FAREY-LORENZ PERMUTATIONS FOR INTERVAL MAPS

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ABSTRACT. Lorenz-like maps arise in models of neuron activity, among other places. Motivated by questions about the pattern of neuron firing in such a model, we study periodic orbits and their itineraries for Lorenz-like maps with nondegenerate rotation intervals. We characterize such orbits for the simplest such case and gain substantial information about the general case.

1. INTRODUCTION

The motivation for this paper comes from the paper [3], where the authors consider a mathematical model of a neuron, and after some simplifications they get an interval map. Under certain assumptions, this map is Lorenz-like.

A *Lorenz-like map* is a map f of an interval $I = [0, 1]$ to itself, for which there exists a point $c \in (0, 1)$ such that f is continuous and increasing (not necessarily strictly) on $[0, c)$ and on $(c, 1]$, and $\lim_{x \to c^-} f(x) = 1$, $\lim_{x \to c^+} f(x) = 0$. There is a minor problem regarding what we should do with $f(c)$. A possible solution is to set $f(c) = 0$ or $f(c) = 1$, but the most common way of looking at it is to say that at c the map f takes two values, both 0 and 1. Then we can say that f is continuous and increasing on both $I_L = [0, c]$ and $I_R = [c, 1]$. Throughout most of the paper we will consider only orbits of f that avoid c, while at a certain moment we will explain what happens if they do not.

If $f(0) > f(1)$, that is, $f(I_L) \cap f(I_R) = \emptyset$, the map is *non-overlapping*. If $f(0) \leq f(1)$, that is, $f(I_L) \cap f(I_R) \neq \emptyset$, the map is *overlapping* (see Figure 1).

For a point $x \in [0, 1]$ and a positive integer n we will denote by $n_R(x)$ the number of integers $i \in \{0, \ldots, n-1\}$ such that $f^i(x) \in I_R$. If the limit

$$\rho(x) = \lim_{n \to \infty} \frac{n_R(x)}{n}$$

exists, we will call it the *rotation number* of x. Observe that if x is a periodic point of f of period p then $\rho(x)$ exists and is equal to $n_R(x)/p$. Moreover, it is known that if $f(0) \geq f(1)$ (in particular, if the map is non-overlapping), then all points have the same rotation number (see [2]). We will denote it $\rho(f)$.

In [3], orbits of Lorenz-like maps are interpreted in a special way. A point in I_L corresponds to a spike of the activity of the neuron, while a point in I_R corresponds to a spike preceded by a small oscillation. Thus, in order to predict the order of spikes and small oscillations (the *MMO signature*) along an orbit, we need to know the *itinerary* of our orbit: a sequence of symbols L and R. The nth symbol is L or
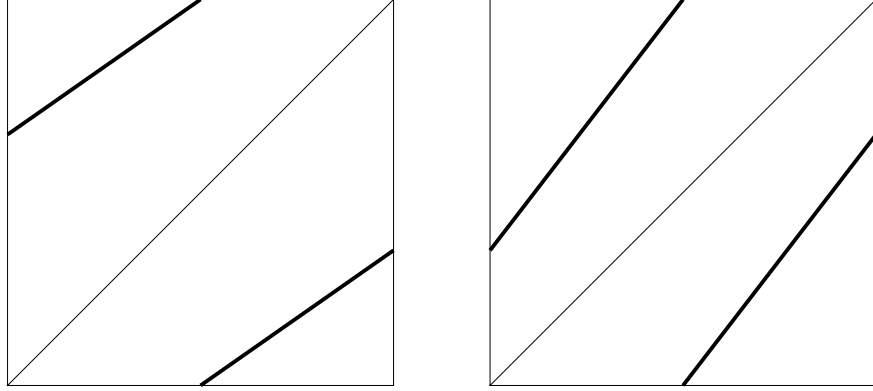

FIGURE 1. Non-overlapping (left) and overlapping (right) Lorenz-like maps.

R, depending on whether the nth point of the orbit is in I_L or I_R. Periodic orbits are of special interest.

As noted in [3], the non-overlapping case is simple. Let us consider the case when the rotation number of the map is a/p, where a and p are coprime. If the numbering of the terms of the periodic itinerary starts with 0, then the kth term is L if and only if $(1 + ka \mod p) \leq p - a$.

Our aim is to obtain similar results for overlapping maps. By "similar results" we mean results of similar simplicity. Precise description of possible itineraries for general overlapping Lorenz-like maps is possible, but it will be quite complicated and not so easy to apply in practice.

In the overlapping case, when the rotation interval is nondegenerate it will contain, or in the simplest case have as endpoints, *Farey neighbors* (fractions $a/p < b/q$, such that $bp - aq = 1$).

2. DEFINITIONS

Let f be a Lorenz-like map. If $t \in f(I_L) \cap f(I_R)$, then we define the *water map at level t* (see Figure 2) by

$$(1) \qquad f_t(x) = \begin{cases} \max(t, f(x)) & \text{if } x \in I_L, \\ \min(t, f(x)) & \text{if } x \in I_R. \end{cases}$$

This map is also Lorenz-like and $f_t(0) = f_t(1)$, so all points have the same rotation number $\rho(f_t)$ for it. It is known that $\rho(f_t)$ is an increasing continuous function of t, and if $f(0) \leq f(1)$, then the set $\text{Rot}(f)$ of the rotation numbers for f of all points having rotation number is equal to the interval $[\rho(f_{f(0)}), \rho(f_{f(1)})]$ (see, e.g., [1]).

If $t \in I \setminus (f(I_L) \cap f(I_R))$, then the map f_t, defined by (1), will be called the *semiwater map at level t*.

We will be considering not only periodic orbits, but also their finite unions (in fact, only unions of two orbits, but we can make definitions slightly more general). We will call them *fupos* (for *finite unions of periodic orbits*). For each fupo we will consider its *permutation*, that is, if a fupo P consists of points $x_1 < \cdots < x_n$, and $f(x_i) = x_{\sigma(i)}$ for $i = 1, \ldots, n$, then σ is the permutation of P. Since we are using indices from 1
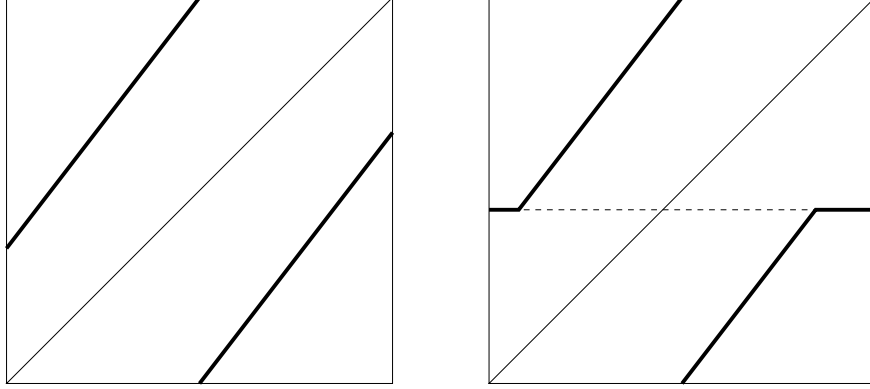
FIGURE 2. A Lorenz-like map and one of its water maps.

to n, rather than from 0 to $n-1$, we will understand that $m \mod n$ is the number $i \in \{1, \ldots, n\}$ such that n divides $m - i$.

Permutations of fupos of Lorenz-like maps have a specific form. Namely, if a fupo has $n > 1$ elements, then there exists $k \in \{1, \ldots, n-1\}$ such that σ is increasing on $\{1, \ldots, k\}$ and on $\{k+1, \ldots, n\}$. We will call such permutations (and the permutation of $\{1\}$) *L-permutations*. If our fupo is a periodic orbit, then its rotation number is $(n-k)/n$. It is clear that for every L-permutation σ there exists a Lorenz-like map f with the fupo P such that P has permutation σ. A canonical model can be built as the "connect the dots" map with the dots

$$\left(0, \frac{\sigma(1)}{n+1}\right), \left(\frac{1}{n+1}, \frac{\sigma(1)}{n+1}\right), \ldots, \left(\frac{k}{n+1}, \frac{\sigma(k)}{n+1}\right), \left(\frac{k+\frac{1}{2}}{n+1}, 1\right),$$

$$\left(\frac{k+\frac{1}{2}}{n+1}, 0\right), \left(\frac{k+1}{n+1}, \frac{\sigma(k+1)}{n+1}\right), \ldots, \left(\frac{n}{n+1}, \frac{\sigma(n)}{n+1}\right), \left(1, \frac{\sigma(n)}{n+1}\right),$$

see Figure 3.

Among L-permutations there are some special ones, which look like cyclic permutations for circle rotations. They are those cyclic L-permutations σ of $\{1, \ldots, n\}$, for which $\sigma(1) > \sigma(n)$. We will call them *twist* permutations, and a periodic orbit with such permutation will be called a *twist orbit*. It is easy to see that a periodic orbit of a Lorenz-like map f is twist if and only if it is also an orbit of some water or semiwater map f_t. It is also easy to describe explicitly a twist permutation of $\{1, \ldots, n\}$ with rotation number j/n. Namely, $\sigma(i) = i + j \mod n$.

We can treat each cycle of an L-permutation as a permutation. Clearly, it will also be an L-permutation. Now we can define the type of permutations that will be the main object in this paper.

Definition 2.1. An L-permutation σ of $\{1, \ldots, p+q\}$ will be called a *Farey-Lorenz permutation* (or *FL-permutation*) if σ consists of two cycles, both of them twist, of period p and q, with rotation numbers a/p and b/q respectively, and $a/p < b/q$ are Farey neighbors, that is, $bp = aq + 1$.
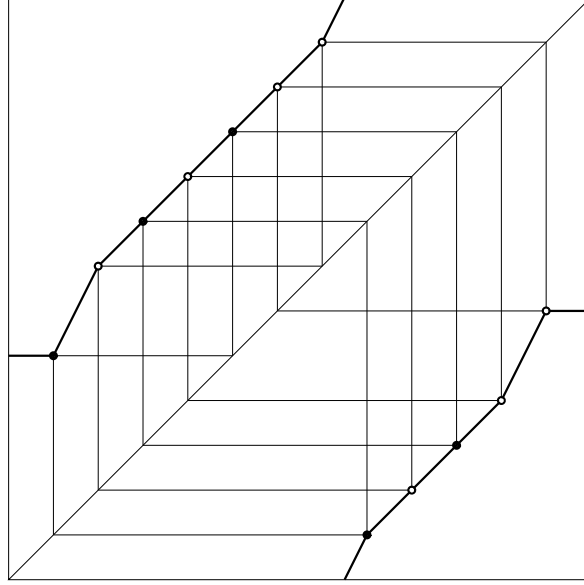
FIGURE 3. The canonical model for a union of twist periodic orbits of rotation numbers $2/5$ and $3/7$.

Note that the rotation numbers of the periodic orbits from Figure 3 are Farey neighbors; the corresponding FL-permutation is

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 5 & 7 & 8 & 9 & 10 & 11 & 12 & 1 & 2 & 3 & 4 & 6 \end{pmatrix}$$

Observe that we can assume that $p < q$. If $p = q$ then $a/p = 0/1$ and $b/q = 1/1$; this is a trivial case which we will not consider in the next two sections. If $p > q$ then we can replace f by g, conjugate to f via the map $x \mapsto 1 - x$. Then the rotation numbers of our orbits will become $(p - a)/p$ and $(q - b)/q$, and we have $(q - b)/q < (p - a)/p$ and $q < p$. The results obtained for this case can be easily translated to the original case.

3. STRUCTURE OF FL-PERMUTATIONS

In this section we assume that σ is an FL-permutation with cycles of rotation numbers a/p and b/q, where $a/p < b/q$ are Farey neighbors and $p < q$. We will call those cycles *slow* and *fast*, respectively. Let f be the canonical model for this permutation, with the corresponding fupo $P \cup Q$, where periodic orbits P and Q correspond to the slow and fast cycles, respectively. We will refer to P as the slow orbit and Q as the fast orbit.

Lemma 3.1. *The slow cycle contains* 1, *and the fast cycle contains* $p + q$.

Proof. We will show that the slow cycle contains 1; the proof that the fast cycle contains $p + q$ is similar.

Let t be the leftmost element of P and s the leftmost element of Q. Then P is an orbit of the water map $f_{f(t)}$ and Q is an orbit of the water map $f_{f(s)}$. Therefore $\rho(f_{f(t)}) = a/p < b/q = \rho(f_{f(s)})$, and since $\rho(f_u)$ is an increasing function of u, we get

$f(t) < f(s)$. Both t and s are to the left of the discontinuity point c, so $f(t) < f(s)$ implies $t < s$. This shows that the leftmost element of the slow cycle is to the left of the leftmost element of the fast cycle, so 1 belongs to the slow cycle. $\qquad \square$

Lemma 3.2. *The fast cycle contains 2.*

Proof. Suppose that 2 belongs to the slow cycle. Let z be the second from the left element of the slow orbit P. Look at the map $f_{f(z)}$. If w is the leftmost element of P, then $f_{f(z)}(w) = f_{f(z)}(z) = f(z)$, and thus, $f_{f(z)}^i(w) = f^i(z)$ for $i = 1, \ldots, k$, where $k < p$ is such a number that $f^k(z) = w$. Therefore, w is a periodic point of $f_{f(z)}$ of period $k < p$. This means that $\rho(f_{f(z)})$ is a rational number with denominator smaller than $p = \min(p, q)$.

If $f(z) \leq f(1)$, then $f_{f(z)}$ is a water map, so its rotation number is contained in $[a/p, b/q]$. However, since a/p and b/q are Farey neighbors, there is no such number in $[a/p, b/q]$, a contradiction.

If $f(z) > f(1)$, then $Q \subset [z, 1]$, so Q is a periodic orbit of $f_{f(z)}$. Therefore, $\rho(f_{f(z)}) = b/q$, also a contradiction. $\qquad \square$

Now we want to know where the points of P are compared to the points of Q. Let the points of P be $x_1 < x_2 < \cdots < x_p$ and the points of Q $y_1 < y_2 < \cdots < y_q$. Denote $J_1 = [x_1, y_1]$ and $J_j = [y_{j-1}, y_j]$ for $j = 2, \ldots, q$. By Lemma 3.1, the definition of J_1 is correct, and by Lemma 3.2, the only element of P in J_1 is x_1. Also, by Lemma 3.1, there are no elements of P to the right of y_q.

Lemma 3.3. *For $i = 1, \ldots, p$, the point $f^i(x_1)$ belongs to $J_{ib \mod q}$.*

Proof. Each interval J_j, with two exceptions, is mapped in a monotone way onto J_ℓ, where $\ell = j + b \mod q$. One exception is the interval J_{q-b+1}, which contains c in its interior. Its left part is mapped to $[y_q, 1]$, while its right part is mapped to $[0, y_1]$. However, there are no points of P in $[y_q, 1]$, and the only point of P in $[0, y_1]$ is x_1. Thus, for the purpose of finding the positions of the points of P, we may say that J_{q-b+1} is mapped onto J_1 (notice that $1 = (q-b+1)+b \mod q$). The other exception is J_1, and at the moment we do not know where it is mapped.

Let k be the integer such that $f(x_1) \in J_k$. Then, by what we said above, $f^i(x_1) \in J_\ell$, where $\ell = k + (i - 1)b \mod q$, for $i = 1, 2, \ldots, p$. In particular, since $f^p(x_1) = x_1$, we get $1 = k + (p - 1)b \mod q$. Since $pb = aq + 1$, we get $1 = k + 1 - b \mod q$, so $k = b$. This completes the proof. $\qquad \square$

The above lemma gives us full information about the permutation of the fupo $P \cup Q$. We can summarize it in the following proposition.

Proposition 3.4. *With the notation we adopted, $f(x_i) = x_{i+a \mod p}$ and $f(y_j) = y_{j+b \mod q}$. The relative order of the points of the orbits P and Q is given by the following rule: $x_1 < y_1$; then for $i = 1, \ldots, p - 1$, if $j = 1 + ia \mod p$ and $\ell = ib \mod q$, then $y_{\ell-1} < x_j < y_\ell$.*

4. MARKOV GRAPHS OF FL-PERMUTATIONS

Now we want to describe the Markov graph for the fupo considered in the preceding section. The vertices of this graph are *basic intervals*, that is, the closures of the intervals into which $P \cup Q$ divides I. There is an arrow from a vertex K_1 to K_2 if

and only if $K_2 \subset f(K_1)$. Then the trajectories of f correspond to the infinite paths in this (directed) graph.

We are interested only in periodic trajectories, so we can remove from the graph all vertices that do not contribute to any periodic trajectories (except perhaps P or Q). We will call the resulting subgraph the *essential Markov graph*.

The first vertices that we can remove are intervals $[0, x_1]$ and $[y_q, 1]$, because f is constant on each of them. We are left with basic intervals of three types.

> **Fast-fast intervals:** Both endpoints belong to Q. They are intervals J_j that do not contain any point of P.
>
> **Fast-slow intervals:** The left endpoint belongs to Q, and the right one to P. Such an interval is contained in some J_j, and we will denote it by J_j^-.
>
> **Slow-fast intervals:** The left endpoint belongs to P, and the right one to Q. Such an interval is contained in some J_j, and we will denote it by J_j^+.

With two exceptions, each of the intervals listed above is mapped by f onto one basic interval listed above, of the same type and with index larger by b (modulo q).

The first exception is the slow-fast interval J_1. It is mapped onto $[x_{1+a}, y_{1+b}]$, which is the union of the slow-fast interval J_b^+ and the interval J_{b+1}.

Lemma 4.1. *Interval J_{b+1} is a basic fast-fast interval.*

Proof. We have to show that there are no elements of P in J_{b+1}. Suppose there is one. Then $b + 1 = ib \mod q$ for some $i \in \{1, \ldots, p\}$. Therefore $1 = kb \mod q$ for some $k \in \{0, \ldots, p-1\}$. However, $bp = aq + 1$, so also $1 = pb \mod q$. Therefore, q divides $(p-k)b$, so, since q and b are coprime, q divides $p - k$. This is impossible, since $1 \le p - k \le p < q$, and we get a contradiction. $\square$

The second exception is that the fast-slow interval J_{q+1-b} (which contains the discontinuity point c) maps only to the two intervals we removed.

Now we see that every fast-slow basic interval maps onto another fast-slow one, etc., until it maps onto J_{q+1-b}, which maps to intervals we removed. Therefore no fast-slow interval in the essential Markov graph, and we remove them all.

The remaining basic intervals are all the fast-fast ones and all the slow-fast ones. Each of them is contained in one of the intervals J_j; then call it K_j. On the other hand, each interval J_j is either a fast-fast basic interval (so $K_j = J_j$), or it contains a unique slow-fast interval, $K_j = J_j^+$. Hence, we get the following theorem.

Theorem 4.2. *The vertices of the essential Markov graph are the intervals K_j, $j = 1, \ldots, q$. The arrows are from K_j to $K_{j+b \mod q}$, and there is an additional arrow from K_1 to K_b.*

In order to find all periodic orbits of f, we have to understand the structure of the loops of the essential Markov graph. The above theorem shows that there are two loops, one long, not using the additional arrow, and the other one shorter, using the additional arrow. In fact, we are interested not only in the periods of the periodic points, but also in their itineraries. The *itinerary* of a point x is the sequence of symbols L and R, whose nth term is L is $f^n(x) < c$ and R if $f^n(x) > c$ (at the moment we assume that the orbit of x does not pass through c). If a periodic orbit S of period n has itinerary A^∞, where A is a block of length n, then we will call A

a *periodic itinerary* of S. Of course, all periodic itineraries of S can be obtained by "rotating" A. In particular, A cannot be the repetition of a shorter block.

Remark 4.3. Since all intervals given by vertices of this graph have as their right endpoint an element of Q, and for the only interval J_j that contains c, the corresponding K_j lies entirely to the right of c, we conclude that as we traverse the long loop, the corresponding part of the itinerary is a periodic itinerary of Q.

We want now to show a similar property for the short loop.

Lemma 4.4. *As we traverse the short loop, the corresponding part of the itinerary is a periodic itinerary of P. In particular, the length of this loop is p.*

Proof. The interval K_b is slow-fast. Therefore all its images are slow-fast intervals, until we get to the exceptional interval K_1. Along the short loop, K_1 goes immediately back to K_b. Thus, in the short loop all left endpoints of the intervals are elements of P, so the itineraries are the same as for P. $\square$

In view of Theorem 4.2, Remark 4.3, and Lemma 4.4, we get the following theorem.

Theorem 4.5. *Any periodic orbit of f has a periodic itinerary, which is a concatenation of finitely many periodic itineraries (starting at x_1 or y_1) of P and Q, and is not a repetition of a shorter periodic itinerary. Conversely, for each such concatenation there is a periodic orbit of f, having it as a periodic itinerary.*

To apply this theorem in practice, we need a simple rule to write down the periodic itineraries of P and Q, starting at x_1 and y_1, respectively. The rule is simple. If the numbering of the terms of the periodic itinerary starts with 0, then the kth term is L if and only if $(1 + ka \mod p) \leq p - a$. Similarly, for Q, the kth term is L if and only if $(1 + kb \mod q) \leq q - b$.

If $a/p < b/q$, but $p > q$, then we replace a/p by $(p - a)/p$ and b/q by $(q - b)/q$. Then, in the itineraries we replace symbols L by R and vice versa.

Example 4.6. Let f be the canonical model for a union of twist periodic orbits of rotation numbers $2/5$ and $3/7$ (Figure 3).

Then the essential Markov graph described in Theorem 4.2 has seven vertices, given by the intervals $K_1 = [1/13, 2/13]$, $K_2 = [3/13, 4/13]$, $K_3 = [5/13, 6/13]$, $K_4 = [6/13, 7/13]$, $K_5 = [8/13, 9/13]$, $K_6 = [10/13, 11/13]$, and $K_7 = [11/13, 12/13]$. The first four of these will contribute an L to an itinerary, while the last three will contribute an R. The essential Markov graph consists of the loop $K_1 \to K_4 \to K_7 \to K_3 \to K_6 \to K_2 \to K_5 \to K_1$, together with an extra edge $K_1 \to K_3$ which reduces the length of the loop from seven to five. The periodic orbit Q of length seven (the open points in Figure 3) has periodic itinerary $LLRLRLR$, while the periodic orbit P of length five (the solid points in Figure 3) has periodic itinerary $LLRLR$. The periodic itineraries of all other periodic orbits of f are obtained from finite concatenations of these two words.

5. Rotation intervals

Up to now we were thinking about a map f which is a canonical model for an FL-permutation. Let us now only assume that the rotation interval of a Lorenz-like map
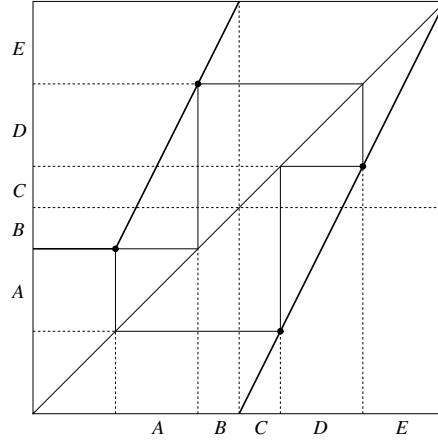
FIGURE 4. The map from Example 5.1.

g contains the interval $[a/p, b/q]$, where $a/p < b/q$ are Farey neighbors, and $p < q$. As we noted at the end of Section 2, this last condition is not restrictive.

It is known (see, e.g., [1]) that if a number r/s is in the rotation interval of g, and r, s are coprime, then g has a twist periodic orbit of period s and rotation number r/s. Thus, provided the two periodic orbits we get this way are disjoint, g has a fupo with the FL-permutation considered in the preceding section. Its Markov graph is as we described there, so we get periodic orbits with all itineraries described in Theorem 4.5.

However, it can happen that two distinct periodic orbits are not disjoint, since g takes two values at the discontinuity point. We will consider this situation more closely in the next section.

Observe that we cannot claim that there are no other itineraries of periodic orbits than those from Theorem 4.5, even if the rotation interval is equal to $[a/p, b/q]$. This follows from [1], and we can illustrate it on the following simple example.

Example 5.1. Let $f : [0, 1] \to [0, 1]$ be a Lorenz-like map, given by

$$f(x) = \begin{cases} \frac{2}{5} & \text{if } 0 \leq x \leq \frac{1}{5}, \\ 2x & \text{if } \frac{1}{5} \leq x \leq \frac{1}{2}, \\ 2x - 1 & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

see Figure 4.

Then $\{1/5, 2/5, 4/5, 3/5\}$ is a periodic orbit of period 4 and rotation number $1/2$, while $\{1\}$ is a fixed point of rotation number 1. Thus, the rotation interval of f contains $[1/2, 1]$. To see that it is equal to $[1/2, 1]$, first notice that there cannot be an orbit of rotation number larger than 1. Then observe that the water map at level $f(0) = 2/5$ has a periodic orbit of period 2 and rotation n umber $1/2$ (see Figure 5.

Now the existence of a periodic orbit of period $4 = 2 \cdot 2$ implies (by [1]) that there are periodic orbits of rotation number $1/2$ and all even periods. We can also find them by looking at the Markov graph of f for the partition by the points $0, 1/5, 2/5, 1/2, 3/5, 4/5, 1$. Let us denote $A = [1/5, 2/5]$, $B = [2/5, 1/2]$, $C = [1/2, 3/5]$, $D = [3/5, 4/5]$ and $E = [4/5, 1]$. Then we get our periodic orbits by going
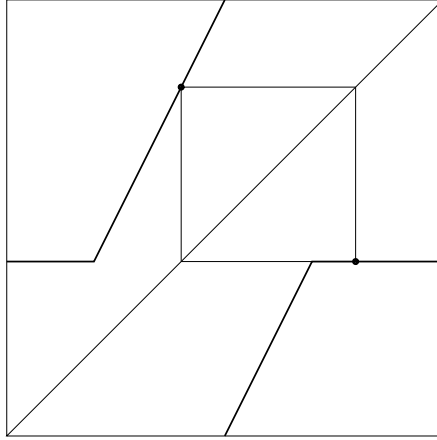
FIGURE 5. The water map at the level $f(0)$ for the map from Example 5.1.

around the concatenations of the loops $D \to A \to D$ and $D \to A \to B \to E \to D$. Those periodic orbits are not the ones predicted by Theorem 4.5, because all orbits predicted by Theorem 4.5 have rotation numbers larger than $1/2$.

If the rotation interval J of a Lorenz-like map f is nondegenerate, and we want to apply our results to f, there is a question what is the simplest way of finding "reasonable" Farey neighbors in J. The answer to this question is simple, since if $p < q$ and there are no fractions with denominator smaller than or equal to q between a/p and b/q, then a/p and b/q are Farey neighbors. Thus, we have to find the two smallest denominators $p < q$ of the fractions from J, look at the fractions with those denominators in J, and choose two that are neighbors in the usual ordering, with distinct denominators.

6. INTERSECTING ORBITS

As we already noted, at the point c of discontinuity, we may agree that a Lorenz-like map f takes two values: 1 from the left and 0 from the right. Another way of looking at this is to agree that the point c consists of two halves: a left c^- and a right c^+. Then $f(c^-) = 1$ and $f(c^+) = 0$. Thus, it may happen that two periodic orbits can both pass through c, each through a different half of it. However, it may happen also that before they come to c, they both pass together through several preimages of c. It helps to think that those preimages also consist of two half-points each.

Let us see how this may happen for our periodic orbits P and Q, considered in Section 4. The fast-slow interval J_{q+1-b} gets replaced by the "interval" $[c^-, c^+]$, that is the singleton $\{c\}$. Then we proceed by induction. Each fast-slow interval, except one, is the image of another fast-slow interval. If the target interval is replaced by a singleton, then the other one also may be replaced by a singleton. Another possibility is that the map f is constant on it (we are thinking at this moment about a more general Lorenz-like map; not necessarily a canonical model). Thus, non-disjointness of the two orbits results in replacing one or more fast-slow intervals by singletons. However, we did not include the fast-slow intervals in the Markov graph of f in the first place, so this will not change anything in Theorem 4.5 or its applications.

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