# W-LIKE MAPS WITH VARIOUS INSTABILITIES OF ACIM'S

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ABSTRACT. This paper generalizes the results of [13] and then provides an interesting example. We construct a family of W-like maps  $\{W_a\}$  with a turning fixed point having slope  $s_1$  on one side and  $-s_2$  on the other. Each  $W_a$  has an absolutely continuous invariant measure  $\mu_a$ . Depending on whether  $\frac{1}{s_1} + \frac{1}{s_2}$  is larger, equal or smaller than 1, we show that the limit of  $\mu_a$  is a singular measure, a combination of singular and absolutely continuous measure or an absolutely continuous measure, respectively. It is known that the invariant density of a single piecewise expanding map has a positive lower bound on its support. In Section 4 we give an example showing that in general, for a family of piecewise expanding maps with slopes larger than 2 in modulus and converging to a piecewise expanding map, their invariant densities do not necessarily have a positive lower bound on the support.

### 1. INTRODUCTION

In practice, due to external noise, or roundoff errors in computation, there is a natural interest in the stability of properties of chaotic dynamical systems under small perturbations. If we consider a family of piecewise expanding maps  $\tau_a : I \to I$ , a > 0 with absolutely continuous invariant measures (acim's)  $\mu_a$ , converging to a piecewise expanding map  $\tau_0$  with acim  $\mu_0$ , then under general assumptions  $\mu_a$ 's converge to  $\mu_0$ . One such assumption is that  $\inf |\tau'_a| > 2$  for all a > 0 (see [1], [6], [7] or [10]). This is useful in the study of the metastable systems [15], or to approximate the invariant densities [8].

Keller [9] introduced the family of  $\{W_a\}$  maps that are piecewise expanding, ergodic transformations with a "stochastic singularity", i.e.,  $\mu_a$ 's converge to a singular measure. This occurs because of the existence of diminishing invariant neighborhoods of the turning fixed point. The slopes of the Keller's  $W_a$  maps converge to 2 and -2 on the left and right hand sides of the turning fixed point, respectively.

Given two numbers,  $s_1$  and  $s_2$ , greater than 1, we consider a W-like map with one turning fixed point having slope  $s_1$  on one side and  $-s_2$  on the other. In [13], the authors considered the special case where  $s_1 = s_2 = 2$ . Their perturbed maps  $W_a$  are piecewise expanding with slopes strictly greater than 2 in modulus and are exact with their acim's supported on all of [0, 1]. The standard bounded variation method [2] cannot be applied in this setting as the slopes of the maps in that family are not uniformly bounded away from 2. Other methods, for example, those studied in [3], [12] and [14] cannot be applied either. Using the main result of [5], it can

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be shown that the  $\mu_a$ 's converge to  $\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}$ , where  $\delta_{(\frac{1}{2})}$  is the Dirac measure at point 1/2 and  $\mu_0$  is the acim of the  $W_0$  map. Thus, the family of measures  $\mu_a$ approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map. Similar instability was also shown in [4] for a countable family of transitive Markov maps approaching Keller's  $W_0$  map.

In this paper, we construct a family of maps for which the instability of the acim's has a global character, not a local one. In the more general case considered in this paper, with  $s_1$ ,  $s_2$  not necessarily equal to 2, we will discuss the limits of the acim's  $\mu_a$  of the  $\{W_a\}$  maps. We have three cases:

(I) If 
$$\frac{1}{s_1} + \frac{1}{s_2} > 1$$
, then  $\mu_a$ 's converge \*-weakly to  $\delta_{(\frac{1}{2})}$ .

(II) If 
$$\frac{1}{s_1} + \frac{1}{s_2} = 1$$
, then  $\mu_a$ 's converge \*-weakly to

$$\frac{(qs_1 + ps_2 - p - q)(s_2 + 2)}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2}\mu_0 + \frac{2rs_1s_2^2}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2}\delta_{(\frac{1}{2})}$$

where p, q and r are parameters defining our family of maps.

(III) If 
$$\frac{1}{s_1} + \frac{1}{s_2} < 1$$
, then  $\mu_a$ 's converge to  $\mu_0$ 

Additionally, in Theorem 2, we prove that in case (III) the densities of the  $\mu_a$ 's are uniformly bounded. The first case of our result contains the example in which Keller [9] obtained the "stochastic singularity." In the second case, the limit measure is a combination of an absolutely continuous and a singular measure, and this combination is varying according to p, q and r for fixed  $s_1$  and  $s_2$ . This is a generalization of the result of [13]. In the third case, we have a map with a stable acim.

At the end of the paper, we use our main results to provide an interesting example. Keller [11] and Kowalski [12] proved that for a piecewise expanding map  $\tau: I \to I$  with  $\frac{1}{|\tau'(x)|}$  being a function of bounded variation, the density of the acim of  $\tau$  has a uniform positive lower bound on its support. We construct a family of piecewise expanding, piecewise linear maps  $\tau_n$  such that  $\tau_n$  are exact on [0, 1],  $\tau_n$ converge to  $\tau = W_0$  ( $s_1 = s_2 = 2$ ),  $|\tau'_n| > 2$  for all n but the densities of the acims  $\mu_n$ 's do not have a uniform positive lower bound.

In Section 2, we introduce our family of  $W_a$  maps and state the main result. In Section 3 we present the proofs. In Section 4, we show the example related to the results of Keller [11] and Kowalski [12].

# 2. Family of $W_a$ maps and the main result

Let  $s_1, s_2 > 1$  and p, q, r > 0. We consider the family  $\{W_a : 0 \le a\}$  of maps of [0, 1] onto itself defined by

(1) 
$$W_{a}(x) = \begin{cases} 1 - \frac{2(s_{1}+pa)}{s_{1}-1+pa-2ra}x, & \text{for } 0 \leq x < \frac{1}{2} - \frac{\frac{1}{2}+ra}{s_{1}+pa};\\ (s_{1}+pa)(x-1/2) + 1/2 + ra, & \text{for } \frac{1}{2} - \frac{\frac{1}{2}+ra}{s_{1}+pa} \leq x < 1/2;\\ -(s_{2}+qa)(x-1/2) + 1/2 + ra, & \text{for } 1/2 \leq x < \frac{1}{2} + \frac{\frac{1}{2}+ra}{s_{2}+qa};\\ 1 + \frac{2(s_{2}+qa)}{s_{2}-1+qa-2ra}(x-1), & \text{for } \frac{1}{2} + \frac{\frac{1}{2}+ra}{s_{2}+qa} \leq x \leq 1. \end{cases}$$

For each choice of  $s_1$ ,  $s_2 > 1$ , p, q, r > 0, we consider only a > 0 such that  $0 \le W_a(x) \le 1$  for  $x \in [0, 1]$ .

An example of a  $W_a$  map is shown in Fig.1. Fig.1(a) is the unperturbed  $W_0$  map with turning fixed point at 1/2 and  $s_1 = 3/2$ ,  $s_2 = 3$ . Fig.1(b) is the perturbed map  $W_a$ , with a = 0.05, r = 2, p = 3, q = 2. The slope of the second branch is  $s_1 + pa = 1.65$ , the slope of the third branch is  $s_2 + qa = 3.1$ , and  $W_{0.05}(1/2) = 1/2 + ra = 0.6$ .



FIGURE 1. The *W*-like maps with  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ : (a)  $W_0$  with  $s_1 = 3/2$  and  $s_2 = 3$ , (b)  $W_a$  with  $s_1 = 3/2$ ,  $s_2 = 3$ ; a = 0.05; r = 2, p = 3, q = 2; also several initial points of the trajectory of 1/2.

Every  $W_a$  has a unique absolutely continuous invariant measure  $\mu_a$  since all the slopes are greater than 1 in modulus. We will show later that, for  $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$ ,  $\mu_a$  is supported on [0,1] and for  $\frac{1}{s_1} + \frac{1}{s_2} > 1$  it is supported on a subinterval around 1/2.  $W_a$  is an exact map with the measure  $\mu_a$ . Let  $h_a$  denote the normalized density of  $\mu_a$ ,  $a \geq 0$ . Since the  $W_0$  map is a Markov one, it is easy to check that

(2) 
$$h_0 = \begin{cases} \frac{2s_1(s_2+1)}{2s_1s_2+s_1-s_2} & \text{for } 0 \le x < 1/2 ; \\ \frac{2s_2(s_1-1)}{2s_1s_2+s_1-s_2} & \text{for } 1/2 \le x \le 1 . \end{cases}$$

Our main result is the following theorem

**Theorem 1.** As  $a \to 0$  the measures  $\mu_a$  converge \*-weakly to the measure (I)  $\delta_{(\frac{1}{2})}$ , if  $\frac{1}{s_1} + \frac{1}{s_2} > 1$ ; (II)  $\frac{(qs_1+ps_2-p-q)(s_2+2)}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\mu_0 + \frac{2rs_1s_2^2}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\delta_{(\frac{1}{2})}$ , if  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ ; (III)  $\mu_0$ , if  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ , where  $\delta_{(\frac{1}{2})}$  is the Dirac measure at point 1/2.

The proof relies on the general formula for invariant densities of piecewise linear maps [5] and direct calculations. Most objects and quantities we use depend on the parameter a. We suppress a from the notation to make it simpler.

In case (III), we actually prove a little more:

**Theorem 2.** If  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ , then the normalized invariant densities  $\{h_a\}$  are uniformly bounded for given p, q and r. Consequently, we obtain Theorem 1(III).

## 3. Proofs

This section contains the proofs of Theorems 1 and 2, divided into a number of steps.

3.1. Assume  $\frac{1}{s_1} + \frac{1}{s_2} > 1$ . Let

$$x_l^* = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)}$$

and

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$$x_r^* = \frac{s_2s_1 - s_2 + (2rs_1 - q + ps_2 + qs_1)a + (2rp + pq)a^2}{2(s_1 - 1 + pa)(s_2 + qa)}$$

 $x_l^*$  is the fixed point on the second branch of  $W_a$ , and  $x_r^*$  is the preimage of  $x_l^*$  under the third branch of  $W_a$ . Both  $x_r^*$  and  $x_l^*$  converge to  $\frac{1}{2}$  as a approaches 0. For small a, we have

$$W_a(1/2) - x_r^* = \frac{ra\left[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)\right]}{(s_1 - 1 + pa)(s_2 + qa)} < 0$$

In this case, we have  $W_a([x_l^*, x_r^*]) \subseteq [x_l^*, x_r^*]$ .  $W_a|_{[x_l^*, x_r^*]}$  is a skewed tent map with  $W_a(1/2) > 1/2$ ; it is known that with acim  $\mu_a$ , it is exact on  $[x_l^*, W_a(1/2)]$ . Since  $\mu_a$  is concentrated on  $[x_l^*, x_r^*]$ , we conclude that  $\mu_a$  converge \*-weakly to  $\delta_{(\frac{1}{2})}$ . This proves Theorem 1(I).

Fig.2 shows an example with  $a = 0.05, r = 2, p = 3, q = 2; s_1 = 4/3, s_2 = 5/2.$ 



FIGURE 2. The  $W_a$  map with  $\frac{1}{s_1} + \frac{1}{s_2} > 1$ 

3.2. Formula for the non-normalized invariant density of  $W_a$  if  $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$ . An example of a map  $W_a$  is shown in Fig.1. We have the following proposition.

**Proposition 1.** For  $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$ , the map  $W_a$  has an absolutely continuous invariant measure  $\mu_a$  supported on [0,1] and the map  $W_a$  with respect to  $\mu_a$  is exact.

*Proof.*  $W_a$  is a piecewise expanding transformation. From the general theory (see for example [2]), it follows that it is enough to show that the images  $W_a^n(J)$  grow to cover all [0,1] as  $n \to \infty$ , for any interval  $J \subset [0,1]$ . Since  $W_a$  is expanding,  $W_a^n(J)$ grow until some image  $W_a^{n_0}(J)$  contains an internal partition point. If this point is not 1/2, then  $W_a^{n_0+2}(J)$  contains the repelling fixed point 1. Then its images grow

to cover all of [0, 1]. If this point is 1/2, we proceed as follows. First, assume that  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ . Consider a small neighborhood  $J = (z_1, z_2)$  around 1/2 with length  $\ell$ , then

$$\min_{z_2-z_1=\ell} \max\left\{ (\frac{1}{2}-z_1)(s_1+pa), (z_2-\frac{1}{2})(s_2+qa) \right\} = \frac{1}{\frac{1}{s_1+pa}+\frac{1}{s_2+qa}} \ell > \ell.$$

Thus, the interval J will grow until its image covers two partition points of  $W_a$ . Then the second iteration afterward will cover [0, 1]. Therefore,  $W_a$  is exact with respect ot  $\mu_a$ .

Assume  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ . If  $a \neq 0$ , then  $\frac{1}{\frac{1}{s_1 + pa} + \frac{1}{s_2 + qa}} > 1$ , which implies  $W_a$  is exact with respect to  $\mu_a$ . In the case a = 0, we first note that 1/2 is a turning fixed point. Take again a small interval  $J = (z_1, z_2) \ni 1/2$ . Its image is an interval (z, 1/2). It will grow under iteration and its iterations still contain 1/2. It will grow until its image covers another partition point of  $W_a$ . Then, the second iteration afterward will covers all of [0, 1]. Thus,  $W_a$  is again exact with respect to  $\mu_a$ .

We adapt the general formulas of [5] to our case and obtain the following lemma:

Lemma 1. (I) N=4, K=2, L=0; (II)  $\alpha = (1, 1/2 + ra, 1/2 + ra, 1)$ ,  $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)$ , where  $\beta_1 = -\frac{2(s_1+pa)}{s_1-1+pa-2ra}$ ,  $\beta_2 = s_1 + pa$ ,  $\beta_3 = -(s_2 + qa)$  and  $\beta_4 = \frac{2(s_2+qa)}{s_2-1+qa-2ra}$ ,  $\gamma = (0, 0, 0, 0)$ ; (III) The digits  $A = (a_1, a_2, a_3, a_4)$ , where  $a_1 = -1, a_2 = \frac{s_1-1+pa-2ra}{2}$ ,  $a_3 = -\frac{s_2+1+qa+2ra}{2}$ ,  $a_4 = \frac{s_2+1+qa+2ra}{s_1-1+pa-2ra}$ ; (IV) There are two  $c_i$ 's, which are  $c_1 = (1/2, 2)$  and  $c_2 = (1/2, 3)$ , and  $j(c_1) = 2$ ,

(IV) There are two  $c_i$ 's, which are  $c_1 = (1/2, 2)$  and  $c_2 = (1/2, 3)$ , and  $j(c_1) = 2$ ,  $j(c_2) = 3$ . Then,  $W_u = \{c_1, c_2\}, W_l = \emptyset$ ,  $U_l = \{c_2\}, U_r = \{c_1\}$ ; (V)  $\beta(c_1, 1) = s_1 + pa$  since  $j(c_1) = 2$ , then  $\beta(c_1, 2) = -(s_1 + pa)(s_2 + qa)$  and  $\beta(c_1, k) = -(s_2 + qa)(s_1 + pa)^{k-1}$  up to some k which is the first moment j when the  $W_a^j(1/2)$  is less than  $\frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}$ , and is the same one defined in Lemma 4; (VI)  $\beta(c_2, 1) = -(s_2 + qa)$  since  $j(c_2) = 3$ , then  $\beta(c_2, 2) = (s_2 + qa)^2$  and  $\beta(c_2, k) = (s_2 + qa)^2(s_1 + pa)^{k-2}$  up to the same k in part (e),  $W_a^n(c_1) = W_a^n(c_2)$  for all n; (VII) Based on (VI), we have the following for the matrix  $S = (S_{i,j})_{1 \le i,j \le 2}$ : For  $c_1 \in U_r$ 

$$\begin{split} S_{1,1} &= \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1,n)>0))\delta(W_a^n(c_1)>1/2) + \delta(\beta((c_1,n)<0))\delta(W_a^n(c_1)<1/2)}{|\beta(c_1,n)|} \\ S_{1,2} &= \sum_{n=1}^{\infty} \frac{\delta(\beta((c_1,n)>0))\delta(W_a^n(c_1)>1/2) + \delta(\beta((c_1,n)<0))\delta(W_a^n(c_1)<1/2)}{|\beta(c_1,n)|} \\ For \ c_2 \in U_l \\ S_{2,1} &= \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2,n)<0))\delta(W_a^n(c_2)>1/2) + \delta(\beta((c_2,n)>0))\delta(W_a^n(c_2)<1/2)}{|\beta(c_2,n)|} \\ \end{split}$$

$$S_{2,2} = \sum_{n=1}^{\infty} \frac{\delta(\beta((c_2, n) < 0))\delta(W_a^n(c_2) > 1/2) + \delta(\beta((c_2, n) > 0))\delta(W_a^n(c_2) < 1/2)}{|\beta(c_2, n)|}$$

**Remark 1.** It follows from (V, VI) of Lemma 1 that

$$S_{1,1} = S_{1,2}$$
,  $S_{2,1} = S_{2,2}$  and  $S_{1,1} = \frac{s_2 + qa}{s_1 + pa} S_{2,2}$ .

Let Id be the  $2 \times 2$  identity matrix and let V = [1,1]. Then, for the solution,  $D = [D_1, D_2], of the system :$ 

$$\left(-S^T + Id\right)D^T = V^T,\tag{1}$$

we have  $D_1 = D_2$ . Let us denote them by  $\Lambda$ .

Let  $I_1, I_2, I_3, I_4$  be the partition of I = [0, 1] into maximal intervals of monotonic-ity of  $W_a$ :  $I_1 = [0, \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa)}), I_2 = (\frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa)}, 1/2), I_3 = (1/2, \frac{s_2 + 1 + qa + 2ra}{2(s_2 + qa)})$ and  $I_4 = (\frac{s_2 + 1 + qa + 2ra}{2(s_2 + qa)}, 1]$ . We define the following index function:

$$j(x) = j$$
 for  $x \in I_j, j = 1, 2, 3, 4$ ,

and

$$j(c_1) = 2, j(c_2) = 3.$$

We define the cumulative slopes for iterates of points as follows:

$$\beta(x,1) = \beta_{j(x)}, \text{ and } \beta(x,n) = \beta(x,n-1) \cdot \beta_{j(W_a^{n-1}(x))}, \quad n \ge 2.$$

In particular, we have

$$\beta(1/2,n) = (s_1 + pa) \cdot W'_a(W_a(1/2)) \cdot W'_a(W_a^2(1/2)) \cdots W'_a(W_a^{n-1}(1/2)) ,$$

which is the cumulative slope along the n steps of trajectory of 1/2. Recall that k is the first moment j when the  $W_a^j(1/2)$  is less than  $\frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}$ . Let  $k_1 = [\frac{2}{3}k]$  (the integer part of 2k/3). Note that  $k_1 \to \infty$  as  $a \to 0$ . Let

$$\chi^{s}(t,x) = \begin{cases} \chi_{[0,x]} & \text{for } t > 0 ;\\ \chi_{[x,1]} & \text{for } t < 0 . \end{cases}$$

Now, we can obtain the following formula for  $f_a$ :

Lemma 2. Let

$$f_a = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda\left(\sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|}\right)$$

Then  $f_a$  is  $W_a$  invariant non-normalized density. Furthermore, for small a > 0, we have:

(I) If  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ , then  $\Lambda < -1$ ; (II) If  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ , the sign of  $\Lambda$  depends on  $s_1$  and  $s_2$ , can be either positive or negative depending on the sign of  $\vartheta = 1 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right) = 1 - \frac{s_1 + s_2}{s_1 s_2} \left(1 + \frac{s_1}{s_2(s_1 - 1)}\right)$ . The case when  $\vartheta = 0$  is discussed at the end of Section 3.

*Proof.* By the Theorem 2 in [5], it follows from (IV, V, VI) of Lemma 1 that:

$$\begin{aligned} f_a &= 1 + D_1 \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1, n), W_a^n(c_1))}{|\beta(c_1, n)|} + D_2 \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2, n), W_a^n(c_2))}{|\beta(c_2, n)|} \\ &= 1 + \Lambda \sum_{n=1}^{\infty} \frac{\chi^s(\beta(c_1, n), W_a^n(1/2))}{|\beta(c_1, n)|} + \Lambda \sum_{n=1}^{\infty} \frac{\chi^s(-\beta(c_2, n), W_a^n(1/2))}{|\beta(c_2, n)|} \\ &= 1 + (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda \left( \sum_{n=1}^{\infty} \frac{\chi^s(\beta(1/2, n), W_a^n(1/2))}{|\beta(1/2, n)|} \right). \end{aligned}$$

Since

$$S_{1,1} \geq \frac{1}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{n=1}^{k_1 - 1} \frac{1}{(s_1 + pa)^n} = \frac{1}{s_1 + pa} + \frac{1}{s_2 + qa} \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1}$$

$$S_{1,1} \leq \frac{1}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{n=1}^{\infty} \frac{1}{(s_1 + pa)^n} = \frac{1}{s_1 + pa} + \frac{1}{s_2 + qa} \frac{1}{s_1 + pa - 1} ,$$

and  $\Lambda = \frac{1}{1 - \frac{s_1 + s_2 + pa + qa}{s_2 + qa} S_{1,1}}$ , we have

(3) 
$$\Lambda_l = \frac{1}{1 - (\kappa + \eta (1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}))} \le \Lambda \le \frac{1}{1 - (\kappa + \eta)} = \Lambda_h ,$$

where  $\kappa = \frac{s_1+s_2+p_a+q_a}{(s_1+p_a)(s_2+q_a)}$ ,  $\eta = \frac{s_1+s_2+p_a+q_a}{(s_2+q_a)^2(s_1+p_a-1)}$ . To obtain the upper bound of  $S_{1,1}$ , we assume  $s_1 < s_2$ . For  $s_1 > s_2$  the calculations differ slightly.

(I) Note that for small a both estimates  $\Lambda_l$  and  $\Lambda_h$  are smaller than -1 since both  $\kappa$  and  $\eta$  are smaller than 1 and close to 1. Furthermore, as a approaches 0, both  $\kappa$  and  $\eta$  approach 1.

(II) As a approaches 0,  $\kappa$  and  $\eta$  approach  $\frac{s_1+s_2}{s_1s_2}$  and  $\frac{s_1+s_2}{s_2^2(s_1-1)}$ , respectively. Again, note that for small a, estimates  $\Lambda_l$  and  $\Lambda_h$  can be either positive or negative, and they have the same sign.  $\square$ 

For small positive a, the first image of 1/2 is  $W_a(1/2) = 1/2 + ra$  and the next one falls just below the fixed point  $x_l^*$  slightly less than 1/2. The following images form a decreasing sequence until they go below  $\frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}$ . Since k is the first iteration j when the  $W_a^j(1/2)$  is less than  $\frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}$ , the consecutive cumulative slopes of 1/2 are

$$(s_1 + pa), -(s_1 + pa)(s_2 + qa), -(s_1 + pa)^2(s_2 + qa), \dots, -(s_1 + pa)^{k-1}(s_2 + qa)$$
,  
and

and

(4) 
$$f_a = 1 + (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda \left( \frac{\chi_{[0, W_a(1/2)]}}{(s_1 + pa)} + \sum_{j=2}^k \frac{\chi_{[W_a^j(1/2), 1]}}{(s_1 + pa)^{j-1}(s_2 + qa)} + \dots \right).$$

3.3. Estimates, normalizations and integrals on  $f_a$  for  $\frac{1}{s_1} + \frac{1}{s_2} \leq 1$ . Remembering that  $k = \min\{j \ge 1 : W_a^j(1/2) \le \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}\}$  and  $k_1 = [\frac{2}{3}k]$  (the integer part of 2k/3), we will give the estimates on  $f_a$ .

Let us define

$$g_l = \frac{\chi_{[0,W_a(1/2)]}}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1 + pa)^{j-1}} ,$$

and

$$g_h = g_l + \frac{1}{s_2 + qa} \sum_{j=0}^{\infty} \frac{1}{(s_1 + pa)^{j+k_1}} = g_l + \frac{1}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1 - 1}}$$

Also, let  $\chi_1 = \chi_{[0,1/2+ra]}, \chi_j = \chi_{[W^j_a(1/2),1/2+ra]}, j = 2, 3, \dots, k_1, \chi_c = \chi_{(1/2+ra,1]}.$ 

3.3.1. Estimates on  $f_a$  if  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ . We have the following lemma:

**Lemma 3.** For the family of  $W_a$  maps, if  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ , we have (I)  $W_a(1/2) = 1/2 + ra$ ,  $W_a^2(1/2) = -ra(s_2 + qa) + 1/2 + ra$ , and for  $3 \le m \le k$ , we have  $W_a^m(1/2) = -a^2(s_1 + pa)^{m-2} \frac{r(qs_1 + ps_2 - p - q) + rpq_a}{s_1 + pa - 1} + \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa - 1)}$ ;  $(II) \lim_{a \to 0} ak = 0;$  $(III) \lim_{a \to 0} \frac{1}{a(s_1 + pa)^k} = 0;$   $(IV) \lim_{a \to 0} \frac{1}{a(s_1 + pa)^{k_1}} = 0;$   $(V) \lim_{a \to 0} a^2 (s_1 + pa)^{k_1} = 0;$ (VI)  $\lim_{a \to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{2}.$ 

*Proof.* Suppose (I) is true. Let us first prove that (II) and (III) are true. By the definition of k, we have:

(5)  

$$0 \le -a^2(s_1 + pa)^{k-2} \frac{r(qs_1 + ps_2 - p - q) + rpqa}{s_1 + pa - 1} + \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa - 1)} \le \frac{1}{2} - \frac{1/2 + ra}{s_1 + pa}$$

The first inequality of (5) implies that  $(s_1 + pa)^{k-2} \le \frac{s_1 - 1 + pa - 2ra}{2a^2(r(qs_1 + ps_2 - p - q) + rpqa)}$ , thus  $\ln(s_1 - 1 + pa - 2ra) - \ln 2 - 2\ln a - \ln(r(qs_1 + ps_2 - p - q) + rpqa)$ 

$$ak \le a \frac{\ln(s_1 - 1 + pa - 2ra) - \ln 2 - 2\ln a - \ln(r(qs_1 + ps_2 - p - q) + rpqa)}{\ln(s_1 + pa)} + 2a,$$

$$a \le \frac{\sqrt{s_1 - 1 + pa - 2ra}(s_1 + pa)}{\sqrt{2(r(qs_1 + ps_2 - p - q) + rpqa)}(s_1 + pa)^{k/2}},$$

$$a^2(s_1 + pa)^{k_1} \le \frac{(s_1 - 1 + pa - 2ra)(s_1 + pa)^2}{2(r(qs_1 + ps_2 - p - q) + rpqa)(s_1 + pa)^{k - k_1}},$$
so we obtain (V) and since line along a properties of the pr

so we obtain (V), and since  $\lim_{a\to 0} a \ln a = 0$ , we obtain (II). The second inequality of (5) implies

$$\frac{1}{a(s_1+pa)^{k-2}} \le \frac{2a(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)}{s_1-1+pa-2ra}$$

Therefore,

(6) 
$$\frac{1}{a(s_1+pa)^k} \le \frac{2a(r(qs_1+ps_2-p-q)+rpqa)}{(s_1-1+pa-2ra)(s_1+pa)}$$

and as  $a \to 0$ , we obtain (III).

On the other hand, (6) implies

$$\frac{1}{a(s_1+pa)^{k_1}} \leq \frac{2a(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)^{k-k_1}}{(s_1+pa-2ra-1)(s_1+pa)} \\ \leq \frac{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)^{k-k_1}}}{\sqrt{s_1+pa-2ra-1}(s_1+pa)^{k/2}} \\ = \frac{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)}}{\sqrt{s_1+pa-2ra-1}(s_1+pa)^{k_1-k/2}}.$$

By the definition of  $k_1$ , we obtain (IV). (VI) follows from (V).

Now, let us prove (I).

The fixed point slightly less than 1/2 is  $x_l^* = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)}$ , and

$$x_l^* - W_a^2(1/2) = \frac{ra^2(q(s_1 - 1) + p(s_2 - 1) + apq)}{s_1 - 1 + pa} > 0,$$

which implies that  $W_a^m(1/2)$  are all in the domain of the second branch of  $W_a$  for  $3 \le m \le k$ . For a linear map  $T(x) = m_0 x + b_0$ , we have  $T^n(x) = m_0^n x + \frac{m_0^n - 1}{m_0 - 1} b_0$ . This proves (I).

Using (4) and (3) we see that for the functions  $f_l = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_l g_h$  and  $f_h = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_h g_l$ , we have

(7) 
$$f_l \le f_a \le f_h \; .$$

Now, we will represent functions  $f_l$  and  $f_c$  as combinations of functions  $\chi_j$ ,  $j = 1, \ldots, k_1$  and  $\chi_c$ . After some calculations, we obtain

$$\begin{split} f_l &= 1 + (1 + \frac{s_1 + pa}{s_2 + qa}) \Lambda_l \left( \frac{\chi_{[0, W_a(1/2)]}}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2), 1]}}{(s_1 + pa)^{j-1}} \right. \\ &+ \frac{1}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1 - 1}} \right) \\ &= \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_l + 1 \right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_l \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}} \right. \\ &+ \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + pa)^2} \Lambda_l \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + 1 \right) \chi_c \\ &+ \frac{\frac{s_1 + s_2 + pa + qa}{s_2 + qa}}{(s_2 + qa)(s_1 + pa - 1)(s_1 + pa)^{k_1 - 1}} \,, \end{split}$$

$$\begin{split} f_h &= 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_h \bigg(\frac{\chi_{[0, W_a(1/2)]}}{s_1 + pa} + \frac{1}{s_2 + qa} \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2), 1]}}{(s_1 + pa)^{j-1}} \bigg) \\ &= \left(\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)}\Lambda_h + 1\right)\chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2}\Lambda_h \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}} \\ &+ \left(\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2}\Lambda_h \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + 1\right)\chi_c \; . \end{split}$$

In the case we are considering, (3) implies that both  $\Lambda_l$ ,  $\Lambda_h$  are smaller than -1. Using this, one can show that all the coefficients in the representation of  $f_l$  and  $f_h$  are negative for sufficiently small a. For example, let us consider the coefficient of  $\chi_1$  in  $f_h$ :

$$\frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)}\Lambda_h + 1 = \frac{\kappa}{1 - (\kappa + \eta)} + 1 = \frac{1 - \eta}{1 - (\kappa + \eta)} < 0.$$

3.3.2. Normalizations and integrals if  $\frac{1}{s_1} + \frac{1}{s_2} = 1$ . Let us define  $J_1 = [0, W_a^{k_1}(1/2)]$ ,  $J_2 = (W_a^{k_1}(1/2), 1/2 + ra]$ ,  $J_3 = (1/2 + ra, 1]$ . We will calculate integrals of  $f_h$  over each of these intervals  $J_1$ ,  $J_2$  and  $J_3$ , and use them to normalize  $f_h$ . We have

$$\begin{split} C_1 &= \int_{J_1} f_h \ d\lambda = \int_{J_1} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \ d\lambda \\ &= \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] W_a^{k_1} (\frac{1}{2}) = \left[ \frac{\kappa}{1 - (\kappa + \eta)} + 1 \right] W_a^{k_1} (\frac{1}{2}) \\ &= \left[ \frac{a(2qs_1s_2 + ps_2^2 - 2qs_2 - p - q)}{(1 - (\kappa + \eta))(s_2 + qa)^2(s_1 + pa - 1)} \right] \\ &+ \frac{a^2(2pqs_2 - q^2 + q^2s_1) + pq^2a^3}{(1 - (\kappa + \eta))(s_2 + qa)^2(s_1 + pa - 1)} \right] W_a^{k_1} (\frac{1}{2}) \ . \end{split}$$

Using Lemma 3, we obtain

$$\lim_{a \to 0} \frac{C_1}{a} = -\frac{2qs_1s_2 + ps_2^2 - 2qs_2 - p - q}{2s_2^2(s_1 - 1)} = -\frac{2qs_1 + ps_2^2 - p - q}{2s_2s_1}$$

In the same way, we can see that for any  $0 < \theta < 1/2$ , we obtain

$$\lim_{a \to 0} \frac{1}{a} \int_0^{\theta} f_h d\lambda = -\frac{2qs_1 + ps_2^2 - p - q}{s_2 s_1} \theta \; .$$

On the interval  $J_2$ , the integral of  $f_h$  is:

$$\begin{aligned} C_2 &= \int_{J_2} f_h \, d\lambda &= \int_{J_2} \left[ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right] \chi_1 \, d\lambda \\ &+ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \int_{J_2} \frac{\chi_j}{(s_1 + a)^{j-1}} \, d\lambda \\ &= \frac{1 - \eta}{1 - (\kappa + \eta)} \left( \frac{1}{2} + ra - W_a^{k_1} \left( \frac{1}{2} \right) \right) \\ &+ \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \left[ \frac{ra(s_2 + qa)}{s_1 + pa} + \frac{ra(1 - \frac{1}{(s_1 + pa)^{k_1 - 2}})}{(s_1 + pa - 1)^2} \right] \\ &+ \frac{a^2(k_1 - 2)}{s_1 + pa} \frac{r(qs_1 + ps_2 - p - q) + rpqa}{s_1 + pa - 1} \right]. \end{aligned}$$

Using Lemma 3, we obtain

$$\lim_{a \to 0} \frac{C_2}{a} = -\frac{s_1 + s_2}{s_2^2} \left[ \frac{rs_2}{s_1} + \frac{r}{(s_1 - 1)^2} \right] = -rs_2$$

On the interval  $J_3$ , the integral of  $f_h$  is:

$$\begin{split} C_3 &= \int_{J_3} f_h \ d\lambda &= \int_{J_3} \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + 1 \right) \chi_c \ d\lambda \\ &= \left[ \left( 1 - \frac{1}{(s_1 + pa)^{k_1 - 1}} \right) \frac{\eta}{1 - (\kappa + \eta)} + 1 \right] (\frac{1}{2} - ra) \\ &= \frac{\frac{a(qs_1 + ps_2 - p - q) + pqa^2}{(s_1 + pa)(s_2 + qa)} - \frac{\eta}{(s_1 + pa)^{k_1 - 1}}}{1 - (\kappa + \eta)} (\frac{1}{2} - ra) \ . \end{split}$$

Using Lemma 3, we obtain

$$\lim_{a \to 0} \frac{C_3}{a} = -\frac{qs_1 + ps_2 - p - q}{2s_1 s_2} \; .$$

In the same way, we can see that for any  $0 < \theta < 1/2$ , we obtain

$$\lim_{a \to 0} \frac{1}{a} \int_{1/2+\theta}^{1} f_h d\lambda = -\frac{qs_1 + ps_2 - p - q}{s_1 s_2} \left(\frac{1}{2} - \theta\right)$$

If we define  $B = C_1 + C_2 + C_3$ , then  $\frac{f_h}{B}$  is a normalized density. We see that

$$\lim_{a \to 0} \frac{B}{a} = -\frac{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1s_2^2}{2s_1s_2}$$

Our calculations show that the normalized measures  $\{(f_h/B) \cdot \lambda\}$  converge \*-weakly to the measure

$$\frac{(qs_1+ps_2-p-q)(s_2+2)}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\mu_0 + \frac{2rs_1s_2^2}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\delta_{(\frac{1}{2})}$$

Now, we will show the same holds for the normalized measure defined by  $f_l$ . To this end, let us notice that

$$f_{h} - f_{l} = \left(1 + \frac{s_{1} + pa}{s_{2} + qa}\right)\Lambda_{h}g_{l} - \left(1 + \frac{s_{1} + pa}{s_{2} + qa}\right)\Lambda_{l}g_{h}$$

$$= \left(1 + \frac{s_{1} + pa}{s_{2} + qa}\right)\left(\Lambda_{h} - \Lambda_{l}\right)g_{l} - \Lambda_{l}\frac{1 + \frac{s_{1} + pa}{s_{2} + qa}}{(s_{2} + qa)(s_{1} + pa - 1)(s_{1} + pa)^{k_{1} - 1}}$$

$$= \left(1 + \frac{s_{1} + pa}{s_{2} + qa}\right)\frac{\frac{\eta}{(s_{1} + pa)^{k_{1} - 1}}}{[1 - (\kappa + \eta)][1 - \kappa - \eta(1 - \frac{1}{(s_{1} + pa)^{k_{1} - 1}})]}g_{l}$$

$$-\Lambda_{l}\frac{1 + \frac{s_{1} + pa}{s_{2} + qa}}{(s_{2} + qa)(s_{1} + pa - 1)(s_{1} + pa)^{k_{1} - 1}},$$

where  $|g_l| \leq \frac{2}{s_1}$  and  $\lim_{a\to 0} \Lambda_l = -1$ . Using Lemma 3 once again, we can show that for any subinterval  $J \subset [0, 1]$ , we have

$$\lim_{a \to 0} \frac{1}{a} \int_J (f_h - f_l) d\lambda = 0 \; .$$

For J = [0, 1] this means that the normalizations of  $f_l$  and  $f_h$  are asymptotically the same. With this, the limit for a general J means in particular that the \*-weak limit of normalized measures defined using  $f_l$  is the same as for those defined using  $f_h$ . In view of inequality (7), this proves Theorem 1(II).

3.3.3. Estimates on  $f_a$  if  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ . We have the following lemma:

Lemma 4. For the family of  $W_a$  maps, if  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ , we have  $(I) W_a(1/2) = 1/2 + ra, W_a^2(1/2) = -ra(s_2 + qa) + 1/2 + ra, and for <math>3 \le m \le k$ , we have  $W_a^m(1/2) = -a(s_1 + pa)^{m-2} \frac{r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}{s_1 + pa - 1} + \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa - 1)};$   $(II) \lim_{a \to 0} ak = 0;$   $(III) \lim_{a \to 0} a(s_1 + pa)^{k_1} = 0;$  $(IV) \lim_{a \to 0} W_a^{k_1}(\frac{1}{2}) = \frac{1}{2}.$  *Proof.* Suppose (I) is true. Let us first prove that (II) and (III) are true. By the definition of k, we have:

(8)  
$$0 \le -a(s_1 + pa)^{k-2} \frac{r \left[s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)\right]}{s_1 + pa - 1} + \frac{s_1 - 1 + pa - 2ra}{2(s_1 + pa - 1)}.$$

The inequality (8) implies  $a(s_1 + pa)^{k-2} \le \frac{s_1 - 1 + pa - 2ra}{2r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)]}$ , thus

$$ak \leq a \frac{\ln(s_1 - 1 + pa - 2ra) - \ln 2 + 2\ln(s_1 + pa) - \ln r - \ln a}{\ln(s_1 + pa)} - \frac{\ln(2r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)])}{\ln(s_1 + pa)},$$
$$a(s_1 + pa)^{k_1} \leq \frac{(s_1 - 1 + pa - 2ra)(s_1 + pa)^2}{2r[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)](s_1 + pa)^{k - k_1}},$$

and since  $\lim_{a\to 0} a \ln a = 0$ , we obtain (II) and (III). (IV) follows from (III).

Now, let us prove (I).

The fixed point slightly less than 1/2 is  $x_l^* = \frac{s_1 - 1 + pa - 2ra}{2(s_1 - 1 + pa)}$ , and

$$x_l^* - W_a^2(1/2) = \frac{ra\left[s_1s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa)\right]}{s_1 - 1 + pa} > 0,$$

which implies that  $W_a^m(1/2)$  are all in the domain of the second branch of  $W_a$  for  $3 \le m \le k$ . Now, (I) follows by the same reasoning as in Lemma 3.

**Lemma 5.** If the normalized densities  $\{h_a\}_{a < a_0}$ , for some  $a_0 > 0$ , are uniformly bounded, then  $h_a \to h_0$  in  $L^1$ .

*Proof.* The uniform boundedness implies  $\{h_a\}_{a < a_0}$  is a weakly precompact set in  $L^1$ . Thus, any limit of  $\{h_a\}_{a < a_0}$  is a invariant density by Proposition 11.3.1 [2]. At the same time, this limit is an  $L^1$  function, thus defines an absolutely continuous invariant measure. Since the map  $W_0$  is exact and has only one acim, we conclude that  $h_a \to h_0$  in  $L^1$ .

Now, we will prove Theorem 2:

The main idea of the proof is the following: since non-normalized densities  $\{f_a\}$  are uniformly bounded (formulas (9, 10, 11)), it is enough to show that  $\{\int_0^1 f_a d\lambda\}$  are uniformly separated from zero.

For small a, by Lemma 2,  $\Lambda$  (and then both  $\Lambda_l$  and  $\Lambda_h$ ) can be either positive or negative. Thus, we can have the following cases.

Case (i):  $\Lambda_l < 0$ :

Comparing with (4) and (3), we see that for the functions  $\hat{f}_l = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_l g_h$ and  $\hat{f}_h = 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda_h g_l$ , we have

(9) 
$$\widehat{f}_l \le f_a \le \widehat{f}_h \ .$$

Note that  $\hat{f}_l$  and  $\hat{f}_h$  have the same form as  $f_l$  and  $f_h$  in Section 3.3.1, so their representations as combinations of functions  $\chi_j$ ,  $j = 1, \ldots, k_1$  and  $\chi_c$  are similar to

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that of  $f_l$  and  $f_h$ . At the same time, now we have  $\frac{1}{s_1} + \frac{1}{s_2} < 1$ , so the representation is as follows:

$$\begin{aligned} \widehat{f}_{l} &= \left(\frac{s_{1}+s_{2}+pa+qa}{(s_{2}+qa)(s_{1}+pa)}\Lambda_{l}+1\right)\chi_{1}+\frac{s_{1}+s_{2}+pa+qa}{(s_{2}+qa)^{2}}\Lambda_{l}\sum_{j=2}^{k_{1}}\frac{\chi_{j}}{(s_{1}+pa)^{j-1}} \\ &+ \left(\frac{s_{1}+s_{2}+pa+qa}{(s_{2}+pa)^{2}}\Lambda_{l}\frac{1-\frac{1}{(s_{1}+pa)^{k_{1}-1}}}{s_{1}+pa-1}+1\right)\chi_{c} \\ &+ \frac{\frac{s_{1}+s_{2}+pa+qa}{s_{2}+qa}}{(s_{2}+qa)(s_{1}+pa-1)(s_{1}+pa)^{k_{1}-1}} ,\end{aligned}$$

$$\widehat{f}_h = \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} \Lambda_h + 1 \right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}} \\ + \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + 1 \right) \chi_c .$$

(3) implies that all the coefficients in the representation of  $\hat{f}_l$  and  $\hat{f}_h$  are negative for sufficiently small a.

We use the same notations  $J_1$ ,  $J_2$  and  $J_3$  as in Section 3.3.2. First, we do the calculations assuming that  $\vartheta = 1 - \left(\frac{s_1+s_2}{s_1s_2} + \frac{s_1+s_2}{s_2^2(s_1-1)}\right) \neq 0$ . We will calculate the integrals of  $\hat{f}_h$  over each of  $J_1$ ,  $J_2$  and  $J_3$ , and use them to normalize  $\hat{f}_h$ . We have

$$\begin{split} \widehat{C}_{1} &= \int_{J_{1}} \widehat{f}_{h} \ d\lambda = \int_{J_{1}} \left[ \frac{s_{1} + s_{2} + pa + qa}{(s_{2} + qa)(s_{1} + pa)} \Lambda_{h} + 1 \right] \chi_{1} \ d\lambda \\ &= \left[ \frac{s_{1} + s_{2} + pa + qa}{(s_{2} + qa)(s_{1} + pa)} \Lambda_{h} + 1 \right] W_{a}^{k_{1}}(\frac{1}{2}) = \left[ \frac{\kappa}{1 - (\kappa + \eta)} + 1 \right] W_{a}^{k_{1}}(\frac{1}{2}) \\ &= \left[ \frac{s_{1}s_{2}^{2} - s_{1} - s_{2} - s_{2}^{2}}{1 - (\kappa + \eta))(s_{2} + qa)^{2}(s_{1} + pa - 1)} \right. \\ &+ \frac{a(2qs_{1}s_{2} + ps_{2}^{2} - 2qs_{2} - p - q)}{(1 - (\kappa + \eta))(s_{2} + qa)^{2}(s_{1} + pa - 1)} \\ &+ \frac{a^{2}(2pqs_{2} - q^{2} + q^{2}s_{1}) + pq^{2}a^{3}}{(1 - (\kappa + \eta))(s_{2} + qa)^{2}(s_{1} + pa - 1)} \right] W_{a}^{k_{1}}(\frac{1}{2}) \ . \end{split}$$

Using Lemma 4, we have

$$\lim_{a \to 0} \widehat{C}_1 = \frac{1}{2} \frac{\frac{s_1 s_2^2 - s_1 - s_2 - s_2^2}{s_2^2(s_1 - 1)}}{1 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right)} = \frac{1}{2} \frac{1 - \frac{s_1 + s_2}{s_2^2(s_1 - 1)}}{1 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right)}$$

On the interval  $J_2$ , the integral of  $\hat{f}_h$  is:

$$\hat{C}_{2} = \int_{J_{2}} \hat{f}_{h} d\lambda = \int_{J_{2}} \left[ \frac{s_{1} + s_{2} + pa + qa}{(s_{2} + qa)(s_{1} + pa)} \Lambda_{h} + 1 \right] \chi_{1} d\lambda$$
$$+ \frac{s_{1} + s_{2} + pa + qa}{(s_{2} + qa)^{2}} \Lambda_{h} \sum_{j=2}^{k_{1}} \int_{J_{2}} \frac{\chi_{j}}{(s_{1} + pa)^{j-1}} d\lambda$$

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$$= \frac{1-\eta}{1-(\kappa+\eta)} \left(\frac{1}{2} + ra - W_a^{k_1}(\frac{1}{2})\right) \\ + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \left[\frac{ra(s_2 + qa)}{s_1 + pa} + \frac{ra(1 - \frac{1}{(s_1 + pa)^{k_1 - 2}})}{(s_1 + pa - 1)^2} \\ + \frac{a(k_1 - 2)}{s_1 + pa} \frac{r(s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q + pqa))}{s_1 + pa - 1}\right]$$

Using Lemma 4, we have  $\lim_{a\to 0} \hat{C}_2 = 0$ .

On the interval  $J_3$ , the integral of  $\hat{f}_h$  is:

$$\begin{aligned} \widehat{C}_3 &= \int_{J_3} \widehat{f}_h \ d\lambda &= \int_{J_3} \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \Lambda_h \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + 1 \right) \chi_c \ d\lambda \\ &= \left[ \left( \left( 1 - \frac{1}{(s_1 + pa)^{k_1 - 1}} \right) \frac{\eta}{1 - (\kappa + \eta)} + 1 \right] \left( \frac{1}{2} - ra \right) \right] \\ &= \frac{\frac{s_1 s_2 - s_1 - s_2 + a(qs_1 + ps_2 - p - q) + pqa^2}{(s_1 + pa)(s_2 + qa)} - \frac{\eta}{(s_1 + pa)^{k_1 - 1}}}{1 - (\kappa + \eta)} (\frac{1}{2} - ra) \end{aligned}$$

Using Lemma 4 once again, we have

$$\lim_{a \to 0} \widehat{C}_3 = \frac{1}{2} \frac{1 - \frac{s_1 + s_2}{s_1 s_2}}{1 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right)}$$

Note that if we define  $\widehat{B} = \widehat{C}_1 + \widehat{C}_2 + \widehat{C}_3$ , then

$$\lim_{a \to 0} \widehat{B} = \frac{1}{2} \frac{2 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right)}{1 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right)}$$

which is not 0. Since  $\{\hat{f}_h\}$  are uniformly bounded, we conclude that the normalized  $\{\hat{f}_h\}$  are also uniformly bounded.

Now, we will show that the normalized  $\{\widehat{f}_l\}$  are also uniformly bounded. To this end, let us notice that

$$\begin{aligned} \widehat{f}_{h} - \widehat{f}_{l} &= (1 + \frac{s_{1} + pa}{s_{2} + qa})\Lambda_{h}g_{l} - (1 + \frac{s_{1} + pa}{s_{2} + qa})\Lambda_{l}g_{h} \\ &= (1 + \frac{s_{1} + pa}{s_{2} + qa})(\Lambda_{h} - \Lambda_{l})g_{l} - \Lambda_{l}\frac{1 + \frac{s_{1} + pa}{s_{2} + qa}}{(s_{2} + qa)(s_{1} + pa - 1)(s_{1} + pa)^{k_{1} - 1}} \\ &= (1 + \frac{s_{1} + pa}{s_{2} + qa})\frac{\frac{\eta}{(s_{1} + pa)^{k_{1} - 1}}}{[1 - (\kappa + \eta)][1 - \kappa - \eta(1 - \frac{1}{(s_{1} + pa)^{k_{1} - 1}})]}g_{l} \\ &-\Lambda_{l}\frac{1 + \frac{s_{1} + pa}{s_{2} + qa}}{(s_{2} + qa)(s_{1} + pa - 1)(s_{1} + pa)^{k_{1} - 1}}, \end{aligned}$$

where  $|g_l| \leq \frac{1}{s_1} + \frac{1}{s_2(s_1-1)}$  and  $\lim_{a \to 0} \Lambda_l = \frac{1}{1 - \left(\frac{s_1+s_2}{s_1s_2} + \frac{s_1+s_2}{s_2^2(s_1-1)}\right)}$ . Thus,  $\lim_{a \to 0} \widehat{f}_h - \widehat{f}_l = 0$ .

We conclude that the normalized  $\{\hat{f}_l\}$  are uniformly bounded since the normalized  $\{\hat{f}_h\}$  are uniformly bounded. Thus, after normalization,  $\{f_a\}$  are also uniformly bounded.

Case (ii):  $\Lambda_l > 0$ :

This case implies that  $f_a$  given by (4) has the following properties:

(10) 
$$f_a \ge 1 ,$$

and all the coefficients of the characteristic functions appearing in (4) are positive. We note that  $\Lambda$  is always positive for small *a*. Thus,

(11) 
$$f_a \le 1 + (1 + \frac{s_1 + p_a}{s_2 + q_a}) \Lambda \sum_{n=1}^{\infty} \frac{1}{|\beta(1/2, n)|} ,$$

which is finite since our maps  $\{W_a\}$  are expanding. In view of (10), we conclude that the normalized  $\{f_a\}$  are uniformly bounded.

If  $\vartheta = 1 - \left(\frac{s_1 + s_2}{s_1 s_2} + \frac{s_1 + s_2}{s_2^2(s_1 - 1)}\right) = 0$ , then we have  $\lim_{a \to 0} \frac{1}{\Lambda_l} = \lim_{a \to 0} \frac{1}{\Lambda_h} = 0$ ,  $\Lambda_l$  and  $\Lambda_h$  are still of the same sign. We can renormalize  $f_a$ . Let us take the  $\hat{f}_h$  as an example. Multiplying it by  $\frac{1}{\Lambda_h}$ , we obtain

$$\frac{1}{\Lambda_h} \widehat{f}_h = \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)(s_1 + pa)} + \frac{1}{\Lambda_h} \right) \chi_1 + \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1 + pa)^{j-1}} \\ + \left( \frac{s_1 + s_2 + pa + qa}{(s_2 + qa)^2} \frac{1 - \frac{1}{(s_1 + pa)^{k_1 - 1}}}{s_1 + pa - 1} + \frac{1}{\Lambda_h} \right) \chi_c .$$

Note that the coefficients of  $\chi_1$  and  $\chi_c$  converge to  $\frac{s_1+s_2}{s_1s_2}$  and  $\frac{s_1+s_2}{s_2^2(s_1-1)}$ , respectively. Thus,  $\{\int_0^1 \frac{1}{\Lambda_h} \hat{f}_h \ d\lambda\}$  are separated from 0. This implies  $\{\frac{1}{\Lambda_h} \hat{f}_h\}$  are uniformly bounded. A similar procedure can be applied to  $\hat{f}_l$ . We conclude that  $\{\frac{1}{\Lambda} f_a\}$  are uniformly bounded.

### 4. Example

One of the important properties of a piecewise expanding transformation of an interval is that its invariant density is bounded away from 0 on its support. The following result was proved, by Keller [11] and by Kowalski [12].

**Theorem 3.** Let a transformation  $\tau : I \to I$  be piecewise expanding with  $\frac{1}{|\tau'(x)|}$ a function of bounded variation, and let f be a  $\tau$ -invariant density which can be assumed to be lower semicontinuous. Then there exists a constant c > 0 such that  $f|_{supp} f > c$ .

We provide an example showing that this result cannot be generalized to a family of expanding maps, even if they all have this property and converge to a limit map also with this property. Let  $d(\cdot, \cdot)$  be the metric on the weak topology of measures.

# **Example 1.** Let us fix

$$s_1 = s_2 = 2, \ p = q = 1.$$

For small a > 0, let  $W_{a,r}$  denote the  $W_a$  maps with varying parameter r, and let  $\mu_{a,r}$  denote the absolutely continuous invariant measure of  $W_{a,r}$ . We know that  $\mu_{a,r}$  is supported on [0,1] and  $W_{a,r}$  with  $\mu_{a,r}$  is exact. Using Theorem 1, we know that  $\{\mu_{a,r}\}$  converge \*-weakly to the measure

$$\mu_{0,r} = \frac{1}{1+2r}\mu_0 + \frac{2r}{1+2r}\delta_{\frac{1}{2}}.$$

Let  $r_n = n$ ,  $n = 1, 2, 3, \cdots$ . Also, let  $\{a_n\}_1^\infty$  satisfy  $r_n a_n < 1/2$  and be so small that

$$d(\mu_{a_n,r_n},\mu_{0,r_n}) < \frac{1}{n}$$
.

Now, for the family of maps  $\tau_n = W_{a_n,r_n}$ ,  $n = 1, 2, 3, \dots, \tau_n$  converge to  $W_0$  with  $|\tau'_n(x)| > 2$ , but the invariant densities  $\mu_{a_n,r_n}$  converge to  $\delta_{(\frac{1}{2})}$ . This implies that the invariant densities  $\{f_{a_n,r_n}\}$  corresponding to  $\{\mu_{a_n,r_n}\}$  have no uniform positive lower bound.

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