G., & Ryashko, L., [2010] "Analysis of stochastic cycles in the Chen system," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **20**, 1439–1450.

Cafagna, D., & Grassi, G.Y., [2008] "Bifurcation and chaos in the fractional-order Chen system via a time-domain approach," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **18**, 1845–1863.

Cai, N., Jing, Y.W., & Zhang, S.Y., [2009] "Anti-synchronization between hyper-chaotic Chen system and hyper-chaotic Lorenz system," *J. Northeast. Univ. at. Sci.* **30**, 313–317.

Cao, J., Chen, C., & Zhang, X., [2008] "The Chen system having an invariant algebraic surface," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **18**, 3753–3758.

Chen, G., & Ueta, T., [1999] "Yet another chaotic attractor," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **9**, 1465–1466.

Chen, L., & Wang, D.S., [2007] "Adaptive tracking control of the Chen system," *Acta Phys. Sinica* **56**, 5661–5664.

Chen, T., & Zhou, S.F., [2009] "Synchronization in lattices of coupled non–autonomous Chen system via Lyapunov function," *J. Shanghai Univ.* **13**, 242–247.

Chowdhury, M.S., & Hashim, I., [2009] "Application of mutistage homotopy-perturbation method for the solutions of the Chen system," *Nonlinear Anal. Real World Appl.* **10**, 381–391.

Darboux, G., [1878] "Mémoire sur les équations différentielles du primer ordre et du premier degrée (Mélanges)," *Bull. Sci. Math. 2éme Série* **2**, 60–96, 123–144, 151–200.

Denquan, L., & Zhixiang, Y., [2009] "Connecting the Lorenz and Chen systems via nonlinear control," *Commun. Nonlinear Sci. Numer. Simul.* **14**, 655–667.

Fallahi, K., Raoufi, R., & Khoshbin, H., [2008] "An application of Chen system for secure chaotic communication based on extended Kalmar filter and multi-shift cipher algorithm," *Commun. Nonlinear Sci. Numer. Simul.* **13**, 763–781.

Forsyth, A.R., [1900] *Theory of differential Equations* Cambridge University Press, Cambridge.

Hou, Z., Kang, N., Kong, X., & Chen, G., [2008] "On the equivalence of Lorenz system and Chen system," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **13**, 763–781.

Jouanolou, J.P., [1979] *Equations de Plaff algébriques* Lecture Notes in Mathematics 708 Springer, New York.

Llibre, J., & Zhang, X., [2002] "Invariant algebraic surfaces of the Lorenz system," *J. Math. Phys.* **43**, 1622–1645.

Lü, T., [2007] "Exponential factors and non–Darbouxian integrability of the Chen system," *Ann. of Diff. Eqs.* **23**, 304–311.

Lü, T., [2009] "Darboux polynomials and non-algebraic integrability of the Lü system," *Ann. Differential Equations* **25**, 420–431.

Lü, J. & Chen, G., [2002] "A new chaotic attractor coined," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **12**, 659–661.

Lü, T. & Zhang, X., [2007] "Darboux polynomials and algebraic integrability of the Chen system," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **17**, 2739–2748.

Mahmoud, G.M., Mahmoud, E.E., & Ahmed, M.E., [2007] "A hyperchaotic complex Chen and its dynamics," *Internat. J. Appl. Math. Stat.* **12**, 90–100.

Poincaré, H., [1891] "Sur l'intégration des équations différentielles du primer ordre et du primer degreée I," *Rendicontin del Circolo Matematico di Palermo* **5**, 161–191.

Yao, H., & Liu, Z., [2010] "The application of contraction theory in synchronization of coupled Chen systems," *Int. J. Nonlinear Sci.* **9**, 72–77.