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## POPULATION DYNAMICS WITH NONLINEAR DELAYED CARRYING CAPACITY

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We consider a class of evolution equations describing population dynamics in the presence of a carrying capacity depending on the population with delay. In an earlier work, we presented an exhaustive classification of the logistic equation where the carrying capacity is linearly dependent on the population with a time delay, which we refer to as the "linear delayed carrying capacity" model. Here, we generalize it to the case of a nonlinear delayed carrying capacity. The nonlinear functional form of the carrying capacity characterizes the delayed feedback of the evolving population on the capacity of their surrounding by either creating additional means for survival or destroying the available resources. The previously studied linear approximation for the capacity assumed weak feedback, while the nonlinear form is applicable to arbitrarily strong feedback. The nonlinearity essentially changes the behavior of solutions to the evolution equation, as compared to the linear case. All admissible dynamical regimes are analyzed, which can be of the following types: punctuated unbounded growth, punctuated increase or punctuated degradation to a stationary state, convergence to a stationary state with sharp reversals of plateaus, oscillatory attenuation, everlasting fluctuations, everlasting up-down plateau reversals, and divergence in finite time. The theorem is proved that, for the case characterizing the evolution under gain and competition, solutions are always bounded, if the feedback is destructive. We find that even a small noise level profoundly affects the position of the finite-time singularities. Finally, we demonstrate the feasibility of predicting the critical time of solutions having finite-time singularities from the knowledge of a simple quadratic approximation of the early

time dynamics.

*Keywords*: Differential delay equations; Population dynamics; Functional carrying capacity; Punctuated evolution; Finite-time singularities; Prediction of divergence time

#### 1. Introduction

There exists a number of evolution equations characterizing population dynamics, which have been applied to numerous concrete systems, ranging from populations of human and biological species to the development of firms and banks (see review articles [Kapitza, 1996; Hern, 1999; Korotayev, 2007; Yukalov et al., 2012a]). The mathematical structure of these equations usually represents some generalizations of the logistic equation. Such equations can be classified into three main classes, depending on the nature of the dynamics of the carrying capacity. The first class, independently of whether the growth rate is a nonlinear function without or with delay of the population, assumes that the carrying capacity is a constant quantity given once for all that describes the total resources available to the population, in agreement with the initial understanding of the carrying capacity (e.g., [Haberl & Aubauer, 1992; Varfolomeyev & Gurevich, 2001; Hui & Chen, 2005; Gabriel et al., 2005; Berezowski & Fudala, 2006; Arino et al., 2006]). The second class allows the carrying capacity to change as a function of time, but for exogenous reasons, either by explicitly prescribing its evolution or by specifying its own independent dynamics for instance also given by a logistic equation [Dolgonosov & Naidenov, 2006; Pongvuthithum & Likasiri, 2010].

The third class of equations interprets the carrying capacity as a functional of the population itself, implying that the population does influence the carrying capacity, either by producing additional means for survival or by destroying the available resources [Yukalov et al., 2009, 2012a,b]. This feedback makes it possible to describe the regime of punctuated evolution, which is often observed in a variety of biological, social, economic, and financial systems [Yukalov et al., 2009, 2012a].

In our previous articles [Yukalov et al., 2009, 2012a], the carrying capacity was approximated by a linear dependence on the population variable, which, strictly speaking, assumes that the population influence should be smaller than the initial given capacity. In the present paper, we generalize the approach by accepting a nonlinear carrying capacity that allows us to consider a population influence of arbitrary strength. Moreover, the population variable enters the carrying capacity with a time delay, since to either create or destroy resources requires time.

The outline of the present paper is as follows. In Sec. 2, we explain the deficiency that is typical of the linear capacity specification and suggest the way of generalizing its form to a nonlinear expression, by the use of the theory of self-similar approximations. The existence and general conditions for the stability of evolutionary stationary states are formulated in Sec. 3. The temporal behavior of solutions essentially depends on the system parameters characterizing different prevailing situations, when the main features are described as gain and competition (Sec. 4), loss and cooperation (Sec. 5), loss and competition or gain and cooperation (Sec. 6). The equations display a rich variety of dynamical regimes, including punctuated unbounded growth, punctuated increase or punctuated degradation to a stationary state, convergence to a stationary state with sharp reversals of plateaus, oscillatory convergence, everlasting fluctuations, everlasting up-down plateau reversals, and divergence in finite time. All admissible dynamical regimes are studied and illustrated. The role of noise on the dynamics is investigated in Sec. 7. The possibility of predicting finite-time singularities by observing only the initial stage of motion is discussed in Sec. 8. Section 9 concludes.

#### 2. Evolution equation with functional delayed carrying capacity

Here and in what follows, we use the dimensionless variable for the population x(t) as a function of time t. The reduction of the dimensional equation to the dimensionless form has been explained in full details in our previous papers [Yukalov et al., 2009, 2012a] and we do not repeat it here.

#### 2.1. Singular solutions for linear delayed carrying capacity

The general expression for the evolution equation with functional carrying capacity, in dimensionless notation, reads as

$$\frac{dx(t)}{dt} = \sigma_1 x(t) - \sigma_2 \frac{x^2(t)}{y(x)}, \qquad (1)$$

where the carrying capacity functional

$$y(x) = y[x(t-\tau)] \tag{2}$$

depends on the population at an earlier time, with a constant delay time  $\tau$ , which embodies that any influence of the population on the capacity requires time in order to either create additional means or to destroy the given resources. Here and in what follows, we use the term *population*, although the variable x can characterize either population, or firm assets, or other financial and economic indices [Yukalov et al., 2009, 2012a].

The coefficients  $\sigma_i$  describe the prevailing features in the balance between gain (birth) or loss (death) and competition versus cooperation. There exist four situations characterized by these coefficients:

$$\begin{aligned}
\sigma_1 &= 1, \quad \sigma_2 = 1 & (gain \& competition), \\
\sigma_1 &= -1, \quad \sigma_2 = -1 & (loss \& cooperation), \\
\sigma_1 &= -1, \quad \sigma_2 = 1 & (loss \& competition), \\
\sigma_1 &= 1, \quad \sigma_2 = -1 & (gain \& cooperation).
\end{aligned}$$
(3)

Various admissible interpretations of the equation and possible applications have been described in the published papers [Yukalov et al., 2009, 2012a].

If we fix  $\sigma_1 = \sigma_2 = 1$  and take a constant capacity  $y_0 = y(0)$ , we come back to the logistic equation. In our previous papers [Yukalov et al., 2009, 2012a], we modelled the influence of population on the carrying capacity by the linear approximation

$$y_1(x) = 1 + b_1 x(t - \tau) , \qquad (4)$$

with the parameter  $b_1$  describing either destructing action of population on the resources, when  $b_1 < 0$ , or creative population activity, if  $b_1 > 0$ .

In the case of destructive action, it may happen that the capacity (4) reaches zero and becomes even negative, with the effect of the growing population. It crosses zero at a critical time  $t_c$  defined by the equation

$$1 + b_1 x(t_c - \tau) = 0$$
.

Being in the denominator of the second term of (1), the vanishing capacity leads to the appearance of divergent or non-smooth solutions. In some cases, having to do with financial and economic applications, the arising negative capacity can be associated with the leverage effect [Yukalov et al., 2012a]. However, in the usual situation, the solution divergencies, caused by the zero denominator, look rather unrealistic, reminding of mathematical artifacts. Therefore, it would be desirable to define a carrying capacity that would not cross zero in finite time.

#### 2.2. Evolution equation with nonlinear delayed carrying capacity

It would be possible to replace the linear form (4) by some nonlinear function. This, however, would be a too arbitrary and ambiguous procedure, since it would be always unclear why this or that particular function has been chosen. In order to justify the choice of a nonlinear function, we propose the following procedure to select the form of the nonlinear carrying capacity.

Strictly speaking, the linear approximation presupposes that the second term in the r.h.s of expression (4) is smaller than one. Generally, a function y(x) can be expanded in powers of its variable x according to the series

$$y(x) \simeq 1 + b_1 x + b_2 x^2 + \dots$$
, (5)

#### 4 V.I. Yukalov, E.P. Yukalova, D. Sornette

where different terms describe the influence with different action intensity. Since expression (4) can be interpreted as the first-order term in the general series expansion (5) with, in principle, infinite many terms, it is convenient to think of it as the general expansion of some nonlinear function to be determined by a suitable summation. We thus propose to construct a nonlinear extension of (4) by defining an effective sum of these series (5). A standard way to realize the summation (5) is via Padé approximants [Baker & Graves-Moris, 1996]. However, as is well known, the Padé approximants are very often plagued by the occurrence of artificial zeroes and divergencies, which makes them inappropriate for the summation of a quantity that is required to be finite and positive. For this purpose, it is more appropriate to resort to the method of self-similar approximations [Yukalov, 1990a,b, 1991, 1992; Yukalov & Yukalova, 1996]. This is a mathematical method allowing for the construction of effective sums of power-law series, even including divergent series. According to this method, a series (5) is treated as a trajectory of a dynamical system, whose fixed point represents the sought effective sum of the series. In the vicinity of a fixed point, the trajectory becomes self-similar, which gives its name to the method of self-similar approximations.

Employing this method, under the restriction of getting a positive effective sum of the series, which is done by using the self-similar exponential approximants [Yukalov & Gluzman, 1997, 1998], we obtain the effective sum

$$y(x) = \exp\{bx(t-\tau)\}.$$
(6)

Here the *production parameter* b characterizes the type of the influence of the population on the carrying capacity. In the case of creative activity of population, producing additional means for survival, the creation parameter is positive, b > 0. And if the population destroys the given carrying capacity, then the destruction parameter is negative (b < 0).

In this way, we come to the evolution equation

$$\frac{dx(t)}{dt} = \sigma_1 x(t) - \sigma_2 x^2(t) \exp\{-bx(t-\tau)\},$$
(7)

with the nonlinear carrying capacity that generalizes the delayed logistic equation (1) with the linear capacity. The proposed generalization (7) allows us to consider population influences of arbitrarily strong intensity. The initial condition to the equation defines the history

$$x(t) = x_0 \qquad (t \le 0) .$$
 (8)

By definition, the population is described by a positive variable, so that we will be looking for only positive solutions x(t) > 0 for t > 0.

#### 3. Existence and stability of stationary states

This section gives the general conditions for the existence and stability of stationary states of the delayed Eq. (7). The details of such conditions depend on the type of the system characterized by the values of  $\sigma_i$ . More specific investigations of the evolutionary stable states as well as the overall dynamical regimes will be analyzed in the following sections.

#### 3.1. Existence of stationary states

The stationary states of Eq. (7) are defined by the solutions to the equation

$$\sigma_1 x^* - \sigma_2 \left( x^* \right)^2 \exp(-bx^*) = 0 .$$
(9)

There always exists the trivial solution

$$x_1^* = 0$$
  $(-\infty < b < \infty, \ \sigma_1 = \pm 1, \ \sigma_2 = \pm 1)$ . (10)

But the nontrivial solutions require the validity of the relation

$$\frac{\sigma_1}{\sigma_2} = x^* \exp(-bx^*) , \qquad (11)$$

which tells us that they may happen only for

$$\sigma_1 = \sigma_2 = \pm 1 \qquad (x^* > 0) .$$
 (12)

Two nontrivial states may exist depending on the value of the production parameter. One of the states exists for negative values of the latter in the range

$$0 < x_2^* \le 1 \qquad (b \le 0) . \tag{13}$$

Below we show that varying the parameters of Eq. (7) generates a number of bifurcations and provides a rich variety of qualitatively different solutions. We give a complete classification of all possible solutions demonstrating how the bifurcation control [Chen et al., 2003] can be realized for this equation.

For positive b, there can occur two states, such that

$$1 < x_2^* < e$$
,  $x_3^* > e$   $\left( 0 < b < \frac{1}{e} \right)$ . (14)

At the bifurcation point b = 1/e, the states coincide:  $x_2^* = x_3^* = e$ . There are no stationary nontrivial states for b > 1/e. The existence of the nontrivial states is illustrated in Fig. 1.

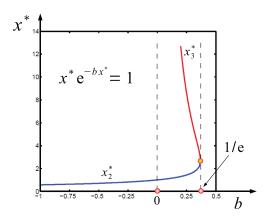


Fig. 1. Existence of nontrivial stationary states depending on the value of the production parameter b.

#### 3.2. Stability of stationary states

Studying the stability of the stationary states for the delay differential equation, we follow the Lyapunovtype procedure developed for delay equations in the book [Kolmanovskii & Myshkis, 1999] and described in detail in Refs. [Yukalov et al., 2009, 2012a]. The steps of this procedure are as follows. We consider a small deviation from the solution of equation (7) by writing  $x = x^* + \delta x$ . Substituting this in (7) yields the linearized equation

$$\frac{d}{dt}\,\delta x(t) = \left(\sigma_1 - 2\sigma_2 x^* e^{-bx^*}\right)\delta x(t) + b\sigma_1 x^* \delta x(t-\tau) , \qquad (15)$$

which is the basis for the stability analysis. The solution of equation (7) is Lyapunov stable, when the solution to (15) is bounded. A fixed point is asymptotically stable, when the solution to (15) converges to zero, as time tends to infinity. Thus, the conditions of fixed-point stability are prescribed by the convergence to zero of the small deviation described by equation (15).

We find that the trivial state  $x_1^* = 0$  is stable when

$$\sigma_1 = -1$$
,  $\sigma_2 = \pm 1$ ,  $-\infty < b < \infty$ ,  $\tau \ge 0$ , (16)

while for  $\sigma_1 = 1$  it is always unstable.

In the case of the nontrivial states, we use relation (11) and reduce Eq. (15) to

$$\frac{d}{dt}\,\delta x(t) = -\sigma_1 \delta x(t) + b\sigma_1 x^* \delta x(t-\tau) \;. \tag{17}$$

#### 6 V.I. Yukalov, E.P. Yukalova, D. Sornette

According to the existence condition (12), we need to study the stability of the nontrivial states only for coinciding  $\sigma_i$ .

The following analysis of the stationary states and the solution of the full evolution equation (7) requires to specify the values of  $\sigma_i$ .

### 4. Dynamics under gain and competition $(\sigma_1 = \sigma_2 = 1)$

#### 4.1. Evolutionary stable states

The stability analysis shows that the nontrivial state  $x_2^*$  is stable either in the range

$$0 < x_2^* \le \frac{1}{e}$$
  $(b \le -e, \ \tau < \tau_2^*)$  (18)

or in the range

$$\frac{1}{e} < x_2^* \le e \qquad \left( -e < b \le \frac{1}{e} , \quad \tau > 0 \right) , \tag{19}$$

where

$$\tau_2^* = \frac{1}{\sqrt{(bx_2^*)^2 - 1}} \, \arccos\left(\frac{1}{bx_2^*}\right) \,. \tag{20}$$

Close to the boundary  $b \to -e$  one has

$$x_2^* \simeq \frac{1}{e} \left( 1 + \frac{b+e}{2e} \right) , \qquad \tau_2^* \simeq \frac{\pi \sqrt{e}}{|b|-e} \qquad (b \to -e) .$$

The state  $x_3^* > e$  is always unstable under  $\sigma_1 = \sigma_2 = 1$ .

Thus, there can exist just one stable stationary state  $x_2^* \in (0, e)$ , whose region of stability is given by Eqs. (18) to (20) and shown in Fig. 2.

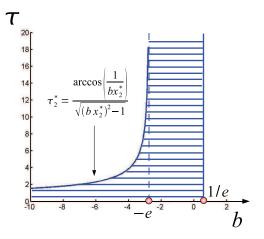


Fig. 2. Stability region (shadowed) for the stationary state  $x_2^*$  under  $\sigma_1 = \sigma_2 = 1$ .

For some values of the production parameter b, the basin of attraction of  $x_2^*$  is not the whole positive semiline of  $x_0$ , but a limited interval. This happens for  $b \in (0, 1/e)$ , when the basin of attraction is defined by the inequalities

$$0 < x_0 < x_3^*$$
  $\left( 0 < b \le \frac{1}{e} \right)$  . (21)

For these b values, the solution x tends to infinity, as  $t \to \infty$ , for the history  $x_0 > x_3^*$ , even for the parameters b and  $\tau$  in the region of stability of  $x_2^*$ .

#### 4.2. Boundedness of solutions for semi-negative production parameters

When  $\sigma_1 = \sigma_2 = 1$  and when the production parameter b is non-positive, this means that the population does not produce its carrying capacity but rather destroys it or, in the best case, retains the given capacity value. In this case, an important result for the overall temporal behavior of solutions can be derived rigorously: the *population growth has to be limited*.

**Proposition 1.** The solution x(t) to the evolution equation (7), under the condition  $\sigma_1 = \sigma_2 = 1$ , for  $b \leq 0$ , any finite  $\tau \geq 0$ , and any history  $x_0 \geq 0$ , is bounded for all times  $t \geq 0$ , and, for b < 0, there exists a time  $t_0 = t_0(x_0, \tau)$  such that

$$0 \le x(t) \le 1$$
  $(t \ge t_0)$ . (22)

*Proof.* When b = 0, the explicit solution is

$$x(t) = \frac{x_0}{x_0 + (1 - x_0)e^{-t}}$$
  $(b = 0)$ .

If  $x_0 < 1$ , then  $x \to 1$  from below as  $t \to \infty$ . If  $x_0 = 1$ , then x = 1 for all t > 0. If  $x_0 > 1$ , then  $x \to 1$  from above, as  $t \to \infty$ . So, the solution is always bounded.

When b is arbitrary and  $t \leq \tau$ , then the explicit solution

$$x(t) = \frac{x_0 e^{bx_0 + t}}{e^{bx_0} + x_0(e^t - 1)}$$

is evidently bounded.

For b < 0, any finite  $\tau \ge 0$  and all  $t \ge 0$ , the evolution equation (7) reads as

$$\frac{dx}{dt} = x - x^2 \exp\{|b|x(t-\tau)\}.$$
(23)

If, at some moment of time t > 0, it happens that  $x \ge 1$ , then the above equation defines a semi-negative derivative

$$\frac{dx}{dt} \le x(1 - \exp\{|b|x(t-\tau)\}) \le 0 ,$$

implying that x decreases or does not grow.

If the history is such that  $x_0 < 1$ , then either x stays always below one, or it grows and reaches one at some moment of time t > 0. But x cannot cross the line x = 1, since, as is shown above, at the time when x would become  $\ge 1$ , it has to either stay on this line x = 1 or has to decrease. The solution cannot stay forever on the unity line, as far as x = 1 is not a stable stationary state, which is  $x_2^* < 1$  for b < 0. This means that there is a moment of time  $t_0 < +\infty$  such that the solution x has to go down for  $t > t_0$ .

For  $x_0 = 1$ , again the solution cannot rise, having a non-positive derivative, and cannot stay forever at this value, which is not a stable fixed point. The sole possibility is that x starts diminishing beyond a finite  $t_0$ .

When  $x_0 > 1$ , then its derivative is non-positive. The solution cannot grow and cannot stay forever at the value that is larger than the fixed point  $x_2^* < 1$ . Hence x must decrease. When diminishing, it reaches the value x = 1, where it cannot stay forever, but has to go down beyond a finite time  $t_0$ .

Thus, for b < 0, there always exists such a moment of time, when the solution goes below the value x = 1 and can never cross this line from below.

#### 4.3. Punctuated unbounded growth

A different situation happens for positive production parameters, when the solutions may be unbounded. For example, when b is outside of the stability region of  $x_2^*$ , so that

$$b > \frac{1}{e}$$
,  $\tau \ge 0$ ,  $x_0 > 0$ , (24)

#### 8 V.I. Yukalov, E.P. Yukalova, D. Sornette

then x tends by steps to infinity, as  $t \to \infty$ . A similar unbounded punctuated growth occurs when b is inside the stability region, but  $x_0$  is outside of the attraction basin of  $x_2^*$ , which takes place if

$$0 < b < \frac{1}{e}$$
,  $\tau \ge 0$ ,  $x_0 > x_3^*$ . (25)

This behavior is illustrated by Fig. 3.

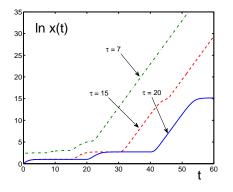


Fig. 3. Punctuated unbounded growth, under  $\sigma_1 = \sigma_2 = 1$ , for the parameters: b = 1 > 1/e,  $\tau = 20$ ,  $x_0 = 1$  (solid line); b = 1,  $\tau = 15$ ,  $x_0 = 1$  (dashed line); b = 0.25,  $\tau = 7$ ,  $x_0 = 10 > x_3^* = 8.613$  (dashed-dotted line).

The meaning of such a behavior is clear: for either sufficiently high production parameter or for sufficiently high startup and creative activity, the population (or firm development) can demonstrate unlimited punctuated growth with time.

#### 4.4. Punctuated convergence to stationary state

For positive production parameters, the solutions can also tend to the stationary state  $x_2^*$  by punctuated steps. They tend to  $x_2^*$  from below if  $x_0 < x_2^*$ , and from above if  $x_0 > x_2^*$ . This happens when

$$0 < b < \frac{1}{e}, \qquad \tau \ge 0, \qquad x_0 < x_3^*.$$
 (26)

When the production parameter is negative, the approach to the stationary state becomes quite nonmonotonic, with sharp reversals after almost horizontal plateaus. This regime arises when

$$-e < b < 0$$
,  $\tau \ge 0$ ,  $x_0 > 0$ . (27)

With decreasing further the negative destruction parameter, the plateaus shorten and the dynamics reduce to strongly fluctuating convergence to the focus  $x_2^*$ , which occurs for

$$b < -e, \qquad \tau < \tau_2^*, \qquad x_0 > 0,$$
 (28)

where the critical time lag is given by (20).

Different regimes of punctuated convergence to the stationary state  $x_2^*$  are shown in Fig. 4, which include punctuated growth, punctuated decay, and convergence with plateau reversals. Strongly oscillatory convergence to  $x_2^*$  is demonstrated in Fig. 5.

#### 4.5. Everlasting oscillations

When the destructive action is rather strong, increasing the time delay leads to the switch from the oscillatory convergence to a stationary state to the regime of everlasting oscillations, which happens for

$$b < -e, \qquad \tau \ge \tau_2^*, \qquad x_0 > 0.$$
 (29)

Figure 6 illustrates this effect of changing the dynamical regime with increasing the time lag above  $\tau_2^*$  that plays the role of a critical point.

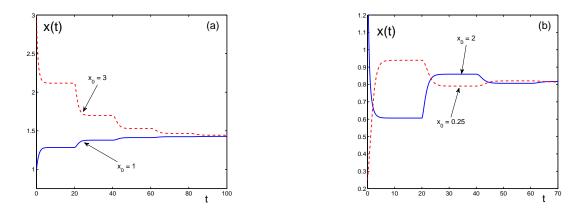


Fig. 4. Different types of punctuated convergence to the stationary state  $x_2^*$ , under  $\sigma_1 = \sigma_2 = 1$ , for different parameters: (a) punctuated growth for b = 0.25,  $\tau = 20$ ,  $x_0 = 1 < x_2^* = 1.43$  (solid line); punctuated decay for b = 0.25,  $\tau = 20$ ,  $x_0 = 3 > x_2^* = 1.43$  (dashed line); (b) convergence with plateau reversals under negative b = -0.25 > -e and  $\tau = 20$  for  $x_0 = 2 > x_2^* = 0.816$  (solid line) and  $x_0 = 0.25 < x_2^* = 0.816$  (dashed line).

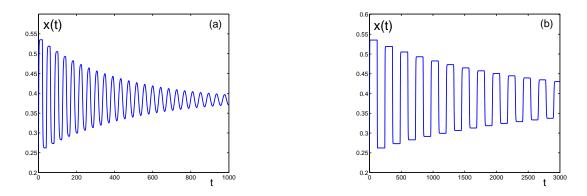


Fig. 5. Strongly oscillatory convergence to the stationary state  $x_2^* = 0.383$ , under  $\sigma_1 = \sigma_2 = 1$ , with b = -2.5 < -e and  $x_0 = 0.25$ , for different time lags: (a)  $\tau = 20$ ; (b)  $\tau = 120$ .

A rather exotic regime develops when the feedback action on the carrying capacity is strongly destructive and the time lag is very long. Then, there appears the regime of everlasting up-down reversals of plateaus located at zero and one, as is shown in Fig. 7.

### 5. Dynamics under loss and cooperation ( $\sigma_1 = \sigma_2 = -1$ )

#### 5.1. Existence of finite-time singularities

Contrary to the previous case of gain and competition, now there can appear unbounded solutions diverging at a finite time. For instance, in the region  $t < \tau$ , there exists the critical time

$$t_c = -\ln\left(1 - \frac{e^{bx_0}}{x_0}\right) , \qquad (30)$$

where x hyperbolically diverges, as  $t \to t_c$ , for some b and  $x_0$ . The relation between b and  $x_0$ , for which the divergence occurs, is shown in Fig. 8.

The occurrence of a finite-time singularity is associated with the development of an instability of the system, and the critical time  $t_c$  corresponds to the time of a change of regime. Concrete interpretations for various dynamical systems with such singularities, corresponding to population growth, mechanical ruptures or fractures, and economic or financial bubbles, have been discussed in detail in many articles

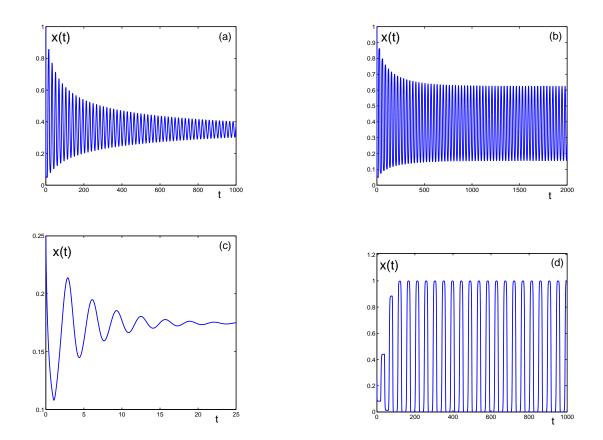


Fig. 6. Regime switch from fluctuating convergence to everlasting oscillations, under  $\sigma_1 = \sigma_2 = 1$ , with increasing the time lag over its critical value: (a) fluctuating convergence to  $x_2^* = 0.35$  for b = -3 < -e,  $x_0 = 1$ ,  $\tau = 8 < \tau_2^* = 8.854$ ; (b) everlasting oscillations for b = -3,  $x_0 = 1$ ,  $\tau = 15 > \tau_2^* = 8.854$ ; (c) fluctuating convergence to  $x_2^* = 0.175$  for b = -10 < -e,  $x_0 = 0.25$ ,  $\tau = 1 < \tau_2^* = 1.524$ ; (d) everlasting oscillations for b = -10,  $x_0 = 0.25$ ,  $\tau = 20 > \tau_2^* = 1.524$ .

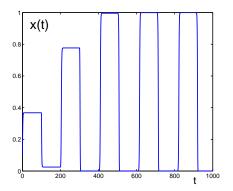


Fig. 7. Everlasting up-down plateau reversals,  $\sigma_1 = \sigma_2 = 1$ , for the destruction parameter  $b = -10 \ll -e$ , history  $x_0 = 0.1$ , and the time lag  $\tau = 100 \gg \tau_2^* = 1.524$ .

[Kapitza, 1996; Hern, 1999; Korotayev, 2007; Yukalov et al., 2004, 2009, 2012a; Johansen & Sornette, 2001; Fogedby & Poutkaradze, 2002; Sornette & Andersen, 2002, 2006; Andersen & Sornette, 2004, 2005].

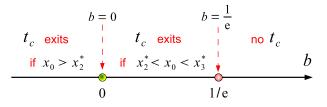


Fig. 8. Conditions for the existence of singular solutions under  $\sigma_1 = \sigma_2 = -1$ , depending on the relation between b and  $x_0$ .

#### 5.2. Evolutionary stable states

Under loss and cooperation, the state  $x_2^*$  is always unstable. But the trivial stationary state  $x_1^* = 0$  is also stable for all b and  $\tau$ . The nontrivial state  $x_3^* > e$  is stable for

$$0 < b < \frac{1}{e}, \qquad \tau < \tau_3^*,$$
 (31)

where

$$\tau_3^* = \frac{1}{\sqrt{(bx_3^*)^2 - 1}} \, \arccos\left(\frac{1}{bx_3^*}\right) \,. \tag{32}$$

On the boundary of stability, when b approaches 1/e, then

$$x_3^* \simeq e[1 + \sqrt{2(1 - be)}], \qquad \tau_3^* \simeq 1 - \frac{2}{3}\sqrt{2(1 - be)} \qquad \left(b \to \frac{1}{e}\right).$$

The stability region of  $x_3^*$  is presented in Fig. 9. Because the state  $x_1^*$  is stable everywhere, hence the stability region of  $x_3^*$  corresponds to the region of bistability.

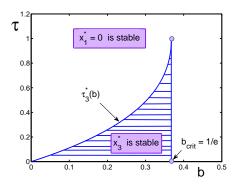


Fig. 9. Stability region of the stationary state  $x_3^*$  (shadowed), under  $\sigma_1 = \sigma_2 = -1$ . Since  $x_1 = 0$  is always stable, the shadowed region is also the region of bistability.

#### 5.3. Dynamical regimes of evolution

In the case of loss and cooperation, depending on the parameters b,  $\tau$ , and the history  $x_0$ , there can occur the following dynamical regimes: monotonic convergence to a stationary state, convergence with oscillations, everlasting oscillations, and finite-time singularity. All these regimes are demonstrated in Figs. 10 to 13. The summary of all possible solution types is illustrated in the scheme of Fig. 14.

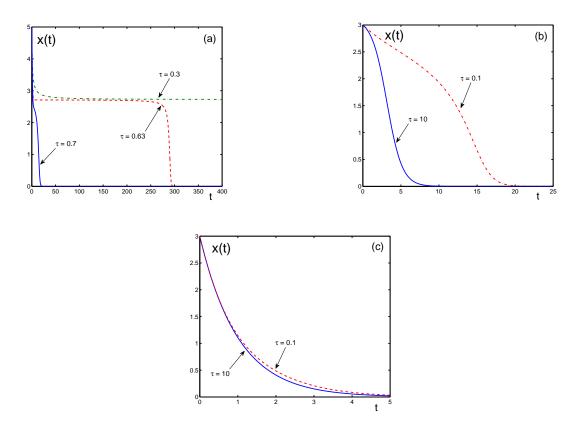


Fig. 10. Different types of monotonic convergence to a stationary state, under  $\sigma_1 = \sigma_2 = -1$ , for different parameters: (a) b = 1/e,  $x_0 = 5$ ,  $\tau = 0.7 > \tau^* = 0.627$  (solid line),  $\tau = 0.63 > \tau^* = 0.627$  (dashed line), and  $\tau = 0.3 < \tau^* = 0.627$  (dashed-dotted line); (b) b = 0.38,  $x_0 = 3$ ,  $\tau = 0.1$  (dashed line) and  $\tau = 10$  (solid line); (c) b = 2,  $x_0 = 3$ ,  $\tau = 10$  (solid line) and  $\tau = 0.1$  (dashed line).

### 6. Dynamics under loss and competition or gain and cooperation $(\sigma_1 \sigma_2 = -1)$

Contrary to the previous cases, where  $\sigma_1$  and  $\sigma_2$  were equal, now they are of opposite signs, so that  $\sigma_1 \sigma_2 = -1$ .

#### 6.1. Decay under loss and competition

In such an unfortunate situation, when

$$\sigma_1 = -1 , \qquad \sigma_2 = 1 , \qquad (33)$$

there is only the stationary state  $x_1^* = 0$  that is stable for all production parameters  $b \in (-\infty, \infty)$ , any time lag  $\tau \ge 0$ , and arbitrary history  $x_0 \ge 0$ . The solutions always monotonically decay to zero, as shown in Fig. 15.

# 6.2. Finite-time singularity or unbounded growth under gain and cooperation $(\sigma_1 = 1, \sigma_2 = -1)$

When  $\sigma_1 = 1, \sigma_2 = -1$ , the solutions always grow with time, exhibiting either unbound increase as  $t \to \infty$ , or a finite-time singularity at a critical time  $t_c^*$ .

If b < 0, the solutions exhibit finite-time singularities. For  $\tau \ge t_c$ , where

$$t_c = \ln\left(1 + \frac{e^{bx_0}}{x_0}\right) , \qquad (34)$$

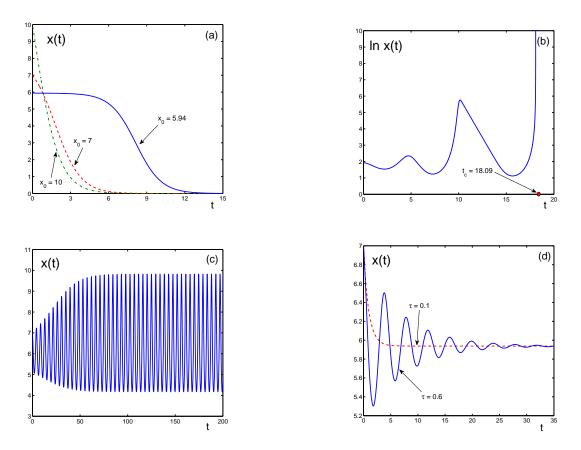


Fig. 11. Change of dynamical regimes, under  $\sigma_1 = \sigma_2 = -1$ , for the history  $x_0 > x_3^* = 5.938$ , with varying time lags: (a) convergence to the stationary state  $x_1^* = 0$  for b = 0.3,  $\tau = 20$ ,  $x_0 = 5.94$  (solid line),  $x_0 = 7$  (dashed line), and  $x_0 = 10$  (dashed-dotted line); (b) finite-time singularity at  $t_c = 18.09$  for b = 0.3,  $x_0 = 7$ ,  $\tau = 0.8$  in the interval  $\tau_1 < \tau < \tau_2$ , with  $\tau_1 = 0.777$ ,  $\tau_2 = 1.398$ ; (c) everlasting oscillations for b = 0.3,  $x_0 = 7$ ,  $\tau = 0.68 > \tau_3^* = 0.661$ ; (d) transformation of the oscillatory convergence to the state  $x_3^* = 5.938$  for b = 0.3,  $x_0 = 7$  and  $\tau = 0.6 > \tau_3 = 0.31$  (solid line) to the monotonic decay to the same state for  $\tau = 0.1 < \tau_3 = 0.31$  (dashed line).

the point of singularity is  $t_c^* = t_c(x_0, b)$ . But if  $\tau < t_c$ , then the singularity point, defined numerically, is  $t_c^*(x_0, b, \tau) \ge t_c$ , such that  $t_c^* \to t_c + 0$ , as  $\tau \to t_c - 0$ . The corresponding finite-time singularities are illustrated in Fig. 16a.

When b > 0, there can occur either unbounded growth as  $t \to \infty$  or a finite-time singularity. If  $0 \le \tau < \tau^*$ , where  $\tau^* = \tau^*(x_0, b)$  is defined numerically, the solution unboundedly grows as  $t \to \infty$ . For  $\tau^* \le \tau < t_c$ , there exists a critical time  $t_c^* = t_c^*(\tau) > t_c$  at which the solution diverges. And when  $\tau \ge t_c$ , the divergence happens at  $t_c^* = t_c$ , where  $t_c$  is given by expression (38). The change of the regime for the same production parameter b and history  $x_0$ , but a varying time lag  $\tau$  is illustrated in Fig. 16b.

#### 7. Influence of noise

#### 7.1. Stochastic differential equation

In the presence of noise, the evolution equation (7) becomes the stochastic differential equation

$$dx = g(x, t)dt + \alpha dW_t , \qquad (35)$$

where  $\alpha = \sqrt{2D}$  characterizes the noise strength, with D being the diffusion coefficient, and where

$$g(x,t) = \sigma_1 x - \sigma_2 x^2 \exp\{-bx(t-\tau)\}.$$
(36)

We consider Eq. (35) in the sense of Ito, where  $W_t$  is the standard Wiener process.

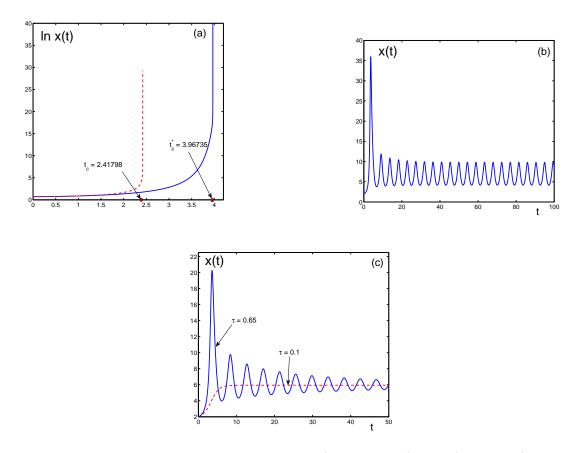


Fig. 12. Dynamical regimes, under  $\sigma_1 = \sigma_2 = -1$ , for the history  $x_2^* < x_0 = 2 < x_3^*$ , with  $x_2^* = 1.631$ ,  $x_3^* = 5.938$ , the fixed production parameter b = 0.3, varying only the time lag: (a) finite-time singularity at  $t_c = 2.42$  for  $\tau = 10 > \tau_2 = 0.713277$  (dashed line) and  $\tau = 0.713278 > \tau_2$ , with the singularity at  $t_c = 3.967$  (solid line); (b) everlasting oscillations for  $\tau_3^* < \tau = 0.68 < \tau_2$ , with  $\tau_3^* = 0.661$ ,  $\tau_2 = 0.713277$ ; (c) convergence to the stationary state  $x_3^*$  for  $\tau_1 < \tau = 0.65 < \tau_3^*$ , with  $\tau_1 = 0.285$  (solid line) and  $\tau = 0.1 < \tau_1$  (dashed line).

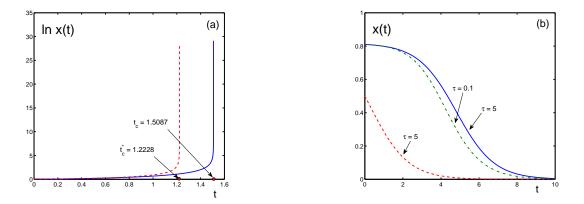


Fig. 13. Dependence of dynamical regimes, under  $\sigma_1 = \sigma_2 = -1$ , in the case of the destruction parameter b = -0.25, on the history and time lags: (a) finite-time singularity at  $t_c = 1.509$  for  $x_0 = 1 > x_2^* = 0.816$ ,  $\tau = 10 > t_c$  (solid line) and the singularity at  $t_c = 1.223$  for  $\tau = 0.1 < t_c$  (dashed line); (b) monotonic degradation to  $x_1^* = 0$  for the same b = -0.25, but for  $x_0 = 0.81 < x_2^*$ ,  $\tau = 5$  (solid line), for  $x_0 = 0.5 < x_2^*$ ,  $\tau = 0.1$  (dashed line), and  $x_0 = 0.81 < x_2^*$ ,  $\tau = 0.1$  (dashed-dotted line).

The addition of the noise does not influence much those solutions that do not exhibit finite-time singularities, while the latter can be strongly influenced by even weak noise. Therefore we concentrate our

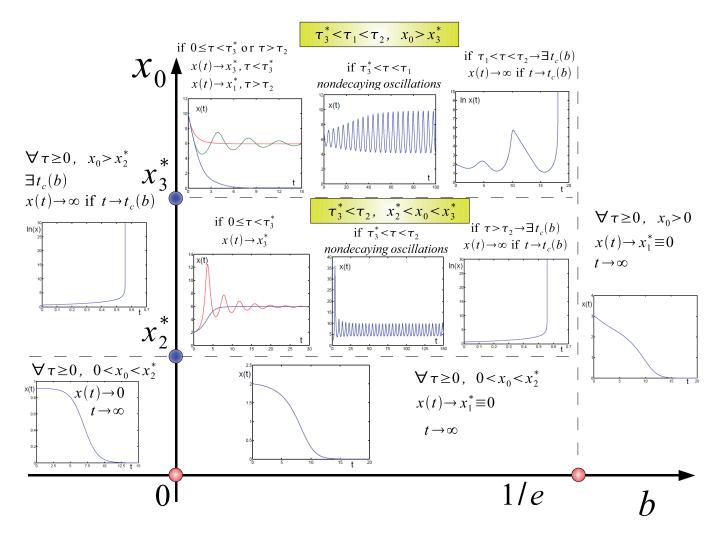


Fig. 14. Classification of all possible dynamical regimes, under  $\sigma_1 = \sigma_2 = -1$ , depending on the production parameter b, time lag  $\tau$ , and history  $x_0$ .

15

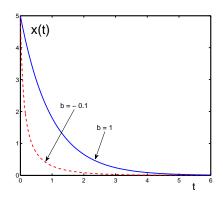


Fig. 15. Monotonic decay to zero, under loss and competition ( $\sigma_1 = -1$  and  $\sigma_2 = 1$ ), with the history  $x_0 = 5$  and time lag  $\tau = 10$ , for b = 1 (solid line) and b = -0.1 (dashed line).

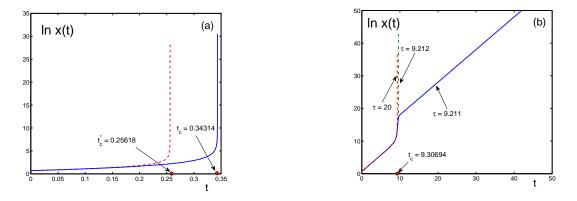


Fig. 16. Fig. 16. Behavior of the solutions in logarithmic scale, under  $\sigma_1 = 1$ ,  $\sigma_2 = -1$ , exhibiting either finite-time singularities or unbounded growth at  $t \to \infty$ : (a) finite-time singularity, with the divergence at  $t_c = 0.343$  for b = -0.1,  $x_0 = 2$ ,  $\tau = 10 > t_c$  (solid line) and with the divergence at  $t_c^* = 0.256$  for b = -0.1,  $x_0 = 2$ ,  $\tau = 0.01 < t_c$  (dashed line); (b) the change of regime for b = 5,  $x_0 = 2$  and varying time lag, when the finite-time singularity at  $t_c = 9.307$  for  $\tau = 20 > t_c$  (dashed line), or at  $t_c^* = 9.655 > t_c$  for  $\tau^* < \tau = 9.212 < t_c$ , where  $\tau^* = 9.2117$  (dashed-dotted line), changes to the unbounded growth as  $t \to \infty$  for  $\tau = 9.211 < \tau^*$  (solid line).

attention on the most interesting case of the noise influence on the solutions with finite-time singularities.

#### 7.2. Influence of noise on finite-time singularities

Recall that a finite-time singularity can occur only in the case of cooperation, when  $\sigma_2 = -1$ . Even rather weak noise can essentially shift the singularity point. Moreover the same noise strength, in different stochastic realizations, shifts the singularity point in a random way. Under the occurrence of a finite-time singularity, the influence of noise turns out to be more important than the variation of the time lag. This is in agreement with the Mao theorem [Mao, 1996], according to which there can exist a finite range of time lags for which the solution to the differential delay equation is close to that of the related ordinary differential equation. Figure 17 illustrates the influence of noise on the singularity point.

# 7.3. Fokker-Planck equation and condition for existence of a stationary probability distribution

The stochastic differential equation (35) corresponds to the Fokker-Planck equation

$$\frac{\partial}{\partial t} P(x,t) = -\frac{\partial}{\partial x} [g(x,t)P(x,t)] + D \frac{\partial^2}{\partial x^2} P(x,t)$$
(37)

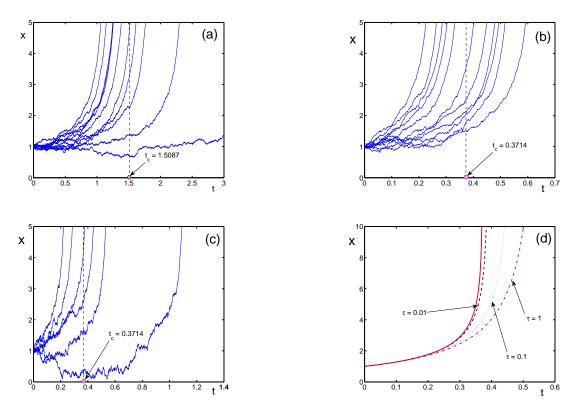


Fig. 17. Influence of noise, in the case of b = -0.25 and  $x_0 = 1$ , on the point of the finite-time singularity: (a) several realizations of stochastic trajectories, under  $\sigma_1 = \sigma_2 = -1$  and  $\tau > t_c = 1.509$  for the same noise strength  $\alpha = 0.25$ ; (b) stochastic trajectories, under  $\sigma_1 = 1$ ,  $\sigma_2 = -1$ , with  $\tau \ll t_c = 0.371$ , for the same noise strength  $\alpha = 0.5$ ; (c) stochastic trajectories, with the parameters as in (b), but for the larger noise strength  $\alpha = 1$ ; (d) singular solutions for the parameters as in (b), but with  $\alpha = 0$ , for different time lags,  $\tau = 0$  (solid line),  $\tau = 0.01$  (dashed line),  $\tau = 0.1$  (dotted line), and  $\tau = 1$  (dashed-dotted line).

for the distribution function P(x,t), with  $D = \alpha^2/2$ . Looking for a solution that would be defined for all  $x \in [0,\infty)$  and all  $t \ge 0$  requires the existence of a stationary distribution

$$P(x) \equiv \lim_{t \to \infty} P(x, t) . \tag{38}$$

The latter is defined by the equation

$$D \frac{\partial^2}{\partial x^2} P(x) - \frac{\partial}{\partial x} [g(x)P(x)] = 0 , \qquad (39)$$

in which

$$g(x) = \sigma_1 x - \sigma_2 x^2 \exp(-bx) .$$
(40)

Equation (39) has to be complemented by boundary conditions, which are usually taken in the form of absorbing boundary conditions

$$\lim_{x \to \infty} P(x) = 0 , \qquad \lim_{x \to \infty} g(x)P(x) = 0 , \qquad (41)$$

since these conditions allow for the normalization of the distribution as

$$\int_{0}^{\infty} P(x) \, dx = 1 \,. \tag{42}$$

The solution to Eq. (39) is

$$P(x) = C \exp\{-\beta U(x)\}, \qquad (43)$$

#### 18 V.I. Yukalov, E.P. Yukalova, D. Sornette

with the normalization constant C, the effective temperature  $\beta \equiv 1/D = 2/\alpha^2$  and the effective potential

$$U(x) = -\frac{\sigma_1}{2} x^2 - \frac{\sigma_2}{b^3} \left(2 + 2bx + b^2 x^2\right) e^{-bx} .$$
(44)

To satisfy the boundary condition for P(x), it is necessary that

$$U(x) \to \infty \qquad (x \to \infty) .$$
 (45)

From the asymptotic expression

$$U(x) \simeq -\left(\frac{\sigma_1}{2} + \frac{\sigma_2}{b} e^{-bx}\right) x^2 \qquad (x \to \infty) , \qquad (46)$$

we find that condition (45) is valid in the following cases.

For positive production parameters, the limit (46) yields

$$U(x) \simeq -\frac{\sigma_1}{2} x^2$$
  $(b > 0, x \to \infty)$ 

which means that condition (45) is satisfied provided that

 $\sigma_1 = -1$ ,  $\sigma_2 = \pm 1$ , b > 0. (47)

When the feedback is destructive, then the limit (46) gives

$$U(x) \simeq \frac{\sigma_2}{|b|} e^{|b|x} \qquad (b < 0, \ x \to \infty) ,$$

hence condition (45) implies that

$$\sigma_1 = \pm 1 , \qquad \sigma_2 = 1 , \qquad b < 0 .$$
 (48)

If b = 0, then Eqs. (39) and (40) show that the limit (45) requires the same conditions (48) as for b < 0.

The stationary distribution (43) possesses maxima when the effective potential (44) displays minima. The latter correspond to the stable fixed points  $x^*$  of the differential equation for x(t). Combining the analysis of the conditions for the existence of the distribution P(x) with the conditions for the existence of stable fixed points of the delay differential equation (7), we find the following: When the solution x(t), for any history  $x_0$  and  $\tau \to 0$ , converges to a fixed point, then P(x) exists. Conversely, when there is, at least for some history  $x_0$  and  $\tau \to 0$ , an unbound solution x(t), diverging either at a finite-time singularity or at increasing time  $t \to \infty$ , then P(x) does not exist. Summarizing, we come to the conclusion.

**Statement**. The necessary and sufficient condition for the existence of the distribution P(x) is the convergence, for any history  $x_0 \ge 0$  and  $\tau \to 0$ , of the solution x(t) to a fixed point.

**Remark**. Note the importance of the limit  $\tau \to 0$ . As follows from the analysis of the previous sections, P(x) can exist, though x(t) diverges for some finite  $\tau > 0$ .

Different shapes of the distribution P(x), as a function of x, are shown for the case of a single fixed point  $x_1^* = 0$  in Fig. 18 and for either the occurrence of the bistability region, with two fixed points  $x_1^* = 0$ and  $x_3^* > 0$ , or for the case of one nontrivial fixed point  $x_2^* > 0$  in Fig. 19.

#### 8. Prediction of finite-time singularities

When the solution diverges at a finite time, this is called a *finite-time singularity*. As has been discussed above, as well as in a series of papers [Kapitza, 1996; Hern, 1999; Korotayev, 2007; Yukalov et al., 2004, 2009, 2012a; Johansen & Sornette, 2001; Fogedby & Poutkaradze, 2002; Sornette & Andersen, 2002, 2006; Andersen & Sornette, 2004, 2005], a finite-time singularity represents a kind of a critical point, where the system experiences a transformation to another dynamic state, similarly to the occurrence of phase transitions in statistical systems [Yukalov & Shumovsky, 1990; Sornette, 2006]. Such critical points for complex systems, described by the evolution equations of type (7), depending on applications, can correspond to the points of overpopulation, firm ruin, market crash towards the end of a bubble, earthquakes, and so on.

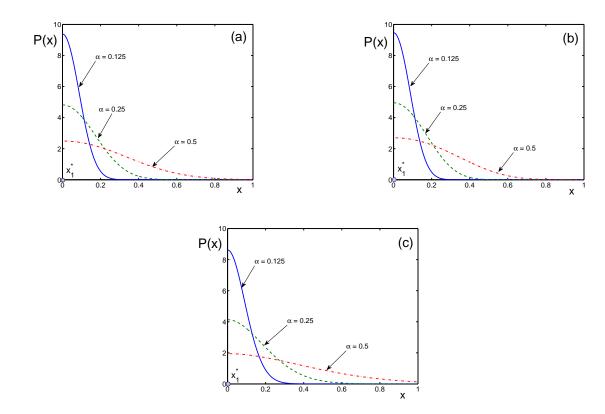


Fig. 18. Distribution P(x), as a function of x, in the case of a single fixed point  $x_1^* = 0$ , for the noise strengths  $\alpha = 0.125$  (solid line),  $\alpha = 0.25$  (dashed line), and  $\alpha = 0.5$  (dashed-dotted line), for different parameters: (a)  $\sigma_1 = -1$ ,  $\sigma_2 = 1$ , b = 1; (b)  $\sigma_1 = -1$ ,  $\sigma_2 = 1$ , b = -1; (c)  $\sigma_1 = -1$ ,  $\sigma_2 = -1$ , b = 1.

It is evident that the possibility of predicting such disasters would be of great importance. Here, for the case of finite-time singularities, we show that indeed such a prediction can be feasible.

Suppose, we make observations for the behavior of the function of interest x(t), starting from an initial condition  $x_0$ , during a short period of time, when this function can be well characterized by the simple asymptotic form

$$x(t) \simeq x_0 + c_1 t + c_2 t^2 \qquad (t \to 0) .$$
 (49)

In real life, the coefficients  $c_i$  can be defined from the observed data. And for the considered equation, they are found by substituting expression (49) into the evolution equation. Thus for the case  $\tau > t_c$ , we get

$$c_1 = \sigma_1 x_0 - \sigma_2 x_0^2 \exp(-bx_0) ,$$
  
$$c_2 = \frac{c_1}{2} \left[ \sigma_1 - \sigma_2 (2 - bx_0) x_0 \exp(-bx_0) \right]$$

If  $c_1$  is negative, this means that, in the vicinity of the initial condition  $x_0$ , the function x(t) decreases, hence in the near future, we do not expect the occurrence of a singularity, where x(t) would quickly rise. If  $c_1$  is positive, then x(t) increases, and the singularity is not excluded. To understand whether it really happens, we need to extrapolate the asymptotic series (49) to longer times. A powerful method for extrapolating asymptotic series has been developed in Refs. [Yukalov et al., 2003; Gluzman et al., 2003; Yukalov & Yukalova, 2007a], being termed the *method of self-similar factor approximants*. This method has been proved to be accurate for predicting critical points of different nature, including the critical points for dynamical systems [Yukalov et al., 2003; Gluzman et al., 2003; Yukalov & Yukalova, 2007a,b; Yukalova et al., 2008].

In the framework of self-similar factor approximants, the second-order factor approximant reads

$$x^*(t) = x_0 (1 + At)^n . (50)$$

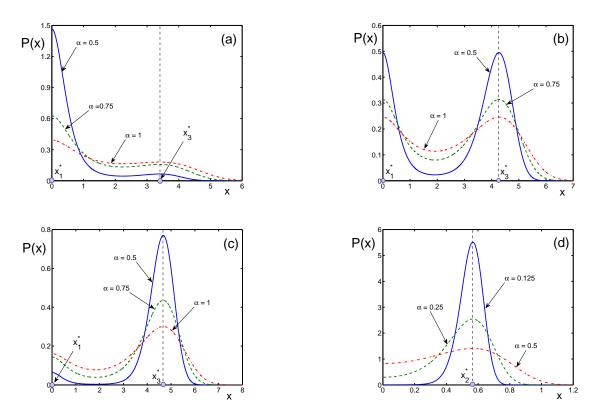


Fig. 19. Distribution P(x), as a function of x, in the case of either the bistability region with two fixed points  $x_1^* = 0$  and  $x_3^* > 0$  or for one nontrivial fixed point  $x_2^* > 0$ . The parameters are: (a)  $\sigma_1 = \sigma_2 = -1$ , b = 0.36, with  $x_3^* = 3.397$ , for  $\alpha = 0.5$  (solid line),  $\alpha = 0.75$  (dashed line), and  $\alpha = 1$  (dashed-dotted line); (b)  $\sigma_1 = \sigma_2 = -1$ , b = 0.34, with  $x_3^* = 4.268$ , for the same noise strengths as in (a); (c)  $\sigma_1 = \sigma_2 = -1$ , b = 0.33, with  $x_3^* = 4.671$ , for the same noise strengths as in (a); (d)  $\sigma_1 = \sigma_2 = 1$ , b = -1, with  $x_2^* = 0.567$ , for the noise strengths  $\alpha = 0.125$  (solid line),  $\alpha = 0.25$  (dashed line), and  $\alpha = 0.5$  (dashed-dotted line).

The parameters A and n are defined by expanding (50) in powers of t and comparing the expansion with the asymptotic form (49), which yields

$$A = \sigma_2 x_0 (1 - bx_0) \exp(-bx_0) , \qquad n = \frac{\frac{\sigma_1}{\sigma_2} - x_0 \exp(-bx_0)}{x_0 (1 - bx_0) \exp(-bx_0)}$$

Let us recall that, as the analysis of the previous sections shows, for the considered case of  $\tau > t_c$ , the finite-time singularity happens under one of the following conditions: either for  $\sigma_1 = 1, \sigma_2 = -1, b < 0$  and for any history  $x_0$ , or for  $\sigma_1 = \sigma_2 = -1, b < 0$  and  $x_0 > x_2^* < e$ . In both these cases, we find that A < 0 and n < 0, which allows us to rewrite Eq. (50) as

$$x^*(t) = \frac{x_0}{(1 - |A|t)^{|n|}} \,. \tag{51}$$

The latter expression shows that the point of singularity is given by

$$t_c^{app} = \frac{1}{|A|} = \frac{\exp(bx_0)}{\sigma_2 x_0 (1 - bx_0)} \,. \tag{52}$$

We have investigated the behavior of formula (52) for the different situations studied in the previous sections. We find that, when the evolution equation gives a solution x(t) diverging at a finite time, then the predicted value (52) does approximate the real divergence point  $t_c$ . When the solution x(t) is bounded, approaching a stationary state, then either A or n is positive, so that the factor approximant (50) does not predict singularities. And, if the solution x(t) tends to infinity for  $t \to \infty$ , then the factor approximant (50) either does not show a finite-time singularity or, in some cases exhibits its appearance. Such artificial singularities can be removed by constructing the factor approximants of higher orders. In order not to complicate the consideration, here we limit ourselves to the second-order factor approximant that does predict the singularity when it really happens for x(t).

To estimate the accuracy of the prediction, we calculate the predicted  $t_c^{app}$  for different values of the parameters and compare it with the  $t_c$  given by the evolution equation. The results in Fig. 20 demonstrate that the predicted singularity point  $t_c^{app}$  is close to the real  $t_c$ . The accuracy can be more precisely characterized by the absolute and relative errors

$$\Delta \equiv | t_c^{app} - t_c | , \qquad \varepsilon \equiv \frac{\Delta}{t_c} \times 100\% .$$

In Table 1, we present such error metrics, with fixed  $x_0 = 1$ , for varying parameters b.

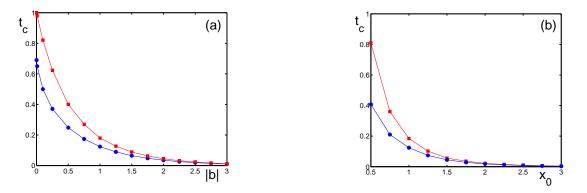


Fig. 20. Predicted singularity times  $t_c^{app}$  (red line with squares) and real  $t_c$  (blue line with circles), as functions of the parameter b = -|b| and history  $x_0$ , under  $\sigma_1 = 1$ ,  $\sigma_2 = -1$  for: (a) varying |b|, with fixed  $x_0 = 1$ ; (b) varying  $x_0$ , with fixed b = -1.

Table 1. The singularity time  $t_c^{app}$  predicted by the self-similar factor approximant (52), as compared to the exact singularity time  $t_c$  following from the evolution equation

b	$t_c$	$t_c^{app}$	$\varepsilon\%$	Δ
-0.0001	0.692	1.00	45%	0.308
-0.01	0.654	0.980	50%	0.326
-0.1	0.501	0.823	64%	0.322
-0.5	0.249	0.404	62%	0.155
-1.0	0.124	0.184	48%	0.060
-1.5	0.065	0.089	37%	0.024
-2.0	0.035	0.045	29%	0.010
-2.5	0.019	0.024	26%	0.005
-3.0	0.010	0.012	20%	0.0020
-3.5	0.0057	0.0067	18%	0.0010
-4.0	0.0032	0.0037	14%	0.0005

Note that  $t_c^{app}$  is systematically larger than the true singularity time  $t_c$ , as can be expected from the fact that the sole information used in the prediction is the quadratic asymptotic representation (49), which necessarily underestimates the full strength of the nonlinear feedback leading to the singularity. Taking third and higher order terms into account would lead to significant improvement of the prediction accuracy. But we believe that using the quadratic asymptotic form (49) is a realistic proxy for the capture of early time dynamics in real life situations. While the relative errors are significant (from 14% to 64% in the investigated cases), we believe that these predictions are useful to provide an approximate estimation of

#### 22 REFERENCES

the critical time of the singularity. Taking into account more terms in the asymptotic expansion for x(t) would improve the accuracy of the prediction, however involving more complicated expressions for the critical time.

#### 9. Conclusion

We have considered the evolution equation describing the population dynamics with functional delayed carrying capacity. The linear delayed carrying capacity, advanced earlier by the authors, has been generalized to the case of a nonlinear delayed carrying capacity. This allowed us to treat the delayed feedback of the evolving population on the capacity of their surrounding, by either creating additional means for survival or destroying the available resources, when the feedback can be of arbitrary strength. This is contrary to the linear approximation for the capacity, which assumes weak feedback. The nonlinearity essentially changes the behavior of solutions to the evolution equation, as compared to the linear case.

The justification for the exponential form of the nonlinearity is based on the derivation of an effective limit of expansion (5) for the carrying capacity by invoking the self-similar approximation theory.

All admissible dynamical regimes have been analyzed, which happen to be of the following types: punctuated unbounded growth, punctuated increase or punctuated degradation to a stationary state, convergence to a stationary state with sharp reversals of plateaus, oscillatory attenuation, everlasting fluctuations, everlasting up-down plateau reversals, and divergence in finite time. The theorem has been proved that, for the case of gain and competition, the solutions are always bounded, when the feedback is destructive.

We have studied the influence of additive noise in two cases: (i) on the solutions exhibiting finite-time singularities and (ii) in the presence of stationary solutions. For the former case, we found that even a small noise level profoundly affects the position of the finite-time singularities. For the later case, we have used the Fokker-Planck equation and derived the general condition for the existence of a stationary distribution function.

Finally, we showed that the knowledge of a simple quadratic asymptotic behavior of the early time dynamics of a solution exhibiting a finite-time singularity provides already sufficient information to predict the existence of a critical time, where the solution diverges.

It is necessary to stress that taking into account the nonlinear delayed carrying capacity not merely changes quantitatively the behavior of the solutions to the evolution equation, but also removes artificial finite-time divergence and finite-time death that exist in the equation with the linear form of the carrying capacity. For example, the linear carrying capacity can lead to the appearance of finite-time singularity or finite-time death even in the case of prevailing competition ( $\sigma_2 = 1$ ), as is found in [Yukalov et al., 2009, 2012a]. But with the nonlinear carrying capacity, as used in the present paper, these finite-time critical phenomena are excluded. Now, finite-time singularity can occur only in the logically clear case of cooperation ( $\sigma_2 = -1$ ). The reason why the linear approximation for the carrying capacity leads to such artificial singularities and deaths has been explained in Sec. 2.1 of the present paper.

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#### 24 REFERENCES

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