The maximal number of limit cycles in a family of polynomial systems^{*}

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Abstract. The main objective of this paper is to study the number of limit cycles in a family of polynomial systems. Using bifurcation methods, we obtain the maximal number of limit cycles in global bifurcation. Keywords. limit cycle, bifurcation, Melnikov function

1 Introduction and main results

In the qualitative theory of real planar differential systems, the main open problem is to determine the number and location of limit cycles. A classical way to produce limit cycles is by perturbing a system which has a center in such a way that limit cycles bifurcate in the perturbed system from some of the periodic orbits of the center for the unperturbed system. For instance, consider a planar system of the form

$$\dot{x}(t) = H_y + \varepsilon f(x, y, \varepsilon, a), \dot{y}(t) = -H_x + \varepsilon g(x, y, \varepsilon, a),$$
(1)

where H, f, g are C^{∞} functions in a region $G \subset R^2$, $\varepsilon \in R$ is a small parameter and $a \in D \subset R^n$ with D compact. For $\varepsilon = 0$, (1) becomes a Hamiltonian system with the Hamiltonian function H(x, y). Suppose there exists a constant $H_0 > 0$ such that for $0 < h < H_0$, the equation H(x, y) = h defines a smooth closed curve $L_h \subset G$ surrounding the origin and shrinking to the origin as $h \to 0$. Hence H(0, 0) = 0 and for $\varepsilon = 0$ (1) has a center at the origin.

Let

$$\Phi(h,a) = \oint_{L_h} (gdx - fdy)_{\varepsilon=0}$$

$$= \oint_{L_h} (H_yg + H_xf)_{\varepsilon=0} dt,$$
(2)

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which is called the first order Melnikov function or Abelian integral of (1). This function plays an important role in the study of limit cycle bifurcation. In the case that (1) is a polynomial system, a well-known problem is to study the least upper bound of the number of zeros of Φ . This is called the weakened Hilbert 16th problem, see [Arnold 1983; Ye, 1986].

In this paper, we first state some preliminary lemmas which can be used to find the maximal number of limit cycles by using zeros of Φ . These lemmas are already known or easy corollaries of known results. Then we study the global bifurcations of limit cycles for some polynomial systems, and obtain the lower upper bound of the number of limit cycles. This is the main part of the paper.

Now we give some lemmas. First, for Hopf bifurcation we have the following:

Lemma 1.1([Han, 2000]) Let $H(x, y) = K(x^2 + y^2) + O(|x, y|^3)$ with K > 0 for (x, y) near the origin. Then the function Φ is of class C^{∞} in h at h = 0. If $\Phi(h, a_0) = K_1(a_0)h^{k+1} + O(h^{k+2})$, $K_1(a_0) \neq 0$ for some $a_0 \in D$, then (1) has at most k limit cycles near the origin for $|\varepsilon| + |a - a_0|$ sufficiently small.

The following lemma is well-known(see [Ye, 1986] for example).

Lemma 1.2 If $\Phi(h, a_0) = K_2(a_0)(h-h_0)^k + O(|h-h_0|^{k+1})$, $K_2(a_0) \neq 0$ for some $a_0 \in D$ and $h_0 \in (0, H_0)$, then (1) has at most k limit cycles near L_{h_0} for $|\varepsilon| + |a - a_0|$ sufficiently small.

Let L_0 denote the origin and set

$$S = \bigcup_{0 \le h < H_0} L_h. \tag{3}$$

It is obvious that S is a simply connected open subset of the plane. We suppose that the function Φ has the following form

$$\Phi(h,a) = I(h)N(h,a), \tag{4}$$

where $I \in C^{\infty}$ for $h \in [0, H - 0)$ and satisfies

$$I(0) = 0, \ I'(0) \neq 0 \ and \ I(h) \neq 0 \ for \ h \in (0, H_0).$$
 (5)

Using above two lemmas, we can prove(see [Xiang & Han, 2004])

Lemma 1.3 Let (4) and (5) hold. If there exists a positive integer k such that for every $a \in D$ the function N(h, a) has at most k zeros in $h \in [0, H_0)$ (multiplicities taken into account), then for any given compact set $V \subset S$, there exists $\varepsilon_0 = \varepsilon_0(V) > 0$ such that for all $0 < |\varepsilon| < \varepsilon_0$, $a \in D$ the system (1) has at most k limit cycles in V.

Remark 1.1 As we known, if there exists $a_0 \in D$ such that the function N(h, a) has exactly k simple zeros $0 < h_1 < \cdots < h_k < H_0$ with $N(0, a_0) \neq 0$, then for any compact set V satisfying $L_{h_k} \subset intV$ and $V \subset S$, there exists $\varepsilon_0 > 0$ such that for all $0 < |\varepsilon| < \varepsilon_0$, $|a - a_0| < \varepsilon_0$, (1) has precisely k limit cycles in V.

Remark 1.2 The conclusion of lemma 1.1 and lemma 1.2 are local with respect to both parameter a and the set S while the conclusion of lemma 1.3 is global because it holds in any compact set of S and uniformly in $a \in D$.

In this paper, we consider a real planar polynomial system of the form

$$\dot{x} = y(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2},
\dot{y} = -x(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2},$$
(6)

where m_1 , m_2 are positive integers and α_1 , α_2 are real constants which satisfy $\alpha_1 \cdot \alpha_2 \neq 0$. We shall prove that if we perturb above system by the polynomial systems of degree n we can obtain up to first order in ε at most $4(\lfloor \frac{n+1}{2} \rfloor + m_1 + m_2) - 7$ limit cycles.

On the region $\Omega = \{(x, y) | (1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2} \neq 0\}$, the perturbed system by the polynomial systems of degree n of (6) is equivalent to

$$\dot{x} = y + \frac{\varepsilon}{(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2}} \sum_{\substack{0 \le i + j \le n \\ 0 \le i + j \le n}} a_{ij} x^i y^j,$$

$$\dot{y} = -x + \frac{\varepsilon}{(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2}} \sum_{\substack{0 \le i + j \le n \\ 0 \le i + j \le n}} b_{ij} x^i y^j,$$
(7)

where $|a_{ij}| \leq K$, $|b_{ij}| \leq K$ with K a positive constant and $B_K = \{(a_{ij}, b_{ij}) | |a_{ij}| \leq K, |b_{ij}| \leq K\}$.

Let $\Phi(h)$ denote the first order Melnikov function of (7) for $0 \leq h < H_0$, $H_0 = min(\frac{1}{\alpha_1^2}, \frac{1}{\alpha_2^2})$. Then we have the following main results.

Theorem 2.1 Suppose $\alpha_1 \neq \alpha_2$. For any K > 0 and compact set V in Ω , if $\Phi(h)$ is not identically zero for (a_{ij}, b_{ij}) varying in a compact set D in B_K , then there exists an $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$, $(a_{ij}, b_{ij}) \in D$, the system (7) has at most $4(\lfloor \frac{n+1}{2} \rfloor + m_1 + m_2) - 7$ limit cycles in V.

Theorem 2.2 Suppose $\alpha_1 = \alpha_2$. For any K > 0 and compact set V in Ω , if $\Phi(h)$ is not identically zero for (a_{ij}, b_{ij}) varying in a compact set D in B_K , then there exists an $\varepsilon_0 > 0$ such that for $0 < |\varepsilon| < \varepsilon_0$, $(a_{ij}, b_{ij}) \in D$, the system (7) has at most n limit cycles in V.

2 Proof of the theorems

Before proving the theorems in Section 1, we give some lemmas first.

Let

$$I_{i,j} = \oint_{L_h} \frac{x^i y^j}{(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2}} dt, \quad i \ge 0, j \ge 0,$$
(8)

$$I_{i,j}^{(k)} = \oint_{L_h} \frac{x^i y^j}{(1 - \alpha_1 x)^k} dt, \quad k = 1, 2, 3, \cdots, i \ge 0, j \ge 0,$$
(9)

$$\Phi_{i,j} = a_{ij}I_{i+1,j} + b_{ij}I_{i,j+1}, \quad i \ge 0, j \ge 0,$$
(10)

$$\Phi_{i,j}^{(k)} = a_{ij}I_{i+1,j}^{(k)} + b_{ij}I_{i,j+1}^{(k)}, \quad k = 1, 2, 3, \cdots, i \ge 0, j \ge 0,$$
(11)

where

$$L_h: x = \sqrt{h} \sin t, y = \sqrt{h} \cos t.$$

Let

$$r_1 = \sqrt{1 - \alpha_1^2 h}, \ r_2 = \sqrt{1 - \alpha_2^2 h}.$$

The following results can be seen in the paper [Xiang & Han, 2004].

Lemma 2.1 For $m \ge 1$ it holds that

$$I_{0,0}^{(m)} = \frac{1}{r_1^{2m-1}} \sum_{j=0}^{\left[\frac{m-1}{2}\right]} C_j r_1^{2j},$$
(12)

where $C_j (j \ge 0)$ are constants which $C_j \ne 0$. [·] denotes the integer part function.

Lemma 2.2 For $0 \le k < m$, we have

$$I_{k,0}^{(m)} = \sum_{j=0}^{k} (-1)^{j} C_{k}^{j} I_{0,0}^{(m-j)}$$
$$= \frac{1}{r_{1}^{2m-1}} \sum_{j=0}^{\left[\frac{m-k-1}{2}\right]+k} C_{j} r_{1}^{2j}, \qquad (13)$$

and for $k \geq m$ we have

$$I_{k,0}^{(m)} = \frac{1}{r_1^{2m-1}} \sum_{j=0}^{m-1} C_j r_1^{2j} + \sum_{j=0}^{\left[\frac{k-m}{2}\right]} D_j r_1^{2j},$$
(14)

where C_j , D_j are constants.

For the function

$$\frac{x^k}{(1-\alpha_1 x)^{m_1}(1-\alpha_2 x)^{m_2}}$$

we have that

if $k < m_1 + m_2$, there exist real constants $\tilde{A}_{k,j}$, $\tilde{B}_{k,j}$ such that

$$\frac{x^k}{(1-\alpha_1 x)^{m_1}(1-\alpha_2 x)^{m_2}} = \sum_{j=1}^{m_1} \frac{\tilde{A}_{k,j}}{(1-\alpha_1 x)^j} + \sum_{j=1}^{m_2} \frac{\tilde{B}_{k,j}}{(1-\alpha_2 x)^j},$$
(15)

and if $k \ge m_1 + m_2$, there exist real constants $A_{k,j}$, $B_{k,j}$ and $C_{k,j}$ such that

$$\frac{x^k}{(1-\alpha_1 x)^{m_1}(1-\alpha_2 x)^{m_2}} = \sum_{j=1}^{m_1} \frac{A_{k,j}}{(1-\alpha_1 x)^j} + \sum_{j=1}^{m_2} \frac{B_{k,j}}{(1-\alpha_2 x)^j} + \sum_{j=0}^{k-m_1-m_2} C_{k,j} x^j.$$
(16)

Hence from the definition of $I_{k,0}$ and lemma 2.1 for $0 \le k < m_1 + m_2$ we have

$$I_{k,0} = \oint_{L_h} \frac{x^k}{(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2}} dt$$

$$= \sum_{j=1}^{m_1} \oint_{L_h} \frac{\tilde{A}_{k,j}}{(1 - \alpha_1 x)^j} dt + \sum_{j=1}^{m_2} \oint_{L_h} \frac{\tilde{B}_{k,j}}{(1 - \alpha_2 x)^j} dt$$

$$= \frac{1}{r_1^{2m-1}} P_{m_1-1}(h) + \frac{1}{r_2^{2m-1}} P_{m_2-1}(h),$$
 (17)

and for $k \ge m_1 + m_2$ we have

$$I_{k,0} = \sum_{j=1}^{m_1} \oint_{L_h} \frac{A_{k,j}}{(1-\alpha_1 x)^j} dt + \sum_{j=1}^{m_2} \oint_{L_h} \frac{B_{k,j}}{(1-\alpha_2 x)^j} dt + \sum_{j=0}^{k-m_1-m_2} \oint_{L_h} C_{k,j} x^j dt$$
$$= \frac{1}{r_1^{2m-1}} P_{m_1-1}(h) + \frac{1}{r_2^{2m-1}} P_{m_2-1}(h) + P_{[\frac{k-m_1-m_2}{2}]}(h), \qquad (18)$$

where $P_n(h)$ denotes a polynomial of h of degree n, and $h = \frac{1-r_1^2}{\alpha_1^2} = \frac{1-r_2^2}{\alpha_2^2}$.

Using the definition of L_h and $I_{i,j}$, we can prove easily ([Xiang & Han, 2004])

Lemma 2.3 For $i \ge 0, k > 0$, we have

$$I_{i,2k-1} = 0$$

and

$$I_{i,2k} = \sum_{j=0}^{k} (-1)^{j} C_{k}^{j} I_{i+2j,0} h^{k-j}.$$

Lemma 2.4 For k > 0 we have

$$\sum_{i+j=2k-1} \Phi_{ij} = \frac{1}{r_1^{2m_1-1}} P_{m_1-1+k}(h) + \frac{1}{r_2^{2m_2-1}} P_{m_2-1+k}(h) + P_{\left[\frac{2k-m_1-m_2}{2}\right]}(h)$$
(19)

and

$$\sum_{i+i=2k} \Phi_{ij} = \frac{1}{r_1^{2m_1-1}} P_{m_1-1+k}(h) + \frac{1}{r_2^{2m_2-1}} P_{m_2-1+k}(h) + P_{\left[\frac{2k+1-m_1-m_2}{2}\right]}(h).$$
(20)

Proof. By the definition Φ_{ij} and lemma 2.3, we have

$$\sum_{i+j=2l} \Phi_{ij} = \sum_{i=1}^{k} (\Phi_{2k-2i,2i} + \Phi_{2k-2i+1,2i-1}) + \Phi_{2k,0}$$
$$= \sum_{i=0}^{k} \tilde{a}_{2k,i} I_{2k-2i+1,2i}$$
$$= \tilde{b}_{2k,k} I_{1,0} h^{k} + \dots + \tilde{b}_{2k,1} I_{2k-1,0} h + \tilde{b}_{2k,0} I_{2k+1,0}.$$

So (20) follows from (17) and (18). (19) can be proved in the same way.

The proof is completed.

Similarly, we can prove the following formulae by using lemma 2.2 and lemma 2.4 of the paper [Xiang & Han, 2004].

$$\sum_{i+j=2k-1} \Phi_{ij}^{(m)} = I_{0,0}^{(m)} (\tilde{b}_{2k-1,k}h^{k} + \dots + \tilde{b}_{2k-1,1}h + \tilde{b}_{2k-1,0}) + (-1)I_{0,0}^{(m-1)} (C_{2}^{1}\tilde{b}_{2k-1,k}h^{k} + \dots + C_{2k-2}^{1}\tilde{b}_{2k-1,1}h + C_{2k}^{1}\tilde{b}_{2k-1,0}) \vdots + (-1)^{m-1}I_{0,0}^{(1)} (C_{2[\frac{m}{2}]}^{m-1}\tilde{b}_{2k,k-[\frac{m}{2}]}h^{k-[\frac{m}{2}]} + \dots + C_{2k-2}^{m-1}\tilde{b}_{2k-1,1}h + C_{2k}^{m-1}\tilde{b}_{2k-1,0}) + (-1)^{m}C_{2[\frac{m+1}{2}]-1}^{m-1} \left(K_{0}\tilde{b}_{2k-1,k-[\frac{m+1}{2}]} + \dots + K_{2k-2[\frac{m+1}{2}]}\tilde{b}_{2k-1,0}\right)h^{k-[\frac{m+1}{2}]} \\ \vdots + (-1)^{m}C_{2k-3}^{m-1} \left(K_{0}\tilde{b}_{2k-1,1} + K_{2}\tilde{b}_{2k-1,0}\right)h + (-1)^{m}C_{2k-1}^{m-1}K_{0}\tilde{b}_{2k-1,0}$$
(21)

and

$$\sum_{i+j=2k} \Phi_{ij}^{(m)} = I_{0,0}^{(m)}(\tilde{b}_{2k,k}h^{k} + \dots + \tilde{b}_{2k,1}h + \tilde{b}_{2k,0}) \\ + (-1)I_{0,0}^{(m-1)}(C_{2}^{1}\tilde{b}_{2k,k}h^{k} + \dots + C_{2k-1}^{1}\tilde{b}_{2k,1}h + C_{2k+1}^{1}\tilde{b}_{2k,0}) \\ \vdots \\ + (-1)^{m-1}I_{0,0}^{(1)}(C_{2[\frac{m+1}{2}]-1}^{m-1}\tilde{b}_{2k,k-[\frac{m-1}{2}]}h^{k-[\frac{m}{2}]} + \dots + C_{2k-1}^{m-1}\tilde{b}_{2k,1}h + C_{2k+1}^{m-1}\tilde{b}_{2k,0}) \\ + (-1)^{m}C_{2[\frac{m}{2}]}^{m-1}\left(K_{0}\tilde{b}_{2k,k-[\frac{m}{2}]} + \dots + K_{2k-2[\frac{m}{2}]}\tilde{b}_{2k,0}\right)h^{k-[\frac{m}{2}]} \\ \vdots \\ + (-1)^{m}C_{2k-2}^{m-1}\left(K_{0}\tilde{b}_{2k,1} + K_{2}\tilde{b}_{2k,0}\right)h + (-1)^{m}C_{2k}^{m-1}K_{0}\tilde{b}_{2k,0}$$
(22)

Now we are in position to prove the main results.

Proof of Theorem 2.1 In the following we suppose n = 2s first. In this case, by (2) the Melnikov function $\Phi(h)$ of system (7) has the following form

$$\Phi(h) = \oint_{L_h} \frac{1}{(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2}} \sum_{0 \le i + j \le 2s} (a_{ij} x^{i+1} y^j + b_{ij} x^i y^{j+1}) dt$$

$$= \sum_{0 \le i + j \le 2s} \Phi_{ij}$$

$$= \sum_{k=1}^s (\sum_{i+j=2k-1} \Phi_{ij} + \sum_{i+j=2k} \Phi_{ij}) + \Phi_{00}.$$
(23)

From (19) and (20), (23) becomes

$$\Phi(h) = \sum_{k=0}^{s} \left(\frac{1}{r_1^{2m_1-1}} P_{m_1-1+k}(h) + \frac{1}{r_2^{2m_2-1}} P_{m_2-1+k}(h) + P_{\left[\frac{2k+1-m_1-m_2}{2}\right]}(h) \right)$$

$$= \frac{1}{r_1^{2m_1-1}} P_{m_1-1+s}(h) + \frac{1}{r_2^{2m_2-1}} P_{m_2-1+s}(h) + P_{\left[\frac{2s+1-m_1-m_2}{2}\right]}(h), \qquad (24)$$

where $P_k(h)$ is a polynomial of h of degree k.

Obviously all the zeros of (24) satisfy

$$\left[\frac{1}{r_1^{2m_1-1}}P_{m_1-1+s}(h) + \frac{1}{r_2^{2m_2-1}}P_{m_2-1+s}(h)\right]^2 = \left[P_{\left[\frac{2s+1-m_1-m_2}{2}\right]}(h)\right]^2$$

Further the above formula becomes

$$\sqrt{(1-\alpha_1^2h)(1-\alpha_2^2h)P_{2s+2(m_1+m_2)-4}(h)} = Q_{2s+2(m_1+m_2-3)}(h),$$

where $P_{2s+2(m_1+m_2)-4}(h)$ and $Q_{2s+2(m_1+m_2-3)}(h)$ are two real coefficient polynomials of h of degree $2s + 2(m_1 + m_2) - 4$ and $2s + 2(m_1 + m_2) - 3$ respectively. Hence the number of zeros of $\Phi(h)$ are not large than $4s + 4(m_1 + m_2) - 6$.

For the case of n = 2s - 1, similarly we can prove that the number of zeros of $\Phi(h)$ are not large than $4s + 4(m_1 + m_2) - 6$.

Notice that $\Phi(h) = 0$ at h = 0 in (23). From lemma 1.3, we know that there exists an $\varepsilon_0 > 0$ such that when $0 < |\varepsilon| < \varepsilon_0$, $a = (a_{ij}, b_{ij})$ which satisfy $|a_{ij}| \le K$, $|b_{ij}| \le K$, the system (9) has at most $4\left(m_1 + m_2 + \frac{n+1}{2}\right) - 7$ limit cycles. The proof is completed.

Proof of theorem 2.2 We suppose $\alpha_1 = \alpha_2$ and $m_1 + m_2 = m$ in (7).

For the case of n = 2s, from (21) and (22) the Melnikov function $\Phi(h)$ of system (7) has the following form

$$\Phi(h) = I_{(0,0)}^{(m)}(b_s^{(m)}h^s + \dots + b_1^{(m)}h + b_0^{(m)}) + \dots + I_{0,0}^{(1)}(b_{s-[\frac{m-1}{2}]}^{(1)}h^{s-[\frac{m-1}{2}]} + \dots + b_1^{(1)}h + b_0^{(1)}) + (B_{s-[\frac{m}{2}]}h^{s-[\frac{m}{2}]} + \dots + B_1h + B_0),$$

where $b_j^{(i)}$, $B_j(1 \le i \le m, j \ge 0)$ are linear combinations of a_{ij} , b_{ij} with $0 \le i + j \le 2s$. Let $\sqrt{1 - \alpha_1^2 h} = r$, 0 < r < 1. And from (12) and (13), the above formula becomes

$$\Phi(h) = \frac{1}{r^{2m-1}} (c_{2s+m} r^{2s+m} + c_{2s+m-1} r^{2s+m-1} + \dots + c_{2m-1} r^{2m-1} + c_{2m-2} r^{2m-2} + c_{2m-4} r^{2m-4} + \dots + c_2 r^2 + c_0)$$

= $\frac{1}{r^{2m-1}} P_{2s+m}(r),$

where $P_{2s+m}(r)$ is a polynomial of r of degree 2s+m and $P_{2s+m}(r) = 0$ at r = 1. Notice that the polynomial $P_{2s+m}(r)$ has only 2s+2 items. By Rolle theorem $P_{2s+m}(r)$ has at most 2s+1 positive zeros. So the polynomial $\frac{P_{2s+m}(r)}{1-r}$ has at most 2s positive zeros. From lemma 1.3, we know that there exists an $\varepsilon_0 > 0$ such that when $0 < |\varepsilon| < \varepsilon_0$, $a = (a_{ij}, b_{ij})$ which satisfy $|a_{ij}| \leq K$, $|b_{ij}| \leq K$ the system (7) has at most 2s limit cycles.

For the case of n = 2s - 1 we can prove the theorem in the same way. The proof is completed.

Remark 2.1 In fact, for the system

$$\dot{x} = y(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2} \cdots (1 - \alpha_k x)^{m_k}, \dot{y} = -x(1 - \alpha_1 x)^{m_1} (1 - \alpha_2 x)^{m_2} \cdots (1 - \alpha_k x)^{m_k},$$

where m_1, m_2, \dots, m_k are positive integers and $\alpha_1, \alpha_2, \dots, \alpha_k$ are real constants which satisfy $\alpha_1 \cdot \alpha_2 \cdot \dots \cdot \alpha_k \neq 0$. Using the same way we can prove that if we perturb the above system inside the polynomial systems of degree n we can obtain up to first order in ε at most $2^k \left([n \frac{n+1}{2}] + \sum_{j=1}^k m_j - k \right) + 2^{k-1}(k-1) - 1$ limit cycles.

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