THE PITCHFORK BIFURCATION

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ABSTRACT. We give development of a new theory of the Pitchfork bifurcation, which removes the perspective of the third derivative and a requirement of symmetry.

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1. Introduction

The normal form for the pitchfork bifurcation is described usually for one variable (see Guckenheimer and Holmes) by

(1)
$$\frac{dx}{dt} = F(x) = \mu x - x^3, \quad x \in \mathbb{R}^1, \ \mu \in \mathbb{R}^1$$

Note that this equation is invariant under the change of the variable $x \to -x$. That is, F is an odd function. This condition suggests that the Pitchfork bifurcation is generic for problems that have symmetry. To obtain the above form one argues (or assumes) that the second derivative of F(x) is zero. This normal form is standard in the literature on Pitchfork bifurcation [2, 3].

Our own proof [4] for the one variable case is different from the previous literature in that a new uniformity condition is satisfied in place of symmetry or the hypothesis, vanishing of a second derivative. For the new uniformity condition see below. In addition we take the path of using this uniformity together with the Poincaré Hopf Theorem to show that the second derivative must be zero.

We have found little in the literature on the case of more than one variable, except for suggesting that (1) still applies. Kuznetsov [5] has a proof for an n variable case for pitchfork that assumes invariance under an involution. Kuznetsov's hypothesis eliminates a second derivative.

Our paper [4] states an n-dimensional version of the pitchfork theorem. That proof involves a reduction to one dimensional theory using center manifold theory. In this paper we give a proof, which gives new insight especially for more than one variable.

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FIGURE 1. Geometry of isoclines for the normal form (3). Left: a single equilibrium at (1,1) for a < 1; middle: a single equilibrium at (1,1) for a = 1; right: for a > 1, (1,1) becomes unstable, and two stable equilibria emerge flanking (1,1). To view how changing the parameter a changes the isoclines, see: https://www.desmos.com/calculator/x5fz6lavrx

As we will see the one dimensional case with emphasis on the third derivative is misleading. In fact for two variables there is a robust pitchfork bifurcation in a quadratic system (with no third derivative at all). Our previous work [4] is a background for this paper.

2. Normal form

We propose that the **normal form** for the Pitchfork bifurcation for the dimension of the space greater than one be:

(2)
$$\frac{dx}{dt} = y^2 - ay - x$$
$$\frac{dy}{dt} = x^2 - ax - y, \quad x, y \ge 0$$

For each point of the bifurcation parameter a, the "central equilibrium" is (x, y) = (0, 0), and may be described in terms of the two isoclines—the curves in the (x, y) plane where $\frac{dx}{dt} = 0$ (the x isoclines) and where $\frac{dy}{dt} = 0$ (the y isoclines). For given a the equilibria are given as the intersection of these isoclines. The isoclines are described respectively, by the equations:

$$(3a) x = y^2 - ay$$

$$(3b) y = x^2 - ax.$$

Figure 1 shows the curves for selected values of a. For $a \leq 1$, the central equilibrium is the only equilibrium relevant for the bifurcation. For a > 1 there are four points of intersection and three equilibrium points for the model. The transition between these situations occur at a = 1, when the isoclines are tangent to each other at (0,0).

The defining characteristic of a pitchfork bifurcation is the transition from a single stable equilibrium to two new stable equilibria separated by a saddle. The saddle emerges from the old stable equilibrium. The Figure 1 illustrates this characteristic as a increases from less than one to greater than one.

Next we will obtain the intersection points of two isoclines as in Equations 3a and 3b. The curves described by Equations 3a and 3b are quadratic and one can solve their intersection points analytically. If we substitute 3b into 3a, solving for x yields for the solutions x_i as can be checked:

(4a)

$$x_{1} = 0, \text{ central equilibrium}$$

$$x_{2} = \frac{1}{2}(a-1) + \frac{1}{2}\sqrt{(a-1)(a+3)}$$

$$x_{3} = \frac{1}{2}(a-1) - \frac{1}{2}\sqrt{(a-1)(a+3)}$$

$$x_{4} = a+1$$

(4b)

Similarly, substituting Equation 3a into 3b and solving for y yields the solutions y_i :

$$y_1 = 0, \text{ central equilibrium}$$
$$y_2 = \frac{1}{2} (a - 1) - \frac{1}{2} \sqrt{(a - 1) (a + 3)}$$
$$y_3 = \frac{1}{2} (a - 1) + \frac{1}{2} \sqrt{(a - 1) (a + 3)}$$
$$y_4 = a + 1.$$

The pairs (x_i, y_i) , i = 1, ..., 4 describe the equilibria. Note that the intersection point (x_4, y_4) is extraneous to the bifurcation phenomena. Note also that the Equation 4a and 4b show the bifurcation effect at a = 1 and a > 1.

The two isoclines are parabolas and they get translated vertically and horizontally as the parameter a increases. One can see the intersections of these parabolas in terms of simple analytic geometry, and these intersections include the equilibria of the pitchfork. This formalism allows us to see the pitchfork variables in terms of geometry and extend the analysis to the nonsymmetric case, $\frac{dx}{dt} = y^2 - ay - x$, $\frac{dy}{dt} = x^2 - bx - y$.

The Jacobian matrix J of the first partial derivatives of System (2) at the central equilibrium $(x_1, y_1) = (0, 0)$ of Equation 3 is:

(5)
$$J = \begin{pmatrix} -1 & -a \\ -a & -1 \end{pmatrix}.$$

For each a, the eigenvalues are $\lambda_1 = a - 1$, $\lambda_2 = -a - 1$ and the corresponding eigenvectors are (-1, 1) and (1, 1) respectively. When a = 1, $\lambda_1 = 0$ and $\lambda_2 < 0$. When a < 1, both λ_1, λ_2 have negative real parts. Hence the central equilibrium is stable. When a > 1, $\lambda_1 > 0$ and $\lambda_2 < 0$, and the central equilibrium is a saddle. The qualitative structure is robust.

Eigenvalues at the equilibrium (x_2, y_2) are given by: $\lambda_1 = -1 + \sqrt{-(a-1)(a+3)} + 3$ and $\lambda_2 = -1 - \sqrt{-(a-1)(a+3)} + 3$. When 1 < a, both λ_1, λ_2 have negative real parts. Hence the equilibrium is stable, similarly for (x_3, y_3) .

Remark 1: When 1.2361 < a, both λ_1, λ_2 are complex conjugate numbers with negative real parts. The pitchfork phenomena continues after a = 1.2361 using the equilibrium Equations 4a and 4b.

3. A relationship of our normal form (2) to a main biological example

We are motivated by work by Gardner et al. [6] for a "synthetic, bistable generegulatory network ... [to] provide a simple theory that predicts the conditions necessary for bistability." The toggle, as designed and constructed by Gardner et al., is a network of two mutually inhibitory genes that acts as a switch by some mechanism, as a control for switching from one basin to another. Consider the particular setting of Gardner et al.'s circuit design of the toggle switch as:

(6)
$$\frac{dx}{dt} = \frac{\alpha_1}{1+y^m} - x$$
$$\frac{dy}{dt} = \frac{\alpha_2}{1+x^n} - y.$$

If m = n = 0, the equilibrium is $x = \frac{\alpha_1}{2}$, $y = \frac{\alpha_2}{2}$, and the eigenvalues of the Jacobian are negative. If $\alpha_1 < 2 \max(x)$ and $\alpha_2 < 2 \max(y)$, the system has a unique global stable equilibrium [6, 7].

When $\alpha_1 = \alpha_2 = 2$, and m = n > 0, the system in Equation 6 can be written more specifically as in Gardner et al.:

(7)
$$\frac{dx}{dt} = \frac{2}{1+y^m} - x = f(x,y)$$
$$\frac{dy}{dt} = \frac{2}{1+x^m} - y = g(x,y), \quad 0 \le x, y.$$

For this system with $0 \le m \le 2$, every equilibria must be (1,1). We give the Taylor approximation for each m about the equilibrium (1,1). For this approximation consider the derivatives f_x , f_y , f_{xx} , f_{yy} , f_{xy} and similar for g, all at the point (1,1). These can be computed as:

$$f_x = -1, \ f_y = -\frac{1}{2}m, \ g_x = -\frac{1}{2}m, \ g_y = -1$$

 $f_{yy} = g_{xx} = \frac{1}{2}m, \ \text{and the remaining derivatives are zero}$

Therefore, the Taylor approximation about the equilibrium (1, 1) for each $m \leq 2$ is given (deleting the remainder):

(8a)
$$\frac{dx}{dt} = \frac{1}{2}my^2 - \frac{3}{2}my + m + 1 - x$$

(8b)
$$\frac{dy}{dt} = \frac{1}{2}mx^2 - \frac{3}{2}mx + m + 1 - y$$

Recall our normal form Equation 1 written in terms of u and v is:

(9a)
$$\frac{du}{dt} = v^2 - av - u$$

(9b)
$$\frac{dv}{dt} = u^2 - au - v.$$

We wish to compare Equation 8 and 9 with the parameter values at the bifurcation points, namely m = 2 for Equation 8 and a = 1 for Equation 9. Then the transformation x = u + 1 and y = v + 1 gives a correspondence between Equations 8 and 9 at these parameter values. After the bifurcation points m and a vary dependently but we don't know the functional relationship. The point is that each m and aincreasing from the bifurcation value create a pitchfork bifurcation.

4. General pitchfork theorem

Recall some setting from our previous paper [4].

(10)
$$\frac{dx}{dt} = F_{\mu}(x), \ \mu \in \mathbb{R}, \ x \in \mathbb{R}^{n} \text{and} \ |\mu|, |x| < \varepsilon$$

This is associated to a family F_{μ} with bifurcation parameter $\mu \in (-\varepsilon, \varepsilon)$ describing $\frac{dx}{dt} = F_{\mu}(x)$. Here x belongs to a domain X of \mathbb{R}^n , and $F_0(x) = F(x)$. We suppose that the dynamics of F_{μ} is that of a stable equilibrium x_{μ} , basin B_{μ} for $\mu < \mu_0$, and that the bifurcation is at μ_0 . We assume that $x_{\mu} = 0$ is an equilibrium for all μ .

Uniformity condition: "First bifurcation from a stable equilibrium." The equilibrium does not "leave it's basin" in the sense that there is a neighborhood N of x_0 , such that N is contained in B_{μ} for all $\mu < 0$.

Note this implies by a uniform continuity argument, that even at $\mu = 0$, x_0 is a sink in the sense that $x(t) \to x_0$, if the initial point belongs to N. The dynamics of F_{μ} , $\mu = 0$ has a "basin" $B: x_0$ is a "weak sink." It follows that in this space B, the only equilibria of F_{μ} is x_{μ} for all $\mu \leq 0$. One could say that μ_0 is the "first" bifurcation.

Define J_{μ} to be the matrix of partial derivatives of F_{μ} at x_{μ} . The eigenvalues of J_{μ} for $\mu < \mu_0$ all have negative real part, either real, or in complex conjugate pairs. At the bifurcation, one has either a single real eigenvalue becoming zero and then positive with the pitchfork (if det $(J_{\mu_0}) \neq 0$ then a complex conjugate pair of distinct eigenvalues with real parts zero becomes positive after the bifurcation and then the Hopf oscillation occurs, not discussed here).

A pitchfork bifurcation converts a stable equilibrium into two stable equilibria (the Hopf bifurcation converts a stable equilibrium into a stable periodic solution).

The pitchfork bifurcation theorem: In Equation 10, let x_{μ} be a stable equilibrium for μ for all $\mu < \mu_0$. Suppose the uniformity condition is satisfied. If the determinant of $J_{\mu_0} = 0$, then generically the dynamics undergoes a pitchfork bifurcation.

1. If n = 1, this has been proved in our paper [4] as discussed above.

2. For n > 1, generically there is a pitchfork as exemplified by the normal form in Section 2.

Sketched of proof of the pitchfork bifurcation theorem for n > 1.

Note in Equation 10, the equation for the equilibria is $F_{\mu}(x) = 0$ for each μ .

Consider the hypothesis above and consider the equilibrium x_{μ} . At $\mu = \mu_0$, the determinant of J_{μ_0} becomes zero. Then by the local stable manifold theory, this equilibrium changes from a sink $\mu < \mu_0$, to a saddle for $\mu > \mu_0$. This saddle has an n-1 dimensional contracting stable manifold \mathbf{W}^u_{μ} and a one dimensional expanding stable manifold \mathbf{W}^u_{μ} .

We will be using the following.

Poincaré-Hopf index theorem: Suppose $\frac{dx}{dt} = F(x)$, x belongs to X and F : $X \to \mathbb{R}^n$. Suppose X homeomorphic to a closed ball [8] and F(x) points to the

interior of X for each x belonging to the boundary

(11)
$$\sum_{F(x)=0, x \in X} sign\left(\det(J)\right)(x) = (-1)^n$$

where J is the Jacobian of F at x. The formula on the left side of Equation (11) is the Poincaré-Hopf index. In particular generically, in the case n is odd, there is an odd number of equilibria.

Then we observe that the Poincaré-Hopf index at the equilibrium for the sink is $(-1)^n$ for $\mu < \mu_0$. For $\mu > \mu_0$ of this equilibria changes to $(-1)^{n+1}$ as the sink changes to a saddle. This follows from the eigenvalue structure of the saddle. Therefore, by the Poincaré-Hopf index theorem there must be other new equilibria for $\mu > \mu_0$. Generically these new equilibria must be two in number and they are sinks. We have obtained the defining property of the pitchfork.

The above needs to be carried out uniformly for all μ . This procedure follows a suggestion of Mike Shub [9, 10].

We consider the 2-dimensional center manifold, C_M , at $\mu = \mu_0$ and the equilibrium $x = x_0$ with the added equation $\frac{d\mu}{dt} = 0$. This gives a dynamical system of n + 1 equations. The C_M is two dimensional and the stable manifold \mathbf{W}^u_{μ} is n - 1 dimensional, corresponding to the eigenvalues J_{μ_0} with negative real parts. The C_M corresponds to the span of the null space (eigenvector corresponding the zero eigenvalue) and the space of the variable μ . The C_M projects on to μ and the inverse image of μ is the expanding one dimensional manifold of the equilibrium x_0 , for F_{μ} .

5. A biological perspective

In our previous work [4], a cell can be thought as a point in the basin and the cell type can be identified with a basin. Thus, the identity of a specific cell type in our genome dynamics can be defined by characteristic gene expression pattern at the equilibrium. We suggest that the emergence of a new cell type from this original cell type, through differentiation, reprogramming, or cancer a result of pitchfork bifurcation, is a departure from a stable equilibrium and requires cell division. In normal cell division during differentiation or reprogramming, a cell can undergo symmetric or asymmetric division. Let A be the mother cell in the following. In symmetric division, two identical daughter cells arise: A and A (B and B), that have genomes with the same activity [11, 12]. In asymmetric division, two daughter cells arise: A and B, that have genomes with different activity. Both cases above reflect a pitchfork bifurcation. In another type of asymmetric division, two daughter cells arise, B and C, where the activity of both genomes is different from the mother also reflecting a pitchfork bifurcation. One example of this is in cases of abnormal cell division, where chromosomes are mis-segregated, resulting in one daughter with too many and one daughter with too few chromosomes. This type of division may be one of the initiating events in emerging cancer cells [13]. Capturing these events in terms of our bifurcations may give us insight into emergence of a cell type.

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