A New Class of Two-dimensional Chaotic Maps with Closed Curve Fixed Points

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This paper constructs a new class of two-dimensional maps with closed curve fixed points. Firstly, the mathematical model of these maps is formulated by introducing a nonlinear function. Different types of fixed points which form a closed curve are shown by choosing proper parameters of the nonlinear function. The stabilities of these fixed points are studied to show that these fixed points are all non-hyperbolic. Then a computer search program is employed to explore the chaotic attractors in these maps, and several simple maps whose fixed points form different shapes of closed curves are presented. Complex dynamical behaviours of these maps are investigated by using the phase-basin portrait, Lyapunov exponents, and bifurcation diagrams.

Keywords: Two-dimensional map; fixed point; stability; non-hyperbolic fixed point; chaotic attractor.

1. Introduction

Chaotic attractors with a special structure of equilibria in continuous dynamical systems have attracted great attention from many researchers, since these attractors are associated with hidden attractors [Leonov & Kuznetsov, 2013; Dudkowski *et al.*, 2016; Pham *et al.*, 2017] which cannot be found using traditional computational methods. Attractors with no equilibrium or with only one stable equilibrium are considered as "hidden", because their basins of attraction do not intersect with small neighborhoods of the equilibria.

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Due to their "hidden" feature, which is difficult to be found in systems, researchers have extensively explored chaotic attractors with no equilibria [Wei, 2011; Wang *et al.*, 2012; Jafari *et al.*, 2013; Pham *et al.*, 2014; Wei *et al.*, 2014; Lin *et al.*, 2016; Pham *et al.*, 2016a; Wang *et al.*, 2016; Yu *et al.*, 2016; Dantsev, 2018; Zhan *et al.*, 2018] and with only one stable equilibrium [Wei & Yang, 2011; Wang & Chen, 2012; Molate *et al.*, 2013; Wei & Zhang, 2014]. Some elementary quadratic chaotic flows with no equilibria were studied firstly [Wei, 2011; Jafari *et al.*, 2013; Pham *et al.*, 2014; Dantsev, 2018]. Then some hyper-chaotic attractors with no equilibria were investigated [Wang *et al.*, 2012; Wei *et al.*, 2014; Pham *et al.*, 2016a] and multi-wing non-equilibrium attractors were considered [Yu *et al.*, 2016; Wang *et al.*, 2016; Lin *et al.*, 2016; Zhan *et al.*, 2018]. Furthermore, some chaotic flows with one stable equilibrium [Wei & Yang, 2011; Wang & Chen, 2012; Molate *et al.*, 2013] and hyper-chaotic flows with one stable equilibrium [Wei & Zhang, 2014] were investigated. Recently, many works have focused on the attractors with various types of infinite equilibria. For example, Jafari & Sprott [2013] presented some simple flows with a line equilibrium, and then some hyper-chaotic attractors with a line equilibrium were studied [Li & Sprott, 2014; Li *et al.*, 2014a,b, 2015; Ma *et al.*, 2015; Pham *et al.*, 2016b]. Researchers also investigated chaotic attractors with the equilibria in different shapes, including circular curve [Chen & Yang, 2015; Gotthans & Petrzela, 2015; Pham *et al.*, 2016c; Kingni *et al.*, 2016; Barati *et al.*, 2016; Singh & Roy, 2017; Wang *et al.*, 2017], square curve [Gotthans *et al.*, 2016], and rounded square curve [Pham *et al.*, 2016d]. Jafari *et al.* [2016a] investigated simple chaotic flows with a plane of equilibria, and then with many surfaces of equilibria [Jafari *et al.*, 2016b; Singh *et al.*, 2018]. In addition, Wang & Chen [2013] developed a method to construct a chaotic system with any number of equilibria. Based on the three-dimensional Lü chaotic system, Zhou & Yang [2014] introduced a four-dimensional nonlinear system with infinite equilibrium points, and verified the existence of hyperchaos by means of topological horseshoe theory and numerical computation.

More recently, chaotic attractors with a special structure of fixed points in discrete-time maps have received much attention, and the research in this area is still immature and there are many unexplored openings, e.g. [Heatha *et al.*, 2015; Jafari *et al.*, 2016c; Dudkowski *et al.*, 2016; Jiang *et al.*, 2016a,b; Liu *et al.*, 2017; Chen *et al.*, 2017; Panahi *et al.*, 2018; Zhang & Jiang, 2018; Wang *et al.*, 2018]. Jiang *et al.* [2016a,b] studied hidden chaotic attractors with no fixed point and a single stable fixed point in a class of two-dimensional maps and three-dimensional maps, respectively. Liu *et al.* [2017] put forward a new high-dimensional hyperchaotic map with infinite equilibria, which can exhibit complex behaviours, including chaos, hyperchaos, multiple coexisting attractors, and three typical bifurcations. Chen *et al.* [2017] introduced some four-dimensional discrete chaotic maps with one-line equilibria. Panahi *et al.* [2018] proposed two simple chaotic maps without equilibria and investigated their dynamical properties. Zhang & Jiang [2018] presented some elementary quadratic chaotic maps with a single non-hyperbolic fixed point. Wang *et al.* [2018] proposed a new two-dimensional chaotic map with hidden attractors by following the Arnolds cat map. On the other hand, multistability in dynamical systems has been extensively investigated as it widely exists in dynamical systems [Li & Sprott, 2013; Lai & Chen, 2016; Li *et al.*, 2017; Lai *et al.*, 2017, 2018a,b]. For example, Li & Sprott [2013] studied multistability in a butterfly flow. Li *et al.* [2017] proposed a method to diagnose multistability by offset boosting. By constructing new chaotic systems, various types of coexisting attractors have been shown in [Lai & Chen, 2016; Lai *et al.*, 2017, 2018a,b].

This paper aims to explore some elementary quadratic chaotic maps with closed curve fixed points by performing an exhaustive computer search [Sprott, 2000, 2010]. As shown in [Jafari & Sprott, 2013; Chen & Yang, 2015; Jafari *et al.*, 2016a,b], the chaotic maps with closed curve fixed points belong to the category of chaotic maps with hidden attractors, since the knowledge about fixed points does not help in the localization of attractors. It is interesting that all the fixed points to be studied in this paper are non-hyperbolic, and chaotic attractors will be presented using phase-basin portrait and will be numerically analyzed using the Lyapunov exponents and the Kaplan-Yorke dimension. Our investigation will show that there is no repelling fixed point in these maps, so the Marotto's theorem is invalid [Zhang & Jiang, 2018]. Multistability in the quadratic chaotic maps with closed curve fixed points will be shown, including: (i) chaotic attractor, period-2 solution and fixed point coexist; (ii) quasi-periodic solution, period-2 solution and fixed point coexist; (iii) high-periodic solution, period-2 solution and fixed point coexist. Findings of this paper are useful for researchers to get insight into the dynamical mechanisms of discrete-time maps.

The rest of this paper is organized as follows. In Section 2, the mathematical model of this class of two-dimensional maps with closed curve fixed points will be introduced, and the existence and stability of their fixed points will be studied. In Section 3, chaotic attractors with closed curve fixed points will be explored, and finally, some conclusions are drawn in Section 4.

2. System model and fixed points

During the search of hidden attractors in maps, we found a new class of two-dimensional maps which can be written using the following difference equation.

$$\begin{cases} x_{k+1} = x_k + a_1 f(x_k, y_k), \\ y_{k+1} = y_k + f(x_k, y_k)(a_2 x_k + a_3 y_k + a_4 x_k^2 + a_5 y_k^2 + a_6 x_k y_k + a_7), \end{cases} \tag{1}$$

where x_k and y_k $(k = 0, 1, 2, \cdots)$ are system states, a_i $(i = 1, \cdots, 7)$ are system parameters, $f(x, y)$ is a nonlinear function which is chosen as

$$f(x, y) = (\frac{x}{m})^p + (\frac{y}{n})^p - r^2, \tag{2}$$

where $m > 0$, $n > 0$, $p \geq 2$ and $r > 0$ are real parameters which can be specified by researcher, and p is a properly selected integer, e.g. $f(x, y) = 0$ represents a closed curve.

The fixed points (x^*, y^*) of the map (1) must satisfy the following conditions

$$\begin{cases} x^* = x^* + a_1 f(x^*, y^*), \\ y^* = y^* + f(x^*, y^*)(a_2 x^* + a_3 y^* + a_4 (x^*)^2 + a_5 (y^*)^2 + a_6 x^* y^* + a_7). \end{cases} \tag{3}$$

By solving Eq. (3), fixed points are (x^*, y^*) which satisfy $f(x^*, y^*) = 0$. So the fixed points of the map (1) form a closed curve. A summary of different shapes of fixed points based on the nonlinear function $f(x, y)$ is given in Table 1 [Pham *et al.*, 2017].

Table 1. Different shapes of fixed points based on the nonlinear function $f(x, y)$

Cases	Parameters	$f(x, y)$	Fixed points types
A	$m = 1, n = 1, p = 2, r = 1$	$x^2 + y^2 - 1$	Circular fixed points
B	$p = 2, r = 1$	$(\frac{x}{m})^2 + (\frac{y}{n})^2 - 1$	Ellipse fixed points
C	$m = 1, n = 1, p = 12, r = 1$	$x^{12} + y^{12} - 1$	Square-shaped fixed points
D	$p = 12, r = 1$	$(\frac{x}{m})^{12} + (\frac{y}{n})^{12} - 1$	Rectangle-shaped fixed points
E	$m = 1, n = 1, p = 4$	$x^4 + y^4 - r^2$	Rounded-square fixed points

The Jacobian matrix of the map (1) at the fixed points (x^*, y^*) can be written as

$$J = \begin{bmatrix} 1 + a_1 f_x(x^*, y^*) & a_1 f_y(x^*, y^*) \\ J_{21} & J_{22} \end{bmatrix}, \tag{4}$$

where $J_{21} = f_x(x^*, y^*)(a_2 x^* + a_3 y^* + a_4 (x^*)^2 + a_5 (y^*)^2 + a_6 x^* y^* + a_7)$ and $J_{22} = 1 + f_y(x^*, y^*)(a_2 x^* + a_3 y^* + a_4 (x^*)^2 + a_5 (y^*)^2 + a_6 x^* y^* + a_7)$, $f_x(x, y)$ and $f_y(x, y)$ are the partial derivative of $f(x, y)$ with respect to x and y, respectively. The characteristic equation of the Jacobian matrix can be calculated using

$$\det(\lambda I - J) = \lambda^2 - \mathrm{tr}(J)\lambda + \det(J) = 0, \tag{5}$$

where $\det(J)$ and $\mathrm{tr}(J)$ represent the determinant of the Jacobian matrix and the trace of the Jacobian matrix, respectively. From the theory of matrices, the sum of eigenvalues of the Jacobian matrix equals to $\mathrm{tr}(J)$, and the product of eigenvalues of the Jacobian matrix equals to $\det(J)$.

Eigenvalues λ_1, λ_2 of J are called multipliers of the fixed point. Let n_-, n_0 and n_+ be the numbers of multipliers of the fixed point (x^*, y^*) lying inside, on and outside the unit circle $\{\lambda \in C : |\lambda| = 1\}$, respectively. So the following definition is given.

Definition 2.1. ([Kuznetsov, 1998]) A fixed point (x^*, y^*) is called hyperbolic if $n_0 = 0$, that is, if there is no eigenvalue of the Jacobian matrix evaluated at this fixed point on the unit circle. Otherwise, the fixed point is called non-hyperbolic, that is, there is at least one eigenvalue of the Jacobian matrix evaluated at the fixed point on the unit circle.

A fixed point (x^*, y^*) of the map (1) is stable if the roots λ_1, λ_2 of the characteristic equation satisfy that $|\lambda_{1,2}| < 1$. It is easy to check that $1 - \mathrm{tr}(J) + \det(J) = 0$, so there is a real root $\lambda = +1$. Therefore, from Definition 2.1, all the fixed points of the map (1) are non-hyperbolic.

Remark 2.1. The map (1) has fixed points which form a closed curve. However, we cannot determine the stability from the definition. So, some fixed points may be stable, while others may be unstable. In the map (1), other types of attractors may also exist, e.g. chaotic attractors, quasi-periodic solutions and periodic solutions. Thus, multistability can be observed.

3. Dynamical behaviours of chaotic maps with closed curve fixed points

In this study, a computer search program [Sprott, 2010] was used to explore the strange chaotic attractors with closed curve fixed points, and many chaotic maps were found. In this section, several simple chaotic maps with closed curve fixed points will be chosen to show the complex dynamics of this class maps.

Five typical examples of the maps with different types of closed curve fixed points are presented in Table 2, where initial value (x_0, y_0), Lyapunov exponents (Les), and Lyapunov (Kaplan-Yorke) dimension (Dky) of these maps are given. It can be verified that the fixed points of the maps listed in Table 1 locate in closed curves. It also can be seen from Table 2 that all the maximal Lyapunov exponents are positive, so the maps with the given initial values are all chaotic. In addition, the phase portraits of these simple chaotic maps with closed curve fixed points are presented in Fig. 1, where the chaotic attractors are marked by black dots and the red curves represent fixed points of these maps.

Table 2. Several simple chaotic maps with closed curve fixed points

Cases	Maps	(x_0, y_0)	Les	Dky	Fixed points types
CF_a	$\begin{cases} x_{k+1} = x_k + 1.2(x_k^2 + y_k^2 - 1) \\ y_{k+1} = y_k + 2y_k(x_k^2 + y_k^2 - 1) \end{cases}$	$(0.27, 0.28)$	0.1318 −0.1336	1.9865	Circular fixed points
CF_b	$\begin{cases} x_{k+1} = x_k - 8.5((\frac{x}{6})^2 + (\frac{y}{1})^2 - 1) \\ y_{k+1} = y_k - y_k((\frac{x}{6})^2 + (\frac{y}{1})^2 - 1) \end{cases}$	$(-4.91, -0.33)$	0.1754 −0.4117	1.4260	Ellipse fixed points
CF_c	$\begin{cases} x_{k+1} = x_k - 0.2(x_k^{12} + y_k^{12} - 1) \\ y_{k+1} = y_k + 2.4x_k y_k(x_k^{12} + y_k^{12} - 1) \end{cases}$	$(0.26, -0.14)$	0.1666 −0.2707	1.6152	Square-shaped fixed points
CF_d	$\begin{cases} x_{k+1} = x_k + ((\frac{x}{5})^{12} + (\frac{y}{1})^{12} - 1) \\ y_{k+1} = y_k + 2.2y_k((\frac{x}{5})^{12} + (\frac{y}{1})^{12} - 1) \end{cases}$	$(4.93, -0.8)$	0.1631 −0.3018	1.5406	Rectangle-shaped fixed points
CF_e	$\begin{cases} x_{k+1} = x_k + 0.8(x^4 + y^4 - 1) \\ y_{k+1} = y_k - 0.8y_k(x^4 + y^4 - 1) \end{cases}$	$(0.81, -0.14)$	0.1270 −0.2242	1.5665	Rounded-square fixed points

In the following, the map CF_a (when $a_1 = 1.2$, $a_2 = a_4 = a_5 = a_6 = a_7 = 0$, $a_3 = 2$, $m = 1$, $n = 1$, $p = 2$, $r = 1$) is chosen to study the dynamics of the map with closed curve fixed points. Since the fixed points of this map are all non-hyperbolic, one cannot determine their stabilities from eigenvalues of the Jacobian matrix evaluated at the fixed points of the map CF_a directly. So, the basins of attraction of the map CF_a as shown in Fig. 2 were used to evaluate their stabilities, where unbounded basin of attraction is displayed in cyan, chaotic attractors are denoted by black dots with white basin, asymptotically stable fixed points are marked by blue dots with orange basin, period-2 solution is shown by purple dots with green basin, and unstable fixed points are presented using red dots. Here, some of the non-hyperbolic fixed points are stable, while the others are unstable. When $y_k = 0$, the map CF_a is reduced to $x_{k+1} = x_k + 1.2(x_k^2 - 1)$. In this case, there exists a stable period-2 solution, i.e. $(-1.3861, 0)$ and $(-0.28056, 0)$. The Lyapunov exponent of the stable period-2 solution of the difference equation $x_{k+1} = x_k + 1.2(x_k^2 - 1)$ is -0.1732, which was computed by using the definition of Lyapunov exponent of one-dimensional map. It should be noted that the basin of attraction of this period-2 solution is a line of $y = 0$ where $x \in (-1.9, 1.0)$.

In order to show the complex dynamical behaviours of the chaotic map with circular fixed points, random bifurcation and Lyapunov exponent diagrams of the map CF_a were plotted in Fig. 3 by using a_3 as a branching parameter, where 5000 random initial conditions for each value of the parameter a_3 were used. In Fig. 3 (a) and (b), attractors are denoted by black dots, and in Fig. 3 (c) and (d), the largest Lyapunov

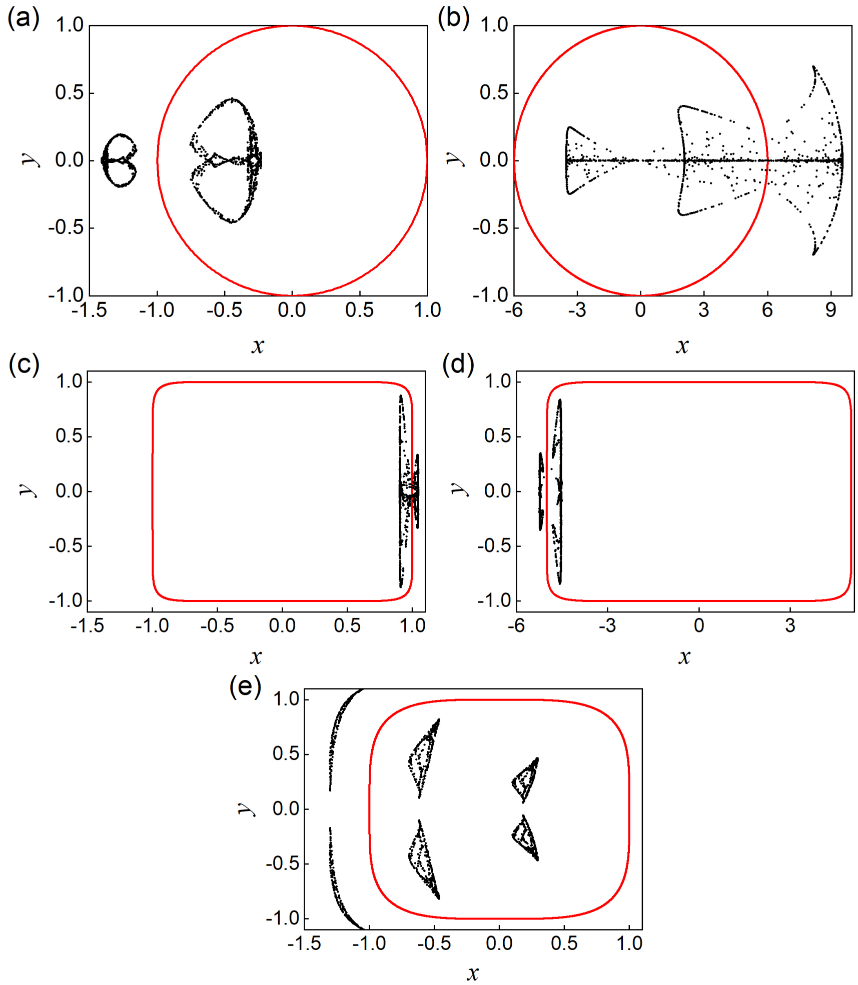

Fig. 1. (Colour online) Phase portraits of chaotic attractors of (a) CF_a: Circular fixed points, (b) CF_b: Ellipse fixed points, (c) CF_c: Square-shaped fixed points, (d) CF_d: Rectangle-shaped fixed points, and (e) CF_e: Rounded-square fixed points listed in Table 2. Chaotic attractors are marked by black dots, and red curves represent fixed points of the map.

exponent (Le1) and second Lyapunov expoent (Le2) are shown, respectively. The Lyapunov exponent of the period-2 solution is shown by a red line. Since all the fixed points of the map CF_a are non-hyperbolic, i.e. one eigenvalue of the Jacobian matrix evaluated at the fixed point is 1, the largest Lyapunov exponent (Le1) of the stable fixed point is 0, which is shown by a blue line. The second Lyapunov expoents (Le2) of fixed points are denoted by blue dots. The Lyapunov exponents of other solutions are denoted in black. Representative random phase portraits of the map CF_a with different parameter a_3 are shown in Fig. 4, where 5000 random initial conditions were used. In Fig. 4, attractors are denoted by black dots, and the unstable fixed points are shown by red dots. From Fig. 3 and Fig. 4, the evolution of complex dynamical behaviours can be described as follows.

When $a_3 = 0$, the map CF_a becomes $x_{k+1} = x_k + 1.2(x_k^2 + y_k^2 - 1)$, $y_{k+1} = y_k$ and y can be considered as an external force of the first difference equation with respective to x. Here the phase portrait is shown in Fig. 4(h). As the parameter a_3 decreases from zero, there are two attractors, the stable fixed points from the unit circle and the stable period-2 solution of the difference equation $x_{k+1} = x_k + 1.2(x_k^2 - 1)$. At $a_3 = -1.55$, a period-4 solution emerges and then becomes a quasi-periodic solution which is shown as

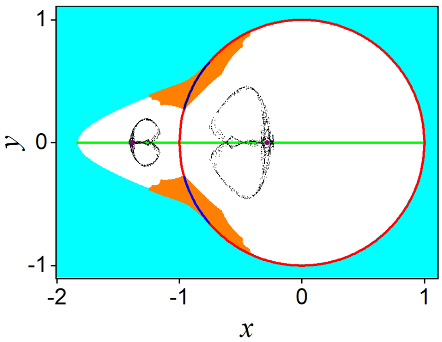

Fig. 2. (Colour online) Basins of attraction of the map CF_a. Unbounded basin of attraction is shown in cyan, the chaotic attractors shown in black dots have a white basin, the asymptotically stable fixed points shown in blue dots have yellow basin, the period-2 solution denoted by purple dots has a green basin, and unstable fixed points are marked by red dots.

four circles in Fig. 4 (e-f) via Neimark-Sacker (NS) bifurcation. These four circles gradually become larger (see Fig.4 (c-d)), and at $a_3 = -1.59$, chaotic attractors are observed for a very tiny range (see Fig.4 (b)). Then the chaotic attractors disappear via chaotic crisis, and the attractors become the fixed points from the unit circle and the stable period-2 solution (see Fig. 4 (a)).

As the parameter a_3 increases from zero, there are two types of attractors, the stable fixed points from the unit circle and the stable period-2 solution of the difference equation $x_{k+1} = x_k + 1.2(x_k^2 - 1)$. At $a_3 = 1.58$, a period-4 solution emerges and then changes to a quasi-periodic solution via NS bifurcation at $a_3 = 1.80$ (see Fig. 4 (j-l)). These four circles gradually become larger, and thereafter chaotic attractors are observed for a small range (see Fig. 4 (n-o)) at where periodic windows exist (see Fig. 4 (m)). Then the chaotic attractors disappear due to chaotic crisis, and the attractors change to the stable fixed points from the unit circle and the stable period-2 solution (see Fig. 4 (p)).

From the bifurcation analysis of the map CF_a, the map with closed curve fixed points can exhibit different dynamical behaviours, including fixed points, periodic solutions, quasi-periodic solution, and chaotic attractors. Some circular fixed points are stable, while others are unstable. On the other hand, many attractors may coexist in the map with closed curve fixed points, e.g. (i) chaotic attractor, period-2 solution and fixed point coexist; (ii) quasi-periodic solution, period-2 solution and fixed point coexist; (iii) high-periodic solution, period-2 solution and fixed point coexist. The appearance and disappearance of chaotic attractors has illustrated the generation mechanisms of the chaotic attractors in the maps.

4. Conclusions

Chaotic dynamics of a class of two-dimensional maps with closed curve fixed points, which belong to hidden attractor category, was studied in this paper. A computer search program was used to explore simple chaotic attractors with closed curve fixed points which were presented by using phase portraits. Numerical methods, including computations of basins of attraction and Lyapunov exponents, and random bifurcation analysis, were used to demonstrate the complex dynamical behaviours of these maps. Future works will focus on the investigation of the high dimensional maps with a special structure of fixed points.

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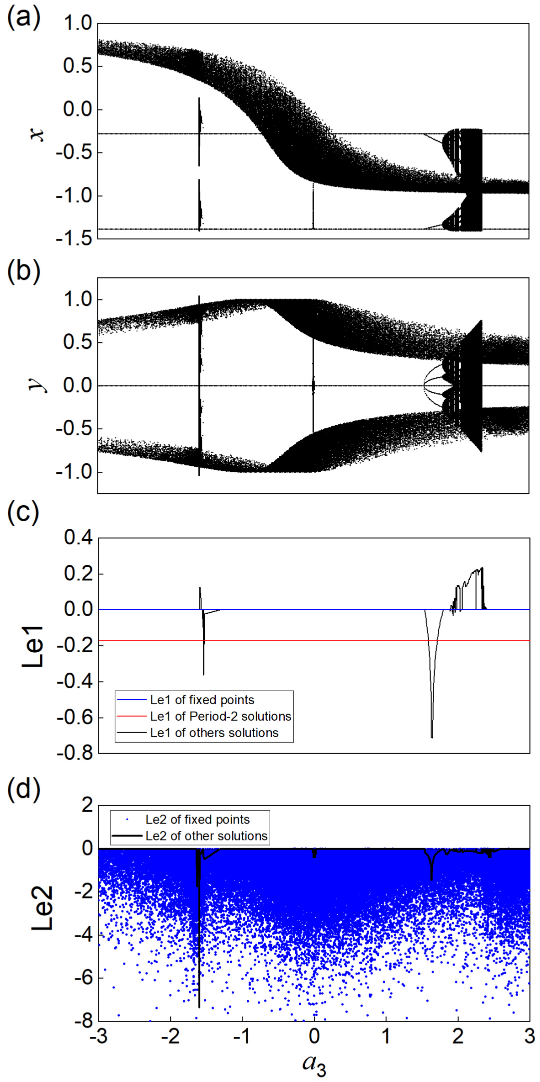

Fig. 3. (Colour online) Random bifurcation diagrams of (a) x, and (b) y, and (c,d) Lyapunov exponents of the map CF_a with respect to the parameter a_3. In Fig. 3 (a) and (b), attractors are denoted by black dots. In Fig. 3 (c) and (d), the largest Lyapunov exponent (Le1) and the second Lyapunov expoent (Le2) are shown, respectively. The Lyapunov exponent of the period-2 solution is -0.1732, which is shown by a red line. The largest Lyapunov exponent (Le1) of fixed points is 0, which is shown by a blue line. The second Lyapunov expoents (Le2) of fixed points are denoted by blue dots. The Lyapunov exponents of other solutions are denoted in black.

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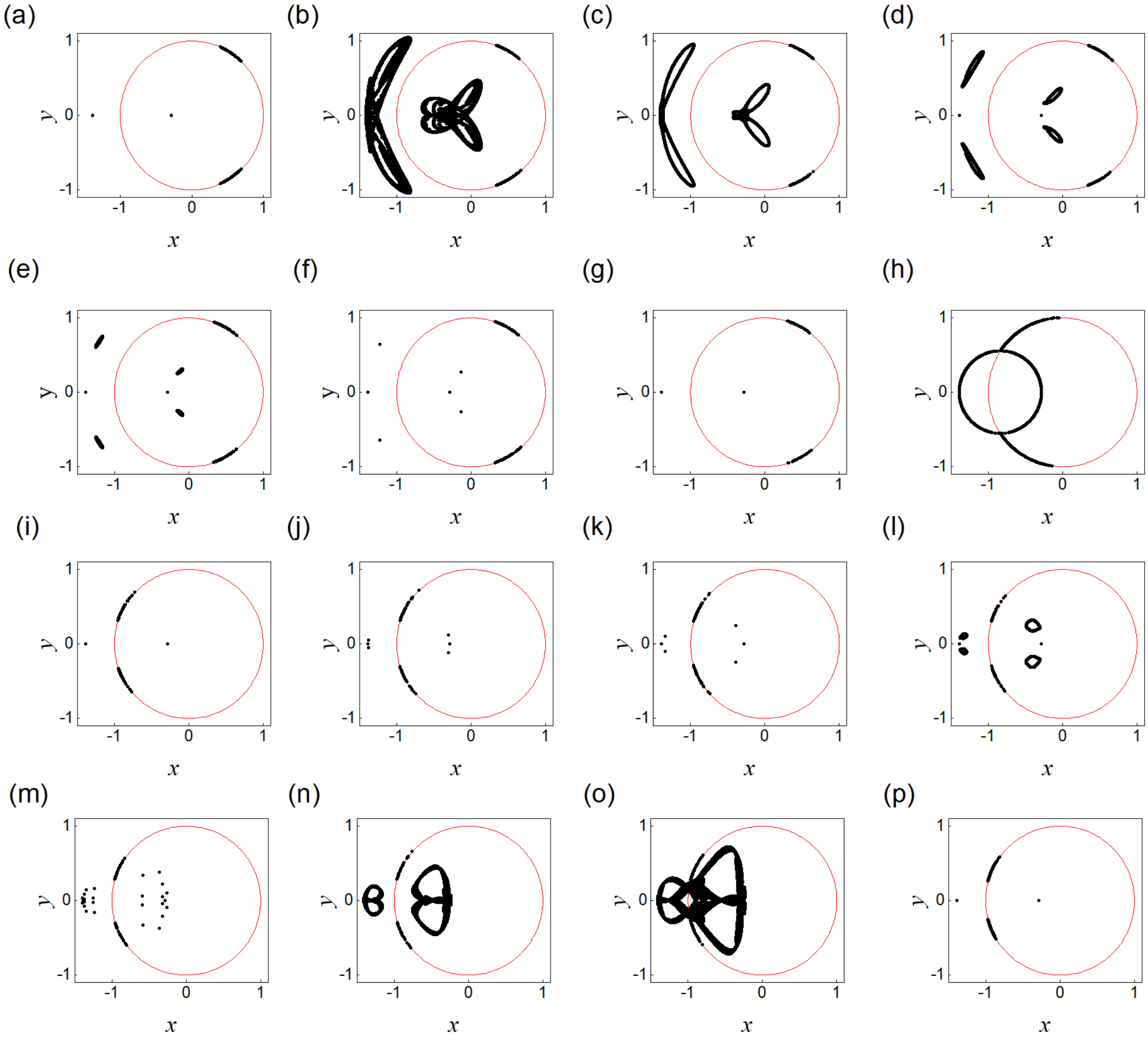

Fig. 4. (Colour online) Random phase portraits of the map CF$_a$ calculated for (a) $a_3 = -1.75$, (b) $a_3 = -1.59$, (c) $a_3 = -1.58$, (d) $a_3 = -1.57$, (e) $a_3 = -1.56$, (f) $a_3 = -1.55$, (g) $a_3 = -1.50$, (h) $a_3 = 0.00$, (i) $a_3 = 1.50$, (j) $a_3 = 1.58$, (k) $a_3 = 1.79$, (l) $a_3 = 1.83$, (m) $a_3 = 1.96$, (n) $a_3 = 2.00$, (o) $a_3 = 2.29$, and (p) $a_3 = 2.37$. Attractors are denoted by black dots, and unstable fixed points are shown by red dots.

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