# $W$-LIKE MAPS WITH VARIOUS INSTABILITIES OF ACIM'S 

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#### Abstract

This paper generalizes the results of 13 and then provides an interesting example. We construct a family of $W$-like maps $\left\{W_{a}\right\}$ with a turning fixed point having slope $s_{1}$ on one side and $-s_{2}$ on the other. Each $W_{a}$ has an absolutely continuous invariant measure $\mu_{a}$. Depending on whether $\frac{1}{s_{1}}+\frac{1}{s_{2}}$ is larger, equal or smaller than 1 , we show that the limit of $\mu_{a}$ is a singular measure, a combination of singular and absolutely continuous measure or an absolutely continuous measure, respectively. It is known that the invariant density of a single piecewise expanding map has a positive lower bound on its support. In Section 4 we give an example showing that in general, for a family of piecewise expanding maps with slopes larger than 2 in modulus and converging to a piecewise expanding map, their invariant densities do not necessarily have a positive lower bound on the support.


## 1. Introduction

In practice, due to external noise, or roundoff errors in computation, there is a natural interest in the stability of properties of chaotic dynamical systems under small perturbations. If we consider a family of piecewise expanding maps $\tau_{a}: I \rightarrow I$, $a>0$ with absolutely continuous invariant measures (acim's) $\mu_{a}$, converging to a piecewise expanding map $\tau_{0}$ with acim $\mu_{0}$, then under general assumptions $\mu_{a}$ 's converge to $\mu_{0}$. One such assumption is that $\inf \left|\tau_{a}^{\prime}\right|>2$ for all $a>0$ (see [1], [6, [7] or [10]). This is useful in the study of the metastable systems [15], or to approximate the invariant densities 8].

Keller [9] introduced the family of $\left\{W_{a}\right\}$ maps that are piecewise expanding, ergodic transformations with a "stochastic singularity", i.e., $\mu_{a}$ 's converge to a singular measure. This occurs because of the existence of diminishing invariant neighborhoods of the turning fixed point. The slopes of the Keller's $W_{a}$ maps converge to 2 and -2 on the left and right hand sides of the turning fixed point, respectively.

Given two numbers, $s_{1}$ and $s_{2}$, greater than 1 , we consider a $W$-like map with one turning fixed point having slope $s_{1}$ on one side and $-s_{2}$ on the other. In [13], the authors considered the special case where $s_{1}=s_{2}=2$. Their perturbed maps $W_{a}$ are piecewise expanding with slopes strictly greater than 2 in modulus and are exact with their acim's supported on all of $[0,1]$. The standard bounded variation method 2] cannot be applied in this setting as the slopes of the maps in that family are not uniformly bounded away from 2 . Other methods, for example, those studied in [3], [12] and [14] cannot be applied either. Using the main result of [5], it can

[^0]be shown that the $\mu_{a}$ 's converge to $\frac{2}{3} \mu_{0}+\frac{1}{3} \delta_{\left(\frac{1}{2}\right)}$, where $\delta_{\left(\frac{1}{2}\right)}$ is the Dirac measure at point $1 / 2$ and $\mu_{0}$ is the acim of the $W_{0}$ map. Thus, the family of measures $\mu_{a}$ approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map. Similar instability was also shown in [4] for a countable family of transitive Markov maps approaching Keller's $W_{0}$ map.

In this paper, we construct a family of maps for which the instability of the acim's has a global character, not a local one. In the more general case considered in this paper, with $s_{1}, s_{2}$ not necessarily equal to 2 , we will discuss the limits of the acim's $\mu_{a}$ of the $\left\{W_{a}\right\}$ maps. We have three cases:
(I) If $\frac{1}{s_{1}}+\frac{1}{s_{2}}>1$, then $\mu_{a}$ 's converge $*$-weakly to $\delta_{\left(\frac{1}{2}\right)}$.
(II) If $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$, then $\mu_{a}$ 's converge $*$-weakly to
$\frac{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)}{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}^{2}} \mu_{0}+\frac{2 r s_{1} s_{2}^{2}}{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}^{2}} \delta_{\left(\frac{1}{2}\right)}$, where $p, q$ and $r$ are parameters defining our family of maps.
(III) If $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$, then $\mu_{a}$ 's converge to $\mu_{0}$.

Additionally, in Theorem 2, we prove that in case (III) the densities of the $\mu_{a}$ 's are uniformly bounded. The first case of our result contains the example in which Keller [9] obtained the "stochastic singularity." In the second case, the limit measure is a combination of an absolutely continuous and a singular measure, and this combination is varying according to $p, q$ and $r$ for fixed $s_{1}$ and $s_{2}$. This is a generalization of the result of [13]. In the third case, we have a map with a stable acim.

At the end of the paper, we use our main results to provide an interesting example. Keller [11 and Kowalski 12 proved that for a piecewise expanding map $\tau: I \rightarrow I$ with $\frac{1}{\left|\tau^{\prime}(x)\right|}$ being a function of bounded variation, the density of the acim of $\tau$ has a uniform positive lower bound on its support. We construct a family of piecewise expanding, piecewise linear maps $\tau_{n}$ such that $\tau_{n}$ are exact on $[0,1], \tau_{n}$ converge to $\tau=W_{0}\left(s_{1}=s_{2}=2\right),\left|\tau_{n}^{\prime}\right|>2$ for all $n$ but the densities of the acims $\mu_{n}$ 's do not have a uniform positive lower bound.

In Section 2, we introduce our family of $W_{a}$ maps and state the main result. In Section 3 we present the proofs. In Section 4 we show the example related to the results of Keller [11] and Kowalski [12].

## 2. FAMILY OF $W_{a}$ MAPS AND THE MAIN RESUlT

Let $s_{1}, s_{2}>1$ and $p, q, r>0$. We consider the family $\left\{W_{a}: 0 \leq a\right\}$ of maps of $[0,1]$ onto itself defined by

$$
W_{a}(x)=\left\{\begin{array}{l}
1-\frac{2\left(s_{1}+p a\right)}{s_{1}-1+p a-2 r a} x, \text { for } 0 \leq x<\frac{1}{2}-\frac{\frac{1}{2}+r a}{s_{1}+p a}  \tag{1}\\
\left(s_{1}+p a\right)(x-1 / 2)+1 / 2+r a, \text { for } \frac{1}{2}-\frac{\frac{1}{2}+r a}{s_{1}+p a} \leq x<1 / 2 \\
-\left(s_{2}+q a\right)(x-1 / 2)+1 / 2+r a, \text { for } 1 / 2 \leq x<\frac{1}{2}+\frac{\frac{1}{2}+r a}{s_{2}+q a} \\
1+\frac{2\left(s_{2}+q a\right)}{s_{2}-1+q a-2 r a}(x-1), \text { for } \frac{1}{2}+\frac{\frac{1}{2}+r a}{s_{2}+q a} \leq x \leq 1
\end{array}\right.
$$

For each choice of $s_{1}, s_{2}>1, p, q, r>0$, we consider only $a>0$ such that $0 \leq W_{a}(x) \leq 1$ for $x \in[0,1]$.

An example of a $W_{a}$ map is shown in Fig 1. Fig (a) is the unperturbed $W_{0}$ map with turning fixed point at $1 / 2$ and $s_{1}=3 / 2, s_{2}=3$. Fig 1 (b) is the perturbed map $W_{a}$, with $a=0.05, r=2, p=3, q=2$. The slope of the second branch is
$s_{1}+p a=1.65$, the slope of the third branch is $s_{2}+q a=3.1$, and $W_{0.05}(1 / 2)=$ $1 / 2+r a=0.6$.


Figure 1. The $W$-like maps with $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$ : (a) $W_{0}$ with $s_{1}=$ $3 / 2$ and $s_{2}=3$, (b) $W_{a}$ with $s_{1}=3 / 2, s_{2}=3 ; a=0.05 ; r=2$, $p=3, q=2$; also several initial points of the trajectory of $1 / 2$.

Every $W_{a}$ has a unique absolutely continuous invariant measure $\mu_{a}$ since all the slopes are greater than 1 in modulus. We will show later that, for $\frac{1}{s_{1}}+\frac{1}{s_{2}} \leq 1, \mu_{a}$ is supported on $[0,1]$ and for $\frac{1}{s_{1}}+\frac{1}{s_{2}}>1$ it is supported on a subinterval around $1 / 2$. $W_{a}$ is an exact map with the measure $\mu_{a}$. Let $h_{a}$ denote the normalized density of $\mu_{a}, a \geq 0$. Since the $W_{0}$ map is a Markov one, it is easy to check that

$$
h_{0}= \begin{cases}\frac{2 s_{1}\left(s_{2}+1\right)}{2 s_{1} s_{2}+s_{1}-s_{2}}, & \text { for } 0 \leq x<1 / 2  \tag{2}\\ \frac{2 s_{2}\left(s_{1}-1\right)}{2 s_{1} s_{2}+s_{1}-s_{2}}, & \text { for } 1 / 2 \leq x \leq 1\end{cases}
$$

Our main result is the following theorem
Theorem 1. As $a \rightarrow 0$ the measures $\mu_{a}$ converge $*$-weakly to the measure
(I) $\delta_{\left(\frac{1}{2}\right)}$, if $\frac{1}{s_{1}}+\frac{1}{s_{2}}>1$;
(II) $\frac{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)}{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}^{2}} \mu_{0}+\frac{2 r s_{1} s_{2}^{2}}{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}} \delta_{\left(\frac{1}{2}\right)}$, if $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$;
(III) $\mu_{0}$, if $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$,
where $\delta_{\left(\frac{1}{2}\right)}$ is the Dirac measure at point $1 / 2$.
The proof relies on the general formula for invariant densities of piecewise linear maps [5] and direct calculations. Most objects and quantities we use depend on the parameter $a$. We suppress $a$ from the notation to make it simpler.

In case (III), we actually prove a little more:
Theorem 2. If $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$, then the normalized invariant densities $\left\{h_{a}\right\}$ are uniformly bounded for given $p, q$ and $r$. Consequently, we obtain Theorem 11(III).

## 3. Proofs

This section contains the proofs of Theorems 1 and 2 divided into a number of steps.
3.1. Assume $\frac{1}{s_{1}}+\frac{1}{s_{2}}>1$. Let

$$
x_{l}^{*}=\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}-1+p a\right)}
$$

and

$$
x_{r}^{*}=\frac{s_{2} s_{1}-s_{2}+\left(2 r s_{1}-q+p s_{2}+q s_{1}\right) a+(2 r p+p q) a^{2}}{2\left(s_{1}-1+p a\right)\left(s_{2}+q a\right)} .
$$

$x_{l}^{*}$ is the fixed point on the second branch of $W_{a}$, and $x_{r}^{*}$ is the preimage of $x_{l}^{*}$ under the third branch of $W_{a}$. Both $x_{r}^{*}$ and $x_{l}^{*}$ converge to $\frac{1}{2}$ as $a$ approaches 0 . For small $a$, we have

$$
W_{a}(1 / 2)-x_{r}^{*}=\frac{r a\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]}{\left(s_{1}-1+p a\right)\left(s_{2}+q a\right)}<0
$$

In this case, we have $W_{a}\left(\left[x_{l}^{*}, x_{r}^{*}\right]\right) \subseteq\left[x_{l}^{*}, x_{r}^{*}\right] .\left.W_{a}\right|_{\left[x_{l}^{*}, x_{r}^{*}\right]}$ is a skewed tent map with $W_{a}(1 / 2)>1 / 2$; it is known that with acim $\mu_{a}$, it is exact on $\left[x_{l}^{*}, W_{a}(1 / 2)\right]$. Since $\mu_{a}$ is concentrated on $\left[x_{l}^{*}, x_{r}^{*}\right]$, we conclude that $\mu_{a}$ converge $*$-weakly to $\delta_{\left(\frac{1}{2}\right)}$. This proves Theorem 1(I).

Fig 2 shows an example with $a=0.05, r=2, p=3, q=2 ; s_{1}=4 / 3, s_{2}=5 / 2$.


Figure 2. The $W_{a}$ map with $\frac{1}{s_{1}}+\frac{1}{s_{2}}>1$
3.2. Formula for the non-normalized invariant density of $W_{a}$ if $\frac{1}{s_{1}}+\frac{1}{s_{2}} \leq 1$. An example of a map $W_{a}$ is shown in Fig 1 . We have the following proposition.

Proposition 1. For $\frac{1}{s_{1}}+\frac{1}{s_{2}} \leq 1$, the map $W_{a}$ has an absolutely continuous invariant measure $\mu_{a}$ supported on $[0,1]$ and the map $W_{a}$ with respect to $\mu_{a}$ is exact.

Proof. $W_{a}$ is a piecewise expanding transformation. From the general theory (see for example [2]), it follows that it is enough to show that the images $W_{a}^{n}(J)$ grow to cover all $[0,1]$ as $n \rightarrow \infty$, for any interval $J \subset[0,1]$. Since $W_{a}$ is expanding, $W_{a}^{n}(J)$ grow until some image $W_{a}^{n_{0}}(J)$ contains an internal partition point. If this point is not $1 / 2$, then $W_{a}^{n_{0}+2}(J)$ contains the repelling fixed point 1 . Then its images grow
to cover all of $[0,1]$. If this point is $1 / 2$, we proceed as follows. First, assume that $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$. Consider a small neighborhood $J=\left(z_{1}, z_{2}\right)$ around $1 / 2$ with length $\ell$, then

$$
\min _{z_{2}-z_{1}=\ell} \max \left\{\left(\frac{1}{2}-z_{1}\right)\left(s_{1}+p a\right),\left(z_{2}-\frac{1}{2}\right)\left(s_{2}+q a\right)\right\}=\frac{1}{\frac{1}{s_{1}+p a}+\frac{1}{s_{2}+q a}} \ell>\ell
$$

Thus, the interval $J$ will grow until its image covers two partition points of $W_{a}$. Then the second iteration afterward will cover $[0,1]$. Therefore, $W_{a}$ is exact with respect ot $\mu_{a}$.

Assume $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$. If $a \neq 0$, then $\frac{1}{\frac{1}{s_{1}+p a}+\frac{1}{s_{2}+q a}}>1$, which implies $W_{a}$ is exact with respect to $\mu_{a}$. In the case $a=0$, we first note that $1 / 2$ is a turning fixed point. Take again a small interval $J=\left(z_{1}, z_{2}\right) \ni 1 / 2$. Its image is an interval $(z, 1 / 2)$. It will grow under iteration and its iterations still contain $1 / 2$. It will grow until its image covers another partition point of $W_{a}$. Then, the second iteration afterward will covers all of $[0,1]$. Thus, $W_{a}$ is again exact with respect to $\mu_{a}$.

We adapt the general formulas of [5] to our case and obtain the following lemma:
Lemma 1. ( $I$ ) $N=4, K=2, L=0$;
(II) $\alpha=(1,1 / 2+r a, 1 / 2+r a, 1), \beta=\left(\beta_{1}, \beta_{2}, \beta_{3}, \beta_{4}\right)$, where $\beta_{1}=-\frac{2\left(s_{1}+p a\right)}{s_{1}-1+p a-2 r a}$, $\beta_{2}=s_{1}+p a, \beta_{3}=-\left(s_{2}+q a\right)$ and $\beta_{4}=\frac{2\left(s_{2}+q a\right)}{s_{2}-1+q a-2 r a}, \gamma=(0,0,0,0)$;
(III) The digits $A=\left(a_{1}, a_{2}, a_{3}, a_{4}\right)$, where $a_{1}=-1, a_{2}=\frac{s_{1}-1+p a-2 r a}{2}, a_{3}=$ $-\frac{s_{2}+1+q a+2 r a}{2}, a_{4}=\frac{s_{2}+1+q a+2 r a}{s_{1}-1+p a-2 r a}$;
(IV) There are two $c_{i}$ 's, which are $c_{1}=(1 / 2,2)$ and $c_{2}=(1 / 2,3)$, and $j\left(c_{1}\right)=2$, $j\left(c_{2}\right)=3$. Then, $W_{u}=\left\{c_{1}, c_{2}\right\}, W_{l}=\emptyset, U_{l}=\left\{c_{2}\right\}, U_{r}=\left\{c_{1}\right\} ;$
$(V) \beta\left(c_{1}, 1\right)=s_{1}+p a$ since $j\left(c_{1}\right)=2$, then $\beta\left(c_{1}, 2\right)=-\left(s_{1}+p a\right)\left(s_{2}+q a\right)$ and $\beta\left(c_{1}, k\right)=-\left(s_{2}+q a\right)\left(s_{1}+p a\right)^{k-1}$ up to some $k$ which is the first moment $j$ when the $W_{a}^{j}(1 / 2)$ is less than $\frac{1}{2}-\frac{1 / 2+r a}{s_{1}+p a}$, and is the same one defined in Lemma 4 ; $(V I) \beta\left(c_{2}, 1\right)=-\left(s_{2}+q a\right)$ since $j\left(c_{2}\right)=3$, then $\beta\left(c_{2}, 2\right)=\left(s_{2}+q a\right)^{2}$ and $\beta\left(c_{2}, k\right)=$ $\left(s_{2}+q a\right)^{2}\left(s_{1}+p a\right)^{k-2}$ up to the same $k$ in part $(e), W_{a}^{n}\left(c_{1}\right)=W_{a}^{n}\left(c_{2}\right)$ for all $n$; (VII) Based on (VI), we have the following for the matrix $S=\left(S_{i, j}\right)_{1 \leq i, j \leq 2}$ :

For $c_{1} \in U_{r}$

$$
\begin{aligned}
& S_{1,1}=\sum_{n=1}^{\infty} \frac{\delta\left(\beta\left(\left(c_{1}, n\right)>0\right)\right) \delta\left(W_{a}^{n}\left(c_{1}\right)>1 / 2\right)+\delta\left(\beta\left(\left(c_{1}, n\right)<0\right)\right) \delta\left(W_{a}^{n}\left(c_{1}\right)<1 / 2\right)}{\left|\beta\left(c_{1}, n\right)\right|} \\
& S_{1,2}=\sum_{n=1}^{\infty} \frac{\delta\left(\beta\left(\left(c_{1}, n\right)>0\right)\right) \delta\left(W_{a}^{n}\left(c_{1}\right)>1 / 2\right)+\delta\left(\beta\left(\left(c_{1}, n\right)<0\right)\right) \delta\left(W_{a}^{n}\left(c_{1}\right)<1 / 2\right)}{\left|\beta\left(c_{1}, n\right)\right|}
\end{aligned}
$$

For $c_{2} \in U_{l}$
$S_{2,1}=\sum_{n=1}^{\infty} \frac{\delta\left(\beta\left(\left(c_{2}, n\right)<0\right)\right) \delta\left(W_{a}^{n}\left(c_{2}\right)>1 / 2\right)+\delta\left(\beta\left(\left(c_{2}, n\right)>0\right)\right) \delta\left(W_{a}^{n}\left(c_{2}\right)<1 / 2\right)}{\left|\beta\left(c_{2}, n\right)\right|}$,
$S_{2,2}=\sum_{n=1}^{\infty} \frac{\delta\left(\beta\left(\left(c_{2}, n\right)<0\right)\right) \delta\left(W_{a}^{n}\left(c_{2}\right)>1 / 2\right)+\delta\left(\beta\left(\left(c_{2}, n\right)>0\right)\right) \delta\left(W_{a}^{n}\left(c_{2}\right)<1 / 2\right)}{\left|\beta\left(c_{2}, n\right)\right|}$.
Remark 1. It follows from (V,VI) of Lemma 1 that

$$
S_{1,1}=S_{1,2}, S_{2,1}=S_{2,2} \text { and } S_{1,1}=\frac{s_{2}+q a}{s_{1}+p a} S_{2,2}
$$

Let Id be the $2 \times 2$ identity matrix and let $V=[1,1]$. Then, for the solution, $D=\left[D_{1}, D_{2}\right]$, of the system :

$$
\begin{equation*}
\left(-S^{T}+I d\right) D^{T}=V^{T} \tag{1}
\end{equation*}
$$

we have $D_{1}=D_{2}$. Let us denote them by $\Lambda$.
Let $I_{1}, I_{2}, I_{3}, I_{4}$ be the partition of $I=[0,1]$ into maximal intervals of monotonicity of $W_{a}: I_{1}=\left[0, \frac{s_{1}-1+p a-2 r a}{2\left(s_{1}+p a\right)}\right), I_{2}=\left(\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}+p a\right)}, 1 / 2\right), I_{3}=\left(1 / 2, \frac{s_{2}+1+q a+2 r a}{2\left(s_{2}+q a\right)}\right)$ and $I_{4}=\left(\frac{s_{2}+1+q a+2 r a}{2\left(s_{2}+q a\right)}, 1\right]$. We define the following index function:

$$
j(x)=j \text { for } x \in I_{j}, j=1,2,3,4
$$

and

$$
j\left(c_{1}\right)=2, j\left(c_{2}\right)=3
$$

We define the cumulative slopes for iterates of points as follows:

$$
\beta(x, 1)=\beta_{j(x)}, \quad \text { and } \beta(x, n)=\beta(x, n-1) \cdot \beta_{j\left(W_{a}^{n-1}(x)\right)}, \quad n \geq 2 .
$$

In particular, we have

$$
\beta(1 / 2, n)=\left(s_{1}+p a\right) \cdot W_{a}^{\prime}\left(W_{a}(1 / 2)\right) \cdot W_{a}^{\prime}\left(W_{a}^{2}(1 / 2)\right) \cdots W_{a}^{\prime}\left(W_{a}^{n-1}(1 / 2)\right)
$$

which is the cumulative slope along the $n$ steps of trajectory of $1 / 2$. Recall that $k$ is the first moment $j$ when the $W_{a}^{j}(1 / 2)$ is less than $\frac{1}{2}-\frac{1 / 2+r a}{s_{1}+p a}$. Let $k_{1}=\left[\frac{2}{3} k\right]$ (the integer part of $2 k / 3$ ). Note that $k_{1} \rightarrow \infty$ as $a \rightarrow 0$. Let

$$
\chi^{s}(t, x)= \begin{cases}\chi_{[0, x]} & \text { for } t>0 \\ \chi_{[x, 1]} & \text { for } t<0\end{cases}
$$

Now, we can obtain the following formula for $f_{a}$ :
Lemma 2. Let

$$
f_{a}=1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda\left(\sum_{n=1}^{\infty} \frac{\chi^{s}\left(\beta(1 / 2, n), W_{a}^{n}(1 / 2)\right)}{|\beta(1 / 2, n)|}\right)
$$

Then $f_{a}$ is $W_{a}$ invariant non-normalized density. Furthermore, for small $a>0$, we have:
(I) If $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$, then $\Lambda<-1$;
(II) If $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$, the sign of $\Lambda$ depends on $s_{1}$ and $s_{2}$, can be either positive or negative depending on the sign of $\vartheta=1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)=1-\frac{s_{1}+s_{2}}{s_{1} s_{2}}\left(1+\frac{s_{1}}{s_{2}\left(s_{1}-1\right)}\right)$. The case when $\vartheta=0$ is discussed at the end of Section 3.

Proof. By the Theorem 2 in [5], it follows from (IV,V,VI) of Lemma 1 that:

$$
\begin{aligned}
f_{a} & =1+D_{1} \sum_{n=1}^{\infty} \frac{\chi^{s}\left(\beta\left(c_{1}, n\right), W_{a}^{n}\left(c_{1}\right)\right)}{\left|\beta\left(c_{1}, n\right)\right|}+D_{2} \sum_{n=1}^{\infty} \frac{\chi^{s}\left(-\beta\left(c_{2}, n\right), W_{a}^{n}\left(c_{2}\right)\right)}{\left|\beta\left(c_{2}, n\right)\right|} \\
& =1+\Lambda \sum_{n=1}^{\infty} \frac{\chi^{s}\left(\beta\left(c_{1}, n\right), W_{a}^{n}(1 / 2)\right)}{\left|\beta\left(c_{1}, n\right)\right|}+\Lambda \sum_{n=1}^{\infty} \frac{\chi^{s}\left(-\beta\left(c_{2}, n\right), W_{a}^{n}(1 / 2)\right)}{\left|\beta\left(c_{2}, n\right)\right|} \\
& =1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda\left(\sum_{n=1}^{\infty} \frac{\chi^{s}\left(\beta(1 / 2, n), W_{a}^{n}(1 / 2)\right)}{|\beta(1 / 2, n)|}\right)
\end{aligned}
$$

Since

$$
\begin{aligned}
S_{1,1} & \geq \frac{1}{s_{1}+p a}+\frac{1}{s_{2}+q a} \sum_{n=1}^{k_{1}-1} \frac{1}{\left(s_{1}+p a\right)^{n}}=\frac{1}{s_{1}+p a}+\frac{1}{s_{2}+q a} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1} \\
S_{1,1} & \leq \frac{1}{s_{1}+p a}+\frac{1}{s_{2}+q a} \sum_{n=1}^{\infty} \frac{1}{\left(s_{1}+p a\right)^{n}}=\frac{1}{s_{1}+p a}+\frac{1}{s_{2}+q a} \frac{1}{s_{1}+p a-1}
\end{aligned}
$$

and $\Lambda=\frac{1}{1-\frac{s_{1}+s_{2}+p a+q a}{s_{2}+q a} S_{1,1}}$, we have

$$
\begin{equation*}
\Lambda_{l}=\frac{1}{1-\left(\kappa+\eta\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}\right)\right)} \leq \Lambda \leq \frac{1}{1-(\kappa+\eta)}=\Lambda_{h} \tag{3}
\end{equation*}
$$

where $\kappa=\frac{s_{1}+s_{2}+p a+q a}{\left(s_{1}+p a\right)\left(s_{2}+q a\right)}, \eta=\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}\left(s_{1}+p a-1\right)}$.
To obtain the upper bound of $S_{1,1}$, we assume $s_{1}<s_{2}$. For $s_{1}>s_{2}$ the calculations differ slightly.
(I) Note that for small $a$ both estimates $\Lambda_{l}$ and $\Lambda_{h}$ are smaller than -1 since both $\kappa$ and $\eta$ are smaller than 1 and close to 1 . Furthermore, as $a$ approaches 0 , both $\kappa$ and $\eta$ approach 1 .
(II) As $a$ approaches $0, \kappa$ and $\eta$ approach $\frac{s_{1}+s_{2}}{s_{1} s_{2}}$ and $\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}$, respectively. Again, note that for small $a$, estimates $\Lambda_{l}$ and $\Lambda_{h}$ can be either positive or negative, and they have the same sign.

For small positive $a$, the first image of $1 / 2$ is $W_{a}(1 / 2)=1 / 2+r a$ and the next one falls just below the fixed point $x_{l}^{*}$ slightly less than $1 / 2$. The following images form a decreasing sequence until they go below $\frac{1}{2}-\frac{1 / 2+r a}{s_{1}+p a}$. Since $k$ is the first iteration $j$ when the $W_{a}^{j}(1 / 2)$ is less than $\frac{1}{2}-\frac{1 / 2+r a}{s_{1}+p a}$, the consecutive cumulative slopes of $1 / 2$ are

$$
\left(s_{1}+p a\right),-\left(s_{1}+p a\right)\left(s_{2}+q a\right),-\left(s_{1}+p a\right)^{2}\left(s_{2}+q a\right), \ldots,-\left(s_{1}+p a\right)^{k-1}\left(s_{2}+q a\right)
$$

and

$$
\begin{equation*}
f_{a}=1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda\left(\frac{\chi_{\left[0, W_{a}(1 / 2)\right]}}{\left(s_{1}+p a\right)}+\sum_{j=2}^{k} \frac{\chi_{\left[W_{a}^{j}(1 / 2), 1\right]}}{\left(s_{1}+p a\right)^{j-1}\left(s_{2}+q a\right)}+\ldots\right) \tag{4}
\end{equation*}
$$

3.3. Estimates, normalizations and integrals on $f_{a}$ for $\frac{1}{s_{1}}+\frac{1}{s_{2}} \leq 1$. Remembering that $k=\min \left\{j \geq 1: W_{a}^{j}(1 / 2) \leq \frac{1}{2}-\frac{1 / 2+r a}{s_{1}+p a}\right\}$ and $k_{1}=\left[\frac{2}{3} k\right]$ (the integer part of $2 k / 3$ ), we will give the estimates on $f_{a}$.

Let us define

$$
g_{l}=\frac{\chi_{\left[0, W_{a}(1 / 2)\right]}}{s_{1}+p a}+\frac{1}{s_{2}+q a} \sum_{j=2}^{k_{1}} \frac{\chi_{\left[W_{a}^{j}(1 / 2), 1\right]}}{\left(s_{1}+p a\right)^{j-1}}
$$

and

$$
g_{h}=g_{l}+\frac{1}{s_{2}+q a} \sum_{j=0}^{\infty} \frac{1}{\left(s_{1}+p a\right)^{j+k_{1}}}=g_{l}+\frac{1}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}}
$$

Also, let $\chi_{1}=\chi_{[0,1 / 2+r a]}, \chi_{j}=\chi_{\left[W_{a}^{j}(1 / 2), 1 / 2+r a\right]}, j=2,3, \ldots, k_{1}, \chi_{c}=\chi_{(1 / 2+r a, 1]}$.
3.3.1. Estimates on $f_{a}$ if $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$. We have the following lemma:

Lemma 3. For the family of $W_{a}$ maps, if $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$, we have
(I) $W_{a}(1 / 2)=1 / 2+r a, W_{a}^{2}(1 / 2)=-r a\left(s_{2}+q a\right)+1 / 2+r a$, and for $3 \leq m \leq k$, we have $W_{a}^{m}(1 / 2)=-a^{2}\left(s_{1}+p a\right)^{m-2} \frac{r\left(q s_{1}+p s_{2}-p-q\right)+r p q a}{s_{1}+p a-1}+\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}+p a-1\right)}$;
(II) $\lim _{a \rightarrow 0} a k=0$;
(III) $\lim _{a \rightarrow 0} \frac{1}{a\left(s_{1}+p a\right)^{k}}=0$;
(IV) $\lim _{a \rightarrow 0} \frac{1}{a\left(s_{1}+p a\right)^{k_{1}}}=0$;
(V) $\lim _{a \rightarrow 0} a^{2}\left(s_{1}+p a\right)^{k_{1}}=0$;
(VI) $\lim _{a \rightarrow 0} W_{a}^{k_{1}}\left(\frac{1}{2}\right)=\frac{1}{2}$.

Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.
By the definition of $k$, we have:
$0 \leq-a^{2}\left(s_{1}+p a\right)^{k-2} \frac{r\left(q s_{1}+p s_{2}-p-q\right)+r p q a}{s_{1}+p a-1}+\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}+p a-1\right)} \leq \frac{1}{2}-\frac{1 / 2+r a}{s_{1}+p a}$.
The first inequality of (5) implies that $\left(s_{1}+p a\right)^{k-2} \leq \frac{s_{1}-1+p a-2 r a}{2 a^{2}\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)}$, thus
$a k \leq a \frac{\ln \left(s_{1}-1+p a-2 r a\right)-\ln 2-2 \ln a-\ln \left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)}{\ln \left(s_{1}+p a\right)}+2 a$,

$$
\begin{gathered}
a \leq \frac{\sqrt{s_{1}-1+p a-2 r a}\left(s_{1}+p a\right)}{\sqrt{2\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)}\left(s_{1}+p a\right)^{k / 2}}, \\
a^{2}\left(s_{1}+p a\right)^{k_{1}} \leq \frac{\left(s_{1}-1+p a-2 r a\right)\left(s_{1}+p a\right)^{2}}{2\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)\left(s_{1}+p a\right)^{k-k_{1}}},
\end{gathered}
$$

so we obtain (V), and since $\lim _{a \rightarrow 0} a \ln a=0$, we obtain (II).
The second inequality of (5) implies

$$
\frac{1}{a\left(s_{1}+p a\right)^{k-2}} \leq \frac{2 a\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)\left(s_{1}+p a\right)}{s_{1}-1+p a-2 r a}
$$

Therefore,

$$
\begin{equation*}
\frac{1}{a\left(s_{1}+p a\right)^{k}} \leq \frac{2 a\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)}{\left(s_{1}-1+p a-2 r a\right)\left(s_{1}+p a\right)} \tag{6}
\end{equation*}
$$

and as $a \rightarrow 0$, we obtain (III).
On the other hand, (6) implies

$$
\begin{aligned}
\frac{1}{a\left(s_{1}+p a\right)^{k_{1}}} & \leq \frac{2 a\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)\left(s_{1}+p a\right)^{k-k_{1}}}{\left(s_{1}+p a-2 r a-1\right)\left(s_{1}+p a\right)} \\
& \leq \frac{\sqrt{2\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)}\left(s_{1}+p a\right)^{k-k_{1}}}{\sqrt{s_{1}+p a-2 r a-1}\left(s_{1}+p a\right)^{k / 2}} \\
& =\frac{\sqrt{2\left(r\left(q s_{1}+p s_{2}-p-q\right)+r p q a\right)}}{\sqrt{s_{1}+p a-2 r a-1}\left(s_{1}+p a\right)^{k_{1}-k / 2}}
\end{aligned}
$$

By the definition of $k_{1}$, we obtain (IV). (VI) follows from (V).
Now, let us prove (I).

The fixed point slightly less than $1 / 2$ is $x_{l}^{*}=\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}-1+p a\right)}$, and

$$
x_{l}^{*}-W_{a}^{2}(1 / 2)=\frac{r a^{2}\left(q\left(s_{1}-1\right)+p\left(s_{2}-1\right)+a p q\right)}{s_{1}-1+p a}>0
$$

which implies that $W_{a}^{m}(1 / 2)$ are all in the domain of the second branch of $W_{a}$ for $3 \leq m \leq k$. For a linear map $T(x)=m_{0} x+b_{0}$, we have $T^{n}(x)=m_{0}^{n} x+\frac{m_{0}^{n}-1}{m_{0}-1} b_{0}$. This proves (I).

Using (44) and (3) we see that for the functions $f_{l}=1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{l} g_{h}$ and $f_{h}=1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{h} g_{l}$, we have

$$
\begin{equation*}
f_{l} \leq f_{a} \leq f_{h} \tag{7}
\end{equation*}
$$

Now, we will represent functions $f_{l}$ and $f_{c}$ as combinations of functions $\chi_{j}, j=$ $1, \ldots, k_{1}$ and $\chi_{c}$. After some calculations, we obtain

$$
\begin{aligned}
f_{l}= & 1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{l}\left(\frac{\chi_{\left[0, W_{a}(1 / 2)\right]}}{s_{1}+p a}+\frac{1}{s_{2}+q a} \sum_{j=2}^{k_{1}} \frac{\chi_{\left[W_{a}^{j}(1 / 2), 1\right]}}{\left(s_{1}+p a\right)^{j-1}}\right. \\
& \left.+\frac{1}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}}\right) \\
= & \left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{l}+1\right) \chi_{1}+\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{l} \sum_{j=2}^{k_{1}} \frac{\chi_{j}}{\left(s_{1}+p a\right)^{j-1}} \\
& +\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+p a\right)^{2}} \Lambda_{l} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+1\right) \chi_{c} \\
& +\frac{\frac{s_{1}+s_{2}+p a+q a}{s_{2}+q a} \Lambda_{l}}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}}, \\
f_{h}= & 1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{h}\left(\frac{\chi_{\left[0, W_{a}(1 / 2)\right]}^{s_{1}+p a}+\frac{1}{s_{2}+q a} \sum_{j=2}^{k_{1}} \frac{\left.\chi_{\left[W_{a}^{j}(1 / 2), 1\right]}^{\left(s_{1}+p a\right)^{j-1}}\right)}{}=}{}\right. \\
& \left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right) \chi_{1}+\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \sum_{j=2}^{k_{1}} \frac{\chi_{j}}{\left(s_{1}+p a\right)^{j-1}} \\
& +\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+1\right) \chi_{c} .
\end{aligned}
$$

In the case we are considering, (3) implies that both $\Lambda_{l}, \Lambda_{h}$ are smaller than -1 . Using this, one can show that all the coefficients in the representation of $f_{l}$ and $f_{h}$ are negative for sufficiently small $a$. For example, let us consider the coefficient of $\chi_{1}$ in $f_{h}$ :

$$
\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1=\frac{\kappa}{1-(\kappa+\eta)}+1=\frac{1-\eta}{1-(\kappa+\eta)}<0
$$

3.3.2. Normalizations and integrals if $\frac{1}{s_{1}}+\frac{1}{s_{2}}=1$. Let us define $J_{1}=\left[0, W_{a}^{k_{1}}(1 / 2)\right]$, $J_{2}=\left(W_{a}^{k_{1}}(1 / 2), 1 / 2+r a\right], J_{3}=(1 / 2+r a, 1]$. We will calculate integrals of $f_{h}$ over each of these intervals $J_{1}, J_{2}$ and $J_{3}$, and use them to normalize $f_{h}$. We have

$$
\begin{aligned}
C_{1}= & \int_{J_{1}} f_{h} d \lambda=\int_{J_{1}}\left[\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right] \chi_{1} d \lambda \\
= & {\left[\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right] W_{a}^{k_{1}}\left(\frac{1}{2}\right)=\left[\frac{\kappa}{1-(\kappa+\eta)}+1\right] W_{a}^{k_{1}}\left(\frac{1}{2}\right) } \\
= & {\left[\frac{a\left(2 q s_{1} s_{2}+p s_{2}^{2}-2 q s_{2}-p-q\right)}{(1-(\kappa+\eta))\left(s_{2}+q a\right)^{2}\left(s_{1}+p a-1\right)}\right.} \\
& \left.\quad+\frac{a^{2}\left(2 p q s_{2}-q^{2}+q^{2} s_{1}\right)+p q^{2} a^{3}}{(1-(\kappa+\eta))\left(s_{2}+q a\right)^{2}\left(s_{1}+p a-1\right)}\right] W_{a}^{k_{1}}\left(\frac{1}{2}\right) .
\end{aligned}
$$

Using Lemma 3, we obtain

$$
\lim _{a \rightarrow 0} \frac{C_{1}}{a}=-\frac{2 q s_{1} s_{2}+p s_{2}^{2}-2 q s_{2}-p-q}{2 s_{2}^{2}\left(s_{1}-1\right)}=-\frac{2 q s_{1}+p s_{2}^{2}-p-q}{2 s_{2} s_{1}}
$$

In the same way, we can see that for any $0<\theta<1 / 2$, we obtain

$$
\lim _{a \rightarrow 0} \frac{1}{a} \int_{0}^{\theta} f_{h} d \lambda=-\frac{2 q s_{1}+p s_{2}^{2}-p-q}{s_{2} s_{1}} \theta
$$

On the interval $J_{2}$, the integral of $f_{h}$ is:

$$
\begin{aligned}
C_{2}=\int_{J_{2}} f_{h} d \lambda= & \int_{J_{2}}\left[\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right] \chi_{1} d \lambda \\
& +\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \sum_{j=2}^{k_{1}} \int_{J_{2}} \frac{\chi_{j}}{\left(s_{1}+a\right)^{j-1}} d \lambda \\
= & \frac{1-\eta}{1-(\kappa+\eta)}\left(\frac{1}{2}+r a-W_{a}^{k_{1}}\left(\frac{1}{2}\right)\right) \\
& +\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h}\left[\frac{r a\left(s_{2}+q a\right)}{s_{1}+p a}+\frac{r a\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-2}}\right)}{\left(s_{1}+p a-1\right)^{2}}\right. \\
& \left.+\frac{a^{2}\left(k_{1}-2\right)}{s_{1}+p a} \frac{r\left(q s_{1}+p s_{2}-p-q\right)+r p q a}{s_{1}+p a-1}\right] .
\end{aligned}
$$

Using Lemma 3, we obtain

$$
\lim _{a \rightarrow 0} \frac{C_{2}}{a}=-\frac{s_{1}+s_{2}}{s_{2}^{2}}\left[\frac{r s_{2}}{s_{1}}+\frac{r}{\left(s_{1}-1\right)^{2}}\right]=-r s_{2}
$$

On the interval $J_{3}$, the integral of $f_{h}$ is:

$$
\begin{aligned}
C_{3}=\int_{J_{3}} f_{h} d \lambda & =\int_{J_{3}}\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+1\right) \chi_{c} d \lambda \\
& =\left[\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}\right) \frac{\eta}{1-(\kappa+\eta)}+1\right]\left(\frac{1}{2}-r a\right) \\
& =\frac{\frac{a\left(q s_{1}+p s_{2}-p-q\right)+p q a^{2}}{\left(s_{1}+p a\right)\left(s_{2}+q a\right)}-\frac{\eta}{\left(s_{1}+p a\right)^{k_{1}-1}}}{1-(\kappa+\eta)}\left(\frac{1}{2}-r a\right) .
\end{aligned}
$$

Using Lemma 3, we obtain

$$
\lim _{a \rightarrow 0} \frac{C_{3}}{a}=-\frac{q s_{1}+p s_{2}-p-q}{2 s_{1} s_{2}}
$$

In the same way, we can see that for any $0<\theta<1 / 2$, we obtain

$$
\lim _{a \rightarrow 0} \frac{1}{a} \int_{1 / 2+\theta}^{1} f_{h} d \lambda=-\frac{q s_{1}+p s_{2}-p-q}{s_{1} s_{2}}\left(\frac{1}{2}-\theta\right) .
$$

If we define $B=C_{1}+C_{2}+C_{3}$, then $\frac{f_{h}}{B}$ is a normalized density. We see that

$$
\lim _{a \rightarrow 0} \frac{B}{a}=-\frac{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}^{2}}{2 s_{1} s_{2}}
$$

Our calculations show that the normalized measures $\left\{\left(f_{h} / B\right) \cdot \lambda\right\}$ converge $*$ weakly to the measure

$$
\frac{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)}{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}^{2}} \mu_{0}+\frac{2 r s_{1} s_{2}^{2}}{\left(q s_{1}+p s_{2}-p-q\right)\left(s_{2}+2\right)+2 r s_{1} s_{2}^{2}} \delta_{\left(\frac{1}{2}\right)} .
$$

Now, we will show the same holds for the normalized measure defined by $f_{l}$. To this end, let us notice that

$$
\begin{aligned}
f_{h}-f_{l}= & \left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{h} g_{l}-\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{l} g_{h} \\
= & \left(1+\frac{s_{1}+p a}{s_{2}+q a}\right)\left(\Lambda_{h}-\Lambda_{l}\right) g_{l}-\Lambda_{l} \frac{1+\frac{s_{1}+p a}{s_{2}+q a}}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}} \\
= & \left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \frac{\frac{\eta}{\left(s_{1}+p a\right)^{k_{1}-1}}}{[1-(\kappa+\eta)]\left[1-\kappa-\eta\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}\right)\right]} g_{l} \\
& -\Lambda_{l} \frac{1+\frac{s_{1}+p a}{s_{2}+q a}}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}},
\end{aligned}
$$

where $\left|g_{l}\right| \leq \frac{2}{s_{1}}$ and $\lim _{a \rightarrow 0} \Lambda_{l}=-1$. Using Lemma 3 once again, we can show that for any subinterval $J \subset[0,1]$, we have

$$
\lim _{a \rightarrow 0} \frac{1}{a} \int_{J}\left(f_{h}-f_{l}\right) d \lambda=0 .
$$

For $J=[0,1]$ this means that the normalizations of $f_{l}$ and $f_{h}$ are asymptotically the same. With this, the limit for a general $J$ means in particular that the $*$-weak limit of normalized measures defined using $f_{l}$ is the same as for those defined using $f_{h}$. In view of inequality (7), this proves Theorem (1).
3.3.3. Estimates on $f_{a}$ if $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$. We have the following lemma:

Lemma 4. For the family of $W_{a}$ maps, if $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$, we have
(I) $W_{a}(1 / 2)=1 / 2+r a, W_{a}^{2}(1 / 2)=-r a\left(s_{2}+q a\right)+1 / 2+r a$, and for $3 \leq m \leq k$, we have $W_{a}^{m}(1 / 2)=-a\left(s_{1}+p a\right)^{m-2} \frac{r\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]}{s_{1}+p a-1}+\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}+p a-1\right)}$;
(II) $\lim _{a \rightarrow 0} a k=0$;
(III) $\lim _{a \rightarrow 0} a\left(s_{1}+p a\right)^{k_{1}}=0$;
(IV) $\lim _{a \rightarrow 0} W_{a}^{k_{1}}\left(\frac{1}{2}\right)=\frac{1}{2}$.

Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.
By the definition of $k$, we have:

$$
\begin{align*}
0 \leq & -a\left(s_{1}+p a\right)^{k-2} \frac{r\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]}{s_{1}+p a-1} \\
& +\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}+p a-1\right)} . \tag{8}
\end{align*}
$$

The inequality (8) implies $a\left(s_{1}+p a\right)^{k-2} \leq \frac{s_{1}-1+p a-2 r a}{2 r\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]}$, thus

$$
\begin{aligned}
a k \leq \quad & a \frac{\ln \left(s_{1}-1+p a-2 r a\right)-\ln 2+2 \ln \left(s_{1}+p a\right)-\ln r-\ln a}{\ln \left(s_{1}+p a\right)} \\
& -a \frac{\ln \left(2 r\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]\right)}{\ln \left(s_{1}+p a\right)}, \\
a\left(s_{1}+p a\right)^{k_{1}} \leq \quad & \frac{\left(s_{1}-1+p a-2 r a\right)\left(s_{1}+p a\right)^{2}}{2 r\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]\left(s_{1}+p a\right)^{k-k_{1}}},
\end{aligned}
$$

and since $\lim _{a \rightarrow 0} a \ln a=0$, we obtain (II) and (III). (IV) follows from (III).
Now, let us prove (I).
The fixed point slightly less than $1 / 2$ is $x_{l}^{*}=\frac{s_{1}-1+p a-2 r a}{2\left(s_{1}-1+p a\right)}$, and

$$
x_{l}^{*}-W_{a}^{2}(1 / 2)=\frac{r a\left[s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right]}{s_{1}-1+p a}>0
$$

which implies that $W_{a}^{m}(1 / 2)$ are all in the domain of the second branch of $W_{a}$ for $3 \leq m \leq k$. Now, (I) follows by the same reasoning as in Lemma 3,

Lemma 5. If the normalized densities $\left\{h_{a}\right\}_{a<a_{0}}$, for some $a_{0}>0$, are uniformly bounded, then $h_{a} \rightarrow h_{0}$ in $L^{1}$.

Proof. The uniform boundedness implies $\left\{h_{a}\right\}_{a<a_{0}}$ is a weakly precompact set in $L^{1}$. Thus, any limit of $\left\{h_{a}\right\}_{a<a_{0}}$ is a invariant density by Proposition 11.3.1 [2]. At the same time, this limit is an $L^{1}$ function, thus defines an absolutely continuous invariant measure. Since the map $W_{0}$ is exact and has only one acim, we conclude that $h_{a} \rightarrow h_{0}$ in $L^{1}$.

Now, we will prove Theorem 2 .
The main idea of the proof is the following: since non-normalized densities $\left\{f_{a}\right\}$ are uniformly bounded (formulas (9) 10, 11) ), it is enough to show that $\left\{\int_{0}^{1} f_{a} d \lambda\right\}$ are uniformly separated from zero.

For small $a$, by Lemma 2] (and then both $\Lambda_{l}$ and $\Lambda_{h}$ ) can be either positive or negative. Thus, we can have the following cases.

Case (i): $\Lambda_{l}<0$ :
Comparing with (4) and (3), we see that for the functions $\widehat{f}_{l}=1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{l} g_{h}$ and $\widehat{f_{h}}=1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{h} g_{l}$, we have

$$
\begin{equation*}
\widehat{f_{l}} \leq f_{a} \leq \widehat{f_{h}} \tag{9}
\end{equation*}
$$

Note that $\widehat{f_{l}}$ and $\widehat{f_{h}}$ have the same form as $f_{l}$ and $f_{h}$ in Section 3.3.1 so their representations as combinations of functions $\chi_{j}, j=1, \ldots, k_{1}$ and $\chi_{c}$ are similar to
that of $f_{l}$ and $f_{h}$. At the same time, now we have $\frac{1}{s_{1}}+\frac{1}{s_{2}}<1$, so the representation is as follows:

$$
\begin{aligned}
\widehat{f}_{l}= & \left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{l}+1\right) \chi_{1}+\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{l} \sum_{j=2}^{k_{1}} \frac{\chi_{j}}{\left(s_{1}+p a\right)^{j-1}} \\
& +\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+p a\right)^{2}} \Lambda_{l} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+1\right) \chi_{c} \\
& +\frac{\frac{s_{1}+s_{2}+p a+q a}{s_{2}+q a} \Lambda_{l}}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}}, \\
\widehat{f_{h}}= & \left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right) \chi_{1}+\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \sum_{j=2}^{k_{1}} \frac{\chi_{j}}{\left(s_{1}+p a\right)^{j-1}} \\
& +\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+1\right) \chi_{c} .
\end{aligned}
$$

(3) implies that all the coefficients in the representation of $\widehat{f_{l}}$ and $\widehat{f_{h}}$ are negative for sufficiently small $a$.

We use the same notations $J_{1}, J_{2}$ and $J_{3}$ as in Section 3.3.2. First, we do the calculations assuming that $\vartheta=1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right) \neq 0$.

We will calculate the integrals of $\widehat{f}_{h}$ over each of $J_{1}, J_{2}$ and $J_{3}$, and use them to normalize $\widehat{f_{h}}$. We have

$$
\begin{aligned}
\widehat{C}_{1}= & \int_{J_{1}} \widehat{f}_{h} d \lambda=\int_{J_{1}}\left[\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right] \chi_{1} d \lambda \\
= & {\left[\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right] W_{a}^{k_{1}}\left(\frac{1}{2}\right)=\left[\frac{\kappa}{1-(\kappa+\eta)}+1\right] W_{a}^{k_{1}}\left(\frac{1}{2}\right) } \\
= & {\left[\frac{s_{1} s_{2}^{2}-s_{1}-s_{2}-s_{2}^{2}}{1-(\kappa+\eta))\left(s_{2}+q a\right)^{2}\left(s_{1}+p a-1\right)}\right.} \\
& +\frac{a\left(2 q s_{1} s_{2}+p s_{2}^{2}-2 q s_{2}-p-q\right)}{(1-(\kappa+\eta))\left(s_{2}+q a\right)^{2}\left(s_{1}+p a-1\right)} \\
& \left.+\frac{a^{2}\left(2 p q s_{2}-q^{2}+q^{2} s_{1}\right)+p q^{2} a^{3}}{(1-(\kappa+\eta))\left(s_{2}+q a\right)^{2}\left(s_{1}+p a-1\right)}\right] W_{a}^{k_{1}}\left(\frac{1}{2}\right) .
\end{aligned}
$$

Using Lemma 4. we have

$$
\lim _{a \rightarrow 0} \widehat{C}_{1}=\frac{1}{2} \frac{\frac{s_{1} s_{2}^{2}-s_{1}-s_{2}-s_{2}^{2}}{s_{2}^{2}\left(s_{1}-1\right)}}{1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)}=\frac{1}{2} \frac{1-\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}}{1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)}
$$

On the interval $J_{2}$, the integral of $\widehat{f_{h}}$ is:

$$
\begin{aligned}
\widehat{C}_{2}=\int_{J_{2}} \widehat{f}_{h} d \lambda= & \int_{J_{2}}\left[\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)} \Lambda_{h}+1\right] \chi_{1} d \lambda \\
& +\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \sum_{j=2}^{k_{1}} \int_{J_{2}} \frac{\chi_{j}}{\left(s_{1}+p a\right)^{j-1}} d \lambda
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1-\eta}{1-(\kappa+\eta)}\left(\frac{1}{2}+r a-W_{a}^{k_{1}}\left(\frac{1}{2}\right)\right) \\
& +\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h}\left[\frac{r a\left(s_{2}+q a\right)}{s_{1}+p a}+\frac{r a\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-2}}\right)}{\left(s_{1}+p a-1\right)^{2}}\right. \\
& \left.+\frac{a\left(k_{1}-2\right)}{s_{1}+p a} \frac{r\left(s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q+p q a\right)\right)}{s_{1}+p a-1}\right] .
\end{aligned}
$$

Using Lemma 4, we have $\lim _{a \rightarrow 0} \widehat{C}_{2}=0$.
On the interval $J_{3}$, the integral of $\widehat{f_{h}}$ is:

$$
\begin{aligned}
\widehat{C}_{3}=\int_{J_{3}} \widehat{f}_{h} d \lambda & =\int_{J_{3}}\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \Lambda_{h} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+1\right) \chi_{c} d \lambda \\
& =\left[\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}\right) \frac{\eta}{1-(\kappa+\eta)}+1\right]\left(\frac{1}{2}-r a\right) \\
& =\frac{\frac{s_{1} s_{2}-s_{1}-s_{2}+a\left(q s_{1}+p s_{2}-p-q\right)+p q a^{2}}{\left(s_{1}+p a\right)\left(s_{2}+q a\right)}-\frac{\eta}{\left(s_{1}+p a\right)^{k_{1}-1}}}{1-(\kappa+\eta)}\left(\frac{1}{2}-r a\right) .
\end{aligned}
$$

Using Lemma 4 once again, we have

$$
\lim _{a \rightarrow 0} \widehat{C}_{3}=\frac{1}{2} \frac{1-\frac{s_{1}+s_{2}}{s_{1} s_{2}}}{1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)} .
$$

Note that if we define $\widehat{B}=\widehat{C}_{1}+\widehat{C}_{2}+\widehat{C}_{3}$, then

$$
\lim _{a \rightarrow 0} \widehat{B}=\frac{1}{2} \frac{2-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)}{1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)},
$$

which is not 0 . Since $\left\{\widehat{f}_{h}\right\}$ are uniformly bounded, we conclude that the normalized $\left\{\widehat{f}_{h}\right\}$ are also uniformly bounded.

Now, we will show that the normalized $\left\{\widehat{f}_{l}\right\}$ are also uniformly bounded. To this end, let us notice that

$$
\begin{aligned}
\widehat{f}_{h}-\widehat{f}_{l}= & \left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{h} g_{l}-\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda_{l} g_{h} \\
= & \left(1+\frac{s_{1}+p a}{s_{2}+q a}\right)\left(\Lambda_{h}-\Lambda_{l}\right) g_{l}-\Lambda_{l} \frac{1+\frac{s_{1}+p a}{s_{2}+q a}}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}} \\
= & \left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \frac{\frac{\eta}{\left(s_{1}+p a\right)^{k_{1}-1}}}{[1-(\kappa+\eta)]\left[1-\kappa-\eta\left(1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}\right)\right]} g_{l} \\
& -\Lambda_{l} \frac{1+\frac{s_{1}+p a}{s_{2}+q a}}{\left(s_{2}+q a\right)\left(s_{1}+p a-1\right)\left(s_{1}+p a\right)^{k_{1}-1}},
\end{aligned}
$$

where $\left|g_{l}\right| \leq \frac{1}{s_{1}}+\frac{1}{s_{2}\left(s_{1}-1\right)}$ and $\lim _{a \rightarrow 0} \Lambda_{l}=\frac{1}{1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)}$. Thus, $\lim _{a \rightarrow 0} \widehat{f}_{h}-\widehat{f}_{l}=0$. We conclude that the normalized $\left\{\widehat{f}_{l}\right\}$ are uniformly bounded since the normalized $\left\{\widehat{f}_{h}\right\}$ are uniformly bounded. Thus, after normalization, $\left\{f_{a}\right\}$ are also uniformly bounded.

Case (ii): $\Lambda_{l}>0$ :
This case implies that $f_{a}$ given by (4) has the following properties:

$$
\begin{equation*}
f_{a} \geq 1 \tag{10}
\end{equation*}
$$

and all the coefficients of the characteristic functions appearing in (4) are positive. We note that $\Lambda$ is always positive for small $a$. Thus,

$$
\begin{equation*}
f_{a} \leq 1+\left(1+\frac{s_{1}+p a}{s_{2}+q a}\right) \Lambda \sum_{n=1}^{\infty} \frac{1}{|\beta(1 / 2, n)|} \tag{11}
\end{equation*}
$$

which is finite since our maps $\left\{W_{a}\right\}$ are expanding. In view of (10), we conclude that the normalized $\left\{f_{a}\right\}$ are uniformly bounded.

If $\vartheta=1-\left(\frac{s_{1}+s_{2}}{s_{1} s_{2}}+\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}\right)=0$, then we have $\lim _{a \rightarrow 0} \frac{1}{\Lambda_{l}}=\lim _{a \rightarrow 0} \frac{1}{\Lambda_{h}}=0, \Lambda_{l}$ and $\Lambda_{h}$ are still of the same sign. We can renormalize $f_{a}$. Let us take the $\widehat{f}_{h}$ as an example. Multiplying it by $\frac{1}{\Lambda_{h}}$, we obtain

$$
\begin{aligned}
\frac{1}{\Lambda_{h}} \widehat{f}_{h}= & \left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)\left(s_{1}+p a\right)}+\frac{1}{\Lambda_{h}}\right) \chi_{1}+\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \sum_{j=2}^{k_{1}} \frac{\chi_{j}}{\left(s_{1}+p a\right)^{j-1}} \\
& +\left(\frac{s_{1}+s_{2}+p a+q a}{\left(s_{2}+q a\right)^{2}} \frac{1-\frac{1}{\left(s_{1}+p a\right)^{k_{1}-1}}}{s_{1}+p a-1}+\frac{1}{\Lambda_{h}}\right) \chi_{c}
\end{aligned}
$$

Note that the coefficients of $\chi_{1}$ and $\chi_{c}$ converge to $\frac{s_{1}+s_{2}}{s_{1} s_{2}}$ and $\frac{s_{1}+s_{2}}{s_{2}^{2}\left(s_{1}-1\right)}$, respectively. Thus, $\left\{\int_{0}^{1} \frac{1}{\Lambda_{h}} \widehat{f_{h}} d \lambda\right\}$ are separated from 0 . This implies $\left\{\frac{1}{\Lambda_{h}} \widehat{f}_{h}\right\}$ are uniformly bounded. A similar procedure can be applied to $\widehat{f_{l}}$. We conclude that $\left\{\frac{1}{\Lambda} f_{a}\right\}$ are uniformly bounded.

## 4. Example

One of the important properties of a piecewise expanding transformation of an interval is that its invariant density is bounded away from 0 on its support. The following result was proved, by Keller [11] and by Kowalski 12 .
Theorem 3. Let a transformation $\tau: I \rightarrow I$ be piecewise expanding with $\frac{1}{\left|\tau^{\prime}(x)\right|}$ a function of bounded variation, and let $f$ be a $\tau$-invariant density which can be assumed to be lower semicontinuous. Then there exists a constant $c>0$ such that $\left.f\right|_{\text {supp } f}>c$.

We provide an example showing that this result cannot be generalized to a family of expanding maps, even if they all have this property and converge to a limit map also with this property. Let $d(\cdot, \cdot)$ be the metric on the weak topology of measures.

Example 1. Let us fix

$$
s_{1}=s_{2}=2, p=q=1
$$

For small $a>0$, let $W_{a, r}$ denote the $W_{a}$ maps with varying parameter $r$, and let $\mu_{a, r}$ denote the absolutely continuous invariant measure of $W_{a, r}$. We know that $\mu_{a, r}$ is supported on $[0,1]$ and $W_{a, r}$ with $\mu_{a, r}$ is exact. Using Theorem 11, we know that $\left\{\mu_{a, r}\right\}$ converge $*$-weakly to the measure

$$
\mu_{0, r}=\frac{1}{1+2 r} \mu_{0}+\frac{2 r}{1+2 r} \delta_{\frac{1}{2}} .
$$

Let $r_{n}=n, n=1,2,3, \cdots$. Also, let $\left\{a_{n}\right\}_{1}^{\infty}$ satisfy $r_{n} a_{n}<1 / 2$ and be so small that

$$
d\left(\mu_{a_{n}, r_{n}}, \mu_{0, r_{n}}\right)<\frac{1}{n} .
$$

Now, for the family of maps $\tau_{n}=W_{a_{n}, r_{n}}, n=1,2,3, \cdots, \tau_{n}$ converge to $W_{0}$ with $\left|\tau_{n}^{\prime}(x)\right|>2$, but the invariant densities $\mu_{a_{n}, r_{n}}$ converge to $\delta_{\left(\frac{1}{2}\right)}$. This implies that the invariant densities $\left\{f_{a_{n}, r_{n}}\right\}$ corresponding to $\left\{\mu_{a_{n}, r_{n}}\right\}$ have no uniform positive lower bound.

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