Erratum: Random Attractors of Stochastic Lattice Dynamical Systems Driven by Fractional Brownian Motions. [Int. J. Bifurcation Chaos 23, 1350041 (2013)]

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In the recent paper [1] we have obtained the existence of a random attractor which turns out to be a singleton sets random attractor. The result relies on the theory in [2], that is the existence of a compact pullback absorbing set, which just deals with random dynamical systems in finite dimensional cases.

However, in the first-order lattice dynamical systems driven by fractional Brownian motions, we considered it in the framework of infinite dimensional square-summable sequences space ℓ^2. The main result need few changes because we cannot obtain it based on Theorem 1 in [1] but on the assumptions of the nonlinear function f. The random dynamical system generated by the system has a unique random equilibrium, all solutions converge pathwise to each other, so the random attractor, which consists of a unique random equilibrium, is proven to be a singleton sets random attractor. With the misused theory several important parts of the paper should be corrected as:

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- The last sentence in the Abstract section should be "In our case, the random dynamical system has a unique random equilibrium, which constitutes a singleton sets random attractor."
- In page 7, from line 15 to 29, the main statement should be adjusted to "Assume that the conditions on f are satisfied. Then the random dynamical system φ has a unique random equilibrium, which constitutes a singleton sets random attractor."

References

[1] Gu, A. Random Attractors of Stochastic Lattice Dynamical Systems Driven by Fractional Brownian Motions," *Int. J. Bifurcation Chaos* **23**, 1350041 (2013).

[2] Arnold, L. *Random Dynamical systems*, Springer-Verlag, Berlin (1998).

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RANDOM ATTRACTORS OF STOCHASTIC LATTICE DYNAMICAL SYSTEMS DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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This paper is devoted to considering the stochastic lattice dynamical systems (SLDS) driven by fractional Brownian motions with Hurst parameter bigger than 1/2. Under usual dissipativity conditions these SLDS are shown to generate a random dynamical system for which the existence and unique of a random attractor is established. Furthermore, the random attractor is in fact a singleton sets random attractor.

Keywords: Stochastic lattice dynamical system; fractional Brownian motion; random dynamical systems; random attractor.

1. Introduction

The purpose of this paper is to investigate the long-term behavior for the following SLDS

$$\frac{du_i(t)}{dt} = \kappa(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f_i(u_i) + g_i + \sigma_i \frac{d\beta_i^H(t)}{dt},\ i \in \mathbb{Z}, \tag{1}$$

where $\mathbb{Z}$ denotes the integer set, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, κ and λ are positive constants, f_i are smooth functions satisfying some dissipative conditions, $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2$, $\sigma = (\sigma_i)_{i\in\mathbb{Z}} \in \ell^2$, and $\{\beta_i^H : i \in \mathbb{Z}\}$ are independent two-sided fractional Brownian motions (fBms) with Hurst parameter $H \in (1/2, 1)$.

Recently, the dynamics of infinite lattice dynamical systems have drawn much attention of mathematicians and physicists, see e.g. [Bates et al., 2001, 2006; Zhou , 2002, 2003, 2004; Zhou &Shi , 2006; Wang , 2006; Lv & Sun, 2006a,b; Huang, 2007; Wang et al., 2008; Caraballo & Lu , 2008; Zhao & Zhou, 2009; Wang et al., 2010; Han et al., 2011; Han , 2011a,b] and the references therein. Since most of the realistic systems involve noises which may play an important role as intrinsic phenomena rather than just compensation of defects in deterministic models, SLDS then arise naturally while these random influences or uncertainties are taken into account.

Since Bates et al. [Bates et al., 2006] initiated the study of SLDS, many works have been done regarding the existence of global random attractors for SLDS with white noises on lattices $\mathbb{Z}$ (see e.g. [Lv & Sun, 2006a,b; Huang, 2007; Wang et al., 2008; Caraballo & Lu , 2008; Zhao & Zhou, 2009; Wang et al., 2010]). Later, the existence of global random attractors have been extended to other SLDS with additive white noises, for example, first-order SLDS on $\mathbb{Z}^k$ [Lv & Sun, 2006a], stochastic Ginzburg-Landau lattice equations [Lv & Sun, 2006b], stochastic FitzHugh-Nagumo lattice equations [Huang, 2007; Wang et al., 2008], second-order stochastic lattice systems [Wang et al., 2010] and the first (or second)-order SLDS with a multiplicative white noise [Caraballo & Lu , 2008; Han , 2011a]. Zhao and Zhou [Zhao & Zhou, 2009] gave some sufficient conditions for the existence of a global random attractor for general SLDS in the non-weighted space $\mathbb{R}$ of infinite sequences and provided an application to damped sine-Gordon lattice system with additive noises. Very recently, Han, et al. [Han et al., 2011] provided some sufficient conditions for the existence of global compact random attractors for general random dynamical systems in weighted space ℓ^p_ρ ($p \geqslant 1$) of infinite sequences, and their results are applied to second-order SLDS in [Han , 2011b].

However, as can be seen that all the works above are considered in the frameworks of the classical Itö theory of Brownian motion. There are no results on these systems when they are perturbed by a fractional Brownian motion (fBm) to the best of our knowledge. FBm appears naturally in the modeling of many complex phenomena in applications when the systems are subject to "rough" external forcing. An fBm is a stochastic process which differs significantly from the standard Brownian motion and semi-martingales, and other classically used in the theory of stochastic process. As a centered Gaussian process, it is characterized by the stationarity of its increments and a medium- or long-memory property. It also exhibits power scaling with exponent H. Its paths are Hölder continuous of any order $H' \in (0, H)$. An fBm is not a semi-martingale nor a Markov process. So an fBm is the good candidate to model random long term influences in climate systems, hydrology, medicine and physical phenomena. For more details on fBm, we can refer to the books [Biagini et al. , 2008; Mishura , 2008] for its further development.

The goal of this article is to establish the existence of a random attractor for SLDS with the nonlinearity f under some dissipative conditions and driven by fBms with Hurst parameter $H \in (1/2, 1)$. By borrowing the main ideas of the proofs in [Garrido-Atienza et al., 2009], we firstly define a random dynamical system by using the possibility of a pathwise interpretation of the stochastic integral with respect to the fBms. This method is based on the fact that a stochastic integral with respect to an fBm with Hurst parameter $H \in (1/2, 1)$ can be defined by a generalized pathwise Riemann-Stieltjes integral (see e.g. [Zahle , 1998; Decreusefond & Ustunel , 1999; Nualart & Rascanu, 2002; Tindel et al. , 2003]). And then we show the existence of a pullback absorbing set for the random dynamical system which achieved by means of a fractional Ornstein-Uhlenbeck transformation and Gronwall lemma. Since every trajectory of the solutions of system (1) cannot be differentiated, we have to consider the difference between any two solutions among them, which is pathwise differentiable (see [Garrido-Atienza et al., 2009]). Due to the stationarity of the fractional Ornstein-Uhlenbeck solution, we get the random attractor finally. All solutions converge pathwise to each other, so the random attractor is proven to be a singleton sets random attractor.

The paper is organized as follows. In Sec. 2, we recall some basic concepts on random dynamical systems. In Sec.3, we give a unique solution to system (1) and make sure that the solution generates a random dynamical system. We establish the main result, that is, the random attractor generated by equation (1) turns out to be a singleton sets random attractor in Sec.4.

2. Random dynamical systems and Random attractor

In this section, we introduce some basic concepts related to random dynamical systems and random attractor, which are taken from [Crauel et al., 1997; Arnold, 1998].

Let $(\mathbb{E}, \|\cdot\|_{\mathbb{E}})$ be a separable Hilbert space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Definition 2.1. A metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ with two-sided $\mathbb{R}$ consists of a measurable flow

$$\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F}),$$

where the flow property for the mapping θ holds for the partial mappings $\theta_t = \theta(t, \cdot)$:

$$\theta_t \circ \theta_s = \theta_t \theta_s = \theta_{t+s}, \quad \theta_0 = \mathrm{id}_\Omega$$

for all $s, t \in \mathbb{R}$, and $\theta\mathbb{P} = \mathbb{P}$ for all $t \in \mathbb{R}$.

Definition 2.2. A continuous random dynamical system (RDS) φ on $\mathbb{E}$ over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is a $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{E}), \mathcal{B}(\mathbb{E}))$-measurable mapping and satisfies

(i) $\varphi(0, \omega)$ is the identity on $\mathbb{E}$;

(ii) $\varphi(t + s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$ for all s, $t \in \mathbb{R}^+$, $\omega \in \Omega$;

(iii) $\varphi(t,\omega)$ is continuous on $\mathbb{E}$ for all $(t,\omega) \in \mathbb{R}^+ \times \Omega$.

A universe $\mathcal{D} = \{D(\omega), \omega \in \Omega\}$ is a collection of nonempty subsets $D(\omega)$ of $\mathbb{E}$ satisfying the following inclusion property: if $D \in \mathcal{D}$ and $D'(\omega) \subset D(\omega)$ for all $\omega \in \Omega$, then $D' \in \mathcal{D}$.

Definition 2.3. A family of $\mathcal{A} = \{A(\omega), \omega \in \Omega\}$ of nonempty measurable compact subsets $\mathcal{A}(\omega)$ of $\mathbb{E}$ is called φ- invariant if $\varphi(t,\omega,\mathcal{A}(\omega)) = \mathcal{A}(\theta_t\omega)$ for all $t \in \mathbb{R}^+$ and is called a random attractor if in addition it is pathwise pullback attracting in the sense that

$$H_d^*(\varphi(t,\theta_{-t}\omega, D(\theta_{-t}\omega)), \mathcal{A}(\omega)) \to 0 \quad \text{as} \quad t \to \infty$$

for all $D \in \mathcal{D}$. Here H_d^* is the Hausdorff semi-distance on $\mathbb{E}$.

Definition 2.4. A random variable $r : \Omega \to \mathbb{R}$ is called tempered if

$$\lim_{t\to\pm\infty} \frac{\log|r(\theta_t\omega)|}{|t|} = 0 \;\; \mathbb{P} - a.s.$$

and a random set $\{D(\omega), \omega \in \Omega\}$ with $D(\omega) \in \mathbb{E}$ is called tempered if it is contained in the ball $\{x \in \mathbb{R} : |x| \leq r(\omega)\}$, where r is a tempered random variable.

Here we will always work with the attracting universe given by the tempered random sets.

Definition 2.5. A family $\hat{B} = \{B(\omega), \omega \in \Omega\}$ is said to be pullback absorbing if for every $D(\omega) \in \mathcal{D}$, there exists $T_D(\omega) \geq 0$ such that

$$\varphi(t,\theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \;\; \forall t \geq T_D(\omega). \qquad (2)$$

Theorem 1. *(See [Schmalfuß, 1992; Crauel et al., 1997; Arnold, 1998].) Let (θ,φ) be a continuous RDS on $\Omega \times \mathbb{E}$. If there exists a pullback absorbing family $\hat{B} = \{B(\omega), \omega \in \Omega\}$ such that, for every $\omega \in \Omega$, $B(\omega)$ is compact and $B(\omega) \in \mathcal{D}$, then the RDS (θ,φ) has a random attractor*

$$\mathcal{A}(\omega) = \bigcap_{\tau>0} \overline{\bigcup_{t\geqslant\tau} \varphi(t,\theta_{-t}\omega) B(\theta_{-t}\omega)}.$$

Note that if the random attractor consists of singleton sets, i.e. $\mathcal{A}(\omega) = \{u^*(\omega)\}$ for some random variable u^*, then $u^*(t)(\omega) = u^*(t)(\theta_t\omega)$ is a stationary stochastic process.

3. SLDS with FBms

We now firstly introduce as an example of metric dynamical system a special noise that is called fractional Brownian motion. Given $H \in (0,1)$, a continuous centered Gaussian process $\beta^H(t), t \in \mathbb{R}$, with the covariance function

$$\mathbf{E}\beta^H(t)\beta^H(s) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}), \;\; t,s \in \mathbb{R}$$

is called a two-sided one-dimensional fractional Brownian motion, and H is the Hurst parameter. For $H = 1/2$, β is a standard Brownian motion, while for $H \neq 1/2$, it is neither a semimartingale nor a Markov process. Moreover,

$$\mathbf{E}|\beta^H(t) - \beta^H(s)|^2 = |t-s|^{2H}, \;\; \text{for all} \;\; s,t \in \mathbb{R}.$$

Here, we assume that $H \in (1/2,1)$ throughout the paper. When $H \in (0,1/2)$ we cannot define the stochastic integral by a generalized Stieljes integral and, therefore, dealing with such values of the Hurst parameter seems to be much more complicated. It is worth mentioning that when $H = 1/2$ the fBm becomes the standard Wiener process, the random dynamical system generated by SLDS has been studied in [Bates et al., 2006].

Using the definition of $\beta^H(t)$, Kolmogorov's theorem ensures that β^H has a continuous version, and almost all the paths are Hölder continuous of any order $H' \in (0,H)$ (see [Kunita, 1990]). Thus, we can consider the canonical interpretation of an fBm: denote $\Omega = C_0(\mathbb{R},\ell^2)$, the space of continuous functions on $\mathbb{R}$ with values in ℓ^2 such that $\omega(0) = 0$, equipped with the compact open topology. Let $\mathcal{F}$ be the associated Borel-σ-algebra and $\mathbb{P}$ the distribution of the fBm β^H, and $\{\theta_t\}_{t\in\mathbb{R}}$ be the flow of Wiener shifts such that

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad t \in \mathbb{R}.$$

Due to [Maslowski & Schmalfuß, 2004; Garrido-Atienza et al., 2010], we know that the quadruple $(\Omega,\mathcal{F},\mathbb{P},\theta)$ is a metric dynamical system which is ergodic. Furthermore, it holds that

$$\begin{aligned} \beta^H(\cdot,\omega) &= \omega(\cdot), \\ \beta^H(\cdot,\theta_s\omega) &= \beta^H(\cdot+s,\omega) - \beta^H(s,\omega) \\ &= \omega(\cdot+s) - \omega(s). \end{aligned} \qquad (3)$$

We consider the SLDS

$$\frac{du_i(t)}{dt} = \kappa(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + f_i(u_i) + g_i + \sigma_i \frac{d\omega_i(t)}{dt}, \;\; i \in \mathbb{Z}, \qquad (4)$$

where $\mathbb{Z}$ denotes the integer set, $u = (u_i)_{i\in\mathbb{Z}} \in \ell^2$, κ and λ are positive constants, f_i are smooth functions satisfying some dissipative conditions, $g = (g_i)_{i\in\mathbb{Z}} \in \ell^2$, $\sigma = (\sigma_i)_{i\in\mathbb{Z}} \in \ell^2$, and $\{\omega_i = \beta_i^H : i \in \mathbb{Z}\}$ are independent two-sided fractional Brownian motions with Hurst parameter $H \in (1/2, 1)$. Then, (4) can be understood as the pathwise Riemann-Stieltjes integral equations

$$u_i(t) = u_i(0) + \int_0^t (\kappa(u_{i-1}(s) - 2u_i(s) + u_{i+1}(s)) \\ -\lambda u_i(s) + f_i(u_i(s)) + g_i)ds + \sigma_i\omega_i(t),\ i \in \mathbb{Z}. \tag{5}$$

Let $e^i \in \ell^2$ denote the element having 1 at position i and 0 at other components. Then

$$W(t) \equiv W(t,\omega) = \sum_{i\in\mathbb{Z}} \sigma_i\omega_i(t)e^i \quad \text{with} \quad (\sigma_i)_{i\in\mathbb{Z}} \in \ell^2, \tag{6}$$

is the special noise with values in ℓ^2 defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Assumptions on the nonlinearity f_i. Assume $f_i : \mathbb{R} \to \mathbb{R}$ to be continuously differentiable. Let f be the Nemytski operator associated with f_i, i.e. for all $x = (x_i)_{i\in\mathbb{Z}} \in \ell^2$, let $f(x) = (f_i(x_i))_{i\in\mathbb{Z}}$. Then we have $f(x) \in \ell^2$ (see [Bates et al., 2006]). Assume f satisfies a one-sided dissipative Lipschitz condition

$$\langle x-y, f(x)-f(y)\rangle \le -L|x-y|^2, \quad \text{for all} \quad x, y \in \ell^2, \tag{7}$$

where $L > 0$, and a polynomial growth condition along with its derivative, i.e.,

$$|f(x)| + |Df(x)| \le K(1+|x|^p), \quad \text{for all} \quad x \in \ell^2, \tag{8}$$

$p \ge 1$ and a positive constant K. Here, we remark that we could consider more general dissipativity conditions, which would lead to nontrivial setvalued random attractors. However, to avoid rather technical details, we will restrict here to the dissipativity conditions (7) and (8).

For $x = (x_i)_{i\in\mathbb{Z}} \in \ell^2$, define A, B, B^* to be linear operators from ℓ^2 to ℓ^2 as follows:

$$(Ax)_i = -x_{i-1} + 2x_i - x_{i+1}, \\ (Bx)_i = x_{i+1} - x_i, \quad (B^*x)_i = x_{i-1} - x_i, \quad i \in \mathbb{Z}.$$

It is easy to show that $A = BB^* = B^*B$, $\langle B^*x, x'\rangle = \langle x, Bx'\rangle$ for all $x, x' \in \ell^2$, which implies that $\langle Ax, x\rangle \ge 0$.

The system (4) with initial values $u_0 = (u_{0,i})_{i\in\mathbb{Z}} \in \ell^2$ may be written as an equation in ℓ^2

$$\begin{aligned} u(t) &= u_0 + \int_0^t (-\kappa Au(s) - \lambda u(s) + f(u(s)) + g)ds \\ &\quad +W(t) \\ &:= u_0 + \int_0^t G(u(s))ds + W(t), \\ &\quad t \ge 0,\ \omega \in \Omega, \end{aligned} \tag{9}$$

where $G(u(t)) := -\kappa Au(t) - \lambda u(t) + f(u(t)) + g$.

Proposition 1. *Let the above assumptions on f be satisfied and $T > 0$. Then system (9) has a unique pathwise solution $u = (u(t))_{t\ge0}$. In addition that the solution satisfies*

$$\sup_{t\in[0,T]} |u(t)| \le M(1 + |u_0| + \sum_{i\in\mathbb{Z}} \sup_{\tau\in[0,T]} |\beta_i^H(\tau)| \\ + \sum_{i\in\mathbb{Z}} \sup_{\tau\in[0,T]} |\beta_i^H(\tau)|^p + |g|), \tag{10}$$

for all $T > 0$, where M is a positive constant and is independent of T.

Proof. Let $v(t) = u(t) - W(t), t \ge 0$, then system (9) has a solution $u = (u(t))_{t\ge0}$ for all $\omega \in \Omega$ if and only if the following equation

$$v(t) = u_0 + \int_0^t (-\kappa Av(s) - \lambda v(s) + f(v(s) + W(s)) \\ +g - \kappa AW(s) - \lambda W(s))ds. \tag{11}$$

has a unique pathwise solution for $t \in [0, T]$. However, since the integrand is pathwise continuous, the fundamental theorem of calculus says that the left hand side of (11) is pahtwise differentiable. Thus, for fixed $\omega \in \Omega$, equation (10) is the pathwise random ordinary differential equation (RODE)

$$\begin{aligned} \frac{dv(t)}{dt}(\omega) &= -\kappa Av(t) - \lambda v(t) + f(v(t) + W(t)) \\ &\quad +g - \kappa AW(t) - \lambda W(t) \\ &:= \tilde{G}(t, v(t))(\omega) := \tilde{G}(v(t) + W(t)), \\ &\quad \text{for} \quad t \ge 0, \quad v_0(\omega) = u_0(\omega). \end{aligned} \tag{12}$$

Since $\tilde{G} : [0,\infty) \times \ell^2 \to \ell^2$ is continuous in t and continuous differentiable in v, this RODE has a unique local solution in a small interval $[0, \tau(\omega)]$, which means that (9) has a unique local solution in the same small interval $[0, \tau(\omega)]$, see e.g. Theorem 2.1.4 in [Stuart & Humphries , 1996].

To see that (9) has a unique solution for every $t \ge 0$, we prove at first the a priori estimate (10). Suppose that $u = (u(t))_{t\ge0}$ solves (9) on the interval $[0, T]$. This implies that the process $v(t) = u(t) - W(t)$ solves (11) on the interval $[0, T]$.

By the one-sided dissipative Lipschitz condition and Hölder inequality, we obtain

$$
\begin{aligned}
&2|v(t)|\frac{d|v(t)|}{dt} = \frac{d}{dt}|v(t)|^2 \\
&= 2\langle v(t), \tilde{G}(v(t)+W(t))\rangle \\
&= 2\langle v(t), -\kappa Av(t) - \lambda v(t) + f(v(t)+W(t)) \\
&\quad +g - \kappa AW(t) - \lambda W(t)\rangle \\
&\le -2(\lambda+L)|v(t)|^2 + 2|f(W(t))||v(t)| \\
&\quad +2(|g| - |\kappa AW(t)| - |\lambda W(t)|)|v(t)|,
\end{aligned}
$$

that is

$$
\begin{aligned}
\frac{d|v(t)|}{dt} &\le -(\lambda+L)|v(t)| + c_0(|g| + \sup_{\tau\in[0,T]}|W(\tau)| \\
&\quad + \sup_{\tau\in[0,T]}|W(\tau)|^p), \quad t\in[0,T],
\end{aligned}
$$

where c_0 is a positive constant depends on $\kappa, \lambda, K, \max_{j\in\mathbb{Z}}|\sigma_j|$ and $\max_{j\in\mathbb{Z}}|g_j|$. By Gronwall lemma it yields that

$$
\begin{aligned}
|v(t)| &\le |u_0|e^{-\lambda t} + \frac{c_0}{\lambda}(1-e^{-\lambda t})(|g| + \sup_{\tau\in[0,T]}|W(\tau)| \\
&\quad + \sup_{\tau\in[0,T]}|W(\tau)|^p), \quad t\in[0,T].
\end{aligned}
$$

Since $\lambda > 0$, we have

$$
\begin{aligned}
|v(t)| &\le |u_0| + \frac{c_0}{\lambda}(|g| + \sup_{\tau\in[0,T]}|W(\tau)| \\
&\quad + \sup_{\tau\in[0,T]}|W(\tau)|^p),
\end{aligned}
$$

that is

$$
\begin{aligned}
\sup_{t\in[0,T]}|v(t)| &\le M(|g| + \sup_{\tau\in[0,T]}|W(\tau)| \\
&\quad + \sup_{\tau\in[0,T]}|W(\tau)|^p), \quad t\in[0,T],
\end{aligned}
$$

where the constant M depends on $\kappa, \lambda, K, \max_{j\in\mathbb{Z}}|\sigma_j|$ and $\max_{j\in\mathbb{Z}}|g_j|$. The result of (10) follows then by using the relation $v(t) = u(t) + W(t)$ for $t\in[0,T]$. As a consequence of estimate (10) the local unique solution to (9) can be extended to a global unique solution (see e.g. Theorem 2.1.4 in [Stuart & Humphries , 1996]). ■

We consider (1) with the linear drift term $G(u(t)) = -\lambda u(t)$, that is

$$du(t) = -\lambda u(t)dt + dW(t), \quad \lambda > 0, \tag{13}$$

which is called the fractional Ornstein-Uhlenbeck process, where $W(t)$ denotes a one-dimensional fractional Brownian motion defined in (6). It has the explicit solution

$$u(t) = u_0 e^{-\lambda t} + e^{-\lambda t}\int_0^t e^{\lambda s}dW(s). \tag{14}$$

Taking the pathwise pullback limit, we get the stochastic stationary solution

$$\bar{u}(t) = e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}dW(s), \quad t\in\mathbb{R}, \tag{15}$$

which is called the fractional Ornstein-Uhlenbeck solution. Based on Lemma 2.6 in [Maslowski & Schmalfuß, 2004] and Lemma 1 in [Garrido-Atienza et al., 2009], we have the property

Lemma 1. *For all $\omega\in\Omega$ the Riemann-Stieltjes integrals*

$$e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}dW(s), \quad t\in\mathbb{R},$$

are well defined in ℓ^2. Moreover, for all $\omega\in\Omega$, we have

$$|e^{-\lambda t}\int_{-\infty}^t e^{\lambda s}dW(s)| \le 4\rho(\omega)(1+|t|)^2, \quad t\in\mathbb{R},$$

where the random constant $\rho(\omega) > 0$ and satisfying

$$|W(t)| \le \rho(\omega)(1+|t|^2), \quad \text{for} \ \ t\in\mathbb{R}, \quad \omega\in\bar{\Omega}. \tag{16}$$

Here, $\bar{\Omega}\in\mathcal{F}$ is a $(\theta_t)_{t\in\mathbb{R}}$-invariant set of full measure.

Proof. We know that $\lambda > 0$, $(\sigma_i)_{i\in\mathbb{Z}}\in\ell^2$. The proof is similar to Lemma 2.6 in [Maslowski & Schmalfuß, 2004] and Lemma 1 in [Garrido-Atienza et al., 2009], thus we omit it here. ■

Since the mapping of θ on $\bar{\Omega}$ has the same properties as the original one if we choose the trace σ-algebra with respect to $\bar{\Omega}$ to be denoted also by $\mathcal{F}$, we can change our metric dynamical system with respect to $\bar{\Omega}$, and still denoted by the symbols $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbf{R}})$.

Now, we will verify that the solution of system (1) generates a continuous RDS.

Proposition 2. *The solution of (1) determinants a continuous random dynamical system $\varphi: \mathbb{R}^+\times\Omega\times\ell^2\to\ell^2$, which is given by*

$$
\begin{aligned}
\varphi(t,\omega,u_0) &= u_0 + \int_0^t(-\kappa Au(s) - \lambda u(s) \\
&\quad + f(u(s)) + g)ds + W(t,\omega).
\end{aligned}\tag{17}
$$

Proof. Note that (3) is satisfied for $\omega \in \Omega$ and from the definition of $(\theta_t)_{t\in\mathbb{R}}$, we have the property

$$W(\tau+t,\omega) = W(\tau,\theta_t\omega)+W(t,\omega) \quad \text{for all} \quad t,\tau \in \mathbb{R}. \tag{18}$$

From Proposition 1 we know that φ solves (1), thus φ is measurable and satisfies $\varphi(0,\omega,\cdot) = \mathrm{id}_{\ell^2}$. It remains to verify the cocycle property in Definition 2.2. Let $t,\tau \in \mathbb{R}^+, \omega \in \Omega$ and $u_0 \in \ell^2$, denote $G(u(t))(\omega) := -\kappa Au(t) - \lambda u(t) + f(u(t)) + g$, it yields from (3) that

$$\begin{aligned}
&\varphi(t+\tau,\omega,u_0)\\
&= u_0 + \int_0^{t+\tau} G(u(s))(\omega)ds + W(t+\tau,\omega)\\
&= u_0 + \int_0^{t} G(u(s))(\omega)ds + W(t,\omega)\\
&\quad + \int_t^{t+\tau} G(u(s))(\omega)ds + W(\tau,\theta_t\omega)\\
&= u(t) + \int_0^{\tau} G(u(s))(\theta_t\omega)ds + W(\tau,\theta_t\omega)\\
&= \varphi(\tau,\theta_t\omega,\cdot)\circ\varphi(t,\omega,u_0),
\end{aligned} \tag{19}$$

which completes the proof. ■

4. Existence of a Random attractor

In this section, we will prove the existence of a random attractor for the RDS defined in Proposition 2. The main proof is based on the first part in section 4 in [Garrido-Atienza et al., 2009].

Let u,w be any two solutions of system (1). Their sample paths are not differentiable, but the difference satisfies pathwise

$$\begin{aligned}
u(t)-w(t) = u_0 - w_0 + \int_0^t &(-\kappa A(u(s)-w(s))\\
&-\lambda(u(s)-w(s))\\
&+(f(u(s))-f(w(s))))ds, \quad t\geq 0,
\end{aligned}$$

and again, since the integrand is pathwise continuous, the fundamental theorem of calculus indicates that the left hand side is pathwise differentiable and satisfies

$$\begin{aligned}
&\frac{d}{dt}(u(t)-w(t))\\
&= -\kappa A(u(s)-w(s)) - \lambda(u(s)-w(s))\\
&\quad +(f(u(s))-f(w(s))), \quad t\geq 0.
\end{aligned} \tag{20}$$

Now we use condition (7) again, we obtain from (20) that

$$\begin{aligned}
&\frac{d}{dt}|u(t)-w(t)|^2\\
&= 2\langle u(t)-w(t), -\kappa A(u(s)-w(s))\\
&\quad -\lambda(u(s)-w(s)) + (f(u(s))-f(w(s)))\rangle\\
&\leq -2\lambda|u(t)-w(t)|^2.
\end{aligned}$$

Thus pathwise we have

$$|u(t)-w(t)| \leq |u_0-w_0|e^{-\lambda t} \to 0, \quad \text{as} \quad t\to\infty.$$

That is to say that all solutions converge pathwise forward to each other in time.

Now, we consider the difference $u(t)-\bar{u}(t)$. This is also pathwise differentiable, since the paths are continuous and satisfy the integral equation

$$\begin{aligned}
u(t)-\bar{u}(t) = u_0 - \bar{u}_0 + \int_0^t &(-\kappa Au(s) - \lambda(u(s)\\
&-\bar{u}(s)) + f(u(s)))ds,
\end{aligned}$$

which is equivalent to the pathwise differential equation

$$\begin{aligned}
&\frac{d}{dt}(u(t)-\bar{u}(t))\\
&= -\kappa Au(s) - \lambda(u(s)-\bar{u}(s)) + f(u(s)), t\geq 0.
\end{aligned}$$

By using (7), it follows that

$$\begin{aligned}
\frac{d}{dt}|u(t)-\bar{u}(t)|^2 &= 2\langle u(t)-\bar{u}(t), -\kappa Au(s)\\
&\quad -\lambda(u(s)-\bar{u}(s)) + f(u(s))\rangle\\
&\leq -2\lambda|u(t)-\bar{u}(t)|^2\\
&\quad +2|u(t)-\bar{u}(t)||f(\bar{u}(t))|,
\end{aligned}$$

that is

$$\frac{d}{dt}|u(t)-\bar{u}(t)| \leq -\lambda|u(t)-\bar{u}(t)| + |f(\bar{u}(t))|,$$

and hence

$$\begin{aligned}
&|u(\omega)-\bar{u}(\omega)|\\
&\leq |u_0(\omega)-\bar{u}_0(\omega)|e^{-\lambda t}\\
&\quad +e^{-\lambda t}\int_0^t e^{\lambda s}|f(\bar{u}(s))(\omega)|ds.
\end{aligned} \tag{21}$$

Let us check that the family of balls centered on $\bar{u}_0(\omega)$ with the random radius

$$\varrho(\omega) := 1 + \int_{-\infty}^0 e^{\lambda s}|f(\bar{u}(s))(\omega)|ds \tag{22}$$

is a pullback absorbing family for the random dynamical system generated by system (1).

Due to the assumptions on f and Lemma 1, the radius defined in (22) is well defined. Now, by replacing ω by $\theta_{-t}\omega$ in (21), we get

$$\begin{aligned}
&|u(\theta_{-t}\omega) - \bar{u}(\theta_{-t}\omega)| \\
&\le |u_0(\theta_{-t}\omega) - \bar{u}_0(\theta_{-t}\omega)|e^{-\lambda t} \\
&\quad + \int_0^t e^{\lambda(s-t)}|f(\bar{u}(s))(\theta_{-t}\omega)|ds \\
&= |u_0(\theta_{-t}\omega) - \bar{u}_0(\theta_{-t}\omega)|e^{-\lambda t} \\
&\quad + \int_{-t}^0 e^{\lambda\tau}|f(\bar{u}(\tau))(\omega)|d\tau, \quad t \ge 0. \qquad (23)
\end{aligned}$$

The last term in (23) due to $\bar{u}(s)(\theta_{-t}\omega) = \bar{u}_0(\theta_{s-t}\omega) = \bar{u}(s-t)(\omega)$ which deduced from that $(\bar{u}(t))_{t\in\mathbb{R}}$ is a stationary process. The conclusion now follows for $t \to \infty$.

Because of the stationarity and Lemma 1, we have $e^{-\lambda t}|\bar{u}_0(\theta_{-t}\omega)| = e^{-\lambda t}|\bar{u}(-t)(\omega)| \to 0$ as $t \to \infty$. Then we have the pullback absorption

$$|u(\theta_{-t}\omega)| \le |\bar{u}_0(\omega)| + \varrho(\omega), \quad \forall t \ge T_{\mathcal{D}(\omega)}. \qquad (24)$$

As a consequence of Theorem 1, system (1) has a random attractor $\mathcal{A} = \{A(\omega), \omega \in \Omega\}$. We have know that all solutions converge pathwise to each other, so the random attractor sets are in fact singleton sets $\mathcal{A} = \{\tilde{u}_0(\omega)\}$, i.e. the random attractor is formed by a stationary random process $\tilde{u}(t)(\omega) := \tilde{u}_0(\theta_t\omega)$, which pathwise attracts all other solutions in both forward and pullback senses.

Finally, we are in the position to state the result of existence of a random attractor.

Theorem 2. *Assume that the conditions on f be satisfied. Then the random dynamical system φ has a unique random attractor. Furthermore, the random attractor is in fact a singleton sets random attractor.*

Remark 4.1. Sometimes, for the need of demonstrating the relations between $\frac{dv(t)}{dt}(\cdot)$ (or $G(u(t))(\cdot)$, $\tilde{G}(t, v(t))(\cdot)$) and ω more explicitly, we will write $\frac{dv(t)}{dt}(\omega)$ (or $G(u(t))(\omega)$, $\tilde{G}(t, v(t))(\omega)$) instead if necessary.

Remark 4.2. There are several differences between these two kinds of noises: white noise and fBm for the existence of a random attractor of SLDS. On one hand, in this work, the SLDS perturbed by additive fBms are considered, which allow to transform the SLDS into random systems that can be dealt with in a pathwise way. But for the multiplicative noise, with no transformation into a random system, the existence of random attractor for stochastic differential equations was consider in [Garrido-Atienza et al., 2010] based on considering a suitable sequence of stopping times for the fractional Brownian motion. This method is different from the traditional conjugacy method, which transforms a stochastic equation into a differential equation with random coefficients but without white noise. On the other hand, when SLDS perturbed by either additive white noises (see e.g. [Bates et al., 2006]) or a multiplicative white noise (see e.g. [Caraballo & Lu , 2008])), the existing random attractor is a (nontrivial) compact set of tempered random bounded set, but the random attractor turns out to be a single sets random attractor when the systems disturbed by additive fBms.

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