

Centers and limit cycles of a generalized cubic Riccati system

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Abstract

We obtain condition for existence of a center for a cubic planar differential system, which can be considered as a polynomial subfamily of the generalized Riccati system. We also investigate bifurcations of small limit cycles from the components of the center variety of the system.

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1. Introduction

Consider systems of ordinary differential equations on \mathbb{R}^2 of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \quad (1)$$

where P and Q are polynomials, $\max\{\deg P, \deg Q\} = n$. We view (1) as defining a family of systems parametrized by the coefficients of P and Q . That is, the degree of polynomials P and Q in system (1) is fixed and the coefficients of the polynomials are parameters $\lambda_1, \dots, \lambda_N$, so the N -tuple of the parameters $p = (\lambda_1, \dots, \lambda_N)$ is a point in a Euclidean N -dimensional space \mathcal{E} , and we identify each point in \mathcal{E} with its corresponding system (1). In this paper the space \mathcal{E} of parameters is either \mathbb{R}^N or \mathbb{C}^N .

A singular point (x_0, y_0) of real system (1) is a stable focus if every nearby trajectory spirals towards to it, an unstable focus if every nearby trajectory spirals away from it, and it is a center if every nearby trajectory is an oval.

A singular point $(x_0, y_0) \in \mathbb{R}^2$ of a system $p \in \mathcal{E} \subset \mathbb{R}^N$ is said to have *cyclicity* k with respect to \mathcal{E} if and only if any sufficiently small perturbation of p in \mathcal{E} has at most k limit cycles in a sufficiently small neighborhood of (x_0, y_0) , and k is the smallest number with this property. The problem of cyclicity of a center or a focus of a system of the form (1) is known as the *local 16th Hilbert problem* in (Françoise and Yomdin (1997)), based on its connection to Hilbert's

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still unresolved 16th problem, which in part asks for a bound on the number of limit cycles of system (1) in terms of degree n of the system (see, e.g. survey papers Giné (2007); Li (2003)).

The problem of cyclicity is closely connected to the center problem, that is, the problem of finding systems with centers in a given polynomial family (1). The studies of the center problem dates back to 1908 when Dulac (1908) investigated the case of the quadratic system. The literature devoted to the subject is vast, see e.g. Amel'kin et al. (1982); Christopher and Li (2007); Liu et al. (2008); Romanovski and Shafer (2009); Sibirskii (1976) and the references that they contain.

After a linear transformation and time rescaling any planar polynomial system (1) with an elementary center or weak focus can be written in the form

$$\dot{x} = -y + \tilde{P}(x, y), \quad \dot{y} = x + \tilde{Q}(x, y), \quad (2)$$

where \tilde{P} and \tilde{Q} are polynomials without free and linear terms. By the Poincaré-Lyapunov theorem for real system (2) the origin is a center if and only if in a neighborhood of the origin the system admits a real analytic local first integral of the form

$$\Psi(x, y) = x^2 + y^2 + \sum_{j+k=3} \psi_{jk} x^j y^k. \quad (3)$$

Recently Llibre and Valls (2014, 2015) investigated the planar system

$$\dot{x} = f(y), \quad \dot{y} = g_2(x)y^2 + g_1(x)y + g_0(x), \quad (4)$$

which is called the generalized Riccati system (4), since it becomes the classical Riccati system if $f(x) \equiv 1$.

In this paper we consider a particular subfamily of the generalized Riccati system, namely, the cubic system

$$\begin{aligned} \dot{x} &= -y + a_{02}y^2, \\ \dot{y} &= (b_{02} + b_{12}x)y^2 + (b_{11}x + b_{21}x^2)y + (b_{20}x^2 + b_{30}x^3 + x) \\ &= x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2, \end{aligned} \quad (5)$$

where a_{ij}, b_{ks} are real or complex parameters. We first find conditions for system (5) with complex parameters to have an analytic first integral of the form (3), so, the corresponding real systems have a center at the origin. Then, we study limit cycles bifurcations from the centers of real systems (5).

2. Preliminaries

For system (2) it is always possible to find a function

$$\Phi(x, y) = x^2 + y^2 + \sum_{j+k=3}^{\infty} \phi_{jk} x^j y^k, \quad (6)$$

such that

$$\frac{\partial \Phi}{\partial x}(-y + \tilde{P}(x, y)) + \frac{\partial \Phi}{\partial y}(x + \tilde{Q}(x, y)) = \sum_{m=2}^{\infty} g_{m-1}(u^2 + v^2)^m. \quad (7)$$

The coefficients g_k in (7) are polynomials in parameters of system (7) called the *focus quantities* of the system. Each polynomial g_i represents an obstacle for existing of integral (3), that is, system (2) admits an integral (3) if and only if $g_1 = g_2 = g_3 = \dots = 0$. Thus, the set of all systems in the parametric family (2) with centers (equivalently, the set of systems with a first integral of the form (3)) is the variety¹ $V^{\mathbb{R}} \subset \mathbb{R}^n$ of the ideal $B = \langle g_1, g_2, g_3, \dots \rangle$. By the Hilbert Basis Theorem there is an integer k that $B = \langle g_1, \dots, g_k \rangle$, however there are no regular methods to find such k .

We will also consider system (2) as a system with complex parameters. In such case the polynomials g_i in (7) are polynomials with complex coefficients and then their variety is a complex variety, which we will denote $V^{\mathbb{C}}$. All systems whose parameters belong to the variety $V^{\mathbb{C}}$ admit local analytic first integral of the form (3). We call $V^{\mathbb{R}}$ and $V^{\mathbb{C}}$ the real and complex center varieties of system (2).

It was shown in García et al. (2016) that knowing the complex variety of a real polynomial system can be helpful for investigation of cyclicity of the real system. We also need to work with complex varieties to apply our computational approach for computing conditions of integrability.

To find a real or complex center variety of a polynomial system (2), one computes a few first focus quantities g_1, \dots, g_k of the system and then finds decomposition of the variety of the ideal $B_k = \langle g_1, \dots, g_k \rangle$ obtaining the necessary conditions of existence of integral (3). Then it is necessary to prove the sufficiency of the obtained condition. We recall that one of most powerful methods to prove the existence of integral (3) is the Darboux method, which allows to construct a first integral or integrating factor using Darboux polynomials (see, e.g. surveys Llibre (2011); Llibre and Zhang (2012)). A *Darboux polynomial* of system (1) is a polynomial $f(x, y)$ satisfying

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = Kf,$$

where $K(x, y)$ is a polynomial called the *cofactor* of f . It is easy to see that if f is a Darboux polynomial of system (1), then $f = 0$ is an algebraic invariant curve of the system.

It is easy to see that if system (2) has p irreducible Darboux polynomials f_1, \dots, f_p with the associated cofactors K_1, \dots, K_p , such that $s_1 K_1 + \dots + s_p K_p = 0$, then the function $H = f_1^{s_1} \dots f_p^{s_p}$ is a first integral of the system (called the Darboux first integral) and if

$$s_1 K_1 + \dots + s_p K_p = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y},$$

then the function $\mu = f_1^{s_1} \dots f_p^{s_p}$ is an integrating factor of (1) (called the Darboux integrating factor).

3. Integrability conditions for system (5)

In this section we assume that parameters of system (5) are complex and find the complex variety $V^{\mathbb{C}}$ of the system. The study yields that the variety consists of 7 irreducible components.

Theorem 1. *System (5) has an analytic integral of the form (3) if the 7-tuple $(a_{02}, b_{20}, b_{11}, b_{12}, b_{02}, b_{21}, b_{30})$ of its parameters belong to the variety of one of the following prime ideals:*

¹ We remind that the variety of a given ideal F generated by polynomials $f_1(x_1, \dots, x_m), \dots, f_s(x_1, \dots, x_m)$ over a field k , $F = \langle f_1, \dots, f_s \rangle \subset k[x_1, \dots, x_m]$, is the set $V(F) = \{(x_1, \dots, x_n) \in k^n \mid f(x_1, \dots, x_m) = 0 \text{ for all } f \in F\}$.

$$\begin{aligned}
I_1 &= \langle b_{21}, b_{20}, b_{02} \rangle, \\
I_2 &= \langle b_{30}, b_{12}, b_{02}, b_{11}b_{20} - b_{21} \rangle, \\
I_3 &= \langle b_{30}, b_{21}, b_{12}, -2b_{02}b_{11}^2 + 4b_{02}^2b_{20} - b_{11}^2b_{20}, 2a_{02}b_{11} + b_{11}^2 - 4b_{02}b_{20}, 2a_{02}b_{02} - b_{02}b_{11} - b_{11}b_{20}, 4a_{02}^2 - b_{11}^2 - 4b_{20}^2 \rangle, \\
I_4 &= \langle b_{21}, b_{11}, a_{02} \rangle, \\
I_5 &= \langle a_{02}, b_{02}b_{21} + b_{11}b_{30}, 2b_{02}b_{12} + b_{12}b_{20} + b_{02}b_{30}, b_{02}b_{11} + b_{11}b_{20} - b_{21}, b_{02}^2 + b_{02}b_{20} + b_{30}, b_{12}b_{20}b_{21} - 2b_{11}b_{12}b_{30} - b_{11}b_{30}^2, b_{11}b_{20}b_{21} - b_{21}^2 - b_{11}^2b_{30}, b_{12}b_{20}^2 - 4b_{12}b_{30} - b_{02}b_{20}b_{30} - 2b_{30}^2, b_{11}b_{12}b_{20} - 2b_{12}b_{21} + b_{11}b_{20}b_{30} - b_{21}b_{30}, -(b_{12}b_{21}^2) + b_{11}^2b_{12}b_{30} + b_{11}^2b_{30}^2 \rangle, \\
I_6 &= \langle b_{21}, b_{12}, b_{11}, b_{02} \rangle, \\
I_7 &= \langle b_{21}, b_{12}, b_{30}, 3b_{02} + 5b_{20}, 5a_{02} - b_{11}, 6b_{11}^2 + 25b_{20}^2 \rangle.
\end{aligned}$$

That is, the variety $V^{\mathbb{C}}$ of system has the irreducible decomposition $V^{\mathbb{C}} = \cup_{k=1}^7 V_k^{\mathbb{C}}$, where $V_k^{\mathbb{C}} = \mathbf{V}(I_k)$ ($k = 1, \dots, 7$).

Proof. Necessity. For system (5) we have computed eight first focus quantities g_1, g_2, \dots, g_8 and then tried to find the irreducible decomposition of the variety $\mathbf{V}(I)$ of the ideal

$$I = \langle g_1, g_2, \dots, g_8 \rangle \quad (8)$$

over the field of rational numbers using the routine `minAssGTZ` Decker et al. (2010) (which is based on the algorithm of Gianni et al. (1988)) of the computer algebra system `SINGULAR` Decker et al. (2012), but due to high complexity of calculations we have not succeeded to complete them with our computational facilities. However computing in the polynomial ring

$$\mathbb{Z}_{32003}[a_{02}, b_{02}, b_{11}, b_{12}, b_{20}, b_{21}, b_{30}]^2$$

using the degree reverse lexicographic ordering with $a_{02} > b_{02} > b_{11} > b_{12} > b_{20} > b_{30} > b_{21}$ we have found that in the affine space \mathbb{Z}_{32003}^7 the variety of I consists of 8 irreducible components defined by the following 8 prime ideals:

$$\begin{aligned}
\tilde{I}_1 &= \langle b_{21}, b_{20}, b_{02} \rangle, \\
\tilde{I}_2 &= \langle b_{30}, b_{12}, b_{02}, b_{11}b_{20} - b_{21} \rangle, \\
\tilde{I}_3 &= \langle b_{30}, b_{21}, b_{12}, a_{02}b_{11} - 16001b_{11}^2 - 2b_{02}b_{20}, a_{02}b_{02} + 16001b_{02}b_{11} + 16001b_{11}b_{20}, a_{02}^2 - 8001b_{11}^2 - b_{20}^2, b_{02}b_{11}^2 - 2b_{02}^2b_{20} - 16001b_{11}^2b_{20} \rangle, \\
\tilde{I}_4 &= \langle b_{21}, b_{11}, a_{02} \rangle, \\
\tilde{I}_5 &= \langle a_{02}, b_{02}b_{21} + b_{11}b_{30}, b_{02}b_{12} - 16001b_{12}b_{20} - 16001b_{02}b_{30}, b_{02}b_{11} + b_{11}b_{20} - b_{21}, b_{02}^2 + b_{02}b_{20} + b_{30}, b_{12}b_{20}b_{21} - 2b_{11}b_{12}b_{30} - b_{11}b_{30}^2, b_{11}b_{20}b_{21} - b_{21}^2 - b_{11}^2b_{30}, b_{12}b_{20}^2 - b_{02}b_{20}b_{30} - 4b_{12}b_{30} - 2b_{30}^2, b_{11}b_{12}b_{20} + b_{11}b_{20}b_{30} - 2b_{12}b_{21} - b_{21}b_{30}, b_{11}^2b_{12}b_{30} + b_{11}^2b_{30}^2 - b_{12}b_{21}^2 \rangle, \\
\tilde{I}_6 &= \langle b_{21}, b_{12}, b_{11}, b_{02} \rangle, \\
\tilde{I}_7 &= \langle b_{30}, b_{21}, b_{12}, b_{11} - 15273b_{20}, b_{02} - 10666b_{20}, a_{02} + 3346b_{20} \rangle, \\
\tilde{I}_8 &= \langle b_{30}, b_{21}, b_{12}, b_{11} + 15273b_{20}, b_{02} - 10666b_{20}, a_{02} - 3346b_{20} \rangle.
\end{aligned}$$

Lifting these ideals into the ring $\mathbb{Q}[a_{02}, b_{02}, b_{11}, b_{12}, b_{20}, b_{21}, b_{30}]$ using the rational reconstruction algorithm of Wang et al. (1982) we obtain the ideals I_1, \dots, I_6 given in the statement of Theorem 1 and the ideals

$$\hat{I}_7 = \langle b_{30}, b_{21}, b_{12}, b_{11} + \frac{51}{44}b_{20}, b_{02} + \frac{5}{3}b_{20}, a_{02} + \frac{161}{67}b_{20} \rangle$$

and

$$\hat{I}_8 = \langle b_{30}, b_{21}, b_{12}, b_{11} - \frac{51}{44}b_{20}, b_{02} + \frac{5}{3}b_{20}, a_{02} - \frac{161}{67}b_{20} \rangle.$$

²It appear, for the first time modular computations were used for studies on the center and cyclicity problems in Edneral (1997)

To check the correctness of the obtained decomposition we use the procedure proposed in Romanovski and Prešern (2011).

First, using the Radical Membership Test³ we check if all focus quantities g_i ($i = 1, \dots, 8$) vanish on each of the varieties $\mathbf{V}(I_1), \dots, \mathbf{V}(I_6), \mathbf{V}(\hat{I}_7), \mathbf{V}(\hat{I}_8)$. The calculations show that all polynomials g_i are equal to zero on each of varieties $\mathbf{V}(I_1), \dots, \mathbf{V}(I_6)$, but not on the varieties $\mathbf{V}(\hat{I}_7), \mathbf{V}(\hat{I}_8)$. This means, that $\mathbf{V}(\hat{I}_7)$ and $\mathbf{V}(\hat{I}_8)$ are not correct components of the irreducible decomposition of $\mathbf{V}(I)$. A usual recipe to find the correct components of the decomposition in such situation is to recompute the decomposition over a few fields of larger characteristics (see e.g. Arnold (2003)). However instead of doing this we observe that both ideals \tilde{I}_7 and \tilde{I}_8 contain the polynomials b_{30}, b_{21}, b_{12} and compute with `minAssGTZ` of `SINGULAR` the minimal associate primes of the ideal $\langle I, b_{30}, b_{21}, b_{12} \rangle$ over the field \mathbb{Q} obtaining the component defined by the ideal I_7 of the statement of the theorem.

Now, to check the correctness of the obtained conditions we computed the ideal $\tilde{I} = \cap_{s=1}^7 I_s$, which defines the union of all seven components listed in the statement of the theorem and have checked that reduced Gröbner bases of all ideals $\langle \tilde{I}, 1 - wg_k \rangle$ (where $k = 1, \dots, 8$ and w is a new variable) computed over \mathbb{Q} are $\{1\}$. By the Radical Membership Test it means that

$$\mathbf{V}(\tilde{I}) \subset \mathbf{V}(I). \quad (9)$$

To check the opposite inclusion it is sufficient to check that

$$\langle I, 1 - wf \rangle = \langle 1 \rangle \quad (10)$$

for all polynomials f from a basis of \tilde{I} . Unfortunately, we were not able to perform the check over the field \mathbb{Q} however we have checked that (10) holds over a few fields of finite characteristic. It yields that (10) holds with high probability Arnold (2003).

Sufficiency. We now prove that if the coefficients of the system belong to one of varieties mentioned in the statement of the theorem then the system has an analytic first integral of the form (3).

Usually the center variety of a polynomial system contains components corresponding to Hamiltonian, time-reversible and Darboux integrable systems.

It is easy to see that systems from $V_6^{\mathbb{C}}$ are Hamiltonian with the Hamiltonian

$$H = \frac{x^2 + y^2}{2} + \frac{b_{20}x^3}{3} + \frac{b_{30}x^4}{4} - \frac{a_{02}y^3}{3}.$$

All time-reversible cubic systems were found in Sibirskii (1976); Jarrah et al. (2001). To use the results of Sibirskii (1976); Jarrah et al. (2001) we first complexify system (5) introducing the variable $z = x + iy$ and obtain from (5) after rescaling of time by i the complex differential equation

$$\begin{aligned} \dot{z} = z &+ \frac{1}{4}z^2(ia_{02} - b_{02} - ib_{11} + b_{20}) - \frac{1}{2}z\bar{z}(ia_{02} - b_{02} - b_{20}) + \frac{1}{4}\bar{z}^2(ia_{02} - b_{02} + ib_{11} + b_{20}) \\ &- \frac{1}{8}z^3(b_{12} + ib_{21} - b_{30}) + \frac{1}{8}z^2\bar{z}(b_{12} - ib_{21} + 3b_{30}) + \frac{1}{8}z\bar{z}^2(b_{12} + ib_{21} + 3b_{30}) - \frac{1}{8}\bar{z}^3(b_{12} - ib_{21} - b_{30}). \end{aligned}$$

³ The test says that for a polynomial f and an ideal $I = \langle f_1, \dots, f_m \rangle$ in $\mathbb{C}[x_1, \dots, x_n]$ $f|_{\mathbf{V}(I)} \equiv 0$ if and only if the reduced Gröbner basis of the ideal $\langle 1 - wf, f_1, \dots, f_m \rangle$ (here w is a new variable) is equal to $\{1\}$, see e.g. Cox et al. (1997); Romanovski and Shafer (2009) for more details

Substituting the coefficients of this differential equations into polynomials of Theorem 6 of Jarrah et al. (2001), which define the variety of all time-reversible cubic systems, and then computing with `minAssGTZ` of `SINGULAR` the minimal associate primes of the obtained ideal we get the components V_1^c , V_3^c and V_4^c of the statement of the theorem. Hence, all systems from the components V_1^c , V_3^c and V_4^c are time-reversible and, therefore, admit a first integral of the form (3) (see e.g. Romanovski and Shafer (2009) for more details).

Thus, there remains to prove integrability of systems from the components V_2^c , V_5^c and V_7^c . Systems from the component V_2^c , have the form

$$\dot{x} = -y + a_{02}y^2, \quad \dot{y} = x + b_{20}x^2 + b_{11}xy + b_{11}b_{20}x^2y. \quad (11)$$

Computing we obtain that system (11) admits the Darboux polynomial $f = 1 + b_{11}y$ with the cofactor $K = b_{11}x(1 + b_{20}x)$. Thus, the function $\mu = \frac{1}{f}$ is a Darboux integrating factor of (11), which allows to construct the analytic first integral

$$\Psi = \frac{1}{3} \left(\frac{3b_{11}y(2(a_{02} + b_{11}) - a_{02}b_{11}y) - 6(a_{02} + b_{11}) \log(b_{11}y + 1)}{b_{11}^3} + x^2(2b_{20}x + 3) \right) = x^2 + y^2 + \dots$$

Systems from V_5^c are the so-called reduced Kukles systems. The center problem for such systems has been solved in Jin et al. (1990); Christopher and Lloyd (1990), so by the results of these papers systems from V_5^c admit first integral of the form (3).

Finally, system from the component V_7^c are of the form

$$\dot{x} = -y \pm \frac{ib_{20}y^2}{\sqrt{6}}, \quad \dot{y} = x + b_{20}x^2 \pm \frac{5ib_{20}xy}{\sqrt{6}} - \frac{5b_{20}y^2}{3} \quad (12)$$

The system has the Darboux polynomial

$$f = 1 + \frac{b_{20}^2x^2}{3} - \frac{b_{20}^2y^2}{18} + \frac{4b_{20}x}{3} \pm \frac{1}{3}i\sqrt{\frac{2}{3}}b_{20}^2xy \pm i\sqrt{\frac{2}{3}}b_{20}y$$

with the cofactor $K = \pm \frac{1}{3}ib_{20}(\sqrt{6}x + 4iy)$ which allows to construct the integrating factor $\mu = f^{-5/2}$. \square

Remark. In the statement of theorem the ideals are presented as returned by the routine `minAssGTZ` of `SINGULAR`. However looking for Gröbner bases of I_5 with different ordering of variables we find that the ideal I_5 is the same as the ideal

$$\hat{I}_5 = \langle b_{02}^3 + b_{02}^2b_{20} - 2b_{02}b_{12} - b_{12}b_{20}, -b_{02}b_{11} - b_{11}b_{20} + b_{21}, b_{02}^2 + b_{02}b_{20} + b_{30} \rangle. \quad (13)$$

It is easy to see that the conditions defined by these polynomials are equivalent to conditions (iv) of Theorem 3.6 of Christopher and Lloyd (1990).

4. Limit cycle bifurcations in system (5)

In this section we study bifurcations of limit cycles from each component of the real center variety of system (5) under perturbations inside the family. It is obvious that V_7^c is the empty set in \mathbb{R}^7 . So the real variety $V^{\mathbb{R}}$ of (5) consist of 6 components, $V^{\mathbb{R}} = \cup_{k=1}^6 V_k$, where V_k is the set V_k^c of Theorem 1 restricted to \mathbb{R}^7 .

Let $I = \langle f_1, \dots, f_m \rangle \subset k[x_1, \dots, x_n]$ be an ideal and $V(I)$ be its variety. Assume that a decomposition of $V = V(I)$ is known and let p be a point from V . The tangent space to V at p is defined as $T_p = p + \{v | J_p(I)v = 0\}$, where $J(I)$ is the Jacobian of the polynomials f_1, \dots, f_m and J_p indicates that it is evaluated at p . It follows that $\dim T_p = n - \text{rank}(J_p(I))$. It is said that p is a smooth point of V if $\dim T_p = \dim V_p$. Let C be a component of V of codimension k and assume that $p \in C$, $\text{rank}(J_p(I)) = s$. Then $k \geq s$ and p is a smooth point C if and only if $k = s$; in this case $\text{rank}(J_p(I)) = k$ at any smooth point of C .

The following statement is proved by Christopher (2005).

Theorem 2. *Assume that for system (2) $p \in K$ is a point on the center variety and that the first k of the focus quantities g_i have independent linear parts. Then p lies on a component of the center variety of codimension at least k and there are bifurcations of (2) which produce k limit cycles locally from the center corresponding to the parameter value p .*

If, furthermore, we know that p lies on a component of the center variety of codimension k , then p is a smooth point of the variety, and the cyclicity of the center for the parameter value p is exactly $k - 1$.

In the latter case, $k - 1$ is also the cyclicity of a generic point on this component of the center variety.

According to the theorem in some cases the cyclicity of generic point of a component of the center variety can be easily determined if we know the dimension of the components of center variety. The dimension of a complex variety can be computed using algorithms of computational algebra, since it is equal to the degree of the affine Hilbert polynomial of any ideal defining the variety. However determining dimensions of real varieties is more difficult problem. Nevertheless, it is not difficult to determine the dimension of the components of the center variety of real system (2) and to prove the following result.

Theorem 3. *The cyclicities of generic point of the components V_1, V_2, V_4, V_5, V_6 of system (5) are 2, 3, 2, 3, 3 respectively. The cyclicity of generic point of V_3 is at least 2.*

Proof. It is clear that the codimension of the component V_1 is 3. Computing the minors of the matrix $J(g_1, g_2, g_3)$ evaluated on V_1 we see that for any point of $p \in V_1$ there is a non-zero 3-minor if $f(p) \neq 0$, where

$$f = -140a_{02}^6b_{12} + 61a_{02}^4b_{11}^2b_{12} + 9a_{02}^3b_{11}^3b_{12} - 2a_{02}^2b_{11}^4b_{12} - 36a_{02}^4b_{12}^2 + 9a_{02}^2b_{11}^2b_{12}^2 + 600a_{02}^5b_{11}b_{30} + 450a_{02}^4b_{11}^2b_{30} + 45a_{02}^3b_{11}^3b_{30} - 15a_{02}^2b_{11}^4b_{30} - 100a_{02}^4b_{12}b_{30} + 64a_{02}^3b_{11}b_{12}b_{30} + 37a_{02}^2b_{11}^2b_{12}b_{30} + 11a_{02}b_{11}^3b_{12}b_{30} + 12a_{02}b_{11}b_{12}^2b_{30} - 6b_{11}^2b_{12}^2b_{30} + 750a_{02}^3b_{11}b_{30}^2 + 240a_{02}^2b_{11}^2b_{30}^2 - 15a_{02}b_{11}^3b_{30}^2 - 90a_{02}b_{11}b_{12}b_{30}^2 + 18b_{11}^2b_{12}b_{30}^2 + 210a_{02}b_{11}b_{30}^3.$$

Therefore, by Theorem 2 the cyclicity of a generic point of V_1 is two (more precisely, the cyclicity of point $p \in V_1$ is two if $f(p) \neq 0$).

Similar consideration and conclusion are valid for the component V_4 .

It is also obvious that the codimensions of the components V_2 and V_6 is 4 and the computations show that the rank of $J(g_1, \dots, g_4)$ at a generic point of the component is 4, so by Christopher's theorem the cyclicity of generic point of the component is 3.

To find dimensions of components V_3 and V_5 we look for their parametrizations. The component V_5 can be parametrized as follows:

$$a_{02} = 0, \quad b_{20} = -\frac{t_2(t_2^2 - 2t_3)}{t_2^2 - t_3}, \quad b_{30} = -\frac{t_2^2 t_3}{t_2^2 - t_3}, \quad b_{11} = t_1, \quad b_{21} = \frac{t_1 t_2 t_3}{t_2^2 - t_3}, \quad b_{02} = t_2, \quad b_{12} = t_3. \quad (14)$$

To check this we eliminate from the ideal

$$\langle 1 - w(t_2^2 - t_3), a_{02}, b_{20} + \frac{t_2(t_2^2 - 2t_3)}{t_2^2 - t_3}, b_{30} + \frac{t_2^2 t_3}{t_2^2 - t_3}, b_{11} - t_1, b_{21} - \frac{t_1 t_2 t_3}{t_2^2 - t_3}, b_{02} - t_2, b_{12} - t_3 \rangle$$

of the ring $\mathbb{R}[w, t_1, t_2, t_3, a_{02}, b_{20}, b_{30}, b_{11}, b_{21}, b_{12}, b_{02}]$ the variables w, t_1, t_2, t_3 obtaining the ideal I_5 from the statement of Theorem 1. By Theorem 2 of (Cox et al., 1997, §3.3) it means that (14) gives a rational parametrization of the variety $\mathbf{V}(I_5)$. Now it is easy to conclude that the dimension of the component is 3 and, hence, the codimension is 4. Checking that the rank of $J(g_1, \dots, g_4)$ is four almost everywhere on V_5 we obtain by Theorem 2 that the cyclicity of generic point of the component is 3.

A parametrization of V_3 is given by

$$a_{02} = -\frac{t_1(t_1^2 + 4t_2^2)}{2(t_1 - 2t_2)(t_1 + 2t_2)}, b_{20} = \frac{2t_1^2 t_2}{4t_2^2 - t_1^2}, b_{30} = 0, b_{11} = t_1, b_{21} = 0, b_{02} = t_2, b_{12} = 0.$$

From the parametrization we see that the component is two-dimensional, thus, it is of codimension five. However only first three focus quantities have independent linear parts, so we cannot get an upper bound for cyclicity using Theorem 2 and only conclude that the cyclicity is at least two. \square

Remark. Of course, it is often happens that there many parametrizations of the same variety. Another parametrization of V_5 can be easily obtained using the polynomial basis of I_5 presented in (13).

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