BIRTH OF ISOLATED NESTED CYLINDERS AND LIMIT CYCLES IN 3D PIECEWISE SMOOTH VECTOR FIELDS WITH SYMMETRY

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ABSTRACT. Our start point is a 3D piecewise smooth vector field defined in two zones and presenting a shared fold curve for the two smooth vector fields considered. Moreover, these smooth vector fields are symmetric relative to the fold curve, giving raise to a continuum of nested topological cylinders such that each orthogonal section of these cylinders is filled by centers. First we prove that the normal form considered represents a whole class of piecewise smooth vector fields. After we perturb the initial model in order to obtain exactly $\mathcal{L}$ invariant planes containing centers. A second perturbation of the initial model also is considered in order to obtain exactly k isolated cylinders filled by periodic orbits. Finally, joining the two previous bifurcations we are able to exhibit a model, preserving the symmetry relative to the fold curve, and having exactly $k.\mathcal{L}$ limit cycles.

1. INTRODUCTION

Vector fields tangent to foliations, Hamiltonian systems and first integrals of vector fields are correlated themes very exploit in the literature about Dynamical Systems. In fact the list of papers on these subjects is extremely large and we cite just the books [1, 7, 31, 38] for a brief notion on these issues.

Many authors have used the theoretical aspects about vector fields tangent to foliations, Hamiltonian systems and first integrals of vector fields in order to obtain dynamical properties of models describing some system in applied science. A far from exhaustive list of books in this sense is given by [3, 13, 24].

In recent years, scientists are realizing the importance and applicability of a new branch of dynamical systems that are powerful tools in phenomena where some *"on-off"* phenomena take place. For example, in control theory (see [34]), mechanics models (see [4, 17, 30]), electrical circuits (see [27]), relay systems (see [16, 25]), biological models (with refuge see [28], foraging predators see [33]), among others where an instantaneous change on the system is observed when any barrier is broken. These dynamical systems

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are modeled by "pieces" and are called **piecewise smooth vector fields** (PSVFs for short).

Many authors have contributed to provide a general and consistent theory about PSVFs. We cite here the works [15, 35] where a non familiar reader can found the main definitions, conventions and results on this theory. However, very little have been studied about PSVFs tangent to (piecewise) foliations, Hamiltonian PSVFs and first integrals of PSVFs. Addressing this topic we cite [8, 26, 32].

The present paper deals precisely with PSVFs tangent to piecewise foliations. We found first integrals for them and perform bifurcations on the unstable PSVFs obtained. In fact, a very rich behavior is observed and, which it is very important, an almost fully exploit study area is brought to the surface.

1.1. **Setting the problem and statement of the main results.** Let Σ be a codimension one 3D manifold given by $\Sigma = f^{-1}(0)$, where $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). We call Σ the *switching manifold* that is the separating boundary of the regions $\Sigma^+ = \{q \in \mathbb{R}^3 \mid f(q) \geq 0\}$ and $\Sigma^- = \{q \in \mathbb{R}^3 \mid f(q) \leq 0\}$.

Take $X : \Sigma^+ \to \mathbb{R}^3$ (resp., $Y : \Sigma^- \to \mathbb{R}^3$) smooth vector fields. We combine them in order to constitute the PSVF $Z : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$Z(x,y,z) = \begin{cases} X(x,y,z), & \text{for} \quad (x,y,z) \in \Sigma^+, \\ Y(x,y,z), & \text{for} \quad (x,y,z) \in \Sigma^-. \end{cases}$$

The trajectories of Z are solutions of $\dot{q} = Z(q)$ and we will accept that Z is multi-valued in points of Σ. The basic results of differential equations, in this context, were stated in [20]. We use the notation $Z = (X, Y)$.

Given $p \in \Sigma$, throughout this paper we do not consider the situation where both vector fields X and Y have trajectories arriving (resp. departing) from p transversally. In these cases p is called in the literature as a **sliding** (resp. **escaping**) point. So, here we assume that when an X-trajectory reaches $p \in \Sigma$ transversally, then there is a Y-trajectory starting at p and transversal to Σ, i.e., generically, just **crossing** points will be considered.

In fact, the initial model that we consider is

$$Z_0(x,y,z) = \begin{cases} X_0(x,y,z) = \begin{pmatrix} 0 \\ -1 \\ 2y \end{pmatrix} & \text{if } z \geq 0, \\[2ex] Y_0(x,y,z) = \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} & \text{if } z \leq 0. \end{cases} \tag{1}$$

The phase portrait of (1) is given in Figure 1.

It is patent the symmetry of the trajectories obtained from (1). Moreover, we get that $H_1(x,y,z) = x$ and $H_2(x,y,z) = z + y^2$ (resp., $L_1(x,y,z) = x$ and $L_2(x,y,z) = z - y^2$) are independent first integrals of X_0 (resp., Y_0).

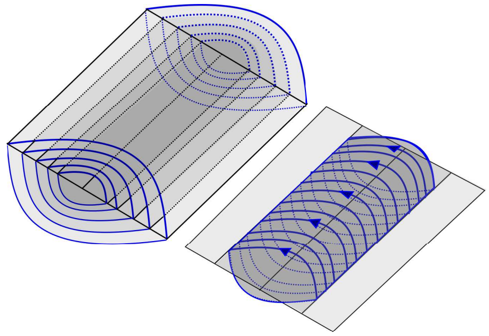

FIGURE 1. Topological cylinders.

The orbits of X_0 are contained in the sets $\{H_1 = c_1\} \cap \{H_2 = c_2\} \cap \{z \geq 0\}$ and the orbits of Y_0 are contained in the sets $\{L_1 = c_3\} \cap \{L_2 = c_4\} \cap \{z \leq 0\}$, with $c_1, c_2, c_3, c_4 \in \mathbb{R}$.

So, a pair of *piecewise first integrals* of (1) is

$$M_1(x,y,z) = x \text{ and } M_2(x,y,z) = \begin{cases} H_2(x,y,z) & \text{if } z \geq 0, \\ L_2(x,y,z) & \text{if } z \leq 0. \end{cases}$$

Of course, the trajectories of X_0 (resp., Y_0) leave at the intersection of the transversal (in fact, orthogonal) foliations H_1 and H_2 (resp., L_1 and L_2) and X_0 (resp. Y_0) is a vector field tangent to both foliations. As consequence, the PSVF $Z_0 = (X_0, Y_0)$ is tangent to the foliation M_1 and the *piecewise* foliation M_2. Moreover, all orbits of Z_0 are closed and topologically equivalent to S^1.

We stress that, in general (where we admit sliding and escaping motion on Σ), is **false** the natural aim: *The piecewise smooth mapping* $H = \frac{h+l}{2} + \text{sign}(z)\frac{h-l}{2}$ *is a first integral of the vector field* $Z = (X, Y)$ *provided that* h *and* l *are smooth first integrals of* X *and* Y *respectively.* See [8] for examples.

Also observe that Z_0 is such that $Z_0(x,y,z) = -Z_0(-x,-y,-z)$ and so, it is φ*-reversible*, where $\varphi(x,y,z) = (-x,-y,-z)$.

Another important definition is the concept of equivalence between two PSVFs.

Definition 1. *Two PSVFs* $Z = (X, Y)$, $\widetilde{Z} = (\widetilde{X}, \widetilde{Y}) \in \Omega$, *where* Ω *be the set of all PSVF endowed with the* C^r *product topology, defined in open sets* $U, \widetilde{U}$ *and with switching manifold* Σ *are* ***Σ-equivalent*** *if there exists an orientation preserving homeomorphism* $h : U \to \widetilde{U}$ *that sends* $U \cap \Sigma$ *to* $\widetilde{U} \cap \Sigma$*, the orbits of* X *restricted to* $U \cap \Sigma^+$ *to the orbits of* $\widetilde{X}$ *restricted to* $\widetilde{U} \cap \Sigma^+$*, and the orbits of* Y *restricted to* $U \cap \Sigma^-$ *to the orbits of* $\widetilde{Y}$ *restricted to* $\widetilde{U} \cap \Sigma^-$.

Now we state the main results of the paper.

Proposition 2. *Let* $Z = (X, Y)$ *be a PSVF defined in a compact* $\mathcal{M}$ *presenting a continuous of topological cylinders filled by periodic orbits, then* Z *is* Σ*-equivalent to* Z_0 *given by* (1).

Theorem A. *Let Z_0 be given by* (1). *For any neighborhood $\mathcal{W} \subset \Omega$ of Z_0 and for any integer $\mathcal{L} > 0$, there exists $\widetilde{Z} \in \mathcal{W}$ such that $\widetilde{Z}$ has $\mathcal{L}$ Z_0-invariant planes. Moreover, in each plane there is a center of Z_0.*

Theorem B. *Let Z_0 be given by* (1). *For any neighborhood $\mathcal{W} \subset \Omega$ of Z_0 and for any integer $k > 0$, there exists $\widetilde{Z} \in \mathcal{W}$ such that $\widetilde{Z}$ has k isolated invariant topological cylinders filled by periodic orbits. The same holds if $k = \infty$.*

Theorem C. *Let Z_0 be given by* (1). *For any neighborhood $\mathcal{W} \subset \Omega$ of Z_0 and for any integers $\mathcal{L} > 0$ and $k > 0$, there exists $\widetilde{Z} \in \mathcal{W}$ such that $\widetilde{Z}$ has $\mathcal{L}.k$ hyperbolic limit cycles. The same holds if $k = \infty$. Moreover, the stability of each limit cycle is obtained. See Figure 2.*

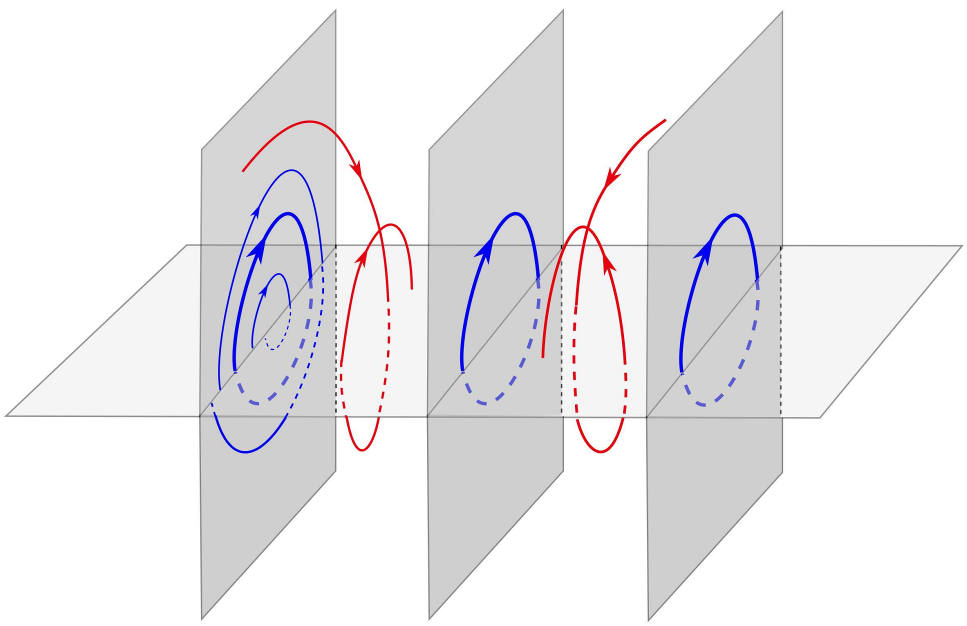

FIGURE 2. The trajectories according to Theorem C.

Moreover, in the previous theorems, we explicitly build families of PSVFs presenting the quoted properties.

The paper is organized as follows. In Section 2 we introduce the terminology, some definitions and the basic theory about PSVFs. Sections 3, 4, 5 and 6 are devoted to prove Proposition 2, Theorem A, Theorem B and Theorem C, respectively.

2. PRELIMINARIES

Definition 3. *Consider $Z \in \Omega$. We say that $q \in \Sigma$ is a* ***Σ-center*** *of Z if $q \in \Sigma$ and there is a codimension one manifold $\mathcal{S}$ such that $\Sigma \bigcap \mathcal{S}$ and there is* *a neighborhood $U \subset \mathbb{R}^3$ of q where $U \cap \mathcal{S}$ is filled by a one-parameter family γ_s of closed orbits of Z in such a way that the orientation is preserved.*

Consider the notation $X.f(p) = \langle \nabla f(p), X(p) \rangle$ and, for $i \geq 2$, $X^i.f(p) = \big\langle \nabla X^{i-1}.f(p), X(p) \big\rangle$, where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^3$. We say that a point $p \in \Sigma$ is a Σ-*fold point* of X if $X.f(p) = 0$ but $X^2.f(p) \neq 0$. Moreover, $p \in \Sigma$ is a *visible* (respectively *invisible*) Σ-fold point of X if $X.f(p) = 0$ and $X^2.f(p) > 0$ (respectively $X^2.f(p) < 0$). We say that $p \in \Sigma$ is a *two-fold singularity* of Z if p is a Σ-fold point for both X and Y. In this work, we consider only two-fold singularities of type invisible-invisible, ie, the fold points are invisible for both, X and Y.

Remark 1. *Since $f(x, y, z) = z$, we conclude from* (1) *that $L = \{(x, 0, 0) \,|\, x \in \mathbb{R}\} \subset \Sigma$ is the curve of invisible fold singularities of both X_0 and Y_0.*

Consider the case when the PSVF $Z = (X, Y)$ has q as two-fold singularity. We can define the *positive half-return map* as $\varphi_X(\rho) = \rho^+$, and the *negative half-return map* as $\varphi_Y(\rho^+) = \rho^-$ (see Figure 3). The complete *return map* associated to Z is given by the composition of these two maps

$$\varphi_Z(\rho) = \varphi_Y(\varphi_X(\rho)). \tag{2}$$

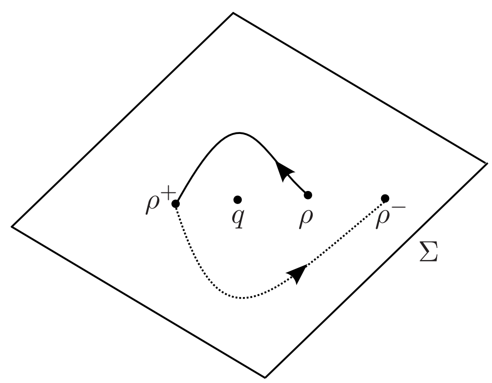

FIGURE 3. Return map of $Z = (X, Y)$.

Proposition 4. *The PSVF $Z_0 = (X_0, Y_0)$ given by* (1) *has a continuous of topological cylinders and, in each cylinder, all orbits are periodic (see Figure 1). Moreover, in each plane $\pi_M = \{(x, y, z) \,|\, x = M\}$, there is a Σ-center.*

Proof. For a direct integration, the trajectories of X_0 and Y_0 are parametrized by

$$\phi_{X_0}(t) = (x_0, -t + y_0, -t^2 + 2ty_0) \tag{3}$$

and

$$\phi_{Y_0}(t) = (x_1, t + y_1, t^2 + 2ty_1), \tag{4}$$

respectively. Note that $\phi_{X_0}(0) = (x_0, y_0, 0)$ and $\phi_{Y_0}(0) = (x_1, y_1, 0)$. Thus the positive half-return map is $\varphi_{X_0}(x, y) = (x, -y)$. Analogously, the negative half-return map is $\varphi_{Y_0}(x, y) = (x, -y)$. Therefore, the complete *return map* associated to Z_0 is given by

$$\varphi_{Z_0}(x, y) = \varphi_{Y_0}(\varphi_{X_0}(x, y)) = (x, y).$$

□

Note that by Proposition 4, we get $\varphi_{Z_0}(x, y) = (\varphi^1_{Z_0}(x), \varphi^2_{Z_0}(y))$, where $\varphi^1_{Z_0}(x) = x$ and $\varphi^2_{Z_0}(y) = y$. In order to obtain isolated Z_0-invariant planes we perturb the map $\varphi^1_{Z_0}(x)$ (see Theorem A), and in order to obtain isolated Z_0-topological cylinders we perturb the map $\varphi^2_{Z_0}(y)$ (see Theorem B). When we perturb both we are able to obtain hyperbolic limit cycles (see Theorem C).

Remark 2.

- *In this work we decide consider only perturbations of* (1) *that keep the straight line* $L = \{(x, y, z) \in \mathbb{R}^3; y = z = 0\}$ *as a two-fold singularity. This assumption is important because in this case the return map is always well defined.*
- *In this sense the return map of all trajectories considered in this paper is given by the composition of two involutions (see* [37]*).*

3. Proof of Proposition 2

In this section we construct homeomorphism that sends orbits of $Z = (X, Y)$, that has a continuous of topological cylinders filled by periodic orbits, to orbits of $Z_0 = (X_0, Y_0)$ given by (1).

Without loss of generality consider that orbits of Z are oriented in an anti-clockwise sense. Let L (respectively, $\overline{L}$) be a set of two-fold singularity of Z_0 (respectively, Z) with length $\mathcal{R}_1 > 0$. By arc length parametrization we identify L with $\overline{L}$. By p (respectively, $\overline{p}$), we mark the line segment $\mathcal{S}_p$ (respectively, $\mathcal{S}_{\overline{p}}$) of length $\mathcal{R}_2$ orthogonal to Σ (see Figure 4). This segment reaches a topological cylinder $\mathcal{M}$ of Z_0 (respectively, $\overline{\mathcal{M}}$ of Z) at a point p^1 (respectively, $\overline{p}^1$).

In each point $\alpha \in L = [p, r]$ (respectively, $\overline{\alpha} \in \overline{L}$) mark the line segment $\mathcal{S}_\alpha$ orthogonal to Σ (respectively, $\mathcal{S}_{\overline{\alpha}}$) with final point in $\mathcal{M}$ (respectively, $\overline{\mathcal{M}}$). Once L and $\overline{L}$ are identified, identify each S_α with $\mathcal{S}_{\overline{\alpha}}$ by arc length parametrization.

By the Implicit Function Theorem (abbreviated by IFT), there exists a smallest time $t_1 < 0$ (respectively, $\overline{t}_1 < 0$), depending on p^1 (respectively, $\overline{p}^1$), such that $\phi_{X_0}(p^1, t_1) := q \in \Sigma(+)$ (respectively, $\phi_X(\overline{p}^1, \overline{t}_1) := \overline{q} \in \overline{\Sigma}(+)$), where $\Sigma(+)$ (respectively, $\overline{\Sigma}(+)$) is the set of all points of Σ situated on the right of L (respectively, $\overline{L}$) and ϕ_W denotes the flow of the vector field W. Identify the orbit arcs $\gamma_q^{p^1}(X_0)$ and $\gamma_{\overline{q}}^{\overline{p}^1}(X)$ of X_0 and

X with initial points q and $\overline{q}$ and final points p^1 and $\overline{p}^1$, respectively, by arc length parametrization. Again by IFT, there exists a smallest time $t_2 > 0$ (respectively, $\overline{t}_2 > 0$), depending on p^1 (respectively, $\overline{p}^1$), such that $\phi_{X_0}(p^1, t_2) := q^1 \in \Sigma(-)$ (respectively, $\phi_X(\overline{p}^1, \overline{t}_2) := \overline{q}^1 \in \overline{\Sigma}(-)$) where $\Sigma(-)$ (respectively, $\overline{\Sigma}(-)$) is the set of all points of Σ situated on the left of L (respectively, $\overline{L}$). Identify the orbit arcs $\gamma_{p^1}^{q^1}(X_0)$ and $\gamma_{\overline{p}^1}^{\overline{q}^1}(X)$ of X_0 and X with initial points p^1 and $\overline{p}^1$ and final points q^1 and $\overline{q}^1$, respectively, by arc length parametrization.

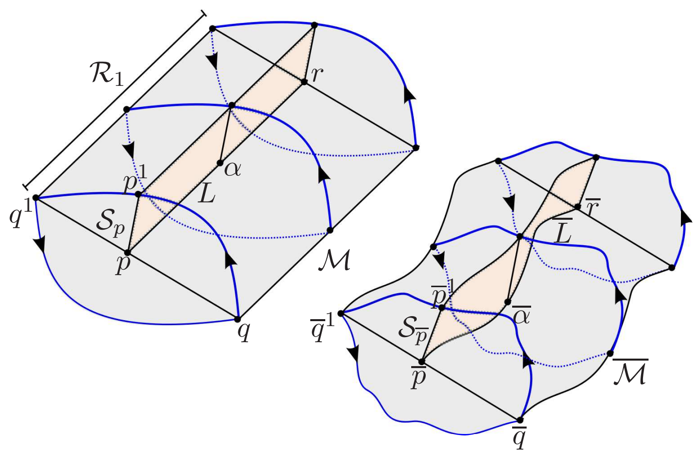

FIGURE 4. Topological cylinders.

Now, since Z_0 (respectively, Z) presents a continuous of topological cylinders, and L (respectively, $\overline{L}$) is an invisible Σ-fold set Y_0 (respectively, Y) by the IFT, there exists a smallest time $t_3 > 0$ (respectively, $\overline{t}_3 > 0$), depending on q^1 (respectively, $\overline{q}^1$), such that $\phi_{Y_0}(q^1, t_3) := q \in \Sigma(+)$ (respectively, $\phi_Y(\overline{q}^1, \overline{t}_3) := \overline{q} \in \overline{\Sigma}(+)$). Identify the orbit arcs $\gamma_{q^1}^{q}(Y_0)$ and $\gamma_{\overline{q}^1}^{\overline{q}}(Y)$ of Y_0 and Y with initial points q^1 and $\overline{q}^1$ and final points q and $\overline{q}$, respectively, by arc length parametrization.

Do the same for all point $\beta \in S_\alpha$ (resp., $\overline{\beta} \in S_{\overline{\alpha}}$), and for all $\alpha \in L$ (resp., $\overline{\alpha} \in \overline{L}$).

4. A PERTURBATION ON THE HORIZONTAL AXIS – PROOF OF THEOREM A

Now we consider a perturbation on the normal form (1) that keeps invariant the nested cylinders and exactly $\mathcal{L}$ planes of the form $\pi_i = \{(x, y, z) \,|\, x = i\mu\}$, where $i \in \{0, 1, 2, \ldots, \mathcal{L} - 1\}$ and $\mu > 0$ is a small real number. In fact, consider

$$\overline{X}_{\mathcal{L}}(x, y, z) = \begin{pmatrix} x(x-\mu)(x-2\mu)\ldots(x-(\mathcal{L}-1)\mu) \\ 0 \\ 0 \end{pmatrix} = \tag{5}$$

$$= \begin{pmatrix} \Pi_{i=0}^{\mathcal{L}-1}(x - i\mu) \\ 0 \\ 0 \end{pmatrix}$$

and

$$Z_{\mathcal{L}}(x,y,z) = \begin{cases} X_{\mathcal{L}}(x,y,z) = \begin{pmatrix} \lambda\Pi_{i=0}^{\mathcal{L}-1}(x - i\mu) \\ -1 \\ 2y \end{pmatrix} & \text{if } z \geq 0, \\ Y_0(x,y,z) = \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} & \text{if } z \leq 0, \end{cases} \tag{6}$$

where $X_{\mathcal{L}}(x,y,z) = X_0(x,y,z) + \lambda\overline{X}_{\mathcal{L}}(x,y,z)$, with λ a sufficiently smal real number and X_0 given in (1).

Remark 3. *There is nothing special in the set $\{0,1,2,\ldots,\mathcal{L}-1\}$ of sequential positive integers and we could take any set of $\mathcal{L}$ integers in the previous consideration.*

Proposition 5. *The topological cylinders obtained in Proposition 4 are $Z_{\mathcal{L}}$-invariant.*

Proof. By Remark 1, $L = \{(x,0,0) \,|\, x \in \mathbb{R}\} \subset \Sigma$ is the curve of invisible fold singularities of both X_0 and Y_0.

The positive half-return map is $\varphi_{X_{\mathcal{L}}}(x,y) = (\varphi^1_{X_{\mathcal{L}}}(x), -y)$ and the negative half-return map is $\varphi_{Y_0}(x,y) = (x,-y)$. For a fixed $\beta \in \mathbb{R}$, take $L_\beta = \{(x,\beta,0) \,|\, x \in \mathbb{R}\} \subset \Sigma$ and let us saturate this straight line by the $Z_{\mathcal{L}}$-flow. In fact, for all $(x,\beta,0) \in L_\beta$ we get

$$\varphi_{Z_{\mathcal{L}}}(x,\beta,0) = \varphi_{Y_0}(\varphi_{X_{\mathcal{L}}}(x,\beta,0)) = (\varphi^1_{X_{\mathcal{L}}}(x),\beta,0) \in L_\beta. \tag{7}$$

□

Proposition 6. *The planes $\pi_i = \{(x,y,z) \,|\, x = i\mu\}$, where $i \in \{0,1,2,\ldots,\mathcal{L}-1\}$, are $Z_{\mathcal{L}}$-invariant.*

Proof. Take $i = i_0$ fixed. When $x = i_0\mu$ we get that the first coordinate of $X_{\mathcal{L}}$ is null. As consequence, the plane π_{i_0} is $X_{\mathcal{L}}$-invariant. The same holds for all $i \in \{0,1,2,\ldots,\mathcal{L}-1\}$. On the other hand, for all $c \in \mathbb{R}$, the plane $\pi_c = \{(x,y,z) \,|\, x = c\}$ is Y_0-invariant. Therefore, each plane π_i is $Z_{\mathcal{L}}$-invariant. □

Proposition 7. *The PSVF $Z_{\mathcal{L}}$ has a Σ-center in each plane π_i, where $i \in \{0,1,2,\ldots,\mathcal{L}-1\}$.*

Proof. The proof is straighforward. Is enough to combine Propositions 5 and 6. □

Proposition 8. *When i is even (resp. odd) the Σ-center π_i behaves like a unstable (resp. stable) center manifold where $i \in \{0,1,2,\ldots,\mathcal{L}-1\}$.*

Proof. From Proposition 6 the planes $\pi_i = \{(x,y,z)\,|\, x = i\mu\}$ are $Z_{\mathcal{L}}$-invariant. As stated in Equation 7, we get $\varphi_{Z_{\mathcal{L}}}(x,y,0) = (\varphi^1_{X_{\mathcal{L}}}(x), y, 0)$. As consequence, the behavior of the complete return map is determined by $\varphi^1_{X_{\mathcal{L}}}(x)$. So, let us consider the first coordinate of $X_{\mathcal{L}}$, i.e., let us consider the differential equation

$$\dot{x} = \Pi_{i=0}^{\mathcal{L}-1}(x - i\mu). \tag{8}$$

Note that each $x = i\mu$, $i \in \{0, 1, 2, \ldots, \mathcal{L}-1\}$, is a solution of (8) and

$$\frac{d}{dx}\Pi_{i=0}^{\mathcal{L}-1}(x - i\mu)|_{x=i\mu}$$

is positive for i even and negative for i odd. The behavior in each solution is given in the Figure 5.

FIGURE 5. The phase portrait of (8) and the graph of $y = \Pi_{i=0}^{\mathcal{L}-1}(x - i\mu)$.

Thus, when i is even (resp. odd) the Σ-center π_i behaves like a unstable (resp. stable) center manifold, where $i \in \{0, 1, 2, \ldots, \mathcal{L}-1\}$. $\square$

Proof of Theorem A. The Propositions 6, 7 and 8 prove Theorem A. $\square$

5. A perturbation of the continuum of cylinders – Proof of Theorem B

In order to prove Theorem B we need some lemmas. Observe that both vector fields X_0 and Y_0 in the normal form (1) are written as $W(x,y,z) = (0, \pm 1, g(y))$ (particularly, $g(y) = 2y$ in such expression). Next lemma gives how are the trajectories of such systems.

Lemma 9. *The trajectories of a vector field $W(x,y,z) = (0,1,g(y))$, in each plane $\pi_c = \{(x,y,z)\,|\, x = c\,;\, c \in \mathbb{R}\}$, are obtained by vertical translations of the graph of $G(y)$, where $\frac{\partial}{\partial y}G(y) = g(y)$.*

Proof. Since $W(x,y,z) = (\dot{x}, \dot{y}, \dot{z}) = (0, 1, g(y)) \in \chi^r$ we obtain that

$$x(t) = c_1,\;\; y(t) = t + c_2 \text{ and } z(t) = \int g(t + c_2)dt = G(t + c_2) + c_3,$$

where $c_1, c_2, c_3 \in \mathbb{R}$ and G is a primitive of g. Now, take $u = t + c_2$ and the trajectories of $W(x,y,z)$ are given by $(c_1, u, G(u) + c_3)$ which in each plane $\pi_{c_1} = \{(x,y,z)\,|\, x = c_1\}$, are vertical translations of the graph of $G(u)$. $\square$

Observe that an analogous result is obtained with $W(x,y,z) = (0,-1,g(y))$.

In what follows, $h : \mathbb{R} \to \mathbb{R}$ will denote the C^{∞}-function given by

$$h(y) = \begin{cases} 0, & \text{if } y \leq 0, \\ e^{-1/y}, & \text{if } y > 0. \end{cases}$$

Lemma 10. *Consider the function* $\xi_\varepsilon^f(y) = \varepsilon h(y)(\varepsilon - y)(2\varepsilon - y)\dots(k\varepsilon - y)$.

(i) *If* $\varepsilon < 0$ *then* ξ_ε^f *does not have roots in* $(0,+\infty)$.

(ii) *If* $\varepsilon > 0$ *then* ξ_ε^f *has exactly* k *roots in* $(0,+\infty)$, *these roots are* $\{\varepsilon, 2\varepsilon \dots, k\varepsilon\}$ *and* $\dfrac{\partial \xi_\varepsilon^f}{\partial y}(j\varepsilon) = (-1)^j \varepsilon^k h(j\varepsilon)(k-j)!(j-1)!$ *for* $j \in \{1,2,\dots,k\}$. *It means that the derivative at the root* $j\varepsilon$ *is positive for* j *even and negative for* j *odd.*

Proof. When $y > 0$, by a straightforward calculation $\xi_\varepsilon^f(y) = 0$ if, and only if, $(\varepsilon - y)(2\varepsilon - y)\dots(k\varepsilon - y) = 0$. So, the roots of $\xi_\varepsilon^f(y)$ in $(0,+\infty)$ are $\varepsilon, 2\varepsilon, \dots, k\varepsilon$. Moreover,

$$\frac{\partial \xi_\varepsilon^f}{\partial y}(y) = \frac{\partial}{\partial y}\Big((j\varepsilon - y)H(y)\Big) = (j\varepsilon - y)\frac{\partial H}{\partial y}(y) - H(y),$$

where $H(y) = \xi_\varepsilon^f(y)/(j\varepsilon - y)$. So,

$$\begin{aligned} \frac{\partial \xi_\varepsilon^f}{\partial y}(j\varepsilon) &= -H(j\varepsilon) = \varepsilon^k h(j\varepsilon)(1-j)\dots((j-1)-j)((j+1)-j)\dots(k-j) \\ &= \varepsilon^k h(j\varepsilon)(-1)^j\Big((j-1)\dots(j-(j-1))\Big)\Big(((j+1)-j)\dots(k-j)\Big) \\ &= (-1)^j\varepsilon^k h(j\varepsilon)(k-j)!(j-1)! \end{aligned}$$

This proves item (ii). Item (i) follows immediately. □

Lemma 11. *Consider the function* $\xi_\varepsilon^i(y) = -h(y)\sin(\pi\varepsilon^2/y)$. *For* $\varepsilon \neq 0$ *the function* ξ_ε^i *has infinity many roots in* $(0,\varepsilon^2)$, *these roots are* $\{\varepsilon^2, \varepsilon^2/2, \varepsilon^2/3, \dots\}$ *and*

$$\frac{\partial \xi_\varepsilon^i}{\partial y}(\varepsilon^2/j) = (-1)^{j+1}(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j) \text{ for } j \in \{1,2,3,\dots\}.$$

It means that the derivative at the root ε^2/j *is positive for* j *even and negative for* j *odd.*

Proof. When $y > 0$, by a straightforward calculation $\xi_\varepsilon^i(y) = 0$ if, and only if, $\sin(\pi\varepsilon^2/y) = 0$. So, the roots of $\xi_\varepsilon^i(y)$ in $(0,\varepsilon^2)$ are $\varepsilon^2, \varepsilon^2/2, \varepsilon^2/3, \dots$. Moreover,

$$\frac{\partial \xi_\varepsilon^i}{\partial y}(y) = -h'(y)\sin(\pi\varepsilon^2/y) - h(y)\cos(\pi\varepsilon^2/y)(-\pi\varepsilon^2/y^2).$$

So,

$$\begin{aligned} \frac{\partial \xi_\varepsilon^i}{\partial y}(\varepsilon^2/j) &= -h'(\varepsilon^2/j)\sin(\pi j) - h(\varepsilon^2/j)\cos(\pi j)(-\pi j^2/\varepsilon^2) \\ &= (-1)^{j+1}(-\pi j^2/\varepsilon^2)h(\varepsilon^2/j). \end{aligned}$$

□

Since h is a C^∞-function, the functions $\xi^f_\varepsilon(y)$ in Lemma 10 and $\xi^i_\varepsilon(y)$ in Lemma 11 are C^∞-functions. So $Z^\rho_\varepsilon \in \Omega$ given by

$$Z^\rho_\varepsilon(x,y,z) = \begin{cases} X_0(x,y,z) = \begin{pmatrix} 0 \\ -1 \\ 2y \end{pmatrix} & \text{if } z \geq 0, \\ Y^\rho_\varepsilon(x,y,z) = \begin{pmatrix} 0 \\ 1 \\ 2y + \frac{\partial \xi^\rho_\varepsilon}{\partial y}(y) \end{pmatrix} & \text{if } z \leq 0, \end{cases} \tag{9}$$

where either $\rho = f$ or $\rho = i$, is a small C^∞-perturbation of Z_0 given by (1) when ε is sufficiently small. Moreover,

$$\lim_{\varepsilon \to 0} Z^\rho_\varepsilon = Z_0. \tag{10}$$

Lemma 12. *Let $\varphi_{Z^\rho_\varepsilon}(x,y) = (\varphi^1_{Z^\rho_\varepsilon}(x), \varphi^2_{Z^\rho_\varepsilon}(y))$ be the return map of Z^ρ_ε where either $\rho = f$ or $\rho = i$. For all $y > 0$ we have that*

$$y^2 - (\varphi^2_{Z^\rho_\varepsilon}(y))^2 - \xi^\rho_\varepsilon(\varphi^2_{Z^\rho_\varepsilon}(y)) = 0.$$

Proof. Let $(x_0, y_0, 0) \in \Sigma$. According to Lemma 9, in each plane $\pi_{x_0} = \{(x,y,z) \,|\, x = x_0\}$, the trajectories of X_0 are the graphs of $F_c(y) = -y^2 + c$ for $c \in \mathbb{R}$. The constant $c \in \mathbb{R}$ that satisfy $F_c(y_0) = 0$ is $c = y_0^2$. The parabola $z = -y^2 + y_0^2$ in the plane π_{x_0} intersects the plane $z = 0$ at the points $(x_0, y_0, 0)$ and $(x_0, -y_0, 0)$. So, $\varphi_{X_0}(x_0, y_0) = (x_0, -y_0)$ and thus $\varphi^2_{X_0}(y_0) = -y_0$. Again by Lemma 9, in each plane π_{x_0}, the trajectories of Y^ρ_ε are the graphs of $G_c(y) = y^2 + \xi^\rho_\varepsilon(y) + c$ for $c \in \mathbb{R}$. The constant $c \in \mathbb{R}$ that satisfy $G_c(-y_0) = 0$ is $c = -y_0^2$. So, in the plane π_{x_0}, the first return $\varphi^2_{Y^\rho_\varepsilon}(-y_0)$ is the first coordinate of the point in Σ given by the intersection of the graph of the function $z = G(y) = y^2 + \xi^\rho_\varepsilon(y) - y_0^2$ with the plane $z = 0$. So $\varphi^2_{Z^\rho_\varepsilon}(y)$ satisfies

$$y^2 - (\varphi^2_{Z^\rho_\varepsilon}(y))^2 - \xi^\rho_\varepsilon(\varphi^2_{Z^\rho_\varepsilon}(y)) = 0, \tag{11}$$

where either $\rho = f$ or $\rho = i$. □

Lemma 13. *Let $\varphi^2_{Z^f_\varepsilon}$ be the second component of return map of Z^f_ε. Then $y > 0$ is a fixed point of $\varphi^2_{Z^f_\varepsilon}$ if, and only if, $y = j\varepsilon$ for $j = 1, 2, \ldots, k$. Moreover, for j even $(\varphi^2_{Z^f_\varepsilon})'(j\varepsilon) < 1$ and for j odd $(\varphi^2_{Z^f_\varepsilon})'(j\varepsilon) > 1$.*

Proof. According to Lemma 12, $y = \varphi^2_{Z^f_\varepsilon}(y)$ if, and only if, $\varphi^2_{Z^f_\varepsilon}(y)$ is a zero of the function $\xi^f_\varepsilon(y)$, i.e., by Lemma 10, $y = j\varepsilon$ for $j = 1, 2, \ldots, k$. Differentiating (11) with respect to y we obtain $2y - 2\varphi^2_{Z^f_\varepsilon}(y)(\varphi^2_{Z^f_\varepsilon})'(y) -$

$\frac{\partial \xi_\varepsilon^f}{\partial y}(\varphi^2_{Z_\varepsilon^f}(y))(\varphi^2_{Z_\varepsilon^f})'(y) = 0$, and so

$$(\varphi^2_{Z_\varepsilon^f})'(j\varepsilon) = \frac{2j\varepsilon}{2j\varepsilon + \frac{\partial \xi_\varepsilon^f}{\partial y}(j\varepsilon)}.$$

According to Lemma 10, if j is even then $\frac{\partial \xi_\varepsilon^f}{\partial y}(j\varepsilon) > 0$ and it implies that $(\varphi_{Z_\varepsilon^f})'(j\varepsilon) < 1$. And if j is odd then $(\varphi_{Z_\varepsilon^f})'(j\varepsilon) > 1$. □

Remark 4. *A similar result obtained in Lemma 13, for the PSVF Z_ε^i, also holds.*

With the previous lemmas we can stated the following proposition.

Proposition 14. *Consider Z_ε^ρ given by* (9). *Then, for $\varepsilon = 0$, $Z_\varepsilon^\rho = Z_0$ given by* (1) *has a continuous of topological cylinders and*

- (I) *For $\rho = f$*
  - (I.i) *Z_ε^f has k isolated topological cylinders when $\varepsilon > 0$,*
  - (I.ii) *The topological cylinder passing through $y = j\varepsilon, z = 0$ is attractor (respectively, repeller) if j is even (respectively, odd), with $j \in \{1, 2, \dots k\}$.*
- (II) *For $\rho = i$*
  - (II.i) *Z_ε^i has infinitely many isolated topological cylinders when $\varepsilon \neq 0$,*
  - (II.ii) *The invariant cylinder passing through $y = \varepsilon^2/j, z = 0$ is attractor (respectively, repeller) if j is even (respectively, odd).*

Proof. According to Lemma 12, $y = \varphi^2_{Z_\varepsilon^\rho}(y)$ if, and only if, $\varphi^2_{Z_\varepsilon^\rho}(y)$ is a zero of the function $\xi_\varepsilon^\rho(y)$.

Therefore when $\rho = f$, by Lemma 10, the fixed points of $\varphi^2_{Z_\varepsilon^f}$ are given by $y = j\varepsilon$ for $j = 1, 2, \dots, k$. Since an isolated fixed point of $\varphi^2_{Z_\varepsilon^f}$ corresponds to a hyperbolic invariant cylinder of Z_ε^f, items (I.i) and (I.ii) follow immediately from Lemma 10 (item (ii)), and Lemma 13.

On other hand when $\rho = i$, by Lemma 11, the fixed points of $\varphi^2_{Z_\varepsilon^i}$ are given by $y = \varepsilon^2/j$ for $j = 1, 2, 3, \dots$. Since an isolated fixed point of $\varphi^2_{Z_\varepsilon^i}$ corresponds to a hyperbolic invariant cylinder of Z_ε^i, items (II.i) and (II.ii) follow immediately from Lemma 11 and Remark 4. □

Finally, we can prove Theorem B.

Proof of Theorem B. Let $\mathcal{W} \subset \Omega$ be an arbitrary neighborhood of Z_0. According to (10), for $\varepsilon > 0$ sufficiently small we have that $Z_\varepsilon^\rho \in \mathcal{W}$. The conclusion of the proof follows from Proposition 14 just taking $\widetilde{Z} = Z_\varepsilon^\rho$. □

6. Combining the two previous perturbations – Proof of Theorem C

Now we combine the perturbations (6) and (9) of the normal form (1) given in the two previous sections in order to obtain Theorem C. In fact, it gives rise to the following PSVF

$$Z_{k\mathcal{L}}(x,y,z)=\begin{cases} X_{\mathcal{L}}(x,y,z)= & \begin{pmatrix} \Pi_{i=0}^{\mathcal{L}-1}(x-i\mu) \\ -1 \\ 2y \end{pmatrix} \quad \text{if } z\geq 0, \\ Y_k(x,y,z)= & \begin{pmatrix} 0 \\ 1 \\ 2y+\frac{\partial \xi_\varepsilon^\rho}{\partial y}(y) \end{pmatrix} \quad \text{if } z\leq 0. \end{cases} \tag{12}$$

where $i\in\{0,1,2,\dots,\mathcal{L}-1\}$, either $\rho=f$ or $\rho=i$, ξ_ε^ρ is given in the previous section and $\mu,\varepsilon\in\mathbb{R}$ are small numbers.

Proof of Theorem C. First of all note that the two perturbations considered are uncoupled.

Theorem A ensures the existence of exactly $\mathcal{L}$ $Z_{k\mathcal{L}}$-invariant planes π_i. Moreover, the Proposition 8 guarantees that these planes are repellers (resp., attractors) for i even (resp., odd).

Theorem B ensures the existence of exactly k $Z_{k\mathcal{L}}$-invariant topological cylinders. Moreover, items I.ii and II.ii of Proposition 14 guarantees that these nested cylinders are repellers (resp., attractors) for j odd (resp., even), where $j=1,2,\dots,k$.

The intersection of the $\mathcal{L}$ planes of Theorem A and the k cylinders of Theorem B, gives rise to the born of $k.\mathcal{L}$ limit cycles. Moreover, Propositions 8 and 14 ensures that these limit cycles are hyperbolic.

The stability of the limit cycle living at the intersection of the plane π_i with the cylinder j is of attractor kind when i is odd and j is even, of repeller kind when i is even and j is odd and of saddle kind otherwise. □

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