Guaranteed estimates for the length of branches of periodic orbits for equivariant Hopf bifurcation

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Abstract

Connected branches of periodic orbits originating at a Hopf bifurcation point of a differential system are considered. A computable estimate for the range of amplitudes of periodic orbits contained in the branch is provided under the assumption that the nonlinear terms satisfy a linear estimate in a ball. If the estimate is global, then the branch is unbounded. The results are formulated in an equivariant setting where the system can have multiple branches of periodic orbits characterized by different groups of symmetries. The non-local analysis is based on the equivariant degree method, which allows us to handle both generic and degenerate Hopf bifurcations. This is illustrated by examples.

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1 Introduction

The theorem of Rabinowitz [20], extending the work of Krasnosel'skii [17], provides a classical topological tool for analysis of global bifurcations. It establishes that a continuous branch of fixed points bifurcating from a trivial solution either extends to infinity or connects back to the trivial solution at another bifurcation point. The global bifurcation theorem of Alexander and Yorke [1] establishes a counterpart of this result for branches of periodic orbits bifurcating from an equilibrium via a Hopf bifurcation (see also [4, 6, 8, 15, 21]). The theorem states that these branches are either unbounded or connect to an equilibrium at another Hopf bifurcation point. The latter of the two alternatives can sometimes

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be excluded by local analysis at the equilibrium point, in which case the theorem guarantees the existence of an unbounded branch of periodic orbits.

It is important to note that in the above theorems branches of periodic orbits are considered in Fuller space, i.e. in the product of the space of periodic functions (solutions) and the space of parameters which include the bifurcation parameter α and the unknown period p of periodic orbits [13]. Thus, a branch is unbounded if it contains elements for α arbitrarily close to the boundary of the parameter interval, or contains periodic orbits of arbitrarily large amplitude (norm $\|x\|$), or orbits of arbitrarily large period p, or several of these possibilities are combined. In particular, it is a non-trivial problem to determine whether the branch is unbounded because it contains large-amplitude periodic orbits or their periods are unbounded. An example of the latter possibility is a branch connecting a Hopf bifurcation point with a homoclinic bifurcation point. In this context, *a priori* estimates of the period play an important role. However, they are typically hard to establish, and the question is further complicated by the fact that periods considered in the theorem are not necessarily the minimal period. If the norm of periodic orbits along the branch is uniformly bounded, the above topological results do not provide any estimate for this norm.

The purpose of this paper is to prove the existence of continuous branches that contain periodic orbits of all amplitudes ranging from zero to $\|x\| = R$ where we can explicitly control R. In particular, under certain conditions, this branch is unbounded and $\|x\|$ ranges from zero to infinity, while the minimal period of the orbits is unifomly bounded. We prove the existence of a non-stationary periodic solution of any given norm $\|x\| = s$ satisfying $0 < s < R$ by showing that the equivariant S^1-degree of a vector field $\mathcal{F}$ associated with the problem is non-trivial on the boundary of some domain Ω containing the periodic orbit. The proof is completed by Kuratowski's lemma which ensures in a standard way that all these solutions are embedded in a connected branch of non-stationary periodic orbits stemming from a Hopf bifurcation point. Because the periodic orbits of interest are neither small nor close to the bifurcation point, the analysis is non-local, and the domain Ω is designed in a special way. In order to compute the S^1-degree, we demand the vector field $\mathcal{F}$ to have a principal linear (with respect to x) component $a(\alpha, p)x$ on a part of the boundary of Ω, in the sense that a certain projection $\mathcal{Q}\mathcal{F}$ of the vector field satisfies $\|\mathcal{Q}\mathcal{F}(\alpha, p, x) - a(\alpha, p)x\| \leq \|a(\alpha, p)x\|$. This condition limits the size of the domain Ω and the maximal amplitude $\|x\| = R$ of the orbits that we can capture, unless the above estimate is global.

In addition, we prove the theorem in the equivariant setting where the system respects a group of spatial symmetries Γ. In other words, we consider a Γ-equivariant Hopf bifurcation. As a typical scenario, this bifurcation can give rise to multiple branches of periodic orbits characterized by different groups of spatio-temporal symmetries. In order to ensure the existence of a branch with a specific symmetry, and estimate the range of norms of its elements, we restrict the opeartor of the problem to the fixed point space of the corresponding symmetry group and apply the non-local topological construction described above to the restricted operator. As we illustrate by examples, this approach

2

allows us to handle not only generic Γ-equivariant systems but also a number of degenerate cases. One of them is the simultaneous Hopf and steady state bifurcations. In another degenerate resonance situation the crossing number is undefined because the linearization has a pair of purely imaginary roots for all the parameter values, and other roots cross the imaginary axis through this pair at the Hopf bifurcation point.

The paper is organized as follows. Section 2 contains a brief account of the S^1-degree [7], which is the main equivariant topological tool used in the proofs (see also [4] for the axiomatic approach to the S^1-degree and [4, 15] for a systematic exposition of the equivariant degree theory and its applications to symmetric Hopf bifurcation). The main result and its proof are presented in Sections 3 and 4. Section 5 contains three examples. Some notation of the latter section is explained in Appendix.

2 Preliminaries

In this section, we provide some equivariant degree background.

2.1 S^1-degree

Let G be a compact Lie group acting on a metric space X (see, for example, [4,5]). For any $x \in X$, put $G(x) = \{gx \in X \ : \ g \in G\}$ and call it the orbit of x. A set $Z \subset X$ is called G-invariant (in short, invariant) if it contains all its orbits. Given a (closed) subgroup $H \leq G$, denote by $X^H := \{x \in X \ : \ hx = x \ \forall h \in H\}$ the set of all H-fixed points of X. Assume G acts on two metric spaces X and Y. A continuous map $f : X \to Y$ is called G-equivariant if $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$. In particular, if the action of G on Y is trivial, then the equivariant map is called G-invariant. We refer to [4,5,9,10] for the equivariant topology and representation theory background frequently used in the present paper.

Let V be an orthogonal S^1-representation. Suppose that an open bounded set $\Omega \subset \mathbb{R} \oplus V$ is invariant with respect to the S^1-action, where we assume that S^1 acts trivially on $\mathbb{R}$. As it is well-known, for any $x \in \Omega$, one has $G_x = S^1$ or $G_x = \mathbb{Z}_k$, where $\mathbb{Z}_k$ stands for a cyclic group of order k.

We say that an equivariant map $f : \overline{\Omega} \to V$ is admissible if $f^{-1}\{0\} \cap \partial\Omega = \emptyset$. In this case, (f, Ω) is called an admissible pair. Similarly, a continuous map $h : [0,1] \times \overline{\Omega} \to V$ is called an admissible (equivariant) homotopy if $h(t, \cdot)$ is admissible for any $t \in [0,1]$. It is possible to axiomatically define a unique function S^1-deg which assigns to each admissible pair a finite linear combination $\sum_{k=1}^{m} n_{l_k}(\mathbb{Z}_{l_k})$, where $n_{l_k} \in \mathbb{Z}$ (cf. [4], pp. 109, 113). The following is a partial list of the axioms:

(A1) (Existence) If S^1-deg $(f, \Omega) = \sum_{k=1}^{m} n_{l_k}(\mathbb{Z}_{l_k})$ and $n_{l_k} \neq 0$ for some k, then there exists an $x \in \Omega$ such that $f(x) = 0$ and $\mathbb{Z}_{l_k} \subset G_x$.

(A2) (Homotopy) If $h : [0,1] \times \overline{\Omega} \to V$ is an admissible equivariant homotopy, then the value of S^1-deg $(h(\mu, \cdot), \Omega)$ is the same for each μ.

(A3)(Additivity) For two invariant open disjoint subsets $\Omega_1, \Omega_2 \subset \Omega$ with $f^{-1}(0) \cap \Omega \subset \Omega_1 \cup \Omega_2$, one has

$$S^1\text{-deg}\,(f, \Omega) = S^1\text{-deg}\,(f, \Omega_1) + S^1\text{-deg}\,(f, \Omega_2).$$

(A4)(Normalization) Denote by $\mathcal{V}_1$ the complex plane equipped with the S^1-action induced by complex multiplication: $\gamma z := \gamma \cdot z$, $\gamma = e^{i\theta} \in S^1$, $z \in \mathbb{C}$. Define a set Ω_0 and a map $b : \mathbb{R} \oplus \mathcal{V}_1 \to \mathcal{V}_1$ by

$$\Omega_0 = \left\{ (t,z) \in \mathbb{R} \oplus \mathcal{V}_1 \; : \; |t| < 1, \; 1/2 < \|z\| < 2 \right\}, \quad b(t,z) = (1 - \|z\| + it) \cdot z.$$

Then, $S^1\text{-deg}\,(b, \Omega_0) = 1 \cdot (\mathbb{Z}_1)$.

(A5)(Multiplicativity) Suppose that $\mathcal{U}$ is a finite-dimensional space with the trivial S^1-representation, U is an open bounded neighborhood of zero in $\mathcal{U}$ and $g : U \to \mathcal{U}$ is a continuous map with no zeros on ∂U . Then,

$$S^1\text{-deg}\,(f \times g, \Omega \times U) = S^1\text{-deg}\,(f, \Omega) \cdot \deg(g, U),$$

where "deg" stands for the Brouwer degree.

(A6)(Suspension) Suppose that $\mathcal{U}$ is an orthogonal S^1-representation and U is an open bounded invariant neighborhood of zero in $\mathcal{U}$. Then,

$$S^1\text{-deg}\,(f \times \mathrm{Id}, \Omega \times U) = S^1\text{-deg}\,(f, \Omega).$$

Remark 2.1. In a standard way, using property **(A3)**, one can deduce the *excision* property of the S^1-degree. Namely, if $f^{-1}(0) \cap \Omega \subset \Omega_0$, where Ω_0 is an invariant open subset of Ω, then $S^1\text{-deg}\,(f, \Omega) = S^1\text{-deg}\,(f, \Omega_0)$.

Combining the equivariant version of the standard Leray-Schauder projection with property **(A6)**, one can define the S^1-degree for S^1-equivariant compact vector fields (see [4, 15] for details). Also, combining the axioms of the S^1-degree with some standard homotopy theory techniques, one can reduce the computation of the S^1-degree of the maps naturally associated with a system undergoing the Hopf bifurcation to the computation of the Brouwer degree. To be more precise, let V be an orthogonal S^1-representation with $V^{S^1} = \{v \in V : \; (\gamma, v) = v \;\; \forall \gamma \in S^1\} = \{0\}$. Take the isotypical decomposition

$$V = V_{k_1} \oplus V_{k_2} \cdots \oplus V_{k_s},$$

where each V_{k_j} is modeled by the k_j-th irreducible representation. Define

$$\mathcal{O} = \{(\lambda, v) \in \mathbb{C} \oplus V : \|v\| < 2 \,, 1/2 < |\lambda| < 4\}. \tag{1}$$

Now, consider a map $a : S^1 \to GL^{S^1}(V)$ and define $a_j : S^1 \to GL^{S^1}(V_{k_j})$ by the formula $a_j(\lambda) = a(\lambda)_{|V_{k_j}}$ (see, [4, p. 284]). Let $f_a : \overline{\mathcal{O}} \to \mathbb{R} \oplus V$ be an S^1-equivariant map defined by

$$f_a(\lambda, v) = \big(|\lambda|(\|v\| - 1) + \|v\| + 1, a\,(\lambda/|\lambda|)\,v\big). \tag{2}$$

4

The following formula (combined with Property (**A5**)) plays an important role in our proofs:

$$S^1\text{-deg}\,(f_a, \mathcal{O}) = \sum_{j=1}^{s} \left(\deg(\det_{\mathbb{C}} \circ\, a_j, B)\right)(\mathbb{Z}_{k_j}), \tag{3}$$

where B stands for the unit ball in $\mathbb{C}$ (cf. [4], Theorem 4.23).

2.2 Spatio-temporal symmetries of periodic functions

If Γ is a finite group and W is a Γ-representation, then for any periodic function $x : \mathbb{R} \to W$, its spatio-temporal symmetries are described by a subgroup $H < \Gamma$ and a homomorphism $\varphi : H \to S^1 = \mathbb{R}/\mathbb{Z}$. This information is encoded in the graph of the homomorphism φ which we will denote by H^φ. To be more specific, if x is a p-periodic function with symmetry group H^φ, then for each $h \in H$, one has $hx(t - \varphi(h)p) = x(t)$ for any t. Clearly, if x is a non-constant function, then H^φ is a finite group. Several twisted subgroups important for the present paper are explicitly described in the Appendix.

3 Main result

Let Γ be a finite group and $V = \mathbb{R}^N$ an orthogonal Γ-representation. Suppose $A : [\alpha_-, \alpha_+] \to L^\Gamma(V; V)$ is a continuous curve in the space of Γ-equivariant linear maps from V to V and $f : [\alpha_-, \alpha_+] \times V \to V$ is a continuous Γ-equivariant map (we assume that Γ acts trivially on $[\alpha_-, \alpha_+]$). We are interested in the existence of branches of periodic solutions with a prescribed spatio-temporal symmetry for the equation

$$\dot{x} = A(\alpha)x + f(\alpha, x), \qquad x \in V. \tag{4}$$

Further, we are interested in effective estimates of the length of these branches. To be more precise, following the standard scheme based on the normalization of the period (see, for example, [4]), instead of looking for p-periodic solutions to (4) with unknown period p, one can introduce p as an additional parameter and reduce the original problem to looking for 2π-periodic solutions. To this end, put $\beta := 2\pi/p$ and apply the change of variables

$$u(t) = x\left(\frac{pt}{2\pi}\right)$$

to obtain the problem

$$\begin{cases} \beta\dot{u} = A(\alpha)u + f(\alpha, u), \\ u(0) = u(2\pi). \end{cases} \tag{5}$$

Denote by $C := C(S^1; V)$ the space of continuous V-valued maps on S^1 equipped with the sup-norm. We naturally identify 2π-periodic functions with the elements of C.

Definition 3.1. *Let $\mathcal{B}$ be a set of non-constant solutions $(\alpha, \beta, x(t)) \subset [\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times C$ of problem (5) such that $[\beta_-, \beta_+] \subset (0, \infty)$. The set $\mathcal{B}$ is called a branch joining the spheres of radius r and R centered at the origin of C if $\mathcal{B}$ is a compact connected set in the space $[\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times C$ equipped with the product norm and $\mathcal{B}$ has a non-empty intersection with each of the sets $[\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times \{\|x\|_C = r\}$ and $[\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times \{\|x\|_C = R\}$.*

Denote $G := \Gamma \times S^1$. To connect symmetric properties of the branches with equivariant spectral properties of $A(\alpha)$, denote by $\widetilde{V} = \mathbb{C}^N$ the complexification of the representation V and extend the complex Γ-representation $\widetilde{V}$ to a real G-representation $^l\widetilde{V}$ by defining the l-folded action of S^1 by $e^{i\theta} \cdot v := e^{il\theta}v$. The following family of finite-dimensional maps $\Delta_l(\alpha, \tau, \beta) \in L^G({}^l\widetilde{V}; {}^l\widetilde{V})$ will play an important role in our considerations (here $L^G({}^l\widetilde{V}; {}^l\widetilde{V})$ stands for G-equivariant linear operators in $^l\widetilde{V}$):

$$\Delta_l(\alpha, \tau, \beta) := l(\tau + i\beta)\mathrm{Id} - A(\alpha). \tag{6}$$

Further, take a twisted subgroup $H^\varphi < G$ (cf. Subsection 2.2) and denote by $^l\widetilde{V}^{H^\varphi}$ the fixed point space of H^φ and by $\Delta_l^{H^\varphi}(\alpha, \tau, \beta)$ the restriction of $\Delta_l(\alpha, \tau, \beta)$ to $^l\widetilde{V}^{H^\varphi}$. With this restriction, we associate the map $\Lambda_l^{H^\varphi} : [\alpha_-, \alpha_+] \times \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ defined by

$$\Lambda_l^{H^\varphi}(\alpha, \tau, \beta) := \begin{cases} \det_{\mathbb{R}}(A(\alpha)|_{V^H}) & \text{if} \quad l = 0, \\ \det_{\mathbb{C}}(\Delta_l^{H^\varphi}(\alpha, \tau, \beta)) & \text{otherwise,} \end{cases} \tag{7}$$

which characterizes symmetric properties of branches of periodic solutions. For a fixed α, the map $\Lambda_l^{H^\varphi}$ can be identified with a polynomial of the complex variable $\tau + i\beta$.

To estimate the length of a branch, the following quantity (which is the operator norm of an operator acting from L_2 to C associated with the periodic problem) will be used:

$$M^{H^\varphi}(\alpha, \beta) := \Big(\sum_{l=0}^{\infty} \Big| (\Delta_l^{H^\varphi}(\alpha, 0, \beta))^{-1} \Big|^2 \Big)^{1/2}, \tag{8}$$

where $| \cdot |$ is the matrix norm induced by the norm in V.

Given a set $\mathcal{P} \subset [\alpha_-, \alpha_+] \times \mathbb{R}_+ \times \mathbb{R}_+$, define

$$\mathcal{P}_\pm := \mathcal{P} \bigcap (\{\alpha_\pm\} \times \mathbb{R}_+ \times \mathbb{R}_+) \quad \text{and} \quad \mathcal{P}_0 := \mathcal{P} \bigcap ([\alpha_-, \alpha_+] \times \{0\} \times \mathbb{R}_+),$$

where $\mathbb{R}_+$ denotes the non-negative semi-axis. We denote by $\partial\Omega$ the boundary of a domain Ω and by $\overline{\Omega}$ the closure of Ω.

We make the following assumptions:

(P0) A and f in (5) depend continuously on their arguments.

(P1) $\Lambda_0^{H^\varphi}(\alpha, 0, 0) \neq 0$ for all $\alpha \in [\alpha_-, \alpha_+]$.

(P2) There exists a bounded domain $\mathcal{P} \subset [\alpha_-, \alpha_+] \times \mathbb{R}_+ \times \mathbb{R}_+$ such that:

(i) $\mathcal{P}$ is homeomorphic to a closed ball;

(ii) $\Lambda_1^{H^\varphi}(\alpha, \tau, \beta) \neq 0$ for all $(\alpha, \tau, \beta) \in \partial\mathcal{P} \setminus (\mathcal{P}_+ \bigcup \mathcal{P}_- \bigcup \mathcal{P}_0)$;

(iii) $\mathcal{P}_+$ and $\mathcal{P}_-$ contain a different number of roots of $\Lambda_1^{H^\varphi}(\alpha, \tau, \beta)$ (counted according to their multiplicities).

(P3) There exists an open set $\mathcal{D} \subset [\alpha_-, \alpha_+] \times \{0\} \times \mathbb{R}_+$ such that

(i) $\overline{\mathcal{D}}$ is homeomorphic to a closed disk;

(ii) $(\Lambda_1^{H^\varphi})^{-1}(0) \bigcap \overline{\mathcal{D}} = (\Lambda_1^{H^\varphi})^{-1}(0) \cap \mathcal{P}_0$;

(iii) $\Lambda_l^{H^\varphi}(\alpha, 0, \beta) \neq 0$ for any $l \in \mathbb{N}$ and any $(\alpha, 0, \beta) \in \partial\mathcal{D}$.

(P4) There exist $N(\alpha)$ and $0 \leq r < R$ such that for each $\alpha \in [\alpha_-, \alpha_+]$,

$$|f(\alpha, x)| \leq N(\alpha) \max\{r, |x|\} \quad \text{for} \quad |x| \leq R.$$

(P5) The following estimate holds:

$$N(\alpha) < \frac{1}{\sqrt{2\pi} M^{H^\varphi}(\alpha, \beta)} \quad \text{for} \quad (\alpha, 0, \beta) \in \partial\mathcal{D}.$$

Remark 3.2. Condition **(P0)** is a mild regularity requirement. Condition **(P1)** guarantees the absence of steady state bifurcation. Assumption **(P2)**(iii) provides the non-triviality of the (isotypical) crossing number, while **(P3)**(iii) is a weak version of the non-resonance condition. Assumptions **(P4), (P5)** ensure that the vector field associated with the problem has a principal linear part on the set difference of balls of radii R and r.

It was observed by J. Ize [14] that the occurrence of the Hopf bifurcation with prescribed symmetry is related to the non-triviality of the equivariant J-homomorphism associated with the equivariant operator equation. This observation gives rise to the following two questions: (a) Under which conditions on the right-hand side of (4) is the J-homomorphism correctly defined? (b) Under which conditions is this homomorphism non-trivial? From this viewpoint, conditions **(P0)**, **(P1)** and **(P3)** are related to (a), while condition **(P2)** is related to (b).

We are now in a position to formulate the main result of the present paper.

Theorem 3.3. *Suppose conditions **(P0)**–**(P5)** are satisfied. Then, there exists a branch of non-constant periodic solutions to system* (4) *joining the sphere of radius r to the sphere of radius R (cf. Definition 3.1). Solutions of this branch have spatio-temporal symmetry at least H^φ.*

Remark 3.4. It will be shown in the proof that the minimal period of the periodic solutions of the branch is uniformly bounded. If $r = 0$ in **(P4)**, then the branch connects to a Hopf bifurcation point. If $R = \infty$ (the estimate in **(P4)** is global), then the branch extends to infinity. A non-equivariant variant of the theorem was proved in [16] for a class of scalar equations in which the nonlinearity satisfies a global sector estimate.

In [4], the Γ-equivariant Hopf bifurcation was studied using an invariant known as the $\Gamma \times S^1$-equivariant twisted degree. Its values are finite linear combinations of the form

$$\sum_i n_{\varphi_i}(H^{\varphi_i}), \tag{9}$$

where $n_{\varphi_i} \in \mathbb{Z}$ and (H^{φ_i}) is a twisted orbit type. Generically, the coefficients n_{φ_i} give an algebraic count of orbits of type (H^{φ_i}) and as such, completely describe the $\Gamma \times S^1$-equivariant J-homomorphism of an operator involved.

In this paper, we do not compute this total invariant. Instead, we just compute the S^1-equivariant twisted degree (in short, S^1-degree) of the associate operator restricted to H^φ-fixed point space (essentially, we show the non-triviality of the corresponding S^1-equivariant J-homomorphism). The advantage of this approach is that in a number of circumstances, which we illustrate by examples, the total $\Gamma \times S^1$-twisted degree is not defined, however we still succeed to detect branches with various symmetric properties.

The usage of the total $\Gamma \times S^1$-equivariant twisted degree is effective for studying *global* behavior of branches of periodic solutions, and in the case when it is defined, each coefficient n_{φ_i} (see (9)) can be recovered by considering the usual crossing numbers related to the restrictions of the operator involved to H^{φ_j}-fixed point subspaces with $(H^{\varphi_j}) > (H^{\varphi_i})$ and applying the so-called Recurrence Formula (see [4], p. 124 and Theorem 4.25).

Remark 3.5. To verify condition **(P2)**(iii), one has to compute multiplicities of the roots of $\Lambda_1^{H^\varphi}(\alpha_\pm, \tau, \beta)$ (cf. (7)). To this end, decompose $^1\widetilde{V}$ into its $\Gamma \times S^1$-isotypical components

$$^1\widetilde{V} = {}^1\widetilde{V}_1 \oplus \cdots \oplus {}^1\widetilde{V}_q,$$

where each $^1\widetilde{V}_k$ is modeled on the irreducible $\Gamma \times S^1$-representation $^1\widetilde{\mathcal{V}}_k$ ($k = 1, ..., q$). Fix α and assume that $\lambda_o = \tau_o + i\beta_o$ is a root of $\Delta_1^{H^\varphi}(\alpha, \tau, \beta)$. Denote by $E(\lambda_o)$ the (generalized) eigenspace of λ_o with respect to $\Delta_1(\alpha, \tau, \beta)$ and let $\mathfrak{m}_k(\lambda) := \dim(E(\lambda_o) \cap {}^1\widetilde{V}_k)/\dim {}^1\widetilde{V}_k$ stand for the $^1\widetilde{\mathcal{V}}_k$-isotypical multiplicity of λ_o, $k = 1, ..., q$ (cf. [4]). Put $d_k^{H^\varphi} := \dim {}^1\mathcal{V}_k^{H^\varphi}$. Then, the multiplicity of λ_o considered as a root of $\Lambda_1^{H^\varphi}(\alpha, \tau, \beta)$ is given by

$$\sum_{k=1}^q d_k^{H^\varphi} \mathfrak{m}_k(\lambda).$$

4 Proof of Theorem 3.3

4.1 Operator reformulation

We consider the space $C^1 = C^1(S^1; V)$ equipped with the standard norm $\|u\|_{C^1} := \{\sup |u(x)| + \sup |\dot{u}(x)| : x \in S^1\}$. Recall that S^1 is identified with the segment $[0, 2\pi]$ and the spaces C, C^1 are identified with the spaces of 2π-periodic functions.

Define the differention operator $L = \frac{d}{dt} : C^1 \to C$ and the projector $K : C^1 \to C$ onto the subspace of constant functions given by

$$Ku(t) = \frac{1}{2\pi} \int_0^{2\pi} u(s) \, ds.$$

We note that the operator $L + K$ maps C^1 onto C and is invertible. Its inverse operator is defined by

$$((L+K)^{-1}u)(t) = \int_0^{2\pi} H(t-s)u(s) \, ds,$$

where

$$H(\tau) = \frac{1}{2\pi}(1 + \pi - \tau), \quad 0 \leq \tau < 2\pi; \qquad H(\tau + 2\pi) = H(\tau), \quad \tau \in \mathbb{R},$$

is the impulse response function of the linear periodic problem

$$\dot{v} + \frac{1}{2\pi} \int_0^{2\pi} v(s) \, ds = u, \qquad u(0) = u(2\pi).$$

In other words, the bounded operator $(L + K)^{-1} : C \to C^1$ is the solution operator of this problem, i.e. $v = (L + K)^{-1}u$.

Rewriting (5) as an equivalent equation

$$\dot{u} + \frac{1}{2\pi} \int_0^{2\pi} u(s) \, ds = \beta^{-1}A(\alpha)u + \beta^{-1}f(\alpha, u) + \frac{1}{2\pi} \int_0^{2\pi} u(s) \, ds$$

with the 2π-periodic boundary conditions, we see that the periodic problem for (4) is equivalent to the fixed point problem

$$u = \mathcal{J}(L+K)^{-1}\left(\beta^{-1}A(\alpha)u + Ku + \beta^{-1}F(\alpha, u)\right) =: T(\alpha, \beta, u) \qquad (10)$$

in the space $\mathbb{R}^2 \oplus C$, where $\mathcal{J}$ is the compact embedding operator from C^1 to C and $F : \mathbb{R} \oplus C \to C$ is given by $F(\alpha, u)(t) := f(\alpha, u(t))$. Also, by condition (P0) and compactness of $\mathcal{J}$, the vector field $\mathrm{Id} - T$ is compact. In addition, formula
$$(g, e^{i\theta})u(t) := gu(t - \theta), \qquad (g, e^{i\theta}) \in \Gamma \times S^1 = G,$$

defines *isometric* Banach G-representations on C and C^1 and $\mathrm{Id} - T : \mathbb{R}^2 \oplus C \to C$ is G-equivariant (we assume that G acts trivially on $\mathbb{R}^2$). In what follows, for any $s \in (r, R)$, we are going to prove the existence of a solution (α, β, u) to (10) such that $\|u\|_C = s$ and $G_u = H^\varphi$. Due to G-equivariance, this is equivalent to studying the solution set of the equation

$$\mathfrak{F}_s(\alpha, \beta, u) := \left(\|u\|_C - s, u - T(\alpha, \beta, u) \right) = 0, \qquad (11)$$

where $(\alpha, \beta) \in \mathbb{R}^2$, $u \in C^{H^\varphi}$, $s \in (r, R)$.

9

4.2 Auxiliary lemmas

It is easy to see that the subspace $C^{H^\varphi} \subset C$ is an isometric S^1-representation. Therefore, solutions to (11) will be studied in the subset $[\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times C^{H^\varphi} \subset \mathbb{R}^2 \oplus C$ using the S^1-degree theory (see Subsection 2.1). As it is common for the application of any (equivariant) degree based methods, our approach includes the following steps:

(a) Construction of an open bounded S^1-invariant domain $\Omega \subset [\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times C^{H^\varphi}$ such that related fields and homotopies are Ω-admissible;

(b) Construction of an Ω-admissible S^1-equivariant deformation of $\mathfrak{F}_s$ to an associated linear field a_s;

(c) Showing that the S^1-degree of a_s is different from zero;

(d) Establishing the existence of (connected) branches of solutions for equation (11).

To simplify our notations, we identify the set $\mathcal{D} \subset [\alpha_-, \alpha_+] \times \{0\} \times \mathbb{R}_+$ with the subset of $[\alpha_-, \alpha_+] \times \mathbb{R}_+$ via $(\alpha, 0, \beta) \to (\alpha, \beta)$ (cf. condition **(P3)**) for which we use the same symbol $\mathcal{D}$. Also, put $\mathfrak{W} := C^{H^\varphi}$ and observe that $\mathfrak{W}$ admits the S^1-isotypical decomposition

$$\mathfrak{W} = \overline{\bigoplus_{l=0}^{\infty} \mathfrak{W}_l}, \tag{12}$$

where $\mathfrak{W}_0$ coincides with V^H-valued constant functions and as such, can be identified with the subspace V^H of the phase space V, while $\mathfrak{W}_l$ can be identified with $^l \widetilde{V}^{H^\varphi}$ (see Section 3).

Remark 4.1. Due to conditions **(P1)** and **(P3)**(iii), the operator $\beta L - A(\alpha) : (C^1)^{H^\varphi} \to \mathfrak{W}$ is invertible for every $(\alpha, \beta) \in \partial \mathcal{D}$. Therefore, applying the same argument as in Subsection 4.1, one can easily show that equation (11) restricted to $\partial \mathcal{D} \times \mathfrak{M}$ is equivalent to

$$\left(\|u\|_C - s, u - \mathcal{J}(\beta L - A)^{-1} F(\alpha, u) \right) = 0,$$

where $(\alpha, \beta) \in \partial \mathcal{D}$, $u \in \mathfrak{W}$, $s \in (r, R)$.

Define $\Omega \subset [\alpha_-, \alpha_+] \times [\beta_-, \beta_+] \times \mathfrak{W}$ by

$$\Omega := \mathcal{D} \times B_R(0), \tag{13}$$

where $B_R(0) := \{u \in \mathfrak{W} : \|u\|_C < R\}$ (cf. conditions **(P3)**–**(P5)**). The following statement is crucial for our considerations.

Lemma 4.2. *Assume that conditions* **(P0)**, **(P1)**, **(P3)**–**(P5)** *are satisfied and Ω is defined by* (13). *Then, for any $\mu \in [0, 1]$ and any $s \in (r, R)$, the equation*

$$\mathcal{F}_s(\alpha, \beta, \mu, u) := \left(\|u\|_C - s, u - \mu \mathcal{J}(\beta L - A)^{-1} F(\alpha, u) \right) = 0 \tag{14}$$

does not have solutions on $\partial \Omega$.

Proof. Due to the restrictions on s, equation (14) does not admit solutions on

$$\partial \mathcal{D} \times \{u \in \overline{B_0(R)} \ : \ \|u\|_C = R \text{ or } 0 \le \|u\|_C \le r\}.$$

For contradiction to the statement of the lemma, assume that (14) admits a solution on

$$\partial \mathcal{D} \times \{u \in \overline{B_R(0)} \ : \ r < \|u\|_C < R\}.$$

Denote $v := F(\alpha, u)$. With this notation, (14) implies

$$u = \mu \mathcal{J}(\beta L - A(\alpha))^{-1} v.$$

Hence,

$$\|u\|_C \le \mu \|(\beta L - A(\alpha))^{-1}\|_{L^2 \to C} \|v\|_{L_2} \le \|(\beta L - A(\alpha))^{-1}\|_{L^2 \to C} \|v\|_{L_2}. \quad (15)$$

On the other hand, according to **(P4)**, the relations $r < \|u\|_C < R$ imply $\|v\|_C = \|F(\alpha, u)\|_C \le N(\alpha)\|u\|_C$. Combining this estimate with (15) and $\|v\|_{L_2} \le \sqrt{2\pi}\|v\|_C$, we obtain

$$\|u\|_C \le \sqrt{2\pi} N(\alpha)\|(\beta L - A(\alpha))^{-1}\|_{L^2 \to C} \|u\|_C. \quad (16)$$

The quantity $\|(\beta L - A(\alpha))^{-1}\|_{L_2 \to C}$ has already been defined in (8) as $M^{H^\varphi}(\alpha, \beta)$. By **(P5)**,

$$q := \sqrt{2\pi} N(\alpha) M^{H^\varphi}(\alpha, \beta) < 1.$$

This together with (16) gives

$$\|u\|_C \le q\|u\|_C < \|u\|_C,$$

which is a contradiction. $\square$

Define the vector field

$$a_{s,\mu}(\alpha, \beta, u) = \Big(\|u\|_C - s, u - J(L+K)^{-1}\big(\beta^{-1}A(\alpha)u + Ku + \mu\beta^{-1}F(\alpha, u)\big)\Big) \quad (17)$$

for $(\alpha, \beta, u) \in \partial\Omega$. We note that each of the vector fields (14) and (17) is equivalent to the periodic problem

$$\begin{cases} \beta\dot{u} = A(\alpha)u + \mu f(\alpha, u), \\ u(0) = u(2\pi), \ \|u\|_C = s. \end{cases}$$

Therefore, as a consequence of Lemma 4.2, one has the following statement.

Corollary 4.3. *Assume that conditions* **(P0)**, **(P1)**, **(P3)**–**(P5)** *are satisfied and Ω is given by (13). Then, for any $s \in (r, R)$, the vector field $\mathfrak{F}_s$ given by (11) is Ω-admissibly homotopic to the vector field*

$$a_s(\alpha, \beta, u) := \Big(\|u\|_C - s, u - \mathcal{J}(L+K)^{-1}\big(\beta^{-1}A(\alpha)u + Ku\big)\Big). \quad (18)$$

In particular, S^1-$\deg(\mathfrak{F}_s, \Omega)$ and S^1-$\deg(a_s, \Omega)$ are correctly defined and coincide (see Subsection 2.1, property **(A2)***).*

4.3 Computation of S^1-deg (a_s, Ω)

Corollary 4.3 essentially reduces studying the solution set of equation (11) to the computation of S^1-deg (a_s, Ω). To this end, it is convenient to identify $\overline{\mathcal{D}}$ with a subset of $\mathbb{C}$ via $(\alpha, \beta) \to \lambda = \alpha + i\beta$, and using **(P3)**(i), to assume without loss of generality that $\overline{\mathcal{D}}$ is a closed disc of radius ε centered at λ_o. Put

$$a(\lambda)u := u - \mathcal{J}(L + K)^{-1}\big(\beta^{-1}A(\alpha)u + Ku\big), \qquad (19)$$

and denote by $a_l(\lambda)$ the restriction of $a(\lambda)$ to $\mathfrak{W}_l$ (see (12)). Also, put

$$\mathfrak{n}_0 := \text{sign}\,(\det(a_0(\lambda))), \quad \mathfrak{n}_l := \deg\,(\det_{\mathbb{C}}(a_l(\cdot)), \mathcal{D}), \qquad (20)$$

where "deg" stands for the usual winding number. By condition **(P1)** (resp. **(P3)**(iii)), $\mathfrak{n}_0$ is independent of $\lambda \in \overline{\mathcal{D}}$ (resp. $\mathfrak{n}_l$ is correctly defined). Observe also that by compactness of the vector field (19), only finitely many $\mathfrak{n}_l$ are different from zero.

Lemma 4.4. *Under the assumptions* **(P0)**, **(P1)**, **(P3)**–**(P5)** *and notations* (13), (18) *and* (20), *one has*

$$S^1\text{-deg}\,(a_s, \Omega) = \mathfrak{n}_0 \sum_{l=1}^{\infty} \mathfrak{n}_l(\mathbb{Z}_l) \qquad (21)$$

for every $r < s < R$.

Proof. We will use a modification of the argument given in [4]. The main strategy is to deform the vector field a_s and to modify Ω in such a way that the computational formula (3) combined with property **(A5)** of the degree (see Subsection 2.1) can be applied. The proof follows three main steps. First, we make a finite-dimensional approximation of the compact vector field a_s. We note that each subspace $\mathfrak{W}_l$ of $\mathfrak{W}$ is invariant for the compact linear map $a(\lambda)$, and so is any subspace $\mathfrak{W}^m := \bigoplus_{l=0}^{m} \mathfrak{W}_l$. We choose a sufficiently large subspace $\mathfrak{W}^m$ and fix a (closed) linear subspace $Y \subset \mathfrak{W}$ complementing $\mathfrak{W}^m$ (without loss of generality, one can assume that Y is also invariant for $a(\lambda)$). Now, we define

$$a^m(\lambda) := a(\lambda)|_{\mathfrak{W}^m} + \text{Id}|_Y, \qquad a_s^m(\lambda, u) := (\|u\|_C - s, a^m(\lambda)u).$$

Due to compactness of a_s, the linear homotopy joining a_s and a_s^m is Ω-admissible for a sufficiently large m. Put

$$\Omega^m := \Omega \cap (\mathbb{C} \oplus \mathfrak{W}^m), \qquad \tilde{a}_s^m := a_s^m|_{\overline{\Omega^m}}.$$

Using properties **(A2)** and **(A6)** (see Subsection 2.1), one obtains:

$$S^1\text{-deg}\,(a_s, \Omega) = S^1\text{-deg}\,(a_s^m, \Omega) = S^1\text{-deg}\,(\tilde{a}_s^m, \Omega^m).$$

Let $P_l : \mathfrak{W}^m \to \mathfrak{W}_l$ be a canonical equivariant projection (see, for example, [4], p. 36). Then, $\tilde{a}_s^m$ is given by

$$\tilde{a}_s^m(\lambda, u) = \Big(\|u\|_C - s, \bigoplus_{l=0}^m a_l(\lambda) P_l u \Big),$$

Since $\overline{\mathcal{D}}$ is contractible to λ_o, there exists a deformation $\mu : \overline{\mathcal{D}} \times [0,1] \to \overline{\mathcal{D}}$ such that $\mu(\lambda, 0) = \lambda$ and $\mu(\lambda, 1) \equiv \lambda_o$. Since $a_0(\lambda)$ is invertible for every $\lambda \in \overline{\mathcal{D}}$, formula

$$\hat{a}(\lambda, u, \nu) := \Big(\|u\|_C - s, a_0(\mu(\lambda, \nu)) P_0 u + \bigoplus_{l=1}^m a_l(\lambda) P_l u \Big)$$

determines an Ω^m-admissible homotopy joining $\tilde{a}_s^m$ with the vector field $\hat{a}$ defined by

$$\hat{a}(\lambda, u) := \Big(\|u\|_C - s, a_0(\lambda_o) P_0 u + \bigoplus_{l=1}^m a_l(\lambda) P_l u \Big).$$

Put

$$B_0 := \{ u \in \mathfrak{W}_0 \ : \ \|u\|_C < R \}, \quad \Omega_* := \Omega^m \cap \Big(\mathbb{C} \oplus \bigoplus_{l=1}^m W_l \Big), \quad a_* := \hat{a}|_{\Omega_*}.$$

Since

$$\hat{a} = a(\lambda_o) \times a_* : \overline{B_0 \times \Omega_*} \to \mathfrak{W}_0 \times \Big(\mathbb{R} \oplus \bigoplus_{l=1}^m \mathfrak{W}_l \Big),$$

one has (thanks to property **(A5)** of the degree, Subsection 2.1):

$$S^1\text{-deg}\,(a_s, \Omega) = n_0 \cdot S^1\text{-deg}\,(a_*, \Omega_*).$$

Finally, to compute $S^1\text{-deg}\,(a_*, \Omega_*)$, take $\xi : \Omega_* \to \mathbb{R} \oplus \bigoplus_{l=1}^m \mathfrak{W}_l$ defined by $\xi(\lambda, u) := |\lambda - \lambda_o|(\|u\|_C - r) + \|u\|_C + \varepsilon r/2$, put $\overline{a} := \bigoplus_{l=1}^m a_l(\lambda) P_l u$ and observe that the field $a_\xi := (\xi, \overline{a})$ is Ω_*-admissibly homotopic to a_*. Since

$$a_\xi^{-1}(0) = \{ (u, \lambda) \in \Omega_* \ : \ u = 0, \quad |\lambda - \lambda_0| = \varepsilon/2 \}, \tag{22}$$

one can combine property **(A2)** of the degree with its excision property (cf. Remark 2.1) to obtain:

$$S^1\text{-deg}\,(a_*, \Omega_*) = S^1\text{-deg}\,(a_\xi, \Omega_*) = S^1\text{-deg}\,(a_\xi, \Omega_1),$$

where

$$\Omega_1 := \{ (u, \lambda) \in \Omega_* \ : \ \varepsilon/4 < |\lambda - \lambda_o| < \varepsilon \}.$$

For any $(\lambda, u) \in \Omega_1$, set

$$\eta(\lambda) := \lambda_o + \frac{\varepsilon(\lambda - \lambda_o)}{2|\lambda - \lambda_o)|}$$

13

and define $b : \Omega_1 \to \mathbb{R} \oplus \mathfrak{W}^m$ by $b(\lambda, u) := (\xi, \bar{a}(\eta(\lambda))u)$. From (22) it follows that $a_\xi^{-1}(0) = b^{-1}(0)$, hence a_ξ and b are Ω_1-admissibly homotopic. To complete the proof, it remans: to take a homemorphism of Ω_1 onto $\mathcal{O}$ (see (1)), replace the above function ξ by $|\lambda|(\|v\|_C - 1) + \|v\|_C + 1$ (see (2)) and apply formula (3). $\qquad\square$

As a consequence of Lemma 4.4, we have the following statement.

Corollary 4.5. *Under the assumptions* **(P0)**–**(P5)** *and notations* (13), (18) *and* (20),
$$S^1\text{-deg}\,(a_s, \Omega) \neq 0.$$

Proof. By condition **(P2)**(i), the local Brouwer degree of $\Lambda_1^{H^\varphi} : \partial\mathcal{P} \to \mathbb{C}$ is correctly defined and equal to zero. Combining this with condition **(P2)**(ii) and excision of the local Brouwer degree yields

$$\deg(\Lambda_1^{H^\varphi}, \partial\mathcal{P}) = \deg(\Lambda_1^{H^\varphi}, \mathcal{P}_+) + \deg(\Lambda_1^{H^\varphi}, \mathcal{P}_-) + \deg(\Lambda_1^{H^\varphi}, \mathcal{P}_0) = 0. \quad (23)$$

Denote by $\mathfrak{t}_\pm$ the number of roots of $\Lambda_1^{H^\varphi}$ in $\mathcal{P}_\pm$ (counted according to their multiplicities). Obviously, $\mathfrak{t}_\pm = \pm \deg(\Lambda_1^{H^\varphi}, \mathcal{P}_\pm)$. Combining this with the $\mathbb{Z}_2$-equivariance of $\Lambda_1^{H^\varphi}$, conditions **(P2)**(iii) and **(P3)**(ii) and formula (23) yields

$$0 \neq \mathfrak{t}_- - \mathfrak{t}_+ = \deg(\Lambda_1^{H^\varphi}, \mathcal{P}_0) = 2\deg(\Lambda_1^{H^\varphi}, \mathcal{D}) = 2\mathfrak{n}_1,$$

and the result follows from (21). $\qquad\square$

4.4 Completion of the proof of Theorem 3.3

Combining Corollaries 4.3 and 4.5 with properties **(A1)** and **(A2)** of the degree (see Subsection 2.1) implies the existence of a solution to equation (11) for each $s \in (0,1)$ and then by compactness also for $s = 0, 1$.

To show that these solutions are not constant, notice that if (α_1, β_1, u_1) is a constant solution of (11) then (α_1, β_*, u_1) is a solution of (11) for any β^*. In particular, we can choose β^* such that $(\alpha_1, \beta^*) \in \partial\mathcal{D}$ which contradicts Lemma 4.2.

Finally, to show that the solution set to equation (10) contains a compact connected branch joining the spheres $\{\|u\|_C = r\}$ and $\{\|u\|_C = R\}$, one can use the standard technique ((see, for example, [4,12] for details) based on the statement following below (see [18], Theorem 3, p. 170).

Lemma 4.6 (Kuratowski)**.** *Let X be a metric space, $A, B \subset X$ two disjoint closed sets, and K a compact set in X such that $K \cap A \neq \emptyset \neq K \cap B$. If the set K does not contain a connected component K_o such that $K_o \cap A \neq \emptyset \neq K_o \cap B$, then there exist two disjoint open sets V_1 and V_2 such that $A \subset V_1$, $B \subset V_2$ and $A \cup B \cup K \subset V_1 \cup V_2$.*

5 Examples

In this section we consider applications of Theorem 3.3. In the first example we consider a system without symmetry and compute estimates of r and R. That is, we guarantee not only the existence of a branch of periodic solutions but also estimate its length. Further examples refer to a number of circumstances where standard genericity assumptions are not satisfied but Theorem 3.3 is applicable. Here estimates of the length of the branch are also possible to obtain but for convenience we consider nonlinearities with sublinear growth and only present results of the form "there exists an R such that there is a branch of solutions joining the trivial equilibrium to the sphere of radius R".

In the second example, we consider a system of coupled oscillators which undergoes a Hopf bifurcation and a steady state bifurcation simultaneously. In this case, restriction to H^φ-fixed point spaces allows us to "separate" these two bifurcations. An additional assumption that the nonlinearity of individual oscillators is odd allows us to refine these results to be more inclusive of various non-generic scenarios.

Finally, in the third example we treat the case when a pair of purely imaginary eigenvalues persists independently of the bifurcation parameter, and other eigenvalues cross through them as the parameter is varied. Again, in this case the restriction to H^φ-fixed point spaces allows us to "separate" these eigenvalues.

5.1 Example 1

Consider a single Van der Pol oscillator given by

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - x_1^2 x_2 + \alpha x_2.$$

In this case, the group H^φ is trivial,

$$A(\alpha) = \begin{bmatrix} 0 & 1 \\ -1 & \alpha \end{bmatrix}$$

and $f(\alpha, x) = (0, -x_1^2 x_2)^T$. To apply the main theorem we begin by identifying that
$$\Lambda_l(\alpha, \tau, \beta) = l^2(\tau + i\beta)^2 - \alpha l(\tau + i\beta) + 1.$$

Given any $\alpha_- < 0 < \alpha_+$, it is easy to see that if we take

$$\mathcal{P} = [\alpha_-, \alpha_+] \times [0, \tau_*] \times [0, \beta_*]$$

with sufficiently large τ_*, β_*, then conditions **(P0)-(P2)** are satisfied. The main challenge is now to construct the set $\mathcal{D}$ in such a way that condition **(P3)** is satisfied and the estimate for R can be optimized. Satisfying condition **(P3)** only requires that $(\alpha, \beta) = (0, 1) \in \mathcal{D}$, while $(0, 1/l) \notin \partial\mathcal{D}$ for $l \geq 2$. In

particular, if $(\alpha, \beta) \neq (0, 1/l)$ for $l \in \mathbb{N}$, then the number (8) is given by

$$M(\alpha, \beta) = \Big(\sum_{l=0}^{\infty} \frac{2(l^2\beta^2 + 1) + \alpha^2}{(l^2\beta^2 - 1)^2 + l^2\alpha^2\beta^2} \Big)^{1/2}.$$

Hence, if we take $\mathcal{D}$ to be any sufficiently small disk surrounding the point $(0, 1)$, then

$$\mathcal{N} := \inf_{(\alpha,\beta) \in \partial\mathcal{D}} \frac{1}{\sqrt{2\pi M(\alpha, \beta)}} > 0.$$

The form of f implies that $|f(\alpha, x)| \leq \mathcal{N}|x|$ for all $x = (x_1, x_2)^T$ with $|x| \leq \sqrt{\mathcal{N}}$. We can therefore see that conditions **(P4)**, **(P5)** are satisfied with $N(\alpha) = \mathcal{N}$, $r = 0$ and any $R < \sqrt{\mathcal{N}}$. Therefore Theorem 3.3 guarantees the existence of a branch of periodic solutions joining the zero equilibrium with the sphere $\{\|x\|_C = \sqrt{\mathcal{N}}\}$.

However we can try to maximize the number $\mathcal{N}$ by choosing an appropriate domain $\mathcal{D}$. For $\mathcal{D}$ to include the point $(0, 1)$ the boundary $\partial\mathcal{D}$ must intersect the line segment $(0, \beta)$, $\beta \in (0, 1)$. If we therefore identify the point $(0, \beta_*)$ which minimizes the function $M(0, \beta)$ along this line segment and take $\partial\mathcal{D}$ to be the level curve of $M(\alpha, \beta)$ passing through that point, then we can maximize the number $\mathcal{N}$. Noting that the function

$$M(0, \beta) = 1 + \left(\frac{\pi}{\beta} \csc \frac{\pi}{\beta} \right)^2$$

achieves its global minimum at the point $\beta_* \approx 0.699$ which is determined as a root of the equation $\tan(\pi/\beta) = \pi/\beta$, in this way we obtain that there exists a branch of periodic solutions joining the trivial equilibrium to the sphere of radius $R \approx 0.3$, see Figure 1. The same scheme can be used to construct the domain $\mathcal{D}$ satisfying the conditions of Theorem 3.3 and obtain an estimate for R for the general non-equivariant system (4) undergoing a generic Hopf bifurcation.

5.2 Example 2

Our second example illustrates the situation where the spectrum of $A(\alpha)$ contains purely imaginary eigenvalues for some value α_0 of the parameter α and, simultaneously, $\det A(\alpha_0) = 0$. This contradicts the usual "absence of the steady state bifurcation" condition. However, we overcome this by considering (7) instead of (6) with properly chosen H^φ and applying Theorem 3.3. In the following example complex eigenvalues cross the imaginary axis transversally at the bifurcation point $\alpha = \alpha_0$. We use the notations adopted in [3] and [11].

We begin with the system

$$\begin{aligned}
\dot{j}_m &= -\frac{R}{L}j_m + \frac{1}{L}u_m, \\
\dot{u}_m &= -\frac{1}{C}j_m + \frac{\alpha}{C}u_m - \frac{\sigma}{C}u_m^3
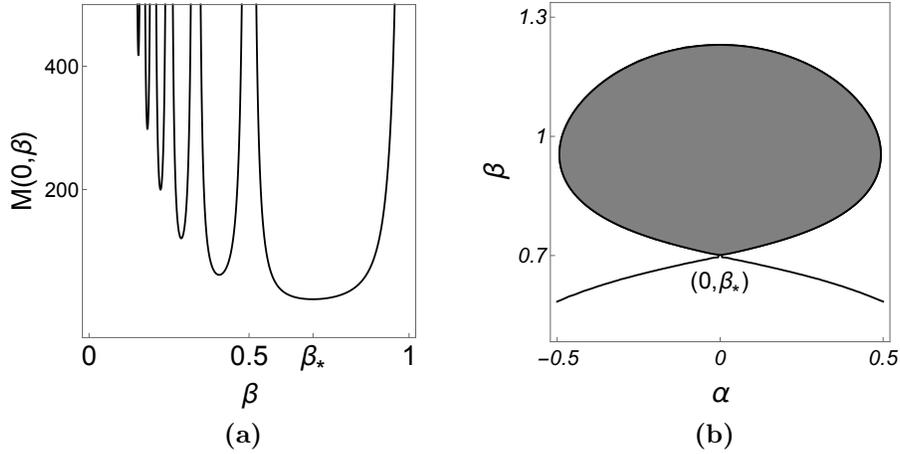\end{aligned} \tag{24}$$

Figure 1: Left: The function $M(0, \beta)$ with the global minimum at the point $\beta_* \approx 0.7$. Right: The domain $\mathcal{D}$ (gray) bounded by the level curve $M(\alpha, \beta) = M(0, \beta_*)$ of the function $M(\alpha, \beta)$.

describing an LCR circuit with a cubic current-voltage characteristic (here α is a bifurcation parameter), which can be rewritten as the classical Van der Pol equation. We further consider a symmetrically coupled system of eight identical oscillators (24), which are arranged in a cube-like configuration. More precisely, using the vector notation $j = (j_1, ..., j_8)^T$, $u = (u_1, ..., u_8)^T$ and $u^3 = (u_1^3, ..., u_8^3)^T$, one can represent the corresponding system as follows:

$$
\begin{aligned}
\dot{j} &= -\frac{R}{L}j + \frac{1}{L}u, \\
\dot{u} &= -\frac{1}{C}j + \frac{\alpha}{C}u - \frac{\sigma}{C}u^3 + \frac{\rho}{2C}\mathcal{K}u,
\end{aligned}
\tag{25}
$$

where

$$
\mathcal{K} = \begin{bmatrix}
-3 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -3 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & -3
\end{bmatrix}
\tag{26}
$$

(we assume that the oscillators are coupled by resistors having the same conductivity ρ as in [3]). Denote by $V := \mathbb{R}^{16}$ the phase space of (25). Clearly, V is an O_4-representation, where $O_4 = S_4 \times O_1$ acts by permuting pairs of coordinates (j_m, u_m), $m = 1, ..., 8$. In addition, system (24) respects the antipodal symmetry, meaning that system (25) is $\Gamma := \mathbb{Z}_2 \times O_4$-equivariant (see Appendix for the explicit description of O_1 and Γ). The Γ-representation V admits the

17

isotypical decomposition

$$V = V_0 \oplus V_1 \oplus V_2 \oplus V_3, \tag{27}$$

where each V_k, $k = 0, 1, 2, 3$, is of isotypical multiplicity two and is modeled on the irreducible Γ-representations $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ respectively, which can be described as follows. Let $\mathcal{B}^+$ and $\mathcal{B}^-$ be the one-dimensional $\mathbb{Z}_2 \times O_1$-representation where $\mathbb{Z}_2$ acts antipodally on both $\mathcal{B}^+$ and $\mathcal{B}^-$ while O_1 acts trivially on $\mathcal{B}^+$ and antipodally on $\mathcal{B}^-$. Let $\mathcal{V}_0$ (resp. $\mathcal{V}_3$) be the one-dimensional trivial (resp. sign) S_4-representation, let $\mathcal{V}_2$ be the natural three-dimensional S_4-representation, where S_4 acts as a subgroup of $SO(3)$, and let $\mathcal{V}_1 := \mathcal{V}_2 \otimes \mathcal{V}_3$. Then,

$$\mathcal{W}_0 = \mathcal{V}_0 \otimes \mathcal{B}^+, \quad \mathcal{W}_1 = \mathcal{V}_1 \otimes \mathcal{B}^-, \quad \mathcal{W}_2 = \mathcal{V}_2 \otimes \mathcal{B}^+, \quad \mathcal{W}_3 = \mathcal{V}_3 \otimes \mathcal{B}^-.$$

Denote by $A = A(\alpha)$ the linearization of the right-hand side of (25) at the origin:

$$A = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & \frac{\alpha}{C} \end{bmatrix} \otimes \mathrm{Id}_8 + \rho \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2C} \end{bmatrix} \otimes \mathcal{K}. \tag{28}$$

By choosing an appropriate basis in V respecting isotypical decomposition (27), one can show (see [3, 11]) that $A(\alpha)$ admits a block diagonal representation with 8 two-by-two blocks

$$A_k(\alpha) = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & \frac{\alpha}{C} - \frac{k\rho}{C} \end{bmatrix}, \tag{29}$$

where $A(\alpha)|_{V_k} = A_k(\alpha)$ for $k = 0, 3$ and $A(\alpha)|_{V_k} = A_k(\alpha) \oplus A_k(\alpha) \oplus A_k(\alpha)$ for $k = 1, 2$. Further, if $R^2 C < L$, then $A(\alpha)$ has purely imaginary eigenvalues when

$$\alpha = \alpha_j^h := RC/L + j\rho, \qquad j = 0, 1, 2, 3,$$

and $A(\alpha)$ is not invertible when $\alpha = \alpha_k^s = 1/R + k\rho$, $k = 0, 1, 2, 3$. Following [3], assume that $R^2 C < L$ and define

$$\mathcal{C} := \frac{1}{\rho R}\left(1 - \frac{R^2 C}{L}\right) > 0.$$

With this notation, the scenario when $\alpha_j^h = \alpha_k^s$ corresponds to

$$\mathcal{C} = j - k > 0 \quad \text{for some} \ k, j = 0, 1, 2, 3. \tag{30}$$

Therefore, if $j = 0, 1, 2, 3$ and $\mathcal{C}$ is *not* an integer satisfying $\mathcal{C} \leq j$, then the steady state bifurcation is *a priori* excluded at the point $\alpha = \alpha_j^h$ (i.e. $\alpha_j^h \neq \alpha_k^s$ because (30) is violated), which implies in a standard way that $\alpha = \alpha_j^h$ is a Hopf bifurcation point for system (25). Moreover, due to symmetries, the Hopf bifurcation points $\alpha = \alpha_1^h$, $\alpha = \alpha_2^h$ give rise to multiple branches of periodic solutions, which can be distinguished by their maximal symmetry group [3, 11]. Table 1 presents spatio-temporal symmetries of multiple branches bifurcationg from the four bifurcation points in this case.

18

An interesting case is when (30) holds for some j, k. This case can be handled by Theorem 3.3. In fact, by direct verification (see Appendix), all the twisted subgroups H^φ appearing in Table 1 have the element $(-1, (), \frac{1}{2})$. Therefore, V^H (which can be identified with H-fixed constant functions) is trivial, hence condition **(P1)** is trivially satisfied. Since all the other assumptions of Theorem 3.3 are obviously also satisfied, we conclude that Table 1 applies to system (25) in the case (30) too. In particular, if $\mathcal{C} = 1, 2, 3$, some of the Hopf bifurcations listed in Table 1 are simulatneous with the steady state bifurcation.

Remark 5.1. Let us consider system (4) with $V = \mathbb{R}^{16}$, where $A(\alpha)$ is given by (26), (28) and $f : \mathbb{R} \times V \to V$ is an arbitrary $\Gamma := \mathbb{Z}_2 \times O_4$-equivariant continuous function satisfying $f(x, \alpha)/|x| \to 0$ as $x \to 0$ for all α. Since the above argument was based on the linearization at zero, Table 1 applies to this system for every $\mathcal{C} > 0$.

Now, let us consider system (4) with the linear part defined by (26), (28) assuming that f is O_4-equivariant but not necessarily $\mathbb{Z}_2 \times O_4$-equivariant (as in Remark 5.1). For example, one can think of a system of eight coupled identical oscillators similar to (25), in which the cubic nonlinearity of an individual oscillator is replaced with a polynomial nonlinearity which is not odd. In this case, the results obtained in [3, 11] imply that if $\mathcal{C} \neq 1, 2, 3$ and $\mathcal{C} > 0$, then Table 1 should be slightly modified. Namely, each twisted subgroup $^+H^\varphi$ (resp. $^-H^\varphi$) is replaced by $^+\overline{H}^\varphi$ (resp. $^-\overline{H}^\varphi$), see Appendix for the explanation of the notation. The question, whether same table applies in the cases (30) when a Hopf bifurcation is simultaneous with a steady state bifurcation (i.e., $\mathcal{C} = 1, 2, 3$) is more subtle than for the $\mathbb{Z}_2 \times O_4$-equivariant system considered above. That is, Theorem 3.3 can still be used in some of the cases (30) but not in all of them.

To be more specific, consider, for example, the branch with symmetry $^+\overline{D}_4^d$ which can potentially bifurcate from the trivial solution at α_2^h. By definition, $^+\overline{D}_4^d$ is a graph of the homomorphism $\varphi : D_4 \times O_1 \to \mathbb{Z}_2 \subset S^1$. By direct computation (cf. (27) and (29)),

$$V^{D_4 \times O_1} = V_0 \quad \text{and} \quad A(\alpha_2^h)|_{V^{D_4 \times O_1}} = A_0(\alpha_2^h) = \begin{bmatrix} -\frac{R}{L} & \frac{1}{L} \\ -\frac{1}{C} & \frac{\alpha_2^h}{C} \end{bmatrix}. \quad (31)$$

Combining (31) with (30) implies that if $\mathcal{C} = 1$, then $\big(\ker A\big) \cap V^{D_4 \times O_1} = \{0\}$, and condition **(P1)** is satisfied. A similar argument shows that **(P1)** is also satisfied for other branches bifurcating from the point α_2^h for $\mathcal{C} = 1$. Hence, Theorem 3.3 ensures that α_2^h is a Hopf bifurcation point giving rise to multiple branches of periodic solutions with symmetries $(^+\overline{D}_4^d), (^+\overline{D}_3), (^+\overline{D}_2^d), (^+\overline{\mathbb{Z}}_4^c), (^+\overline{\mathbb{Z}}_3^t)$ for the O_4-equivariant system (4) with $\mathcal{C} = 1$ and, simultaneously, α_2^h is a steady state bifurcation point. However, if $\mathcal{C} = 2$, then condition **(P1)** is not satisfied at the point $\alpha = \alpha_2^h$.

Similarly, Theorem 3.3 guarantees the Hopf bifurcation of periodic solutions with symmetry $(^-\overline{S}_4^-)$ at the point $\alpha = \alpha_3^h$ (with a simultaneous steady state

Table 1: Symmetries of periodic solutions bifurcating from four Hopf bifurcation points of $\mathbb{Z}_2 \times O_4$-equivariant system (4) with the linear part defined by (26), (28) for any $\mathcal{C} > 0$.

Bifurcation point	Symmetry group of periodic solutions
$\alpha = \alpha_0^h$	$(^+S_4)$
$\alpha = \alpha_1^h$	$(^-D_4^z), (^-D_3^z), (^-D_2^d), (^-\mathbb{Z}_4^c), (^-\mathbb{Z}_3^t)$
$\alpha = \alpha_2^h$	$(^+D_4^d), (^+D_3), (^+D_2^d), (^+\mathbb{Z}_4^c), (^+\mathbb{Z}_3^t)$
$\alpha = \alpha_3^h$	$(^-S_4^-)$

bifurcation) for $\mathcal{C} = 2$ but not for $\mathcal{C} = 1$.

Remark 5.2. Including a ferromagnetic core in an inductor can cause a hysteretic relationship between the magnetic induction B and the magnetic field H. In this case, the instantaneous value of B depends not only on the value of H at the same moment, but also on some previous values of H. Hence, the constitutive relationship between B and H is an operator relationship, which translates into a similar operator relationship between the voltage v_m and the current i_m in an LCR contour with a ferromagnetic-core inductor. The Preisach model is a widely used description of such an operator constitutive relationship, defining the dependence of B on H in ferromagnetic materials (see, for example, [19]). The hysteresis memory is the source of *non-smoothness* and the presence of an infinite dimensional phase space without local linear structure. Hence, the application of the classical methods based on the centre manifold reduction to systems with hysteresis meets serious difficulties. However, following the scheme described in [2,3] and using Theorem 3.3, one can obtain equivariant bifurcation results for networks of LCR circuits with a hysteretic relationship between B and H, which are parallel to the results discussed above.

5.3 Example 3

In this example, we consider system $\dot{x} = A(\rho)x + f(\rho, x)$ with the linearization $A = A(\rho)$ given by (28) but this time we use ρ (the coupling strength) as the bifurcation parameter. The other parameters are fixed. In particular, we assume that $\alpha = RC/L$. In this case, the spectrum of $A(\rho)$ consists of the eigenvalues $\pm i\omega$ of multiplicity 8 for $\rho = 0$ (with $\omega = 1/\sqrt{LC}$). Let the phase space $V := \mathbb{R}^{16}$ of the system be the O_4-representation described in Example 2, and assume again that $f : \mathbb{R} \times V \to V$ is an O_4-equivariant continuous function satisfying $f(\rho, x)/|x| \to 0$ as $x \to 0$. Formula (26) implies

$$\ker \mathcal{K} = \{(x_1, ..., x_8) \ : \ x_1 = \cdots = x_8\} = (\mathbb{R}^8)^{O_4},$$

therefore $A(\rho)$ has the same pair of eigenvalues $\pm i\omega$ corresponding to the eigenspace V_0 (see (27)) for all values of the parameter ρ, while the other seven pairs of complex conjugate eigenvalues of $A(\rho)$ cross the imaginary axis transversely

through the pair $\pm i\omega$ for $\rho = 0$. This is a degenerate situation because the crossing number is not defined, however we can use Theorem 3.3.

The complexification of the phase space is an $O_4 \times S^1$-representation admitting the isotypical decomposition

$$^1\widetilde{V} = {}^1\widetilde{V}_0 \oplus {}^1\widetilde{V}_1 \oplus {}^1\widetilde{V}_2 \oplus {}^1\widetilde{V}_3,$$

where $^1\widetilde{V}_k$ is modeled on the irreducible representation $^1\widetilde{\mathcal{V}}_k$ (recall that all the O_4-isotypical components of V are modeled on irreducible representations of *real* type). In order to apply Theorem 3.3, one needs to choose maximal twisted subgroups H^φ occurring in $^1\widetilde{V}_k$ with $k = 1, 2, 3$ such that

$$^1\widetilde{\mathcal{V}}_0^{H^\varphi} = \{0\}. \tag{32}$$

It is easy to verify that with the exception of D_3, condition (32) is satisfied for all maximal twisted subgroups. Hence, Theorem 3.3 guarantees the existence of bifurcating branches of periodic solutions with symmetries $(^-\overline{D}_4^z), (^-\overline{D}_3^d),$ $(^-\overline{D}_2^d), (^-\overline{\mathbb{Z}}_4^c), (^-\overline{\mathbb{Z}}_3^t), (^+\overline{D}_4^d), (^+\overline{D}_2^d), (^+\overline{\mathbb{Z}}_4^c), (^+\overline{\mathbb{Z}}_3^t), (^-\overline{S}_4^-)$. All these branches bifurcate from the trivial solution at the bifurcation point $\rho = 0$.

6 Appendix

Given a cube with subsequent vertices $1, 2, 3, 4$ on one facet, and subsequent vertices $5, 6, 7, 8$ on the opposite facet, with the vertices 1 and 5 connected by an edge, denote by O_4 the subgroup of the symmetry group S_8 consisting of all symmetries of the above cube, and by S_4 the subgroup of O_4 consisting of all symmetries of the cube preserving its orientation. As is well-known, S_4 can be thought of as the group of permutations of the large diagonals of the cube. Denote by O_1 a subgroup of O_4 generated by the permutation $(17)(28)(35)(46)$. Clearly, O_1 is isomorphic to $\mathbb{Z}_2$ and $O_4 = S_4 \times O_1$.

Define two subgroups $(\mathbb{Z}_2 \times O_1)^o, (\mathbb{Z}_2 \times O_1)^{oz} < \mathbb{Z}_2 \times O_4 \times S^1 =: G$ (both isomorphic to $\mathbb{Z}_2 \times O_1$) by

$$(\mathbb{Z}_2 \times O_1)^o := \Big\{ \big(1, (), 0\big), \big(1, (17)(28)(35)(46), 0\big), \big(-1, (), 1/2\big),$$
$$\big(-1, (17)(28)(35)(46), 1/2\big) \Big\},$$

$$(\mathbb{Z}_2 \times O_1)^{oz} := \Big\{ \big(1, (), 0\big), \big(-1, (17)(28)(35)(46), 0\big), \big(-1, (), 1/2\big),$$
$$\big(1, (17)(28)(35)(46), 1/2\big) \Big\},$$

and two their subgroups (both isomorphic to O_1):

$$(1_{\mathbb{Z}_2} \times O_1)^o := \Big\{ \big(1, (), 0\big), \big(1, (17)(28)(35)(46), 0\big) \Big\},$$

$$(1_{\mathbb{Z}_2} \times O_1)^{oz} := \left\{ (1, (), 0), (1, (17)(28)(35)(46), 1/2) \right\}$$

(here $1_{\mathbb{Z}_2}$ stands for the neutral element in $\mathbb{Z}_2$). Then, for any $H < S_4 \times S^1 \simeq \{1\} \times S_4 \times S^1 < G$, define the subgroups ^+H, ^-H, $^+\overline{H}$, $^-\overline{H} < G$ by

$$^+H := H \cdot (\mathbb{Z}_2 \times O_1)^o, \quad {}^-H := H \cdot (\mathbb{Z}_2 \times O_1)^{oz},$$

$$^+\overline{H} := H \cdot (1_{\mathbb{Z}_2} \times O_1)^o, \quad {}^-\overline{H} := H \cdot (1_{\mathbb{Z}_2} \times O_1)^{oz}.$$

Clearly, for any $H < S_4$, one has $H \cap (\mathbb{Z}_2 \times O_1)^o = H \cap (\mathbb{Z}_2 \times O_1)^{oz} = 1_G$, where 1_G stands for the neutral element in G, therefore ^+H and ^-H are isomorphic to the direct products of their factors. All twisted subgroups of G which we deal with in Example 2 appear as either ^+H, or ^-H, or $^+\overline{H}$, or $^-\overline{H}$, where H is among the following groups:

$$\begin{aligned}
S_4 :=& \Big\{ ((), 0), ((15)(28)(37)(46), 0), ((17)(26)(35)(48), 0), ((12)(35)(46)(78), 0), \\
& ((17)(28)(34)(56), 0), ((14)(28)(35)(67), 0), ((17)(23)(46)(58), 0), \\
& ((13)(24)(57)(68), 0), ((18)(27)(36)(45), 0), ((16)(25)(38)(47), 0), ((254)(368), 0), \\
& ((245)(386), 0), ((163)(457), 0), ((136)(475), 0), ((168)(274), 0), \\
& ((186)(247), 0), ((138)(275), 0), ((183)(257), 0), ((1234)(5678), 0), ((1432)(5876), 0), \\
& ((1265)(3874), 0), ((1562)(3478), 0), ((1485)(2376), 0), ((1584)(2678), 0) \Big\}
\end{aligned}$$

$$\begin{aligned}
D_4^z :=& \Big\{ ((), 0), ((1234)(5678), 0), ((13)(24)(57)(68), 0), ((1432)(5876), 0), \\
& ((17)(26)(35)(48), 1/2), ((18)(27)(36)(45), 1/2), ((15)(28)(37)(46), 1/2), \\
& ((16)(25)(38)(47), 1/2) \Big\}
\end{aligned}$$

$$\begin{aligned}
D_3^z :=& \Big\{ ((), 0), ((254)(368), 0), ((245)(386), 0), ((17)(26)(35)(48), 1/2), \\
& ((17)(28)(34)(56), 1/2), ((17)(23)(46)(58), 1/2) \Big\}
\end{aligned}$$

$$D_2^d := \Big\{ ((), 0), ((17)(26)(35)(48), 1), ((13)(24)(57)(68), 1/2), ((15)(28)(37)(46), 1/2) \Big\}$$

$$\mathbb{Z}_4^c := \Big\{ ((), 0), ((1234)(5678), 1/4), ((13)(24)(57)(68), 1/2), ((1432)(5876), 3/4) \Big\}$$

$$\mathbb{Z}_3^t := \Big\{ ((), 0), ((254)(368), 1/3), ((245)(386), 2/3) \Big\}$$

$$\begin{aligned}
D_4^d :=& \Big\{ ((), 0), ((1234)(5678), 1/2), ((13)(24)(57)(68), 0), ((1432)(5876), 1/2), \\
& ((17)(26)(35)(48), 0), ((18)(27)(36)(45), 1/2), ((15)(28)(37)(46), 0), \\
& ((16)(25)(38)(47), 1/2) \Big\}
\end{aligned}$$

$$\begin{aligned}
D_3 :=& \{ ((), 0), ((254)(368), 0), ((245)(386), (17)(26)(35)(48), 0), \\
& ((17)(28)(34)(56), 0), ((17)(23)(46)(58), 0) \Big\}
\end{aligned}$$

$$S_4^- := \Big\{ \big((),0\big), \big((15)(28)(37)(46),1/2\big), \big((17)(26)(35)(48),1/2\big), \big((12)(35)(46)(78),1/2\big),$$
$$\big((17)(28)(34)(56),1/2\big), \big((14)(28)(35)(67)1/2\big), \big((17)(23)(46)(58),1/2\big),$$
$$\big((13)(24)(57)(68),0\big), \big((18)(27)(36)(45),0\big), \big((16)(25)(38)(47),0\big), \big((254)(368),0\big),$$
$$\big((245)(386),0\big), \big((163)(457),0\big), \big((136)(475),0\big), \big((168)(274),0\big), \big((186)(247),0\big),$$
$$\big((138)(275),0\big), \big((183)(257),0\big), \big((1234)(5678),1/2\big), \big((1432)(5876),1/2\big),$$
$$\big((1265)(3874),1/2\big), \big((1562)(3478),1/2\big), \big((1485)(2376),1/2\big), \big((1584)(2678),1/2\big) \Big\}$$

Acknowledgments

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